C^*-ALGEBRAS ASSOCIATED TO PRODUCT SYSTEMS OF HILBERT BIMODULES

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ABSTRACT. Let (G, P) be a quasi-lattice ordered group and let X be a compactly aligned product system over P of Hilbert bimodules in the sense of Fowler. Under mild hypotheses we associate to X a C^*-algebra which we call the Cuntz-Nica-Pimsner algebra of X. Our construction generalises a number of others: a sub-class of Fowler's Cuntz-Pimsner algebras for product systems of Hilbert bimodules; Katsura's formulation of Cuntz-Pimsner algebras of Hilbert bimodules; the C^*-algebras of finitely aligned higher-rank graphs; and Crisp and Laca's boundary quotients of Toeplitz algebras. We show that for a large class of product systems X, the universal representation of X in its Cuntz-Nica-Pimsner algebra is isometric.

1. INTRODUCTION

In this article we introduce and begin to analyse a class of C^*-algebras, which we call Cuntz-Nica-Pimsner algebras, associated to product systems of Hilbert bimodules. This work draws on and generalises a substantial body of previous work in a number of related areas: results of [14, 16, 18, 24, 26] on C^*-algebras associated to Hilbert bimodules; the study of C^*-algebras associated to product systems in [9, 10, 12, 15]; the theory of C^*-algebras associated to higher-rank graphs [19, 28, 30]; and the theory of Toeplitz algebras (and quotients thereof) associated to quasi-lattice ordered groups [6, 7, 22, 25]. Consequently, putting our results in context and indicating their significance requires some discussion.

1.1. C^*-algebras associated to Hilbert bimodules.

In [26], Pimsner associated to each Hilbert A–A bimodule X two C^*-algebras $\mathcal{O}_X$ and $\mathcal{T}_X$. He showed that, as the notation suggests, the C^*-algebras $\mathcal{O}_X$ generalise the Cuntz-Krieger algebras $\mathcal{O}_A$ associated to $\{0, 1\}$-matrices A in [8], and the algebras $\mathcal{T}_X$ generalise their Toeplitz extensions $\mathcal{T}\mathcal{O}_A$. According to [26], a representation of X in a C^*-algebra B is a pair (π, ψ) where $\pi : A \to B$ is a C^*-homomorphism, $\psi : X \to B$ is linear, and the pair carries the Hilbert A–A bimodule structure on X to the natural Hilbert B–B bimodule structure on B. Pimsner proved that $\mathcal{T}_X$ is universal for representations of X, and that $\mathcal{O}_X$ is universal for representations satisfying a covariance condition now known as Cuntz-Pimsner covariance, and in particular is a quotient of $\mathcal{T}_X$.

In the spirit of Coburn's Theorem for the classical Toeplitz algebra [5], Fowler and Raeburn proved a uniqueness theorem for $\mathcal{T}_X$ [16, Theorem 3.1]. A key example in

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their work, and an informative application of Pimsner's ideas, is related to graph C^*-algebras. A directed graph consists of a set E^0 of vertices, a set E^1 of edges, and maps $r, s : E^1 \to E^0$, called the range and source maps, which give the edges their direction: the edge e is directed from $s(e)$ to $r(e)$. When we discuss graphs and their C^*-algebras here, we follow the conventions of [27].

Fowler and Raeburn used their analysis of $\mathcal{T}_X$ to investigate graph algebras by associating a Hilbert $c_0(E^0)$–$c_0(E^0)$ bimodule $X(E)$ to each directed graph E. Previously [20, 21] C^*-algebras had been associated to directed graphs E which are row-finite (each vertex receives at most finitely many vertices) and have no sources (each vertex receives at least one edge). In [16], Fowler and Raeburn showed that for such graphs, $\mathcal{T}_{X(E)}$ can naturally be thought of as a Toeplitz extension of the graph C^*-algebra $C^*(E)$ of [20]. They observed that E is row-finite precisely when the homomorphism $\phi : A \to \mathcal{L}(X(E))$ which implements the left action takes values in the algebra $\mathcal{K}(X(E))$ of generalised compact operators on $X(E)$ and that E has no sources precisely when ϕ is injective. As Pimsner's theory does not require that the left action be by compact operators, Fowler and Raeburn's results suggested what is now the accepted definition of the graph C^*-algebra of a non-row-finite graph [13]. Results of Exel and Laca [11] were used to prove a version of the Cuntz-Krieger uniqueness theorem for arbitrary graph C^*-algebras in [13], and direct methods were used to extend a number of other graph C^*-algebraic results to the non-row-finite setting in [2].

Via the connection between graph C^*-algebras and Cuntz-Pimsner algebras discussed above, the uniqueness theorems of [2, 13] suggested an alternate approach to Cuntz-Pimsner algebras when ϕ is not injective. Specifically, when interpreted in terms of representations of the bimodule $X(E)$, the gauge-invariant uniqueness theorem of [2] suggested that one could weaken Pimsner's covariance condition to obtain a covariance condition for which the universal C^*-algebra satisfies the following two criteria: that the universal representation is injective on A (in Pimsner's theory, this requires that ϕ is injective); and that any representation of the universal C^*-algebra which respects the gauge action of $\mathbb{T}$ and is injective on A is faithful.

In [18], Katsura identified such a covariance condition: it is the defining relation for a relative Cuntz-Pimsner algebra (see [14, 24]) with respect to a certain ideal of A. Katsura's universal algebra $\mathcal{O}_X$ satisfies the two criteria set forth in the preceding paragraph, and under Katsura's definition, $\mathcal{O}_{X(E)} \cong C^*(E)$ for arbitrary graphs E.

1.2. **C^*-algebras associated to product systems.** Let P be a semigroup with identity e. Informally, a product system over P of Hilbert A–A bimodules is a semigroup $X = \bigsqcup_{p \in P} X_p$ such that each X_p is a right-Hilbert A–A bimodule, and $x \otimes_A y \mapsto xy$ determines an isomorphism of $X_p \otimes_A X_q$ onto X_{pq} for all $p, q \in P \setminus \{e\}$. These objects were introduced in this generality by Fowler in [12] as generalisations of the continuous product systems of Hilbert spaces introduced by Arveson in [1] and their discrete analogues introduced by Dinh in [9, 10].

In [12], Fowler considered a class of product systems X over semigroups P arising in quasi-lattice ordered groups (G, P) which he calls *compactly aligned* product systems (see (2.1)). Inspired by work of Nica [25] and of Laca and Raeburn [22] on Toeplitz algebras associated to quasi-lattice ordered groups, Fowler introduced and studied what he called Nica covariant representations of X and the associated universal C^*-algebra

$\mathcal{T}_{\mathrm{cov}}(X)$. When $P = \mathbb{N}$ Nica covariance is automatic and $\mathcal{T}_{\mathrm{cov}}(X)$ coincides with Pimsner's $\mathcal{T}_{X_1}$. Fowler's main theorem [12, Theorem 7.2] gave a spatial criterion for faithfulness of a representation of $\mathcal{T}_{\mathrm{cov}}(X)$. This theorem generalised both Laca and Raeburn's uniqueness theorem [22, Theorem 3.7] for $C^*(G, P)$ and Fowler and Raeburn's uniqueness theorem [16, Theorem 3.1] for the Toeplitz algebra of a single Hilbert bimodule.

In [12] Fowler also proposed a Cuntz-Pimsner covariance condition for representations of compactly aligned product systems and an associated universal C^*-algebra $\mathcal{O}_X$. Fowler defined a representation ψ of a product system X to be Cuntz-Pimsner covariant if its restriction to each X_p is Cuntz-Pimsner covariant in Pimsner's sense. Fowler showed that his Cuntz-Pimsner covariance condition implies Nica covariance under the hypotheses that: each pair of elements of P has a common upper bound; each $X_p = \overline{\phi_p(A)X_p}$; and each $\phi_p(A) \subset \mathcal{K}(X_p)$.

Results of [28] generalised the construction of a bimodule $X(E)$ from a graph E to the construction of a product system of bimodules $X(\Lambda)$ over $\mathbb{N}^k$ from a k-graph Λ. In particular, [28, Theorem 5.4] identifies the k-graphs Λ for which $X(\Lambda)$ is compactly aligned; such k-graphs are said to be finitely aligned. An analysis of the Toeplitz algebra $\mathcal{T}C^*(\Lambda)$ of such a finitely-aligned k-graph Λ based on Fowler's results [28] led to the formulation in [30] of a Cuntz-Krieger relation for finitely aligned k-graphs. Direct methods were used in [30] to prove versions of the standard uniqueness theorems for higher-rank graph C^*-algebras.

Just as the uniqueness theorems of [2] informed the work of Fowler, Muhly and Raeburn [14] and of Katsura [17, 18], the uniqueness theorems of [30] provide an informative model for C^*-algebras associated to compactly aligned product systems. In particular, the gauge-invariant uniqueness theorem of [30] prompts us to seek a C^*-algebra $\mathcal{NO}_X$ satisfying two criteria:

(A) the universal homomorphism j_X restricts to an injection on $X_e = A$; and
(B) any representation of $\mathcal{NO}_X$ which is faithful on $j_X(A)$ is faithful on the fixed-point algebra $\mathcal{NO}_X^\delta$ (where δ is the canonical gauge coaction of the enveloping group G of P on $\mathcal{NO}_X$).

To see the analogy of criterion (B) with the corresponding criterion given above for the Cuntz-Pimsner algebra of a single Hilbert module, consider the case where $P = \mathbb{N}^k$. Averaging over the gauge action δ of $\mathbb{T}^k$ on $\mathcal{NO}_X$ determines a faithful conditional expectation of $\mathcal{NO}_X$ onto $\mathcal{NO}_X^\delta$. A standard argument then proves that every representation which is faithful on $\mathcal{NO}_X^\delta$ and respects δ is faithful on all of $\mathcal{NO}_X$. So our criterion (B) implies that if a representation of $\mathcal{NO}_X$ restricts to an injection of $j_X(A)$ and respects δ, then it is injective.

Kumjian and Pask's results regarding row-finite k-graphs with no sources suggest that we can expect Fowler's $\mathcal{O}_X$ to satisfy (A) and (B) when the left action on each fibre is injective and by compact operators and $P = \mathbb{N}^k$. However, $\mathcal{O}_X$ will not always fit the bill. If $a \in A$ acts trivially on the left of some X_p, then Fowler's relation forces $j_X(a) = 0$. Moreover examples of [30, Appendix A], when interpreted in terms of product systems, show that if A does not act compactly on the left of every X_p, Fowler's $\mathcal{O}_X$ may not satisfy criterion (B). Indeed, the examples of [30, Appendix A] show that the same problems persist even if, in Fowler's definition of $\mathcal{O}_X$, we replace Pimsner's covariance condition with Katsura's. The situation is less clear when pairs in P need not have a common upper bound because in this case we have fewer guiding examples. We take the approach, different from Fowler's, that $\mathcal{NO}_X$ should always be a quotient of $\mathcal{T}_{\mathrm{cov}}(X)$. The situation is then clearer: the C^*-algebra which is universal for representations satisfying both Nica covariance and Fowler's Cuntz-Pimsner covariance will not satisfy (A) if there exist $p, q \in P$ with $p \vee q = \infty$.

The approach that $\mathcal{NO}_X$ should be a quotient of $\mathcal{T}_{\mathrm{cov}}(X)$ is justified by examples in isometric representation theory for quasi-lattice ordered groups. Fowler [12] showed that when each X_p is a 1-dimensional Hilbert space and multiplication in X is implemented by multiplication of complex numbers, his $\mathcal{T}_{\mathrm{cov}}(X)$ agrees with the C^*-algebra $C^*(G, P)$ universal for Nica covariant representations of P [6, 22, 25]. Crisp and Laca have recently studied what they call boundary quotients of Toeplitz algebras associated to quasi-lattice ordered groups [7]. Their results show that the relationship between their boundary quotient and $C^*(G, P)$ is often analogous to the relationship between the Cuntz algebras and their Toeplitz extensions. In particular if $G = \mathbb{F}_n$ is the free group on n generators, and $P = \mathbb{F}_n^+$ is its positive cone, then $C^*(G, P)$ is isomorphic to the Toeplitz extension $\mathcal{TO}_n$ of $\mathcal{O}_n$ and Crisp and Laca's boundary quotient is isomorphic to $\mathcal{O}_n$ itself.

More generally, [7, Theorem 6.7] shows that when G is a right-angled Artin group with trivial centre (and P is its positive cone), the associated boundary quotient is simple and purely infinite. Regarded as a statement about a C^*-algebra associated to the product system over P with 1-dimensional fibres, simplicity is equivalent to (B). Hence boundary quotients provide important motivation and a good test-case for our theory when $G \neq \mathbb{Z}^k$. In particular, the results of Section 5.4 suggest that we are on the right track for fairly general quasi-lattice ordered groups.

1.3. **Outline of the paper.** In this paper we combine ideas of [12, 14, 18] with intuition drawn from the theory of k-graph C^*-algebras [30] to associate what we call a Cuntz-Nica-Pimsner algebra $\mathcal{NO}_X$ to a broad class of compactly aligned product systems X of Hilbert bimodules. This $\mathcal{NO}_X$ is universal for a class of representations which we refer to as Cuntz-Nica-Pimsner covariant (or CNP-covariant for short). By definition these are the representations which are Nica covariant in the sense of Fowler, and also satisfy a Cuntz-Pimsner covariance relation which looks substantially different from Fowler's (but agrees with Fowler's under a number of additional hypotheses on X). Our ultimate aim is to verify that $\mathcal{NO}_X$ satisfies both (A) and (B) above.

Our main result, Theorem 4.1, shows that our C^*-algebra $\mathcal{NO}_X$ satisfies (A). In Sections 5.2, 5.3 and 5.4, we present evidence that it also satisfies (B). In Section 5.2, we prove that our construction agrees with Katsura's when $P = \mathbb{N}$, and in Section 5.3, we show that given a k-graph Λ and the associated product system $X(\Lambda)$ as constructed in [28], our $\mathcal{NO}_X$ coincides with the Cuntz-Krieger algebra $C^*(\Lambda)$ of [30]. The gauge-invariant uniqueness theorems [30, Theorem 4.2] and [18, Theorem 6.2] then imply that our definition satisfies (B). In Section 5.4 we prove that our Cuntz-Pimsner covariance relation is compatible with the defining relations for Crisp and Laca's boundary quotients [7]. In particular, Crisp and Laca's uniqueness theorem [7, Theorem 6.7] shows that $\mathcal{NO}_X$ satisfies (B) when G is a right-angled Artin group with trivial centre and X is the product system over P with 1-dimensional Hilbert spaces for fibres and multiplication implemented by multiplication of complex numbers.

We show in Section 5.1 that $\mathcal{NO}_X$ coincides with Fowler's $\mathcal{O}_X$ when [12, Proposition 5.4] suggests that it might — namely when each pair in P has a common upper bound, and each ϕ_p is injective with $\phi_p(A) \subset \mathcal{K}(X_p)$.

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2. Definitions

In this section, we recall the definitions and notation described in [12, Section 1].

2.1. **Hilbert bimodules.** We attempt to summarise only those aspects of Hilbert bimodules of direct relevance to this paper. We refer the reader to [3, 23, 31] for more detail.

Let A be a C^*-algebra, and let X be a complex vector space carrying a right action of A. Suppose that $\langle \cdot, \cdot \rangle_A : X \times X \to A$ is linear in the second variable and conjugate linear in the first variable and, for $x, y \in X$ and $a \in A$, satisfies:

(1) $\langle x, y \rangle_A = \langle y, x \rangle_A^*$;
(2) $\langle x, y \cdot a \rangle_A = \langle x, y \rangle_A a$;
(3) $\langle x, x \rangle_A$ is a positive element of A; and
(4) $\langle x, x \rangle_A = 0 \iff x = 0$.

The formula $\|x\| = \|\langle x, x \rangle_A\|^{\frac{1}{2}}$ defines a norm on X. If X is complete in this norm, we call it a *right-Hilbert A-module.*

Let X_A be a right-Hilbert A-module. A map $T : X \to X$ is said to be *adjointable* if there is a map $T^* : X \to X$ such that $\langle Tx, y \rangle_A = \langle x, T^* y \rangle_A$ for all $x, y \in X$. Every adjointable operator on X is norm-bounded and linear, and the adjoint T^* is unique. The collection $\mathcal{L}(X)$ of adjointable operators on X endowed with the operator norm is a C^*-algebra. Given $x, y \in X$, there is an adjointable operator $x \otimes y^*$ on X determined by the formula $(x \otimes y^*)(z) = x \cdot \langle y, z \rangle_A$. We call operators of this form *generalised rank-1 operators*. The subspace $\mathcal{K}(X) := \overline{\operatorname{span}}\, \{x \otimes y^* : x, y \in X\}$ is an essential ideal of $\mathcal{L}(X)$ whose elements we refer to as *generalised compact operators on X*.

A *right-Hilbert A–A bimodule* is a right-Hilbert A module together with a homomorphism $\phi : A \to \mathcal{L}(X)$. We think of ϕ as implementing a left action of A on X, so we typically write $a \cdot x$ for $\phi(a)x$. Because $\phi(a) \in \mathcal{L}(X)$ for all $a \in A$, we automatically have $a \cdot (x \cdot b) = (a \cdot x) \cdot b$ for all $a, b \in A$ and $x \in X$.

An important special case is the bimodule ${}_A A_A$ with inner product given by $\langle a, b \rangle_A = a^* b$ and right- and left-actions given by multiplication in A. The C^*-algebra $\mathcal{L}({}_A A_A)$ is isomorphic to the multiplier algebra $\mathcal{M}(A)$, and the homomorphism that takes $a \in A$ to left-multiplication by a on X is an isomorphism of A onto $\mathcal{K}({}_A A_A)$.

We form the balanced tensor product $X \otimes_A Y$ of two right-Hilbert A–A bimodules as follows. Let $X \odot Y$ be the algebraic tensor product of X and Y as complex vector spaces. Let $X \odot_A Y$ be the quotient of $X \odot Y$ by the subspace spanned by vectors of the form $x \cdot a \odot y - x \odot a \cdot y$ where $x \in X$, $y \in Y$ and $a \in A$. The formula

$$\langle x_1 \odot y_1, x_2 \odot y_2 \rangle_A = \langle y_1, \langle x_1, x_2 \rangle_A \cdot y_2 \rangle_A$$

determines a bounded sesquilinear form on $X \odot_A Y$. Let $N = \operatorname{span}\{n \in X \odot_A Y : \langle n, n \rangle_A = 0_A\}$. Then $\|z + N\| = \inf\{\|\langle z+n, z+n\rangle_A\|^{1/2} : n \in N\}$ defines a norm on $(X \odot_A Y)/N$, and $X \otimes_A Y$ is the completion of $(X \odot_A Y)/N$ in this norm.

If X and Y are right-Hilbert A–A bimodules and $S \in \mathcal{L}(X)$, then there is an adjointable operator $S \otimes 1_Y$ (with adjoint $S^* \otimes 1_Y$) on $X \otimes_A Y$ determined by $(S \otimes 1_Y)(x \otimes_A y) = Sx \otimes_A y$ for all $x \in X$ and $y \in Y$. The formula $a \mapsto \phi(a) \otimes 1_Y$ therefore determines a homomorphism of A into $\mathcal{L}(X \otimes_A Y)$. The notation $X \otimes_A Y$ always refers to the right-Hilbert A–A bimodule in which the left action is implemented by this homomorphism.

2.2. **Semigroups and product systems of Hilbert bimodules.** Let P be a discrete multiplicative semigroup with identity e, and let A be a C^*-algebra. A *product system over P of right-Hilbert A–A bimodules* is a semigroup $X = \bigsqcup_{p \in P} X_p$ such that:

(1) for each $p \in P$, $X_p \subset X$ is a right-Hilbert A–A bimodule;
(2) the identity fibre X_e is equal to the bimodule ${}_AA_A$;
(3) for $p, q \in P \setminus \{e\}$ there is an isomorphism $M_{p,q} : X_p \otimes_A X_q \to X_{pq}$ satisfying $M_{p,q}(x \otimes_A y) = xy$ for all $x \in X_p$ and $y \in X_q$; and
(4) multiplication in X by elements of $X_e = A$ implements the actions of A on each X_p; that is $ax = a \cdot x$ and $xa = x \cdot a$ for all $p \in P$, $x \in X_p$ and $a \in X_e$.

For $p \in P$, we denote the homomorphism of A to $\mathcal{L}(X_p)$ which implements the left action by ϕ_p, and we denote the A-valued inner product on X_p by $\langle \cdot, \cdot \rangle_A^p$.

By (2) and (4), for $p \in P$, multiplication in X induces maps $M_{p,e} : X_p \otimes_A X_e \to X_p$ and $M_{e,p} : X_e \otimes_A X_p \to X_p$ as in (3). Each $M_{p,e}$ is automatically an isomorphism by [31, Corollary 2.7]. We do not insist that $M_{e,p}$ is an isomorphism as this is too restrictive if we want to capture Pimsner's theory (which does not require that $\overline{\phi(A)X} = X$).

Because multiplication in X is associative, we have $\phi_{pq}(a)(xy) = (\phi_p(a)x)y$ for all $x \in X_p$, $y \in X_q$ and $a \in A$.

Given $p, q \in P$ with $p \neq e$, the isomorphism $M_{p,q} : X_p \otimes_A X_q \to X_{pq}$ allows us to define a homomorphism $\iota_p^{pq} : \mathcal{L}(X_p) \to \mathcal{L}(X_{pq})$ by

$$\iota_p^{pq}(S) = M_{p,q} \circ (S \otimes 1_{X_q}) \circ M_{p,q}^{-1}.$$

We may alternatively characterise ι_p^{pq} by the formula $\iota_p^{pq}(S)(xy) = (Sx)y$ for all $x \in X_p$, $y \in X_q$ and $S \in \mathcal{L}(X_p)$. When $p = e$, we do not have $X_p \otimes X_q \cong X_{pq}$; however, since $X_p = A$, we may define ι_e^q on $\mathcal{K}(X_e) \cong A$ by $\iota_e^q(a) = \phi_q(a)$ for all $a \in A$. As a notational convenience, if $p, r \in P$ and $r \neq pq$ for any $q \in P$, we define $\iota_p^r : \mathcal{L}(X_p) \to \mathcal{L}(X_r)$ to be the zero map $\iota_p^r(S) = 0_{\mathcal{L}(X_r)}$ for all $S \in \mathcal{L}(X_p)$.

We will primarily be interested in semigroups P of the following form. Following Nica [25], we say that (G, P) is a *quasi-lattice ordered group* if: G is a discrete group and P is a subsemigroup of G; $P \cap P^{-1} = \{e\}$; and with respect to the partial order $p \leq q \iff p^{-1}q \in P$, any two elements $p, q \in G$ which have a common upper bound in P have a least upper bound $p \vee q \in P$. We write $p \vee q = \infty$ to indicate that $p, q \in G$ have no common upper bound in P, and we write $p \vee q < \infty$ otherwise.

Let (G,P) be a quasi-lattice ordered group, and let X be a product system over P of right-Hilbert A–A bimodules. We say that X is *compactly aligned* if

(2.1) for all $p,q \in P$ such that $p \vee q < \infty$, and for all $S \in \mathcal{K}(X_p)$ and $T \in \mathcal{K}(X_q)$, we have $\iota_p^{p\vee q}(S)\iota_q^{p\vee q}(T) \in \mathcal{K}(X_{p\vee q})$.

Note that this condition does not imply compactness of either $\iota_p^{p\vee q}(S)$ or $\iota_q^{p\vee q}(T)$.

2.3. **Representations of product systems and Nica covariance.** Let (G,P) be a quasi-lattice ordered group, and let X be a compactly aligned product system over P of right-Hilbert A–A bimodules. Let B be a C^*-algebra, and let ψ be a function from X to B. For $p \in P$, let $\psi_p = \psi|_{X_p}$. We call ψ a *representation of* X if

(T1) each $\psi_p : X_p \to B$ is linear, and ψ_e is a C^*-homomorphism;
(T2) $\psi_p(x)\psi_q(y) = \psi_{pq}(xy)$ for all $p,q \in P$, $x \in X_p$, and $y \in X_q$; and
(T3) $\psi_e(\langle x,y\rangle_A^p) = \psi_p(x)^*\psi_p(y)$ for all $p \in P$, and $x,y \in X_p$.

Remark 2.1. Our definition agrees with Fowler's [12, Definition 2.5]: condition (4) of the definition of a product system together with (T1)–(T3) ensures that each (ψ_e,ψ_p) is representation of X_p in the sense of Pimsner. It then follows from Pimsner's results (see [26, p. 202]) that for each $p \in P$ there is a homomorphism $\psi^{(p)} : \mathcal{K}(X_p) \to B$ which satisfies $\psi^{(p)}(x \otimes y^*) = \psi_p(x)\psi_p(y)^*$ for all $x,y \in X_p$.

We say that a representation ψ of X is *Nica covariant* if

(N) For all $p,q \in P$ and all $S \in \mathcal{K}(X_p)$ and $T \in \mathcal{K}(X_q)$,

$$\psi^{(p)}(S)\psi^{(q)}(T) = \begin{cases} \psi^{(p\vee q)}\big(\iota_p^{p\vee q}(S)\iota_q^{p\vee q}(T)\big) & \text{if } p\vee q < \infty \\ 0 & \text{otherwise.} \end{cases}$$

Results of [12] show that there exist a C^*-algebra $\mathcal{T}_{\mathrm{cov}}(X)$ and a Nica covariant representation i of X in $\mathcal{T}_{\mathrm{cov}}(X)$ which are universal in the sense that:

(1) $\mathcal{T}_{\mathrm{cov}}(X)$ is generated by $\{i(x) : x \in X\}$; and
(2) if ψ is any Nica covariant representation of X on a C^*-algebra B then there is a unique homomorphism $\psi_* : \mathcal{T}_{\mathrm{cov}}(X) \to B$ such that $\psi_* \circ i = \psi$.

3. Cuntz-Nica-Pimsner covariance and the Cuntz-Nica-Pimsner algebra

In this section we present our definition of Cuntz-Pimsner covariance for a compactly aligned product system X. We then introduce the Cuntz-Nica-Pimsner algebra $\mathcal{NO}_X$.

To present our definition we begin by introducing a collection of bimodules $\widetilde{X}_p$, $p \in P$ which we associate to a product system X over P. The $\widetilde{X}_p$ do not form a product system; rather, they play a rôle similar to that played by the sets $\Lambda^{\leq m}$ for a locally convex k-graph in [29]. We first recall some standard notation for direct sums of Hilbert modules.

Let J be a set, let X_j be a right-Hilbert A–A bimodule for each $j \in J$, and let $X = \bigsqcup_{j\in J} X_j$. Let $\Gamma_c(J,X)$ be the space of all finitely-supported sections $x : J \to X$; that is $x(j) \in X_j$ for all $j \in J$. The formula $\langle x,y\rangle_A = \sum_{j\in J}\langle x(j),y(j)\rangle_A$ defines an A-valued inner-product on $\Gamma_c(J,X)$. The completion of $\Gamma_c(J,X)$ in the norm arising from this inner product is called the *direct sum* of the X_j and denoted $\bigoplus_{j\in J} X_j$. Endowed with pointwise left- and right-actions of A, $\bigoplus_{j\in J} X_j$ is itself a right-Hilbert A–A bimodule.

Let X be a right-Hilbert A module, and let I be an ideal of A. Let $X \cdot I$ denote $\{x \cdot a : x \in X, a \in I\}$. It is well-known that

$$X \cdot I = \{x \in X : \langle x, x\rangle_A \in I\} = \overline{\operatorname{span}}\,\{x \cdot a : x \in X, a \in I\}. \tag{3.1}$$

One way to see this is as follows. First, if $x \in X$ and $a \in I$, then $\langle x \cdot a, x \cdot a\rangle_A = a^*(\langle x, x\rangle_A)a \in I$. Now suppose $\langle x, x\rangle_A \in I$. Then the element $a = (\langle x, x\rangle_A)^{1/4}$ also belongs to I. Since [23, Lemma 4.4] implies that $x = w \cdot a$ for some $w \in X$, we then have $x \in X \cdot I$. We have now established the first equality in (3.1). The second follows from [31, Lemma 3.23]: though this lemma is stated for imprimitivity bimodules, the proof of the assertion we are using requires only that X is a right-Hilbert A-module.

Since I is an ideal, (3.1) implies that $X \cdot I$ is itself a right-Hilbert A–A bimodule.

Definition 3.1. Let (G, P) be a quasi-lattice ordered group, and let X be a product system over P of right-Hilbert A–A bimodules. Define $I_e = A$, and for $p \in P$ define $I_p = \bigcap_{e<r\leq p} \ker(\phi_r) \lhd A$. For $q \in P$, we define the right-Hilbert A–A bimodule $\widetilde{X}_q$ by

$$\widetilde{X}_q = \bigoplus_{p\leq q} X_p \cdot I_{p^{-1}q}.$$

We write $\widetilde{\phi}_q$ for the homomorphism from A to $\mathcal{L}(\widetilde{X}_q)$ which implements the left action of A on $\widetilde{X}_q$. That is, $(\widetilde{\phi}_q(a)x)(p) = \phi_p(a)x(p)$ for $p \leq q$.

Lemma 3.2. *Let (G, P) be a quasi-lattice ordered group, and let X be a product system over P of right-Hilbert A–A bimodules. Fix $p, q \in P$ with $p \leq q$. For $x \in X_p$, we have*

$$x \in X_p \cdot I_{p^{-1}q} \iff xy = 0 \text{ for all } e < r \leq p^{-1}q \text{ and } y \in X_r. \tag{3.2}$$

Proof. If $p = e$, then $X_p = A$, and $xy = \phi_r(x)y$ for $r \leq e^{-1}q = q$ and $y \in X_r$. Hence both sides of (3.2) reduce to $x \in I_q$.

Suppose $p \neq e$. By (3.1)

$$\begin{aligned} x \in X_p \cdot I_{p^{-1}q} &\iff \langle x, x\rangle^p_A \in I_{p^{-1}q} \\ &\iff \langle x, x\rangle^p_A \in \ker(\phi_r) \text{ for all } e < r \leq p^{-1}q. \end{aligned} \tag{3.3}$$

Remark 2.29 of [31] implies that a positive adjointable operator T on a right-Hilbert A-module X is equal to zero if and only if $\langle z, Tz\rangle_A = 0$ for all $z \in X$. Since $\langle x, x\rangle^p_A$ is a positive element of A, its image under ϕ_r is positive for any r, so (3.3) implies that

$$x \in X_p \cdot I_{p^{-1}q} \iff \langle y, \langle x, x\rangle^p_A \cdot y\rangle^r_A = 0 \text{ for all } e < r \leq p^{-1}q \text{ and } y \in X_r. \tag{3.4}$$

By definition of the inner product on the internal tensor product of Hilbert bimodules, for $e < r \leq p^{-1}q$ and $y \in X_r$, we have

$$\langle y, \langle x, x\rangle^p_A \cdot y\rangle^r_A = \langle x \otimes_A y, x \otimes_A y\rangle_A = \|x \otimes_A y\|^2.$$

Combining this with (3.4), we have

$$x \in X_p \cdot I_{p^{-1}q} \iff x \otimes_A y = 0 \text{ for all } e < r \leq p^{-1}q \text{ and all } y \in X_r.$$

Since $p \neq e$, each $M_{p,r}$ is an isomorphism, and the result follows. □

Example 3.3. Let Λ be a finitely aligned k-graph and let $X = X(\Lambda)$ be the corresponding product system over $\mathbb{N}^k$ as in [28]. Then $A = c_0(\Lambda^0)$, and for $n \in \mathbb{N}^k$, $I_n = \overline{\operatorname{span}}\,\{\delta_v : v \in \Lambda^0, v\Lambda^m = \emptyset \text{ for all } m \leq n\}$. For $m \leq n \in \mathbb{N}^k$, we therefore have

$$X_m \cdot I_{n-m} = \overline{\operatorname{span}}\,\{\delta_\mu : \mu \in \Lambda^m, s(\mu)\Lambda^{e_i} = \emptyset \text{ whenever } m_i < n_i\}.$$

In the language of [29, 30], this spanning set is familiar: $\delta_\mu \in X_m \cdot I_{n-m}$ if and only if $\mu \in \Lambda^m \cap \Lambda^{\leq n}$. That is, as a vector space, $\widetilde{X}_n = c_0(\Lambda^{\leq n})$.

We thank Sean Vittadello for pointing out a simplification of the proof of the following Lemma.

Lemma 3.4. *Let (G, P) be a quasi-lattice ordered group, let X be a compactly aligned product system over P of right-Hilbert A–A bimodules, and fix $p, q, r \in P$ such that $pr \leq q$ and $p \neq e$. Then $X_{pr} \cdot I_{(pr)^{-1}q}$ is invariant under $\iota_p^{pr}(S)$ for all $S \in \mathcal{L}(X_p)$.*

Proof. If $x \in X_p$, $y \in X_r$ and $i \in I_{(pr)^{-1}q}$, then

$$\iota_p^{pr}(S)(xy \cdot i) = (Sx)(y \cdot i) = ((Sx)y) \cdot i \in X_{pr} \cdot I_{(pr)^{-1}q}$$

by definition of ι_p^{pr}, axiom (3) for product systems, and associativity of multiplication in X. Since vectors of the form $xy \cdot i$ where $x \in X_p$, $y \in X_r$ and $i \in I_{(pr)^{-1}q}$ span a dense subspace of $X_{pr} \cdot I_{(pr)^{-1}q}$, and since $\iota_p^{pr}(S) \in \mathcal{L}(X_{pr})$ is bounded and linear, the result follows. □

Remark 3.5. Since $\iota_e^r = \phi_r$, each $X_r \cdot I_{r^{-1}q}$ is also invariant under $\iota_e^r(a)$ for all $a \in \mathcal{K}(X_e)$.

Notation 3.6. Let (G, P) be a quasi-lattice ordered group, and let X be a compactly aligned product system over P of right-Hilbert A–A bimodules. Recall that ι_p^q is the zero homomorphism from $\mathcal{L}(X_p)$ to $\mathcal{L}(X_q)$ when $p \not\leq q$. Lemma 3.4 implies that for all $p, q \in P$ with $p \neq e$ there is a homomorphism from $\mathcal{L}(X_p)$ to $\mathcal{L}(\widetilde{X}_q)$ determined by

$$S \mapsto \bigoplus_{r \leq q} \iota_p^r(S) \quad \text{for all } S \in \mathcal{L}(X_p).$$

We denote this homomorphism by $\widetilde{\iota}_p^q$; it is characterised by $(\widetilde{\iota}_p^q(S)x)(r) = \iota_p^r(S)x(r)$. As with the ι_p^q, when $p = e$, we write $\widetilde{\iota}_e^p$ for the homomorphism from $\mathcal{K}(X_e)$ to $\mathcal{L}(\widetilde{X}_q)$ obtained from $\widetilde{\phi}_p$ and the isomorphism $\mathcal{K}(X_e) \cong A$.

Remark 3.7. Note that $\widetilde{\iota}_p^q(S)$ is the zero operator on those summands $X_r \cdot I_{r^{-1}q}$ of $\widetilde{X}_q$ such that $p \not\leq r$. Thus $\widetilde{\iota}_p^q$ can alternatively be characterised by

$$\widetilde{\iota}_p^q(S) = \Big(\bigoplus_{r \leq q, p \not\leq r} 0_{\mathcal{L}(X_r \cdot I_{r^{-1}q})}\Big) \oplus \Big(\bigoplus_{p \leq r \leq q} \iota_p^r(S)|_{X_r \cdot I_{r^{-1}q}}\Big).$$

In particular, given $S \in \mathcal{L}(X_p)$ and $q \in P$ such that $p \not\leq q$, we have $\widetilde{\iota}_p^q(S) = 0_{\mathcal{L}(\widetilde{X}_q)}$.

To formulate our Cuntz-Pimsner covariance condition, we require another definition.

Definition 3.8. Let (G, P) be a quasi-lattice ordered group. We say that a predicate statement $\mathcal{P}(s)$ (where $s \in P$) is true *for large* s if: for every $q \in P$ there exists $r \in P$ such that $q \leq r$ and $\mathcal{P}(s)$ holds for all $s \geq r$.

We now present our definition of Cuntz-Pimsner covariance. We give a definition only in the situation that the homomorphisms $\widetilde{\phi}_q : A \to \mathcal{L}(\widetilde{X}_q)$ are all injective. We will see in Lemma 3.15 that this is automatically true for extensive classes of product systems. We will also see in Example 3.16 that this is a necessary assumption for our definition to satisfy criterion (A) of Section 1.2.

Definition 3.9. Let (G, P) be a quasi-lattice ordered group, and let X be a compactly aligned product system over P of right-Hilbert A–A bimodules in which the homomorphisms $\widetilde{\phi}_q$ of Definition 3.1 are all injective. Let ψ be a representation of X in a C^*-algebra B. We say that ψ is *Cuntz-Pimsner covariant* if

(CP) $\sum_{p\in F} \psi^{(p)}(T_p) = 0_B$ for every finite $F \subset P$, and every choice of generalised compact operators $\{T_p \in \mathcal{K}(X_p) : p \in F\}$ such that $\sum_{p\in F} \widetilde{\iota}_p^s(T_p) = 0$ for large s.

Remark 3.10. The idea is that as the indices $q \in P$ become arbitrarily large, the associated $\widetilde{X}_q$ approximate a notional "boundary" of the product system (though, at this point in time, we know of no formal way of making this idea precise). That is (CP) is intended to encode relations that we would expect to hold if we could make sense of the boundary of X and let all the $\mathcal{K}(X_p)$ act on it.

Our primary object of study in this paper will be the C^*-algebra which is universal for representations which are both Nica covariant and Cuntz-Pimsner covariant.

Definition 3.11. We shall call a representation which satisfies both (N) and (CP) a *Cuntz-Nica-Pimsner covariant* (*or CNP-covariant*) *representation.*

Proposition 3.12. *Let (G, P) be a quasi-lattice ordered group, and let X be a compactly aligned product system over P of right-Hilbert A–A bimodules such that the homomorphisms $\widetilde{\phi}_q$ of Definition 3.1 are all injective. Then there exist a C^*-algebra $\mathcal{NO}_X$ and a CNP-covariant representation j_X of X in $\mathcal{NO}_X$ such that:*

1. *$\mathcal{NO}_X = \overline{\operatorname{span}}\,\{j_X(x)j_X(y)^* : x, y \in X\}$; and*
2. *the pair $(\mathcal{NO}_X, j_X)$ is universal in the sense that if $\psi : X \to B$ is any other CNP-covariant representation of X, then there is a unique homomorphism $\Pi\psi : \mathcal{NO}_X \to B$ such that $\psi = \Pi\psi \circ j_X$.*

Moreover, the pair $(\mathcal{NO}_X, j_X)$ is unique up to canonical isomorphism.

Proof. Let $\mathcal{T}_{\rm cov}(X)$ be the universal C^*-algebra generated by a Nica covariant representation i_X of X as in [12]. Let $\mathcal{I} \subset \mathcal{T}_{\rm cov}(X)$ be the ideal generated by

$$\Big\{ \sum_{p\in F} (i_X)^{(p)}(T_p) : F \subset P \text{ is finite}, \quad T_p \in \mathcal{K}(X_p) \text{ for each } p \in F, \text{ and } \sum_{p\in F} \widetilde{\iota}_p^s(T_p) = 0 \text{ for large } s \Big\}.$$

Define $\mathcal{NO}_X = \mathcal{T}_{\rm cov}(X)/\mathcal{I}$, and let q_X denote the quotient map $q_X : \mathcal{T}_{\rm cov}(X) \to \mathcal{NO}_X$. Let $j_X : X \to \mathcal{NO}_X$ denote the composition $j_X = q_X \circ i_X$.

Since i_X is a Nica covariant representation of X and q_X is a homomorphism, j_X satisfies (T1)–(T3) and (N). The definition of q_X ensures that j_X satisfies (CP). Hence j_X is a Cuntz-Pimsner covariant representation of X. Statement (1) follows from the same identity for $\mathcal{T}_{\mathrm{cov}}(X)$ [12, Equation (6.1)]. If $\psi : X \to B$ is an CNP-covariant representation then it is, in particular, a Nica covariant representation of X and it follows from [12, Theorem 6.3] that there is a homomorphism $\psi_* : \mathcal{T}_{\mathrm{cov}}(X) \to B$ such that $\psi = \psi_* \circ i_X$. Since ψ is Cuntz-Pimsner covariant, we have $\mathcal{I} \subset \ker(\psi_*)$, and it follows that ψ_* descends to a homomorphism $\Pi\psi : \mathcal{NO}_X \to B$, establishing (2).

All that remains to be proved is the uniqueness claim, for which we give the following standard argument. If (C, ρ) is another such pair then (2) for $\mathcal{NO}_X$ implies that there is a homomorphism $\Pi\rho : \mathcal{NO}_X \to C$ such that $\Pi\rho(j_X(x)) = \rho(x)$ for all x. Statement (2) for C implies that there is a homomorphism $\Pi j_X : C \to \mathcal{NO}_X$ such that $\Pi j_X(\rho(x)) = j_X(x)$ for all x. Applications of (1) then show that $\Pi\rho$ and Πj_X are surjective and are mutually inverse. □

Remark 3.13. No obvious notation for the homomorphism $\Pi\psi$ of Proposition 3.12(2) occurred to us. We were loathe to re-define Fowler's notation ψ_*: our $\Pi\psi$ is induced from Fowler's ψ_* by regarding $\mathcal{NO}_X$ as a quotient of $\mathcal{T}_{\mathrm{cov}}(X)$, so employing the same notation would lead to confusion in any situation where representations of both $\mathcal{T}_{\mathrm{cov}}(X)$ and $\mathcal{NO}_X$ are discussed. We settled on $\Pi\psi$ on the basis that it might bring to mind the integrated form $\pi \times \psi$ of a representation (π, ψ) of a single bimodule.

Remark 3.14. It should be emphasised that $\mathcal{NO}_X$ may differ from Fowler's $\mathcal{O}_X$ even should our notion of Cuntz-Pimsner covariance and Fowler's coincide. The algebras $\mathcal{NO}_X$ and $\mathcal{O}_X$ will only coincide for product systems such that: the two versions of Cuntz-Pimsner covariance are equivalent; and Cuntz-Pimsner covariance implies Nica covariance.

We conclude this section by showing that the hypothesis that the $\widetilde{\phi}_q$ are all injective is automatic for broad classes of product systems; but we also show by example that there exist product systems for which this hypothesis fails.

Specifically, we shall show that the $\widetilde{\phi}_q$ are all injective whenever the ϕ_q are all injective, and also whenever the quasi-lattice ordered pair (G, P) has the property that every nonempty bounded subset of P contains a maximal element in the following sense.

(3.5) If $S \subset P$ is nonempty and there exists $q \in P$ such that $p \leq q$ for all $p \in S$, then there exists $p \in S$ such that $p \not\leq p'$ for all $p' \in S \setminus \{p\}$.

Lemma 3.15. *Let (G, P) be a quasi-lattice ordered group, and let X be a compactly aligned product system over P of right-Hilbert A–A bimodules. If each ϕ_p is injective, or if P satisfies* (3.5), *then $\widetilde{\phi}_q : A \to \mathcal{L}(\widetilde{X}_q)$ is injective for each $q \in P$.*

Proof. Suppose first that each ϕ_p is injective. Then $\widetilde{X}_p \cong X_p$ for all p and this isomorphism intertwines $\widetilde{\phi}_p$ and ϕ_p, so the $\widetilde{\phi}_p$ are injective.

Now suppose that P satisfies (3.5). Fix $a \in A \setminus \{0\}$ and $q \in P$. We must show that $\widetilde{\phi}_q(a) \neq 0$; that is, $\phi_p(a)|_{X_p \cdot I_{p^{-1}q}} \neq 0$ for some $p \leq q$. We have $\phi_e(a) \neq 0$ because $\phi_e(a)(a^*) = aa^* \neq 0$. Let $S = \{p \in P : p \leq q, \phi_p(a) \neq 0\}$. Then S is nonempty, and is bounded above by q. Since P satisfies (3.5), it follows that S contains a maximal element p. Since $p \in S$, we have $\phi_p(a) \neq 0$, so we may fix $x \in X_p$ such that $\phi_p(a)x \neq 0$.

Since p is maximal in S, if $e < r \leq p^{-1}q$ then $a \in \ker(\phi_{pr})$. So if $y \in X_r$ where $e < r \leq p^{-1}q$, then $(\phi_p(a)x)y = \phi_{pr}(a)(xy) = 0$; that is, $(\phi_p(a)x)y = 0$ for all $e < r \leq p^{-1}q$ and all $y \in X_r$. Lemma 3.2 therefore implies that $\phi_p(a)x \in X_p \cdot I_{p^{-1}q}$.

Let $\{\mu_\lambda : \lambda \in \Lambda\}$ be an approximate identity for $I_{p^{-1}q}$. By the preceding paragraph, $\{\phi_p(a)x \cdot \mu_\lambda\}_{\lambda\in\Lambda}$ is norm-convergent to $\phi_p(a)x \neq 0$, so there exists $\lambda \in \Lambda$ such that $\phi_p(a)x \cdot \mu_\lambda \neq 0$. Setting $y = x \cdot \mu_\lambda$, we have $y \in X_p \cdot I_{p^{-1}q}$ and $\phi_p(a)y \neq 0$. That is $\phi_p(a)|_{X_p \cdot I_{p^{-1}q}} \neq 0$, so $\widetilde{\phi}_q(a) \neq 0$ by definition. □

The hypotheses of Lemma 3.15 are not just an artifact of our proof. To see why, consider the following example.

Example 3.16. Let $G = \mathbb{Z} \times \mathbb{Z}$, and let $P = ((\mathbb{N} \setminus \{0\}) \times \mathbb{Z}) \cup (\{0\} \times \mathbb{N})$. Then P is a subsemigroup of G satisfying $P \cap -P = \{(0,0)\}$. The partial order $\leq$ on G defined by $m \leq n \iff n - m \in P$ is the lexicographic order on $\mathbb{Z} \times \mathbb{Z}$, and in particular (G,P) is a quasi-lattice ordered group. Note that (G,P) does not satisfy (3.5): let S denote the subset $S = \{0\} \times \mathbb{N}$. Then S is bounded above by $(1,0)$, but has no maximal element.

Consider the right-Hilbert $\mathbb{C}^2$ module $\mathbb{C}^2_{\mathbb{C}^2}$ with the usual right action and inner-product. Then $\mathcal{L}(\mathbb{C}^2_{\mathbb{C}^2}) = \mathcal{K}(\mathbb{C}^2_{\mathbb{C}^2}) \cong \mathbb{C}^2$: the element $(z_1, z_2) \in \mathbb{C}^2$ acts by point-wise left multiplication on $\mathbb{C}^2_{\mathbb{C}^2}$. Define homomorphisms $\phi_S, \phi_{P\setminus S} : \mathbb{C}^2 \to \mathcal{L}(\mathbb{C}^2_{\mathbb{C}^2})$ by $\phi_S = \mathrm{id}_{\mathbb{C}^2}$ and $\phi_{P\setminus S}(z_1, z_2) = (z_1, z_1)$. We write X_S (respectively $X_{P\setminus S}$) for $\mathbb{C}^2_{\mathbb{C}^2}$ regarded as a right-Hilbert $\mathbb{C}^2$–$\mathbb{C}^2$ bimodule with left action implemented by ϕ_S (respectively $\phi_{P\setminus S}$).

There are isomorphisms

$$\begin{aligned}
X_S \otimes_{\mathbb{C}^2} X_S &\cong X_S &&\text{determined by } (z_1,z_2) \otimes_{\mathbb{C}^2} (w_1,w_2) \mapsto (z_1w_1, z_2w_2),\\
X_S \otimes_{\mathbb{C}^2} X_{P\setminus S} &\cong X_{P\setminus S} &&\text{determined by } (z_1,z_2) \otimes_{\mathbb{C}^2} (w_1,w_2) \mapsto (z_1w_1, z_1w_2),\\
X_{P\setminus S} \otimes_{\mathbb{C}^2} X_S &\cong X_{P\setminus S} &&\text{determined by } (z_1,z_2) \otimes_{\mathbb{C}^2} (w_1,w_2) \mapsto (z_1w_1, z_2w_2), \text{ and}\\
X_{P\setminus S} \otimes_{\mathbb{C}^2} X_{P\setminus S} &\cong X_{P\setminus S} &&\text{determined by } (z_1,z_2) \otimes_{\mathbb{C}^2} (w_1,w_2) \mapsto (z_1w_1, z_1w_2);
\end{aligned}$$

to see this, one checks that each of these formulae preserves inner-products of elementary tensors.

For $p \in S$, let $X_p = X_S$, and for $p \in P \setminus S$, let $X_p = X_{P\setminus S}$. With multiplication maps defined as above, X is a product system over P. Note that for $p \in P$, $\phi_p = \phi_S$ if $p \in S$, and $\phi_p = \phi_{P\setminus S}$ if $p \notin S$. Since $\mathcal{L}(X_p) = \mathcal{K}(X_p)$ for all p, the left action of $\mathbb{C}^2$ on each fibre is by compact operators, and in particular X is compactly aligned.

We have $(0,1) \leq p$ for all $p \in P \setminus \{(0,0)\}$. Since $\ker(\phi_{(0,1)}) = \{0\}$, it follows that $I_p = \{0\}$ for $p \neq (0,0)$. Hence $\widetilde{X}_q = X_q$ and $\widetilde{\phi}_q = \phi_q$. In particular, $\widetilde{\phi}_{(1,0)} = \phi_{(1,0)} = \phi_{P\setminus S}$ is not injective.

Observe that $\widetilde{\iota}^{s}_{(0,0)}((0,1)) = 0_{\mathcal{L}(\widetilde{X}_s)}$ for large s. Hence every representation ψ of X satisfying (CP) satisfies $\psi_e((0,1)) = 0$. In particular the algebra universal for such representations does not satisfy criterion (A) of Section 1.2.

Remark 3.17. Subject to failure of the hypothesis of Lemma 3.15, Example 3.16 is as well-behaved as possible: (G,P) is countable and totally ordered, and the natural order topology is discrete; A and the X_p are all finite-dimensional, so that in particular the action on each fibre is by compact operators; and each X_p is essential as a left A-module in the sense that $\phi_p(A)X_p = X_p$.

4. Injectivity of the universal CNP-covariant representation

In this section we prove our main theorem.

Theorem 4.1. *Let (G, P) be a quasi-lattice ordered group, and let X be a compactly aligned product system over P of right-Hilbert A–A bimodules. Suppose that the homomorphisms $\widetilde{\phi}_q$ of Definition 3.1 are all injective. Then the universal CNP-covariant representation $j_X : X \to \mathcal{NO}_X$ is isometric: $\|j_X(x)\| = \|x\|$ for all $x \in X$. In particular, the conclusion holde if each ϕ_p is injective, or if P satisfies (3.5).*

We now introduce a modification, based on the $\widetilde{X}_p$, of Fowler's Fock representation.

Notation 4.2. Let (G, P) be a quasi-lattice ordered group, and let X be a product system over P of right-Hilbert A–A bimodules. As on [12, page 340], we let $F(X) = \bigoplus_{p\in P} X_p$, and call it the *Fock space* of X. We also define $\widetilde{F}(X) = \bigoplus_{p\in P} \widetilde{X}_p$, and call it the *augmented Fock space* of X.

Fowler shows [12, page 340] that for $x \in X_p$ there is an adjointable operator $l(x)$ on $F(X)$ determined by

$$(l(x)y)(q) = \begin{cases} x(y(p^{-1}q)) & \text{if } p \le q \\ 0 & \text{otherwise.} \end{cases}$$

He shows further that $l : X \to \mathcal{L}(F(X))$ is a Nica covariant representation of X. The next lemma shows that we obtain a parallel result if we replace $F(X)$ with $\widetilde{F}(X)$.

Lemma 4.3. *Let (G, P) be a quasi-lattice ordered group, and let X be a compactly aligned product system over P of right-Hilbert A–A bimodules.*

1. *Let $p, q, r \in P$, with $q \le r$, and let $x \in X_p$ and $z \in X_q \cdot I_{q^{-1}r}$. Then $xz \in X_{pq} \cdot I_{(pq)^{-1}pr}$.*
2. *For $p \in P$ and $x \in X_p$, there is an adjointable operator $\widetilde{l}(x) \in \mathcal{L}(\widetilde{F}(X))$ which satisfies*
$$(\widetilde{l}(x)y)(q) = \begin{cases} x(y(p^{-1}q)) & \textit{if } p \le q \\ 0 & \textit{otherwise} \end{cases}$$
for all $y \in \widetilde{F}(X)$. Moreover if $y \in \widetilde{F}(X)$ satisfies $y(s) = 0$ for all $s \ge p$, then $\widetilde{l}(x)^*y = 0$.
3. *The map $x \mapsto \widetilde{l}(x)$ of (2) is a Nica covariant representation of X in $\mathcal{L}(\widetilde{F}(X))$.*

Proof. To prove (1), write $z = z' \cdot i$ where $z' \in X_q$ and $i \in I_{q^{-1}r} \subset A = X_e$. By associativity of multiplication in X and condition (4) of the definition of a product system we have $xz = x(z' \cdot i) = (xz') \cdot i \in X_{pq} \cdot I_{(pq)^{-1}(pr)}$.

For (2), observe that each $\widetilde{X}_q$ is a sub-module of $F(X)$, and that the restriction of $l(x)$ to this submodule agrees with $\widetilde{l}(x)$. Since $l(x)$ is an adjointable operator on $F(X)$, the first part of (2) follows. For the second part, observe that by linearity and continuity of $\widetilde{l}(x)^*$, we may assume that y has just one nonzero coordinate. That is, we may assume that there exists $r \in P$ such that $r \not\ge p$ and $y(s) = 0$ for $s \ne r$. For $z \in \widetilde{F}(X)$, we then have

$$\langle \widetilde{l}(x)^*y, z\rangle_A = \langle y, \widetilde{l}(x)z\rangle_A = \sum_{s\in P} \langle y(s), (\widetilde{l}(x)z)(s)\rangle_A^{\widetilde{X}_s} = \langle y(r), (\widetilde{l}(x)z)(r)\rangle_A^{\widetilde{X}_r} = 0$$

by the first part of (2).

For (3), recall that l satisfies (T1)–(T3) and (N). It follows that $\widetilde{l}$ does as well. $\square$

Let B be the C^*-algebra generated by $\{\widetilde{l}(x) : x \in X\} \subset \mathcal{L}(\widetilde{F}(X))$. Let $\mathcal{I}_{\widetilde{F}(X)}$ be the ideal in B generated by

$$\Big\{ \sum_{p\in F} \widetilde{l}^{(p)}(T_p) : F \subset P \text{ is finite}, \qquad T_p \in \mathcal{K}(X_p) \text{ for each } p \in F, \text{ and } \sum_{p\in F} \widetilde{\iota}_p^s(T_p) = 0 \text{ for large } s\Big\}.$$

Let $q_{\widetilde{F}(X)} : B \to B/\mathcal{I}_{\widetilde{F}(X)}$ denote the quotient map. Then $\psi := q_{\widetilde{F}(X)} \circ \widetilde{l}$ is a CNP-covariant representation of X in $B/\mathcal{I}_{F(X)}$.

Proposition 4.4. *Let (G, P) be a quasi-lattice ordered group, and let X be a compactly aligned product system over P of right-Hilbert A–A bimodules. Suppose that each $\widetilde{\phi}_q$ is injective. Then $q_{F(X)} \circ \widetilde{l} : X \to B/\mathcal{I}_{\mathcal{F}(X)}$ is faithful on A.*

Proof. We must show that $\mathcal{I}_{\widetilde{F}(X)} \cap \widetilde{l}(A) = \{0\}$.

For $a \in A$, the restriction $\widetilde{l}(a)|_{\widetilde{X}_q}$ of $\widetilde{l}(a)$ to the q-summand of $\widetilde{F}(X)$ is given by $\widetilde{l}(a)x = \widetilde{\phi}_q(a)x$. Since $\widetilde{\phi}_q$ is injective and hence isometric, it follows that the operator norm $\|\widetilde{l}(a)|_{\widetilde{X}_q}\|$ is equal to the C^*-norm $\|a\|$ of $a \in A$. It therefore suffices to show that

(4.1) for each $\psi \in \mathcal{I}_{\widetilde{F}(X)}$ and each $\varepsilon > 0$ there exists $s \in P$ such that $\|\psi|_{\widetilde{X}_s}\| < \varepsilon$.

We do this in stages. Let $K \subset \mathcal{I}_{\widetilde{F}(X)}$ be the subset

$$K = \Big\{ \sum_{p\in F} \widetilde{l}^{(p)}(T_p) : F \subset P \text{ is finite}, \qquad T_p \in \mathcal{K}(X_p) \text{ for each } p \in F, \text{ and } \sum_{p\in F} \widetilde{\iota}_p^s(T_p) = 0 \text{ for large } s\Big\}.$$

Then K is a subspace of $\mathcal{I}_{\widetilde{F}(X)}$, and a continuity argument shows that for each $\psi \in \overline{K}$, each $\varepsilon > 0$, and each $q \in P$ there exists $r \geq q$ such that $\|\psi|_{\widetilde{X}_s}\| < \varepsilon$ for $s \geq r$.

By [12, Proposition 5.10], we have $B = \overline{\operatorname{span}}\,\{\widetilde{l}(x)\widetilde{l}(y)^* : x, y \in X\}$. Consequently,

$$\mathcal{I}_{\widetilde{F}(X)} = \overline{\operatorname{span}}\,\{\widetilde{l}(x)\widetilde{l}(y)^* k \widetilde{l}(x')\widetilde{l}(y')^* : k \in \overline{K}, x, y, x', y', \in X\}.$$

Fix $k \in \overline{K}$ and $x, y, x', y', \in X$ — say $x \in X_{p(x)}$, $y \in X_{p(y)}$, $x' \in X_{p(x')}$ and $y' \in X_{p(y')}$ — and fix $q \in P$. We claim that there exists $r \geq q$ such that $\|\widetilde{l}(x)\widetilde{l}(y)^* k \widetilde{l}(x')\widetilde{l}(y')^*|_{\widetilde{X}_s}\| < \varepsilon$ for each $s \geq r$. To see this, suppose first that $p(y') \vee q < \infty$. Then there exists $r' \geq p(x')p(y')^{-1}(p(y') \vee q)$ such that $\|k|_{\widetilde{X}_{s'}}\| < \frac{\varepsilon}{\|x\|\|y\|\|x'\|\|y'\|}$ for all $s' \geq r'$. Let $r = p(y')p(x')^{-1}r' \geq q$. Then for each $s \geq r$, we have $p(x')p(y')^{-1}s \geq r'$, and hence

$$\|\widetilde{l}(x)\widetilde{l}(y)^* k \widetilde{l}(x')\widetilde{l}(y')^*|_{\widetilde{X}_s}\| < \varepsilon.$$

Now suppose that $p(y') \vee q = \infty$. Then $s \geq q$ implies $s \not\geq p(y')$, so Lemma 4.3(2) implies that $\widetilde{l}(y')^*|_{\widetilde{X}_s} = 0$. Consequently $r = q$ satisfies $\widetilde{l}(x)\widetilde{l}(y)^* k \widetilde{l}(x')\widetilde{l}(y')^*|_{\widetilde{X}_s} = 0$ for every $s \geq r$.

The assertion (4.1) now follows by linearity and continuity. $\square$

Remark 4.5. Since each summand of $\widetilde{F}(X)$ is a sub-module of $F(X)$, it was not necessary to introduce $\widetilde{F}(X)$ at all. Essentially the argument of Proposition 4.4 is valid if we work instead with Fowler's $l : X \to F(X)$ followed by the appropriate quotient map. However using $\widetilde{F}(X)$ makes the argument clearer, and helps give some intuition for the significance of the $\widetilde{X}_p$.

Proof of Theorem 4.1. Proposition 4.4 together with he universal property of $\mathcal{NO}_X$ implies that j_X is injective on $X_e = A$. As $j_X|_{X_e}$ is a C^*-homomorphism, it follows that j_X is isometric on A. For any $x \in X$, we therefore have

$$\|x\|^2 = \|\langle x, x\rangle_A\| = \|j_X(\langle x, x\rangle_A)\| = \|j_X(x)^* j_X(x)\| = \|j_X(x)\|^2,$$

completing the proof. □

Remark 4.6. Note that $\mathbb{N}^k$, and indeed every right-angled Artin semigroup, satisfies (3.5), so Theorem 4.1 applies when P is any of these semigroups.

Remark 4.7. Theorem 4.1 shows that if each $\widetilde{\phi}_q$ is injective, and in particular if P satisfies (3.5), then $\mathcal{NO}_X$ satisfies criterion (A) of Section 1.2. In the next section, we will show that it also satisfies (B) in a number of motivating examples. We will achieve this indirectly by combining Theorem 4.1 with the uniqueness theorems of [7, 18, 30].

We have not verified (B). This is done in [4] for certain pairs (G, P) using a careful analysis of the fixed-point algebra $\mathcal{NO}_X^\delta = \overline{\operatorname{span}}\,\{j_X^{(p)}(T_p) : p \in P, T_p \in \mathcal{K}(X_p)\}$.

5. Relationships to other constructions

In this section we discuss the relationship between $\mathcal{NO}_X$ and a number of other C^*-algebras. We begin by showing that when each pair in P has a least upper bound and each ϕ_p is injective and takes values in $\mathcal{K}(X_p)$, our $\mathcal{NO}_X$ coincides with Fowler's Cuntz-Pimsner algebra $\mathcal{O}_X$ [12]. We then demonstrate that our $\mathcal{NO}_X$ is compatible with Katsura's $\mathcal{O}_X$ [17, 18], with the Cuntz-Krieger algebras of finitely aligned higher-rank graphs [30], and with Crisp and Laca's boundary quotients of Toeplitz algebras [7].

5.1. **Fowler's Cuntz-Pimsner algebras.** Kumjian and Pask's uniqueness theorems for higher-rank graph C^*-algebras [19] suggest that Fowler's notion of Cuntz-Pimsner covariance determines a universal C^*-algebra which satisfies (A) and (B) of Section 1.2 when $(G, P) = (\mathbb{Z}^k, \mathbb{N}^k)$, and each ϕ_p is injective with $\phi_p(A) \subset \mathcal{K}(X_p)$. Proposition 5.4 of [12] then suggests that we may be able to relax the requirement that $(G, P) = (\mathbb{Z}^k, \mathbb{N}^k)$ and insist only that each pair in P has a common upper bound. The next proposition shows that in this case $\mathcal{NO}_X$ and Fowler's $\mathcal{O}_X$ coincide.

Note that we assume only that each pair in P has a common upper bound, not that each pair in G has a common upper bound. Under the latter hypothesis, we deduce that each pair in G has a least common upper bound, and hence that G is in fact lattice ordered (take $g \wedge h = (g^{-1} \vee h^{-1})^{-1}$). It is not clear to us whether the assumption that each pair in P has a common upper bound implies that P is lattice-ordered, or even that G is.

Recall that a representation ψ of X is Cuntz-Pimsner covariant in the sense of [12, Definition 2.5] if, whenever $\phi_p(a) \in \mathcal{K}(X_p)$, we have $\psi^{(p)}(\phi(a)) = \psi_e(a)$.

Proposition 5.1. *Let (G, P) be a quasi-lattice ordered group, and let X be a compactly aligned product system over P of right-Hilbert A–A bimodules. Suppose that each pair in P has a least upper bound. Suppose that for each $p \in P$, the homomorphism $\phi_p : A \to \mathcal{L}(X_p)$ is injective. Let $\psi : X \to B$ be a representation of X.*

1. *If ψ satisfies (CP), then it is Cuntz-Pimsner covariant in the sense of* [12, Definition 2.5].
2. *If $\phi_p(A) \subset \mathcal{K}(X_p)$ for each $p \in P$, and ψ is Cuntz-Pimsner covariant in the sense of* [12, Definition 2.5], *then ψ satisfies (CP).*

Proof. We begin with some observations which we will use to prove both (1) and (2). Let $a \in A$. Recall that X_e is equal to A, and that $\phi_e(a) : X_e \to X_e$ is the left-multiplication operator $b \mapsto ab$, which we denote by $L_a \in \mathcal{K}(X_e)$. Every element T of $\mathcal{K}(X_e)$ can be written as $T = L_a$ for some $a \in A$ (see [31, Lemma 2.26]). By definition of ι_e^p, we have

$$\iota_e^p(L_a) = \phi_p(a) \quad \text{for all } p \in P,\, a \in A. \tag{5.1}$$

We now prove (1) and (2) separately.

For (1), let $p \in P$ and $a \in A$, and suppose that $\phi_p(a) \in \mathcal{K}(X_p)$. We must show that $\psi_e(a) - \psi^{(p)}(\phi_p(a)) = 0$. Equation (5.1) implies that $\iota_e^s(L_a) - \iota_p^s(\phi_p(a)) = 0_{\mathcal{L}(X_s)}$ for all $s \geq p$. Since each pair in P has a least upper bound, it follows that for each $q \in P$, we have $\iota_e^s(L_a) - \iota_p^s(\phi_p(a)) = 0_{\mathcal{L}(X_s)}$ for all $s \geq p \vee q$.

Since each ϕ_p is injective, we have $I_p = \{0\}$ for $p \neq e$. Hence $\widetilde{X}_p = X_p$ and $\widetilde{\iota}_p^q = \iota_p^q$ for all $p \leq q \in P$. Thus $\widetilde{\iota}_e^s(L_a) - \widetilde{\iota}_p^s(\phi_p(a)) = 0$ for large s. Since ψ satisfies (CP), we deduce that $\psi^{(e)}(L_a) - \psi^{(p)}(\phi_p(a)) = 0$. A straightforward computation using an approximate identity for A and that ψ_e is a homomorphism shows that $\psi^{(e)}(L_a) = \psi_e(a)$. Hence $\psi_e(a) = \psi^{(p)}(\phi_p(a))$, and ψ is Cuntz-Pimsner covariant in the sense of [12, Definition 2.5].

Now for (2), fix a finite subset $F \subset P$ and compact operators $T_p \in \mathcal{K}(X_p)$, $p \in F$ such that $\sum_{p \in F} \widetilde{\iota}_p^s(T_p) = 0$ for large s. An inductive argument using that every pair in P has a least upper bound shows that there is a least upper bound $q = \bigvee F$ for F in P. Since each ϕ_p is injective, we have ι_q^s injective for $s \geq q$, so we must have $\sum_{p \in F} \iota_p^q(T_p) = 0$.

We may assume without loss of generality that $e \in F$, and rearrange to obtain $\sum_{p \in F \setminus \{e\}} \iota_p^q(T_p) = -\iota_e^q(T_e)$. As observed above, $T_e = L_a$ for some $a \in A$, so (5.1) implies that $\iota_e^q(T_e) = \phi_q(a)$. Since the left action of A on X_q is by compact operators, we have $\phi_q(a) \in \mathcal{K}(X_q)$, and hence Fowler's Cuntz-Pimsner covariance condition forces

$$\psi^{(q)}\Big(\sum_{p \in F \setminus \{e\}} \iota_p^q(T_p)\Big) = -\psi_e(a). \tag{5.2}$$

Since A acts compactly on the left of each X_p, [26, Corollary 3.7] shows that each $\iota_p^q(T_p) \in \mathcal{K}(X_q)$, and the argument of [26, Lemma 3.10] shows that $\psi^{(q)}(\iota_p^q(S)) = \psi^{(p)}(S)$. Hence (5.2) implies that

$$\sum_{p \in F \setminus \{e\}} \psi^{(p)}(T_p) = -\psi_e(a).$$

Since $\psi_e(a) = \psi^{(e)}(L_a) = \psi^{(e)}(T_e)$, we have $\sum_{p \in F} \psi^{(p)}(T_p) = 0$. So ψ satisfies (CP). □

Corollary 5.2. *Let (G, P) be a quasi-lattice ordered group, and let X be a compactly aligned product system over P of right-Hilbert A–A bimodules. Suppose that each pair* *in P has a least upper bound. Suppose that each $\phi_p : A \to \mathcal{L}(X_p)$ is injective with $\phi_p(A) \subset \mathcal{K}(X_p)$. Let $\psi : X \to B$ be a representation of X. Then ψ is CNP-covariant if and only if $\psi^{(p)} \circ \phi_p = \psi_e$ for all $p \in P$.*

Proof. Since each $\phi_p(A) \subset \mathcal{K}(X_p)$, the condition $\psi^{(p)} \circ \phi_p = \psi_e$ for all $p \in P$ is precisely the Cuntz-Pimsner covariance condition of [12, Definition 2.5]. Proposition 5.4 of [12] implies that if ψ is Cuntz-Pimsner covariant in this sense, then it is also Nica covariant. Hence the result follows from Proposition 5.1. □

5.2. **Katsura's C^*-algebras associated to Hilbert bimodules.** Let X be a right-Hilbert A–A bimodule. There is a product system $X^{\otimes}$ over $\mathbb{N}$ of right-Hilbert A–A bimodules such that $X^{\otimes}_0 = A$, and $X^{\otimes}_n = X^{\otimes n}$ for $1 \leq n \in \mathbb{N}$. For nonzero m, n, the isomorphism $M_{m,n} : X^{\otimes}_m \otimes X^{\otimes}_n \to X^{\otimes}_{m+n}$ implementing the multiplication in the system is the natural isomorphism $X^{\otimes m} \otimes X^{\otimes n} \cong X^{\otimes m+n}$. As $\mathbb{N}$ is totally ordered, $X^{\otimes}$ is compactly aligned.

If (ψ, π) is a representation of X, then there is a representation $\psi^{\otimes}$ of $X^{\otimes}$ given by $\psi^{\otimes}_e = \pi$ and $\psi^{\otimes}_n = \psi^{\otimes n}$ for $n \geq 1$. This representation is automatically Nica covariant, and every representation of $X^{\otimes}$ is of this form. Proposition 2.11 of [12] says that if the homomorphism $\phi : A \to X$ implementing the left action is injective or takes values in $\mathcal{K}(X)$ then ψ is covariant in Pimsner's sense if and only if $\psi^{\otimes}$ is Cuntz-Pimsner covariant in Fowler's sense. Hence Pimsner's $\mathcal{O}_X$ [26] and Fowler's $\mathcal{O}_{X^{\otimes}}$ [12] coincide.

A key goal of our construction was to achieve the same outcome with respect to Katsura's reformulation of Cuntz-Pimsner covariance for a single Hilbert bimodule. That is, given an arbitrary Hilbert bimodule X, we desire that $\mathcal{NO}_{X^{\otimes}}$ should coincide with Katsura's $\mathcal{O}_X$ [17, 18].

Recall that if $I \lhd A$ is an ideal in a C^*-algebra, then $I^{\perp}$ denotes the ideal $\{a \in A : ab = 0 \text{ for all } b \in I\}$. Recall also that Katsura's $\mathcal{O}_X$ is the universal C^*-algebra generated by a representation (i_A, i_X) of X such that $i^{(1)}_X(\phi(a)) = i_A(a)$ for all $a \in \ker(\phi)^{\perp}$ such that $\phi(a) \in \mathcal{K}(X)$.

Proposition 5.3. *Let X be a right-Hilbert A–A bimodule. Let (i_A, i_X) be the universal representation of X on $\mathcal{O}_X$, and $j_{X^{\otimes}}$ be the universal representation of $X^{\otimes}$ on $\mathcal{NO}_{X^{\otimes}}$.*

1. *There is an isomorphism $\theta : \mathcal{O}_X \to \mathcal{NO}_{X^{\otimes}}$ satisfying $\theta(i_A(a)) = j_{X^{\otimes}}(a)$ and $\theta(i_X(x)) = j_{X^{\otimes}}(x)$ for all $a \in A$ and $x \in X$.*
2. *Let (π, ψ) be a representation of X and let $\psi^{\otimes}$ be the corresponding Nica covariant representation of $X^{\otimes}$. Then (π, ψ) is covariant in the sense of Katsura if and only if $\psi^{\otimes}$ satisfies (CP).*

Proof. Statement (2) follows from (1) and the universal properties of $\mathcal{O}_X$ and $\mathcal{NO}_{X^{\otimes}}$, so it suffices to prove (1).

We have $\ker(\phi) \subset \ker(\phi \otimes 1_{n-1})$ for $n \geq 1$, so I_n is equal to A if $n = 0$ and is equal to $\ker(\phi)$ if $n \neq 0$. Let (j_0, j_1) denote the representation of X determined by the universal representation of $X^{\otimes}$ on $\mathcal{NO}_{X^{\otimes}}$. If $a \in \ker(\phi)^{\perp} \cap \phi^{-1}(\mathcal{K}(X))$, then $\widetilde{\phi}_1(a) = 0_{\ker(\phi)} \oplus \phi(a)$. Let $S = \phi(a) \in \mathcal{K}(X)$. Then $\widetilde{\iota}^1_0(\phi_0(a)) - \widetilde{\iota}^1_1(S) = 0$, and it follows that $\widetilde{\iota}^n_0(\phi_0(a)) - \widetilde{\iota}^n_1(S) = 0$ for all $n \geq 1$. Since $j_{X^{\otimes}}$ satisfies (CP), we therefore have $j_0(a) - j^{(1)}_1(S) = 0$; that is $j_0(a) = j^{(1)}_1(\phi(a))$. Thus (j_0, j_1) is covariant in the sense of Katsura, and the universal property of $\mathcal{O}_X$ implies that there is a homomorphism

$\theta : \mathcal{O}_X \to \mathcal{NO}_X$ determined by $\theta \circ i_A = j_0$ and $\theta \circ i_X = j_1$. Moreover, θ is surjective because

$$\mathcal{NO}_X = \overline{\operatorname{span}}\,\{j_1(x_1)\cdots j_1(x_m)j_1(y_n)^*\cdots j_1(y_1)^* : m,n \in \mathbb{N}, x_i, y_i \in X\}.$$

Theorem 4.1 implies that $\theta \circ i_A = j_0$ is injective, so an application of Katsura's gauge-invariant uniqueness theorem [18, Theorem 6.2] shows that θ is injective. □

5.3. **Cuntz-Krieger algebras of finitely aligned higher-rank graphs.** In this section, we use the notation and conventions of [30] for higher-rank graphs. In [28], a product system $X(\Lambda)$ over $\mathbb{N}^k$ of right-Hilbert $c_0(\Lambda^0)$–$c_0(\Lambda^0)$ bimodules is associated to each k-graph Λ. When Λ is row-finite and has no sources, the homomorphism ϕ_p implementing the left action of $c_0(E^0)$ on $X(\Lambda)_p$ is an injective homomorphism into the compact operators on $X(\Lambda)_p$. Corollary 4.4 of [28] shows that $C^*(\Lambda)$ coincides with $\mathcal{O}_{X(\Lambda)}$ as defined in [12] and hence with $\mathcal{NO}_{X(\Lambda)}$ by Proposition 5.1. A key goal of our construction is to extend this to arbitrary finitely aligned k-graphs and the corresponding compactly aligned product systems of bimodules. We show in this section that we have achieved this aim.

We briefly recall some salient point about k-graphs from [30]. A k-graph is a countable category Λ together with a functor $d : \Lambda \to \mathbb{N}^k$ satisfying the factorisation property: for all $m, n \in \mathbb{N}^k$ and $\lambda \in d^{-1}(m+n)$ there exist unique $\mu \in d^{-1}(m)$ and $\nu \in d^{-1}(n)$ such that $\lambda = \mu\nu$. Each $d^{-1}(n)$ is denoted Λ^n. The elements of Λ^0 are called vertices, and are in bijection with the objects of Λ, so the codomain and domain maps in the category Λ determine maps $r, s : \Lambda \to \Lambda^0$. For $\mu, \nu \in \Lambda$, we write $\operatorname{MCE}(\mu,\nu)$ for the collection $\{\lambda \in \Lambda : d(\lambda) = d(\mu) \vee d(\nu), \lambda = \mu\mu' = \nu\nu' \text{ for some } \mu', \nu' \in \Lambda\}$. The k-graph Λ is said to be *finitely aligned* if $\operatorname{MCE}(\mu,\nu)$ is finite (possibly empty) for every pair μ, ν of paths in Λ.

For $n \in \mathbb{N}^k$, we write $\Lambda^{\leq n}$ for the set $\{\lambda \in \bigcup_{m \leq n} \Lambda^m : d(\lambda) < d(\lambda) + p \leq n \implies r^{-1}(s(\lambda)) \cap \Lambda^p = \emptyset\}$. Given a vertex v, a subset F of $r^{-1}(v)$ is said to be exhaustive if for every $\mu \in r^{-1}(v)$ there exists $\nu \in F$ such that $\operatorname{MCE}(\mu,\nu) \neq \emptyset$. Given a finitely aligned k-graph Λ, a set $\{s_\lambda : \lambda \in \Lambda\}$ of partial isometries is called a Toeplitz-Cuntz-Krieger Λ-family if

(CK1) $\{s_v : v \in \Lambda^0\}$ is a set of mutually orthogonal projections;
(CK2) $s_\mu s_\nu = s_{\mu\nu}$ whenever $s(\mu) = r(\nu)$; and
(CK3) $s_\mu^* s_\nu = \sum_{\mu\mu' = \nu\nu' \in \operatorname{MCE}(\mu,\nu)} s_{\mu'} s_{\nu'}^*$ for all $\mu, \nu \in \Lambda$.

It is called a Cuntz-Krieger Λ-family if it additionally satisfies

(CK4) $\prod_{\lambda \in F}(s_v - s_\lambda s_\lambda^*) = 0$ for all $v \in \Lambda^0$ and all nonempty finite exhaustive sets $F \subset r^{-1}(v)$.

The Cuntz-Krieger algebra $C^*(\Lambda)$ is the universal C^*-algebra generated by a Cuntz-Krieger Λ-family.

Theorem 4.2 and Proposition 6.4 of [28] show that Nica covariant representations of $X(\Lambda)$ are in bijective correspondence with Toeplitz-Cuntz-Krieger Λ-families. To describe this bijection, we must briefly recall the definition of $X(\Lambda)$ (see [28] for details). For $n \in \mathbb{N}^k$, we endow $c_c(\Lambda^n) = \operatorname{span}\{\delta_\lambda : \lambda \in \Lambda^n\}$ with the structure of a pre-Hilbert $c_0(\Lambda^0)$-bimodule via the following formulae: $(a \cdot x \cdot b)(\lambda) := a(r(\lambda))x(\lambda)b(s(\lambda))$; and $\langle x, y\rangle_{c_0(\Lambda^0)}(v) = \sum_{s(\lambda)=v} \overline{x(\lambda)}y(\lambda)$. Then $X(\Lambda)_n$ is the completion of $c_c(\Lambda^n)$ in the norm arising from $\langle\cdot,\cdot\rangle_{c_0(\Lambda^0)}$. The isomorphisms $X(\Lambda)_m \otimes X(\Lambda)_n \cong X(\Lambda)_{m+n}$ are given by $\delta_\mu \otimes_{c_0(\Lambda^0)} \delta_\nu \mapsto \delta_{\mu\nu}$ if $s(\mu) = r(\nu)$ (if $s(\mu) \neq r(\nu)$, then $\delta_\mu \otimes_{c_0(\Lambda^0)} \delta_\nu = 0$ by definition of the balanced tensor product). Given a Nica covariant representation ψ of $X(\Lambda)$, the corresponding Toeplitz-Cuntz-Krieger Λ-family (see [28, Definition 7.1]) $\{t_\lambda : \lambda \in \Lambda\}$ is defined by $t_\lambda = \psi(\delta_\lambda)$; and we can recover ψ from the t_λ by linearity and continuity.

Proposition 5.4. *Let Λ be a finitely aligned k-graph, and let $X = X(\Lambda)$ be the associated Cuntz-Krieger product system of Hilbert bimodules. Let $\{s_\lambda : \lambda \in \Lambda\}$ be the universal Cuntz-Krieger family in $C^*(\Lambda)$, and let j_X be the universal CNP-covariant representation of X in $\mathcal{NO}_X$.*

(1) *There is an isomorphism $\theta : C^*(\Lambda) \to \mathcal{NO}_X$ satisfying $\theta(s_\lambda) = j_X(\delta_\lambda)$ for all $\lambda \in \Lambda$.*
(2) *Let $\psi : X \to B$ be a Nica-covariant representation of X, and let $\{t_\lambda : \lambda \in \Lambda\}$ be the corresponding Toeplitz-Cuntz-Krieger Λ-family $t_\lambda = \psi(\delta_\lambda)$. Then ψ satisfies* Definition 3.9 *if and only if $\{t_\lambda : \lambda \in \Lambda\}$ satisfies the Cuntz-Krieger relation* [30, Definition 2.5(iv)].

Proof. Statement (2) follows from (1) and the universal properties of $C^*(\Lambda)$ and $\mathcal{NO}_X$. So it suffices to prove (1).

For each $\lambda \in \Lambda$, let $j_\lambda = j_X(\delta_\lambda)$, so $\{j_\lambda : \lambda \in \Lambda\}$ is a Toeplitz-Cuntz-Krieger Λ family which generates $\mathcal{NO}_X$. We claim that $\{j_\lambda : \lambda \in \Lambda\}$ is a Cuntz-Krieger Λ-family.

Fix $v \in \Lambda^0$ and a finite exhaustive set $F \subset v\Lambda$. We must show that

$$\prod_{\mu\in F}(j_v - j_\mu j_\mu^*) = 0.$$

For a subset $G \subset F$, we will denote by $\vee d(G)$ the element $\bigvee_{\mu\in G} d(\mu)$ of $\mathbb{N}^k$. If $\lambda, \mu \in \Lambda$ satisfy $\lambda = \mu\mu'$ for some $\mu' \in \Lambda$, we say that λ extends μ. Recall from [28] that for a nonempty subset G of F, $\mathrm{MCE}(G)$ denotes the set $\{\lambda \in \Lambda : d(\lambda) = \vee d(G), \lambda \text{ extends } \mu \text{ for all } \mu \in G\}$. Recall also that $\vee F := \bigcup_{G\subset F} \mathrm{MCE}(G)$ is finite and is closed under minimal common extensions. We have

$$\begin{aligned}\prod_{\mu\in F}(j_v - j_\mu j_\mu^*) &= j_v + \sum_{\substack{\emptyset\neq G\subset F\\ \lambda\in\mathrm{MCE}(G)}} (-1)^{|G|} j_\lambda j_\lambda^* \\ &= j_X^{(e)}(\delta_v \otimes \delta_v^*) + \sum_{\substack{\emptyset\neq G\subset F\\ \lambda\in\mathrm{MCE}(G)}} (-1)^{|G|} j_X^{(\vee d(G))}(\delta_\lambda \otimes \delta_\lambda^*).\end{aligned}$$

Since $j_{X(\Lambda)}$ satisfies (CP) it suffices to show that for each $q \in \mathbb{N}^k$ there exists $r \geq q$ such that for all $s \geq r$ we have

$$\widetilde{\iota}_e^s(\delta_v \otimes \delta_v^*) + \sum_{\substack{\emptyset\neq G\subset F\\ \lambda\in\mathrm{MCE}(G)}} (-1)^{|G|} \widetilde{\iota}_{\vee d(G)}^s(\delta_\lambda \otimes \delta_\lambda^*) = 0.$$

For this, fix $q \in \mathbb{N}^k$, let $r = q \vee (\vee d(F))$ and fix $s \geq r$. Since $\widetilde{X}_s = \oplus_{t\leq s} \overline{\mathrm{span}}\,\{\delta_\tau : \tau \in \Lambda^t \cap \Lambda^{\leq s}\}$ (see Example 3.3), it suffices to show that for $\tau \in \Lambda^{\leq s}$,

$$\Big(\widetilde{\iota}_e^s(\delta_v \otimes \delta_v^*) + \sum_{\substack{\emptyset\neq G\subset F\\ \lambda\in\mathrm{MCE}(G)}} (-1)^{|G|} \widetilde{\iota}_{\vee d(G)}^s(\delta_\lambda \otimes \delta_\lambda^*)\Big)(\delta_\tau) = 0. \tag{5.3}$$

Fix $\tau \in \Lambda^{\leq s}$. For any $\mu \in F$, we have $s \geq d(\mu)$, so

$$\widetilde{\iota}^s_{d(\mu)}(\delta_\mu \otimes \delta^*_\mu)(\delta_\tau) = \begin{cases} \delta_\tau & \text{if } \tau \text{ extends } \mu \\ 0 & \text{otherwise.} \end{cases} \tag{5.4}$$

Fix a nonempty subset G of F. Then

$$\Big(\prod_{\mu \in G} \widetilde{\iota}^s_{d(\mu)}(\delta_\mu \otimes \delta^*_\mu)\Big)(\delta_\tau) = \begin{cases} \delta_\tau & \text{if } \tau \text{ extends each } \mu \text{ in } G \\ 0 & \text{otherwise.} \end{cases}$$

The factorisation property implies that τ extends each μ in G if and only if there exists λ in $\operatorname{MCE}(G)$ such that τ extends λ. The factorisation property also implies that if there does exist such a $\lambda \in \operatorname{MCE}(G)$ then it is necessarily unique. We therefore have

$$\Big(\prod_{\mu \in G} \widetilde{\iota}^s_{d(\mu)}(\delta_\mu \otimes \delta^*_\mu)\Big)(\delta_\tau) = \Big(\sum_{\lambda \in \operatorname{MCE}(G)} \widetilde{\iota}^s_{\vee d(G)}(\delta_\lambda \otimes \delta^*_\lambda)\Big)(\delta_\tau)$$

Since the fixed nonempty subset G of F in the preceding paragraph was arbitrary, we may now calculate:

$$\begin{aligned}
&\Big(\prod_{\mu \in F} \big(\widetilde{\iota}^s_e(\delta_v \otimes \delta^*_v) - \widetilde{\iota}^s_{d(\mu)}(\delta_\mu \otimes \delta^*_\mu)\big)\Big)(\delta_\tau) \\
&\qquad = \Big(\widetilde{\iota}^s_e(\delta_v \otimes \delta^*_v) + \sum_{\emptyset \neq G \subset F} \Big((-1)^{|G|} \prod_{\mu \in G} \widetilde{\iota}^s_{d(\mu)}(\delta_\mu \otimes \delta^*_\mu)\Big)\Big)(\delta_\tau) \\
&\qquad = \Big(\widetilde{\iota}^s_e(\delta_v \otimes \delta^*_v) + \sum_{\substack{\emptyset \neq G \subset F \\ \lambda \in \operatorname{MCE}(G)}} (-1)^{|G|} \widetilde{\iota}^s_{\vee d(G)}(\delta_\lambda \otimes \delta^*_\lambda)\Big)(\delta_\tau).
\end{aligned}$$

Since F is exhaustive, there exists $\nu \in F$ such that $\operatorname{MCE}(\tau, \nu) \neq \emptyset$; say $\tau\tau' \in \operatorname{MCE}(\tau, \nu)$. Since $d(\nu), d(\tau) \leq s$, we have $d(\tau\tau') \leq s$. Since $\tau \in \Lambda^{\leq s}$, this forces $d(\tau') = 0$, so τ extends ν. Thus (5.4) implies that

$$\begin{aligned}
&\Big(\prod_{\mu \in F} \big(\widetilde{\iota}^s_e(\delta_v \otimes \delta^*_v) - \widetilde{\iota}^s_{d(\mu)}(\delta_\mu \otimes \delta^*_\mu)\big)\Big)(\delta_\tau) \\
&\quad = \Big(\prod_{\mu \in F \setminus \{\nu\}} \big(\widetilde{\iota}^s_e(\delta_v \otimes \delta^*_v) - \widetilde{\iota}^s_{d(\mu)}(\delta_\mu \otimes \delta^*_\mu)\big)\Big)\Big(\big(\widetilde{\iota}^s_e(\delta_v \otimes \delta^*_v) - \widetilde{\iota}^s_{d(\nu)}(\delta_\nu \otimes \delta^*_\nu)\big)\Big)(\delta_\tau) = 0,
\end{aligned}$$

establishing (5.3). Hence $\{j_\lambda : \lambda \in \Lambda\}$ is a Cuntz-Krieger Λ-family as claimed.

Since the j_λ generate $\mathcal{NO}_X$, the universal property of $C^*(\Lambda)$ implies that there is a surjective homomorphism $\theta : C^*(\Lambda) \to \mathcal{NO}_X$ satisfying $\theta(s_\lambda) = j_\lambda = j_X(\delta_\lambda)$ for all $\lambda \in \Lambda$. By Theorem 4.1, we have $j_v \neq 0$ for all $v \in \Lambda^0$. Since θ intertwines the gauge actions of $\mathbb{T}^k$ on $\mathcal{NO}_X$ and $C^*(\Lambda)$, the gauge-invariant uniqueness theorem for $C^*(\Lambda)$ [30, Theorem 4.2] therefore implies that θ is an isomorphism. □

5.4. **Boundary quotients of Toeplitz algebras.** In this section, we consider product systems whose fibres are isomorphic to ${}_{\mathbb{C}}\mathbb{C}_{\mathbb{C}}$. Nica covariant representations of such product systems amount to Nica covariant representations of (G, P) in the sense of [6, 22, 25]. This prompts us to explore the connection between our $\mathcal{NO}_X$ and the boundary quotients of $\mathcal{T}(G, P)$ studied by Crisp and Laca in [7].

The first part of the following proposition follows from results of Fowler and Raeburn [15], but we include it for completeness. We first make the following definition: if (G, P) is a quasi-lattice ordered group, we say that a finite subset F of P is a *foundation set* for P if, for every $q \in P$ there exists $p \in F$ such that $p \vee q < \infty$.

Note that what we have called foundation sets are precisely the *boundary relations* of [7, Definition 3.4].

Notation 5.5. Given a semigroup P, we denote by $\mathbb{C}^P$ the product system of right-Hilbert $\mathbb{C}$–$\mathbb{C}$ bimodules determined by $X_p = {}_{\mathbb{C}}\mathbb{C}_{\mathbb{C}}$ for all p with multiplication in X given by multiplication of complex numbers. As a notational convenience, when we are regarding a complex number z as an element of $\mathbb{C}^P_p$, we shall denote it z_p.

Proposition 5.6. *Let (G, P) be a quasi-lattice ordered group. Let $X = \mathbb{C}^P$ be the product system discussed above. Then $\mathcal{T}_{\mathrm{cov}}(X)$ is the universal C^*-algebra generated by a semigroup representation $p \mapsto V_p$ of P by isometries satisfying*

$$V_pV_p^*V_qV_q^* = \begin{cases} V_{p\vee q}V_{p\vee q}^* & \text{if } p \vee q < \infty \\ 0 & \text{otherwise.} \end{cases} \tag{5.5}$$

The images $\{W_p : p \in P\}$ of these isometries under the canonical homomorphism $(j_X)_* : \mathcal{T}_{\mathrm{cov}}(X) \to \mathcal{NO}_X$ satisfy the additional relation

$$\prod_{p\in F}(1 - W_pW_p^*) = 0 \text{ for every foundation set } F \text{ for } P. \tag{5.6}$$

Proof. It is straightforward to check that the elements $V_p = i_X(1_p)$ of $\mathcal{T}_{\mathrm{cov}}(X)$ generate $\mathcal{T}_{\mathrm{cov}}(X)$, determine a semigroup representation of P, and satisfy (5.5). We therefore need only show that their images $W_p = j_X(1_p)$ in $\mathcal{NO}_X$ satisfy (5.6).

For this, fix a foundation set F for P. In what follows, we shall write $\vee H$ for the least upper bound of a finite subset H of P when it exists, and when it does not exist, we shall write $\vee H = \infty$. Since the W_p are the images of the V_p under a homomorphism, relation (5.5) holds amongst the W_p. We have

$$\begin{aligned}\prod_{p\in F}(1 - W_pW_p^*) &= 1 + \sum_{\substack{\emptyset\neq H\subset F\\ \vee H<\infty}} (-1)^{|H|} W_{\vee H}W_{\vee H}^* \\ &= 1 + \sum_{\substack{\emptyset\neq H\subset F\\ \vee H<\infty}} (-1)^{|H|} j_X^{(\vee H)}(1_{\vee H} \otimes 1_{\vee H}^*).\end{aligned}$$

Since $j^{(e)}(1_e \otimes 1_e^*) = W_e = 1$, since $\widetilde{\iota}^s_e(1_e \otimes 1_e^*) = 1_{\mathcal{L}(\widetilde{X}_s)}$, and since j_X is Cuntz-Pimsner covariant, it suffices to show that

$$1_{\mathcal{L}(\widetilde{X}_s)} + \sum_{\substack{\emptyset\neq H\subset F\\ \vee H<\infty}} (-1)^{|H|} \widetilde{\iota}^s_{\vee H}(1_{\vee H} \otimes 1_{\vee H}^*) = 0 \text{ for large } s.$$

For this, fix $q \in P$. Since F is a foundation set for P, we must have $q \vee a < \infty$ for some $a \in F$. An inductive argument then shows that there exists $r \in P$ such that $r \geq q \vee a$ and such that for each $p \in F$ either $r \geq p$ or $r \vee p = \infty$. Fix $s \in P$ with $s \geq r$.

Fix $\emptyset \neq H \subset F$ with $\vee H < \infty$. Then

$$\iota^s_{\vee H}(1_{\vee H} \otimes 1^*_{\vee H}) = \begin{cases} 1_{\mathcal{L}(X_s)} & \text{if } \vee H \leq s \\ 0 & \text{otherwise,} \end{cases}$$

and for $p \in H$, we have

$$\iota^s_p(1_p \otimes 1^*_p) = \begin{cases} 1_{\mathcal{L}(X_s)} & \text{if } p \leq s \\ 0 & \text{otherwise.} \end{cases}$$

Since $\vee H \leq s$ if and only if $p \leq s$ for all $p \in H$, the preceding two displayed equations imply that

$$\iota^s_{\vee H}(1_{\vee H} \otimes 1^*_{\vee H}) = \prod_{p \in H} \iota^s_p(1_p \otimes 1^*_p).$$

Since each ϕ_t is injective, $\widetilde{X}_p = X_p$ and $\widetilde{\iota}^s_p(1_p \otimes 1^*_p) = \iota^s_p(1_p \otimes 1^*_p)$ for all $p, s \in P$. Since the calculations in the previous paragraph are valid for arbitrary nonempty $H \subset F$ with $\vee H < \infty$, we may now calculate

$$\begin{aligned} 1_{\mathcal{L}(\widetilde{X}_s)} + \sum_{\substack{\emptyset \neq H \subset F \\ \vee H \neq \infty}} (-1)^{|H|} \widetilde{\iota}^s_{\vee H}(1_{\vee H} \otimes 1^*_{\vee H}) &= 1_{\mathcal{L}(X_s)} + \sum_{\substack{\emptyset \neq H \subset F \\ \vee H \neq \infty}} (-1)^{|H|} \iota^s_{\vee H}(1_{\vee H} \otimes 1^*_{\vee H}) \\ &= \prod_{p \in F} (1_{\mathcal{L}(X_s)} - \iota^s_p(1_p \otimes 1^*_p)) \\ &= \begin{cases} 1_{\mathcal{L}(X_s)} & \text{if } p \not\leq s \text{ for all } p \in F \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

By choice of r and s, we have $a \leq s$ for some $a \in F$, so

$$1_{\mathcal{L}(\widetilde{X}_s)} + \sum_{\substack{\emptyset \neq H \subset F \\ \vee H \neq \infty}} (-1)^{|H|} \widetilde{\iota}^s_{\vee H}(1_{\vee H} \otimes 1^*_{\vee H}) = 0$$

as required. □

The point of the above proposition is the relationship it suggests with Crisp and Laca's boundary quotient of $\mathcal{T}(G, P)$ (see [7, Definitions 3.1 and 3.4]).

Recall that given an undirected graph Γ with vertex set S, the right-angled Artin group G associated to Γ is the group $G = \langle S \mid st = ts$ whenever s and t are adjacent in $\Gamma\rangle$. We write P for the submonoid of G generated by S. The pair (G, P) is a quasi-lattice ordered group and satisfies (3.5).

For this (G, P), [7, Theorem 6.7] shows that the boundary quotient $C_0(\partial\Omega) \rtimes G$ is simple and is the universal C^*-algebra generated by elements $\{T_s : s \in S\}$ satisfying

(1) $T^*_s T_s = 1$ for all $s \in S$;
(2) $T_s T_t = T_t T_s$ and $T^*_s T_t = T_t T^*_s$ whenever s and t are adjacent in Γ;
(3) $T^*_s T_t = 0$ whenever s and t are adjacent in Γ^{opp}; and
(4) $\prod_{s \in C}(1 - T_s T^*_s) = 0$ when C is the vertex set of any finite connected component of Γ^{opp}.

Corollary 5.7. *Let Γ be an undirected graph with vertex set S, and let (G, P) be the associated right-angled Artin group. Suppose that G has trivial centre. Let $X = \mathbb{C}^P$ be the product system of Notation 5.5. Let $C_0(\partial\Omega) \rtimes G$ be the boundary quotient of [7]. Then $\mathcal{NO}_X \cong C_0(\partial\Omega) \rtimes G$.*

Proof. As in [7], let Γ^{opp} denote the graph which has the same vertex set S such that s and t are adjacent in Γ^{opp} if and only if they are not adjacent in Γ.

For $p \in P$, let $W_p = j_X(1_p) \in \mathcal{NO}_X$. Proposition 5.6 shows that $p \mapsto W_p$ is a semigroup representation of P by isometries which satisfy (5.5) and (5.6). Since S generates P, the set $\{W_s : s \in S\}$ generates $\mathcal{NO}_X$ and satisfies (1) and the first part of (2). If s, t are adjacent in Γ, we have $s \vee t = st = ts$, and so (5.5), (1) and the first part of (2) force $W_s^* W_t = W_s^*(W_s W_s^* W_t W_t^*) W_t = W_s^* W_{st} W_{ts}^* W_t = W_t W_s^*$, so the W_s satisfy (2). If s, t are adjacent in Γ^{opp}, then (5.5) gives $W_s^* W_t = W_s^*(W_s W_s^* W_t W_t^*) W_t = 0$. Hence the W_s satisfy (1)–(3). The final paragraph of the proof of [7, Theorem 6.7] and [7, Definition 3.4] imply that the vertex set C of any finite connected component of Γ^{opp} is a foundation set for P. Hence Proposition 5.6 implies that the W_s satisfy (4).

The universal property of $C(\partial\Omega) \rtimes G$ now implies that there is a homomorphism $\pi : C(\partial\Omega) \rtimes G \to \mathcal{NO}_X$ satisfying $\pi(T_s) = W_s$ for all $s \in S$. Since S generates P and since $\{W_p : p \in P\}$ generates $\mathcal{NO}_X$, π is surjective. Theorem 4.1 implies that $\mathcal{NO}_X \neq \{0\}$. Since $C_0(\partial\Omega) \rtimes G$ is simple, it follows that π is an isomorphism. □

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