RENAULT'S EQUIVALENCE THEOREM FOR REDUCED GROUPOID C^*-ALGEBRAS

AIDAN SIMS AND DANA P. WILLIAMS

ABSTRACT. We use the technology of linking groupoids to show that equivalent groupoids have Morita equivalent reduced C^*-algebras. This equivalence is compatible in a natural way in with the Equivalence Theorem for full groupoid C^*-algebras.

INTRODUCTION

Renault's Equivalence Theorem is one of the fundamental tools in the theory of groupoid C^*-algebras. It states that if G and H are equivalent via a (G, H)-equivalence Z, then the groupoid C^*-algebras $C^*(G)$ and $C^*(H)$ are Morita equivalent via an imprimitivity bimodule X which is a completion of $C_c(Z)$. However, one is often interested in the reduced C^*-algebras $C_r^*(G)$ and $C_r^*(H)$. For example, it is the reduced C^*-algebras that play a role in Baum-Connes theory. Furthermore, it is the reduced algebra — rather than the full one — which arises in many applications because it, and its reduced norm, have much more concrete descriptions than their universal counterparts. It is apparently "well known" to experts that equivalent groupoids have Morita equivalent reduced C^*-algebras. For example, the result is stated without proof immediately following [10, Theorem 3.1].

The purpose of this paper is twofold: firstly to give a precise statement and proof of the equivalence result for reduced groupoid C^*-algebras; and secondly to highlight the role of the linking groupoid, which is the main tool in our proofs. The concept of the linking groupoid L of an equivalence between groupoids G and H goes back to work of Kumjian — see in particular, [3]. The linking groupoid was described in general in Muhly's unpublished notes [6, Remark 5.35]. A missing ingredient up until now has been a Haar system for L. We show that if G and H have Haar systems, then so does L; we may then form $C^*(L)$, and we show that it is isomorphic to the linking algebra $L(\mathsf{X})$ of Renault's imprimitivity bimodule X (Corollary 16). Our main results imply that if G and H are equivalent groupoids, then their reduced groupoid C^*-algebras $C_r^*(G)$ and $C_r^*(H)$ are Morita equivalent via a quotient X_r of X (Theorem 17). Moreover, we show that the Rieffel correspondence associated to X matches up the kernel $I_{C_r^*(G)}$ of the canonical surjection of $C^*(G)$ onto $C_r^*(G)$ with the kernel $I_{C^*r(H)}$ of the surjection of $C^*(H)$

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onto $C_r^*(H)$. Therefore for any representation π of $C^*(H)$ that factors through $C_r^*(H)$, the induced representation $\mathsf{X}\text{-}\operatorname{Ind}\pi$ of $C^*(G)$ factors through $C_r^*(G)$.

Our proof of the Equivalence Theorem for the universal algebras, like existing ones, relies heavily on Renault's Disintegration Theorem ([12, Proposition 4.2]) which is a highly nontrivial result. We have organized our work to illustrate that, by contrast, the Morita equivalence for the reduced algebras can be proved without invoking the Disintegration Theorem. Therefore there is a sense in which the equivalence result for reduced C^*-algebras is a more elementary result than the corresponding result for the universal algebras.

We review the set up of the Equivalence Theorem from [4, §2] in Section 1, and we describe the linking groupoid and its Haar system in Section 2. In Section 3 we review some basic facts about regular representations and the reduced groupoid C^*-algebra. We spend a bit more time than strictly necessary so as to clear up some ambiguities in the literature and to state some results for future reference. In Section 4 we prove our equivalence theorem for the reduced algebras, and then tie this in with the universal constructs in Section 5.

We also include a short appendix to clarify the hypotheses necessary for recently published proofs of the Disintegration Theorem and generalizations. In particular, we show that it is not always necessary to assume the representations involved act on separable spaces.

Because we want to be able to appeal both the original Equivalence Theorem and the Disintegration Theorem, it is convenient, and at times necessary, to require all our groupoids and spaces to be second countable locally compact Hausdorff spaces. As we are interested in C^*-algebras associated to groupoids, all our groupoids are assumed to have Haar systems. By convention, all homomorphisms between C^*-algebras are $*$-preserving, and all representations of C^*-algebras are nondegenerate.

1. Background

Throughout, G and H denote second countable, locally compact Hausdorff groupoids with Haar systems $\{\lambda^u\}_{u\in G^{(0)}}$ and $\{\beta^v\}_{v\in H^{(0)}}$, respectively,

In order to establish our notation, it will be useful to review the statement and set-up of the Equivalence Theorem from [4, §2]. First, recall that if G is a locally compact groupoid, then we say that a locally compact space Z is a *G-space* if there is a continuous, open map $r_Z : Z \to G^{(0)}$ and a continuous map $(\gamma, z) \mapsto \gamma\cdot z$ from $G * Z = \{\, (\gamma, z) \in G\times Z : s_G(\gamma) = r_Z(z)\,\}$ to Z such that $r_X(z)\cdot z = z$ for all z and $(\gamma\eta)\cdot z = \gamma\cdot(\eta\cdot z)$ for all $(\gamma,\eta) \in G^{(2)}$ with $s_G(\eta) = r_Z(z)$. (Hereafter we will often drop the subscripts on all r and s maps and trust that the domain is clear from context.) The action is *free* if $\gamma\cdot z = z$ implies $\gamma = r(z)$ and *proper* if the map $(\gamma, z) \mapsto (\gamma\cdot z, z)$ is a proper map of $G * Z$ into $Z\times Z$. Right actions are dealt with similarly except that the structure map is denoted by s instead of r.

Remark 1. Nowadays, many authors do not require the structure map r_Z of a G-space Z to be open. Since it is critical in the definition of an equivalence (see Definition 2) that both structure maps be open, we include the hypothesis here to avoid ambiguities. It was also part of the definition of G-action in [4].

Definition 2. Let G and H be locally compact groupoids. A (G, H)-*equivalence* is a locally compact space Z such that

(a) Z is a free and proper left G-space,

(b) Z is a free and proper right H-space,
(c) the actions of G and H on Z commute,
(d) r_Z induces a homeomorphism of Z/H onto $G^{(0)}$, and
(e) s_Z induces a homeomorphism of $G\backslash Z$ onto $H^{(0)}$.

If Z is a (G,H)-equivalence, then there is a continuous map $(y,z)\mapsto {}_G[y,z]$ of $Z *_s Z$ to G uniquely determined by ${}_G[y,z]\cdot z = y$ for all $(y,z)\in Z*_s Z$. This map induces a topological groupoid isomorphism of $(Z*_s Z)/H$ onto G. Similarly, there is a continuous map $(y,z)\mapsto [y,z]_H$ satisfying $y\cdot[y,z]_H = z$ for all $(y,z)\in Z*_r Z$, and this map induces an isomorphism of $G\backslash(Z*_r Z)$ onto H. It is shown in [4, §2] that if Z is a (G,H)-equivalence, then $C_c(Z)$ is a $C_c(G)-C_c(H)$-bimodule with actions and pre-inner products given as follows: for $f\in C_c(G)$, $b\in C_c(H)$, and $\phi,\psi\in C_c(Z)$,

$$f\cdot\phi(z)=\int_G f(\gamma)\phi(\gamma^{-1}\cdot z)\,d\lambda^{r(z)}(\gamma), \tag{1}$$

$$\phi\cdot b(z)=\int_H \phi(z\cdot\eta)b(\eta^{-1})\,d\beta^{s(z)}(\eta), \tag{2}$$

$$\langle\phi\,,\,\psi\rangle_\star(\eta)=\int_G \overline{\phi(\gamma^{-1}\cdot z)}\psi(\gamma^{-1}\cdot z\cdot\eta)\,d\lambda^{r(z)}(\gamma) \tag{3}$$

for any $z\in Z$ such that $s(z)=r(\eta)$, and

$${}_\star\langle\phi\,,\,\psi\rangle(\gamma)=\int_H \phi(\gamma\cdot w\cdot\eta)\overline{\psi(w\cdot\eta)}\,d\beta^{s(w)}(\eta) \tag{4}$$

for any $w\in Z$ such that $r(w)=s(\gamma)$.

The content of Renault's Equivalence Theorem ([4, Theorem 2.8]) is that $C_c(Z)$ is a pre-$C_c(G)-C_c(H)$-imprimitivity bimodule with respect to the universal norms on $C_c(G)$ and $C_c(H)$, and that its completion X implements a Morita equivalence between $C^*(G)$ and $C^*(H)$.

We define the opposite space of a (G,H)-equivalence Z to be a homeomorphic copy $Z^{\mathrm{op}}:=\{\overline{z}:z\in Z\}$ of Z with the structure of a (H,G)-equivalence determined by

$$r(\overline{z})=s(z),\quad s(\overline{z})=r(z),\quad \eta\cdot\overline{z}:=\overline{z\cdot\eta^{-1}}\quad\text{and}\quad \overline{z}\cdot\gamma=\overline{\gamma^{-1}\cdot z};$$

and then $C_c(Z^{\mathrm{op}})$ becomes a pre-$C_c(H)-C_c(G)$-imprimitivity bimodule as above. For $\psi\in C_c(Z^{\mathrm{op}})$, define $\psi^*\in C_c(Z)$ by $\psi^*(z):=\overline{\psi(\overline{z})}$. The map $\psi\mapsto\psi^*$ determines an isomorphism from the $C^*(H)-C^*(G)$-imprimitivity bimodule completion of $C_c(Z^{\mathrm{op}})$ to the dual module $\widetilde{\mathsf{X}}$ defined in [13, pp. 49–50].

Since we will sometimes use the bimodules $C_c(Z)$ and $C_c(Z^{\mathrm{op}})$ in close proximity, we will write $\psi{:}f$ and $b{:}\psi$ for the right and left actions on $C_c(Z^{\mathrm{op}})$, respectively, and $\langle\!\langle\cdot\,,\cdot\rangle\!\rangle_\star$ and ${}_\star\langle\!\langle\cdot\,,\cdot\rangle\!\rangle$ for the right and left inner products on $C_c(Z^{\mathrm{op}})$, respectively.

We should mention that there are "one-sided" versions of the equivalence theorems in the literature. Stadler and O'uchi [5] present a definition of a correspondence Z from G to H which implies $C_c(Z)$ can be completed to a $C^*_r(G)-C^*_r(H)$-correspondence Y [5, Theorem 1.4]. That is, Y is a right-Hilbert $C^*_r(H)$-module and there is a homomorphism of $C^*_r(G)$ into the adjointable operators $\mathcal{L}(\mathsf{Y})$ on Y. (A correspondence is also known as a right-Hilbert bimodule.) A (G,H)-equivalence is an example of a Stadler-O'uchi correspondence. While it might be interesting to see if their techniques could be modified to produce a Morita equivalence result, we believe the linking groupoid approach developed in the next section has wider applications.

In addition to the Stadler-O'uochi approach, Renault has another definition of a correspondence Z from G to H in [10, Definition 2.5] which also extends the notion of equivalence.

2. The linking groupoid

Lemma 3. *Suppose that G and H are locally compact Hausdorff groupoids and that Z is a (G,H)-equivalence. Let L be the topological disjoint union*

$$L = G \sqcup Z \sqcup Z^{\mathrm{op}} \sqcup H,$$

and let $L^0 := G^0 \sqcup H^0 \subset L$. Define $r, s : L \to L^0$ to be the maps inherited from the range and source maps on G, Z, Z^{op} and H. Let $L^{(2)} := \{(k,l) \in L \times L : s(k) = r(l)\}$, and let $(k,l) \mapsto kl$ be the map from $L^{(2)}$ to L which restricts to multiplication on G and H and to the actions of G and H on Z and Z^{op}, and satisfies

$$z\overline{y} := {}_G[z,y] \quad \text{for } (z,y) \in Z *_s Z \quad \text{and} \quad \overline{y}z := [y,z]_H \quad \text{for } (y,z) \in Z *_r Z.$$

Define $l \mapsto l^{-1}$ to be the map from L to L which restricts to inversion on G and H and satisfies $z^{-1} = \overline{z}$ and $\overline{z}^{\,-1} = z$ for $z \in Z$. Under these operations, L is a locally compact Hausdorff groupoid, called the linking groupoid *of Z.*

Proof. The inverse map is clearly an involution. Since $[z,z]_H = s(z)$ and ${}_G[z,z] = r(z)$, it is easy to see that the formulas for r and s are satisfied.

The continuity of the inverse map follows from the continuity of the inverse maps on G and H together with the definition of the topology on Z^{op}. The continuity of multiplication follows from continuity of multiplication in G and H, the continuity of the actions of G and H on Z and Z^{op}, and the continuity of $(y,z) \mapsto {}_G[y,z]$ and $(y,z) \mapsto [y,z]_H$.

The associativity of multiplication follows from routine calculations using the associativity of the groupoid operations and actions, and property (c) of the definition of groupoid equivalence. For example, if $x, y, z \in Z$ with $s(x) = s(y)$ and $r(y) = r(z)$, then

$$\begin{aligned}(x\overline{y})z &= {}_G[x,y] \cdot z = {}_G[x,y] \cdot (y \cdot [y,z]_H) \\ &= ({}_G[x,y] \cdot y) \cdot [y,z]_H = x \cdot [y,z]_H \\ &= x(\overline{y}z).\end{aligned}$$

□

Given a (G,H)-equivalence Z, the range map on Z induces a homeomorphism from the orbit space Z/H to $G^{(0)}$. Thus if $u \in G^{(0)}$ and $z \in Z$ with $r(z) = u$, there is a Radon measure σ_Z^u on Z, supported on the orbit $z \cdot H$, determined by

$$\sigma_Z^u(\phi) = \int_H \phi(z \cdot \eta)\, d\beta^{s(z)}(\eta) \quad \text{for } \phi \in C_c(Z). \tag{5}$$

As the notation suggests, σ_Z^u does not depend on the choice of $z \in r^{-1}(u)$: if $y \in Z$ with $r(y) = u$ also, then $y = z \cdot \eta'$ for some $\eta' \in H$ with $r(\eta') = s(z)$, so left-invariance of β gives

$$\int_H \phi(z \cdot \eta)\, d\beta^{s(z)}(\eta) = \int_H \phi(z \cdot \eta'\eta)\, d\beta^{s(\eta')}(\eta) = \int_H \phi(y \cdot \eta)\, d\beta^{s(y)}(\eta).$$

Fix $\phi \in C_c(Z)$. By [4, Lemma 2.9(b)], the map $z \cdot H \mapsto \int_H \phi(z \cdot \eta)\, d\beta^{s(z)}(\eta)$ is continuous on Z/H. Since r induces a homeomorphism of Z/H onto $G^{(0)}$, it follows that there is a continuous function on $C_c(G^{(0)})$ given by

$$u \mapsto \int_Z \phi(z)\, d\sigma_Z^u(z).$$

By symmetry, we can also define a family of measures $\sigma_{Z^{\mathrm{op}}}^v$ on Z^{op} with $\operatorname{supp} \sigma_{Z^{\mathrm{op}}}^v = r_{Z^{\mathrm{op}}}^{-1}(v)$.

Lemma 4. *For each $w \in L^{(0)}$, let κ^w be the Radon measure on L given on $F \in C_c(L)$ by*

$$\kappa^w(F) = \begin{cases} \lambda^w(F|_G) + \sigma_Z^w(F|_Z) & \text{if } w \in G^{(0)}, \text{ and} \\ \sigma_{Z^{\mathrm{op}}}^w(F|_{Z^{\mathrm{op}}}) + \beta^w(F|_H) & \text{if } w \in H^{(0)}. \end{cases}$$

Then $\{ \kappa^w \}_{w \in L^{(0)}}$ is a Haar system for L.

Proof. It is clear that $\operatorname{supp} \kappa^w$ is $r^{-1}(w) = L^w$. Continuity follows from continuity of σ_Z and $\sigma_{Z^{\mathrm{op}}}$ and of the Haar systems λ and β. It only remains to check left invariance.

Thus, we need to establish that for $k \in L$,

$$\int_L F(l)\, d\kappa^{r(k)}(l) = \int_L F(kl)\, d\kappa^{s(k)}(l).$$

For convenience, assume that $r(k) \in G^{(0)}$. (The case where $r(k) \in H^{(0)}$ is similar.) There are two possibilities: $k \in G$, or $k \in Z$. First suppose $k \in G$. Then for any z satisfying $r(z) = s(k)$,

$$\begin{aligned} \int_L F(kl)\, d\kappa^{s(k)}(l) &= \int_G F(k\gamma)\, d\lambda^{s(k)}(\gamma) + \int_H F(k \cdot z \cdot \eta)\, d\beta^{s(z)}(\eta) \\ &= \int_G F(\gamma)\, d\lambda^{r(k)}(\gamma) + \int_H F\big((k \cdot z) \cdot \eta\big)\, d\beta^{s(k \cdot z)}(\eta) \\ &= \int_G F(\gamma)\, d\lambda^{r(k)}(\gamma) + \int_Z F(w)\, d\sigma^{r(k)}(w) \\ &= \int_L F(l)\, d\kappa^{r(k)}(l). \end{aligned}$$

Now suppose that $k \in Z$. Then

$$\int_L F(kl)\, d\kappa^{s(k)}(l) = \int_{Z^{\mathrm{op}}} F(k\overline{z})\, d\sigma_{Z^{\mathrm{op}}}^{s(k)}(\overline{z}) + \int_H F(k \cdot \eta)\, d\beta^{s(k)}(\eta).$$

Since we can evaluate $\sigma_{Z^{\mathrm{op}}}^{s(k)}$ with any $\overline{w}$ such that $r(\overline{w}) = s(k)$, we may in particular take $\overline{w} = \overline{k}$, giving

$$\int_L F(kl)\, d\kappa^{s(k)}(l) = \int_G F\big({}_G[k, \gamma^{-1} \cdot k]\big)\, d\lambda^{r(k)}(\gamma) + \int_Z F(z)\, d\sigma_Z^{r(k)}(z).$$

Since ${}_G[k, \gamma^{-1} \cdot k] = \gamma$ for all γ, we conclude that

$$\int_L F(kl)\, d\kappa^{s(k)}(l) = \int_L F(l)\, d\kappa^{r(k)}(l).$$

□

We will always use the Haar system κ on L, so we will henceforth write $C^*(L)$ in place of $C^*(L,\kappa)$. (Similarly, we will write $C^*(G)$ in place of $C^*(G,\lambda)$ and $C^*(H)$ in place of $C^*(H,\beta)$.)

Recall that there is a unital homomorphism $M : C_b(L^{(0)}) \to M(C^*(L))$ such that for $h \in C_b(L^{(0)})$ and $F \in C_c(L)$,

$$\big(M(h)F\big)(l) = h\big(r(l)\big)F(l) \quad \text{and} \quad \big(FM(h)\big)(l) = F(l)h\big(s(l)\big).$$

In particular, we may regard the characteristic functions p_G and p_H of $G^{(0)}$ and $H^{(0)}$ in $C_b(G^{(0)})$ as complementary projections in $M(C^*(L))$.

For $F \in C_c(L)$, let $F_{11} = F|_G \in C_c(G)$, $F_{12} = F|_Z \in C_c(Z)$, $F_{21} = F|_{Z^{\mathrm{op}}} \in C_c(Z^{\mathrm{op}})$ and $F_{22} = F|_H \in C_c(H)$. We view F as a matrix

$$F = \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix}.$$

The involution on $C_c(L)$ is then given by

$$F^* = \begin{pmatrix} F_{11}^* & F_{21}^* \\ F_{12}^* & F_{22}^* \end{pmatrix},$$

where F_{11}^* and F_{22}^* are the images of F_{11} and F_{22} under the standard involutions on $C_c(G)$ and $C_c(H)$, while $F_{12}^*(\overline{z}) = \overline{F_{12}(z)}$ and $F_{21}^*(z) = \overline{F_{21}(\overline{z})}$ for all $z \in Z$. Straightforward computations show that the convolution product on $C_c(L)$ is given by

$$\begin{aligned} F * K &= \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix} * \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix} \\ &= \begin{pmatrix} F_{11} * K_{11} + \langle\!\langle F_{12}^* \,, K_{21} \rangle\!\rangle_\star & F_{11} \cdot K_{12} + F_{12} \cdot K_{22} \\ F_{21} : K_{11} + F_{22} : K_{21} & \langle F_{21}^* \,, K_{12} \rangle_\star + F_{22} * K_{22} \end{pmatrix} \\ &= \begin{pmatrix} F_{11} * K_{11} + {}_\star\langle F_{12} \,, K_{21}^* \rangle & F_{11} \cdot K_{12} + F_{12} \cdot K_{22} \\ (K_{11}^* \cdot F_{21}^*)^* + (K_{21}^* \cdot F_{22}^*)^* & \langle F_{21}^* \,, K_{12} \rangle_\star + F_{22} * K_{22} \end{pmatrix}. \end{aligned}$$

A routine norm calculation shows that we can identify $C_c(L)$ with a dense subalgebra of the linking algebra $L(\mathsf{X})$.

Lemma 5. *The complementary projections p_G and p_H are full in $M(C^*(L))$.*

Proof. By symmetry, it will suffice to see that p_G is full. For $F, K \in C_c(L)$,

$$\begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix} * p_G * \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix} = \begin{pmatrix} F_{11} * K_{11} & F_{11} \cdot K_{12} \\ F_{21} \cdot K_{11} & \langle F_{21}^* \,, K_{12} \rangle_\star \end{pmatrix}. \tag{6}$$

So it suffices to see that elements of the form appearing on the right-hand side of (6) span a dense subspace of $C^*(L)$ in the inductive-limit topology. That elements of the form $F_{11} * K_{11}$ span a dense subspace of $C_c(G)$ and that elements of the form $F_{11} \cdot K_{12}$ span a dense subspace of $C_c(Z)$ follow from the existence of an approximate identity in $C_c(G^{(0)})$ for the left actions of $C_c(G)$ on both itself and $C_c(Z)$ (see [4, Proposition 2.10]). That elements of the form $F_{21} \cdot K_{11}$ span a dense subspace of $C_c(Z^{\mathrm{op}})$ follows from the corresponding property for $C_c(H)$. That the image of $\langle \cdot \,, \cdot \rangle_\star$ is dense in $C_c(H)$ follows from [4, Proposition 2.10] using standard techniques as in [14, p. 115] (see the proof of [4, Theorem 2.8]). □

Remark 6 (Our proofs of the equivalence theorems). By Lemma 5 and [13, Theorem 3.19], to prove the Equivalence Theorem for the full groupoid C^*-algebras, it suffices to show that $p_G C^*(L) p_G \cong C^*(G)$ and similarly for $C^*(H)$; that is, to show that the norms on $C^*(L)$ and $C^*(G)$ agree on the subalgebra $C_c(G)$. Indeed, let $\|\cdot\|_\alpha$ be any pre-C^*-norm on $C_c(L)$ which is continuous in the inductive-limit topology. Then $\|\cdot\|_\alpha$ is dominated by the universal norm, so the completion $C^*_\alpha(L)$ is a quotient of $C^*(L)$ whose multiplier algebra contains $C_b(L^{(0)})$. The projections p_G and p_H are complementary full projections, and $p_G C^*_\alpha(L) p_G$ is isomorphic to the $\|\cdot\|_\alpha$-completion, $C^*_\alpha(G)$, of $C_c(G)$. A similar statement holds for H. Hence $p_G C^*_\alpha(L) p_H$, which is isomorphic to the $\|\cdot\|_\alpha$-completion of $C_c(Z)$, is a $C^*_\alpha(G) - C^*_\alpha(H)$-imprimitivity bimodule ([13, Theorem 3.19]). So to prove the equivalence theorem for reduced groupoid C^*-algebras, it will suffice to show that the reduced norms on $C^*_r(L)$ and $C^*_r(G)$ agree on the subalgebra $C_c(G)$, and similarly for H.

We will indeed prove (in Proposition 15) that the universal norms on $C^*(L)$ and $C^*(G)$ coincide on $C_c(G)$, and similarly for H. But our proof requires Renault's Disintegration Theorem [8, Theorem 7.8] as well as the basic set-up of [4, Theorem 2.8]. So our proof of the equivalence theorem via the linking groupoid does not substantially simplify the original proof.

By contrast, when we show in Theorem 13 that the reduced norms on $C^*_r(L)$ and $C^*_r(G)$ coincide on $C_c(G)$, we require only the algebraic machinery from [4, Theorem 2.8] and the approximate identity of [4, Proposition 2.10] as required to prove Lemma 5. In particular, our proof of the equivalence theorem for reduced C^*-algebras does not require the Disintegration Theorem.

3. Regular Representations

If μ is a finite Radon measure on $G^{(0)}$, we can form the Radon measure $\nu := \mu \circ \lambda$ on G given on $f \in C_c(G)$ by

$$\nu(f) = \int_{G^{(0)}} \int_G f(\gamma)\, d\lambda^u(\gamma)\, d\mu(u).$$

We write ν^{-1} for the image of ν under inversion. The associated *regular representation* $\operatorname{Ind}\mu$ is the representation on $L^2(G, \nu^{-1})$ given by

$$(\operatorname{Ind}\mu)(f)\xi(\gamma) = \int_G f(\eta)\xi(\eta^{-1}\gamma)\, d\lambda^{r(\gamma)}(\eta) \quad \text{for } f \text{ and } \xi \text{ in } C_c(G).$$

One can check that $\operatorname{Ind}\mu$ is a bounded representation of $C^*(G)$ either by appealing to the general theory of induction as in [2, §2], or — with some effort, but without recourse to the equivalence theorem for full groupoid C^*-algebras upon which [2, §2] depends — by verifying directly verifying directly that $\|(\operatorname{Ind}\mu)(f)\| \le \|f\|_I$ for $f \in C_c(G)$ and extending to the completions.

If $u \in G^{(0)}$ and δ_u is the point mass, then the representation $\operatorname{Ind}\delta_u$ is simply the representation of $C_c(G)$ on $L^2(G_u, \lambda_u)$ given by the convolution formula. By definition, the *reduced norm* on $C_c(G)$ is

$$\|f\|_r = \sup\{\, \|(\operatorname{Ind}\delta_u)(f)\| : u \in G^{(0)} \,\}.$$

So $C^*_r(G)$ is the quotient of $C^*(G)$ by

$$I_{C^*_r(G)} := \bigcap_{u \in G^{(0)}} \ker\bigl(\operatorname{Ind}\delta_u\bigr).$$

Alternatively, one can think of $C_r^*(G)$ as the completion of $C_c(G)$ with respect to the reduced norm $\|\cdot\|_r$.

There is some inconsistency in the literature concerning the definition of $\|\cdot\|_r$. The definition given above coincides with that given in [1, §6.1] and the unpublished notes [6, Definition 2.46]. However, the definition in Renault's original [11, Definition II.2.8] takes the supremum over all $\operatorname{Ind}\mu$. We take a moment just to make sure everyone is talking about the same norm (see Corollary 11). Let X be a second countable free and proper left G-space. Then $G\backslash X$ is a locally compact Hausdorff space, and for each $x \in X$, the map $\gamma \mapsto \gamma \cdot x$ is a homeomorphism of $G_{r(x)}$ onto the orbit $G \cdot x$. Just as for the measures σ_Z^u defined in (5), we define a Radon measure $\rho^{G\cdot x}$ on X with support $G \cdot x$ by

$$\rho^{G\cdot x}(f) = \int_X f(y)\,d\rho^{G\cdot x}(y) := \int_G f(\gamma^{-1}\cdot x)\,d\lambda^{r(x)}(\gamma) \quad \text{for } f \in C_c(X).$$

Our definition is independent of our choice of x in its orbit by left-invariance of the Haar system λ. By [4, Proposition 2.9(b)], the map

$$G\cdot x \mapsto \int_X f(y)\,d\rho^{G\cdot x}(y)$$

is continuous on $G\backslash X$. Given a finite Radon measure μ on $G\backslash X$, we define a Radon measure ρ_μ on X by

$$\rho_\mu(f) = \int_{G\backslash X}\int_X f(y)\,\rho^{G\cdot x}(y)\,d\mu(G\cdot x).$$

View $\mathcal{H}_0 = C_c(X)$ as a dense subspace of $L^2(X, \rho_\mu)$, and let $\operatorname{Lin}(\mathcal{H}_0)$ be the vector space of linear operators on $\mathcal{H}_0$. Right multiplication under the convolution product on $C_c(G)$ determines a homomorphism $R_\mu^X : C_c(G) \to \operatorname{Lin} C_c(X)$, and some tedious computations show that R_μ^X is a homomorphism satisfying the hypotheses of Renault's Disintegration Theorem (see [8, Theorem 7.8]).[1] Hence R_μ^X is bounded and extends to a representation of $C^*(G)$ on $L^2(X, \rho_\mu)$ also denoted by R_μ^X. Of course, the regular representations $\operatorname{Ind}\mu$ above are special cases of the R_μ^X obtained by letting $X = G$.

Remark 7 (The κ_w). We will need to use the Radon measures $\{\kappa_w\}_{w\in L^{(0)}}$ on L, where κ_w is the forward image of the measure κ^w of Lemma 3 under inversion. It is not hard to check that for $F \in C_c(L)$ we have

$$\kappa_w(F) = \begin{cases} \lambda_w(F|_G) + \rho^w_{Z^{\mathrm{op}}}(F|_{Z^{\mathrm{op}}}) & \text{if } w \in G^{(0)}, \text{ and} \\ \rho^w_Z(F|_Z) + \beta_w(F|_H) & \text{if } w \in H^{(0)}. \end{cases}$$

Example 8. Let μ be the point mass $\delta_{G\cdot x_0}$. Then $L^2(X, \rho_\mu) \cong L^2(G\cdot x_0, \rho^{G\cdot x_0})$ and the homeomorphism $\gamma \mapsto \gamma\cdot x_0$ of $G_{r(x_0)}$ onto $G\cdot x_0$ induces a unitary which intertwines $R^X_{\delta_{G\cdot x_0}}$ and $\operatorname{Ind}\delta_{r(x_0)}$.

Example 9. Let X be any second countable free and proper left G-space, let μ be a finite Radon measure on $G\backslash X$ and let $\rho^{G\cdot x}$ and ρ_μ be as above. Let $\mathscr{H} = \coprod_{G\cdot x\in G\backslash X} L^2(X, \rho^{G\cdot x})$. If $\{f_i\}$ is a countable set in $C_c(X)$ which is dense in the inductive-limit topology, then each f_i defines a section of $\mathscr{H}$ by $f_i(G\cdot x)(y) = f(y)$.

[1] We called R_μ^X a *pre-representation* in [7, Definition 4.1]. See Appendix A for the definition and more details.

Then [14, Proposition F.8] implies that there is a Borel Hilbert bundle $(G\backslash X)*\mathscr{H}$ such that $\{f_i\}$ is a fundamental sequence (see [14, Definition F.1]) with the property that $L^2(X,\rho_\mu)$ is isomorphic to $L^2((G\backslash X)*\mathscr{H},\mu)$. Furthermore, the representation R^X_μ is equivalent to the direct integral

$$\int_{G\backslash X}^{\oplus} R^X_{\delta_{G\cdot x}}\,d\mu(G\cdot x).$$

Part of the point of Examples 8 and 9 is the following observation.

Lemma 10. *If X is a second countable free and proper left G-space and if μ is a finite Radon measure on $G\backslash X$, then the representations R^X_μ factor through $C^*_r(G)$.*

Proof. Using the direct integral realization of R^X_μ in Example 9 (and the fact that the map $r : X \to G^{(0)}$ is surjective), we clearly have

$$\ker R^X_\mu \supset \bigcap_{G\cdot x\in G\backslash X} \ker R^X_{\delta_{G\cdot x}} = \bigcap_{x\in X} \ker \operatorname{Ind}\delta_{r(x)} = \bigcap_{u\in G^{(0)}} \ker \operatorname{Ind}\delta_u = I_{C^*_r(G)}. \quad \square$$

Since we obtain the $\operatorname{Ind}\mu$ as examples of the R^X_μ (by taking $X = G$), we obtain the following.

Corollary 11. *Suppose G is a second countable locally compact Hausdorff groupoid. Then for all $f \in C_c(G)$,*

$$\|f\|_r = \sup\{\, \|(\operatorname{Ind}\mu)(f)\| : \mu \text{ is a finite Borel measure on } G^{(0)} \,\}.$$

Remark 12. Alternatively, we could take the supremum of the $\|R^X_\mu(f)\|$ ranging over all second countable free and proper G-spaces X, and all finite Radon measures on $G\backslash X$.

4. The equivalence theorem for reduced groupoid C^*-algebras

As mentioned in Remark 6, now that we have the linking groupoid together with its Haar system, the proof that an equivalence induces a Morita equivalence of the reduced algebras is fairly close to the surface and does not require the full power of the equivalence result for the universal algebras.

Theorem 13. *Suppose that G and H are second countable locally compact Hausdorff groupoids with Haar systems as above, and suppose that Z is a (G,H)-equivalence. If $f \in C_c(G)$, and*

$$F := \begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix} \in C_c(L),$$

then $\|F\|_{C^*_r(L)} = \|f\|_{C^*_r(G)}$. In particular, the completion X_r of $C_c(Z)$ in the norm $\|x\| := \|\langle x\ ,\ x\rangle_\star\|^{1/2}_{C^*_r(G)}$, equipped with the actions and inner products given in (1)–(4), is a $C^*_r(G)$–$C^*_r(H)$-imprimitivity bimodule isometrically isomorphic to $p_G C^*_r(L) p_H$. Hence $C^*_r(G)$ and $C^*_r(H)$ are Morita equivalent.

Remark 14. In the proof of Theorem 13 we will use the notation $\rho^u_{Z^{\mathrm{op}}}$ for the Radon measure on Z^{op} which is the image of σ^u_Z on Z under inversion. Although we don't need to describe $\rho^u_{Z^{\mathrm{op}}}$ for the proof of the theorem, for the sake of symmetry, we note that it is the Radon measure on Z^{op} supported on Z^{op}_u such that for all $\psi \in C_c(Z^{\mathrm{op}})$

$$\rho^u_{Z^{\mathrm{op}}}(\psi) = \int_H \psi(\eta^{-1}\cdot\overline{z_0})\,d\beta^{r(\overline{z})}(\eta),$$

for any $\overline{z_0}$ such that $s(\overline{z_0}) = u$. Thus after identifying $H \cdot \overline{z_0}$ with u, $\rho^u_{Z^{\mathrm{op}}}$ is the measure on the free and proper left H-space Z^{op} defined in Section 3.

Proof. Fix $f \in C_c(G)$ and let F be the corresponding element of $p_G C_c(L) p_G \subset C_c(L)$. The theorem follows from Remark 6 once we establish that $\|F\|_{C^*_r(L)} = \|f\|_{C^*_r(G)}$.

For $u \in G^{(0)}$, we have $L_u = G_u \sqcup Z^{\mathrm{op}}_u$, where $Z^{\mathrm{op}}_u := \{\, \overline{z} \in Z^{\mathrm{op}} : s(\overline{z}) = r(z) = u \,\}$. By definition, $\operatorname{Ind}^L \delta_u$ acts on $L^2(L_u, \kappa_u)$. Following Remark 7, $L^2(L_u, \kappa_u) = L^2(G, \lambda_u) \oplus L^2(Z, \rho^u_{Z^{\mathrm{op}}})$, and with respect to this decomposition, $(\operatorname{Ind}^L \delta_u)(F) = (\operatorname{Ind}^G \delta_u)(f) \oplus 0$. It follows that

$$\begin{aligned} \|F\|_{C^*_r(L)} &:= \max\Big\{ \sup_{u \in G^{(0)}} \|(\operatorname{Ind}^L \delta_u)(F)\|, \sup_{v \in H^{(0)}} \|(\operatorname{Ind}^L \delta_v)(F)\| \Big\} \\ &= \max\Big\{ \|f\|_{C^*_r(G)}, \sup_{v \in H^{(0)}} \|(\operatorname{Ind}^L \delta_v)(F)\| \Big\}. \end{aligned} \tag{7}$$

For $v \in H^{(0)}$, let $Z_v = \{\, z \in Z : s(z) = v \,\}$. Then $L_v = Z_v \sqcup H_v$. Furthermore, $L^2(L_v, \kappa_v) = L^2(Z, \rho^v_Z) \oplus L^2(H, \beta_v)$. Here ρ^v_Z is the image of $\sigma^v_{Z^{\mathrm{op}}}$ under inversion. It is the Radon measure on Z with support Z_v given on $\phi \in C_c(Z)$ by

$$\rho^v_Z(\phi) = \int_G \phi(\gamma^{-1} \cdot z_0) \, d\lambda^{r(z_0)}(\gamma)$$

for any $z_0 \in Z$ such that $s(z_0) = v$. Thus, the identification of $H^{(0)}$ and $G \backslash Z$ induced by the source map on Z carries ρ^v_Z to the measure on the free and proper G-space Z defined in Section 3. Hence $(\operatorname{Ind}^L \delta_v)(F) = R^Z_{\delta_{G \cdot x_0}}(f) \oplus 0$. By Example 8, we have $\|R^Z_{\delta_{G\cdot x_0}}(f)\| \le \|f\|_{C^*_r(G)}$. It follows from (7) that $\|F\|_{C^*_r(L)} = \|f\|_{C^*_r(G)}$. □

5. The universal norm and the linking algebra

Proposition 15. *Suppose that G and H are second countable locally compact groupoids with Haar systems, and that Z is a (G, H)-equivalence. Let L be the linking groupoid. If $f \in C_c(G)$ and*

$$F := \begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix}$$

is the corresponding element of $C_c(L)$, then $\|F\|_{C^*(L)} = \|f\|_{C^*(G)}$.

Proof. Since every representation of $C_c(L)$ restricts to a representation of $C_c(G)$ (possibly on a subspace of the original representation), we certainly have $\|F\|_{C^*(L)} \le \|f\|_{C^*(G)}$.

To obtain the reverse inequality, let π be a faithful representation of $C^*(G)$ on $\mathcal{H}_\pi$. By the universal properties of the tensor product, there is a sesquilinear form $(\cdot \mid \cdot)_\pi$ on the algebraic tensor product $\mathcal{H}_{00} := C_c(L) p_G \odot \mathcal{H}_\pi$ such that for F and K in $C_c(L)$ we have

$$\begin{aligned} (F * p_G \otimes \xi \mid K * p_G \otimes \zeta)_\pi &= \big(\pi(p_G * K^* * F * p_G)\xi \mid \zeta\big) \\ &= \big(\pi\big(K^*_{11} * F_{11} + \langle\!\langle K_{12} \,,\, F_{21} \rangle\!\rangle_\star\big)\xi \mid \zeta\big). \end{aligned}$$

We want to see that $(\cdot \mid \cdot)_\pi$ is positive. Fix $t = \sum_{i=1}^n F^i \otimes \xi_i \in \mathcal{H}_{00}$. Since [4, Theorem 2.8] applied to the (H, G)-equivalence Z^{op} implies that $\langle\!\langle \cdot \,,\, \cdot \rangle\!\rangle_\star$ makes $C_c(Z^{\mathrm{op}})$ into a pre-Hilbert $C^*(G)$-module, [13, Lemma 2.65] implies that the matrix

$M = \big(\langle\!\langle F_{21}^i\ ,\ F_{21}^j\rangle\!\rangle_{\star}\big)_{ij}$ is positive in $M_n\big(C^*(G)\big)$. Hence $M = D^*D$ for some $D \in M_n(C^*(G))$, so there are elements $d_{ij} \in C^*(G)$ such that

$$\langle\!\langle F_{21}^i\ ,\ F_{21}^j\rangle\!\rangle_{\star} = \sum_{i=1}^{n} d_{ki}^* d_{kj}.$$

Since $((F^j)^*)_{12} = (F_{21}^j)^*$,

$$\begin{aligned}(t \mid t)_\pi &= \sum_{ij} \big(\pi\big((F_{11}^i)^* * F_{11}^j + \langle\!\langle F_{21}^i\ ,\ F_{21}^j\rangle\!\rangle_{\star}\big)\xi_j \mid \xi_i\big)\\ &= \sum_{ij} \big(\pi(F_{11}^j)\xi_j \mid \pi(F_{11}^i)\xi_i\big) + \sum_{ijk} \big(\pi(d_{kj})\xi_j \mid \pi(d_{ki})\xi_i\big)\\ &= \Big(\sum_i \pi(F_{11}^i)\xi_i \mid \sum_i \pi(F_{11}^i)\xi_i\Big) + \sum_k \Big(\sum_i \pi(d_{ki})\xi_i \mid \sum_i \pi(d_{ki})\xi_i\Big) \geq 0.\end{aligned}$$

Therefore $(\cdot \mid \cdot)_\pi$ is a pre-inner product on $\mathcal{H}_{00}$. Let $\mathcal{N}$ denote the subspace $\{\xi \in \mathcal{H}_{00} : (\xi \mid \xi)_\pi = 0\}$. Then the Cauchy-Schwarz inequality (as in [9, §3.1.1]) implies that $(\cdot \mid \cdot)_\pi$ descends to a bona fide inner product on the quotient $\mathcal{H}_0 = \mathcal{H}_{00}/\mathcal{N}$. Furthermore, for each $F \in C_c(L)$, we can define a linear map $R(F) : \mathcal{H}_{00} \to \mathcal{H}_{00}$ such that

$$R(F)(K \otimes \xi) := F * K \otimes \xi.$$

Another application of the Cauchy-Schwarz inequality shows that $R(F)$ defines an operator on $\mathcal{H}_0$. An easy calculation shows that

$$(R(F)t \mid t')_\pi = (t \mid R(F^*)t')_\pi \quad \text{for } t, t' \in \mathcal{H}_{00}. \tag{8}$$

Furthermore, since π is continuous in the inductive-limit topology, it is not hard to see that

$$F \mapsto (R(F)t \mid t')_\pi \tag{9}$$

is also continuous in the inductive-limit topology. Since $C_c(L)$ has an approximate unit for the inductive-limit topology,

$$\operatorname{span}\{\, R(F)t : F \in C_c(L) \text{ and } t \in \mathcal{H}_{00} \,\} \tag{10}$$

is dense in $\mathcal{H}_{00}$. Equations (8), (9) and (10) imply that $R : C_c(L) \to \operatorname{Lin}(\mathcal{H}_0)$ satisfy the hypotheses of the Disintegration Theorem [4, Theorem 2.8] as outlined in Appendix A, and therefore R is a bounded representation of $C^*(L)$ on the completion $\mathcal{H}_R$ of $\mathcal{H}_0$.

Since π is faithful, it suffices to show that

$$\|R(F)\|_{C^*(L)} \geq \|\pi(f)\| = \|f\|_{C^*(G)}. \tag{11}$$

Fix $\epsilon \in (0, \|f\|)$ and fix $\xi \in \mathcal{H}_\pi$ such that $\|\xi\| = 1$ and $\|\pi(f)\xi\|^2 > \|\pi(f)\|^2 - \epsilon$. Let $\{\, k_\alpha \,\}$ be an approximate identity in $C_c(G)$ for the inductive-limit topology, and let

$$K_\alpha = \begin{pmatrix} k_\alpha & 0 \\ 0 & 0 \end{pmatrix}$$

be the corresponding functions in $C_c(L)$. Then, since π is nondegenerate,

$$\lim_\alpha \|K_\alpha \otimes \xi\|^2 = \lim_\alpha \big(\pi(k_\alpha^* * k_\alpha)\xi \mid \xi\big) = \lim_\alpha \|\pi(k_\alpha)\xi\|^2 = 1.$$

It follows that

$$\|R(F)\|^2 \geq \limsup_\alpha \|R(F)(K_\alpha) \otimes \xi\|^2 = \limsup_\alpha \big(\pi\big(f^* * k_\alpha^* * k_\alpha * f\big)\xi \mid \xi\big)$$

$$= \lim_{\alpha} \|\pi(k_\alpha)\pi(f)\xi\|^2 = \|\pi(f)\xi\|^2 > \|\pi(f)\|^2 - \epsilon.$$

Since ϵ is arbitrary, (11) holds. This completes the proof. □

As an immediate consequence of Proposition 15 and Remark 6, we get the following.

Corollary 16. *Suppose that G and H are second countable locally compact groupoids with Haar systems, and that Z is a (G, H)-equivalence. If X is the corresponding $C^*(G)-C^*(H)$-imprimitivity bimodule and if L is the linking groupoid, then $C^*(L)$ is isomorphic to the linking algebra $L(\mathsf{X})$.*

Recall that if X is an $A-B$-imprimitivity bimodule, then the Rieffel correspondence provides a lattice isomorphism $\mathsf{X}\text{-}\operatorname{Ind}$ from the lattice of ideals $\mathscr{I}(B)$ of B and the lattice of ideals $\mathscr{I}(A)$ in A [13, Theorem 3.22]. We can now prove the second part of our main result.

Theorem 17. *Suppose that G and H are second countable locally compact groupoids with Haar systems, and that Z is a (G, H)-equivalence. Let X be the associated $C^*(G)-C^*(H)$-imprimitivity bimodule. Then $\mathsf{X}\text{-}\operatorname{Ind}(I_{C^*_r(H)}) = I_{C^*_r(G)}$. Furthermore if X_r is the $C^*_r(G)-C^*_r(H)$-imprimitivity bimodule of Theorem 13, then the identity map from $C_c(Z) \subset \mathsf{X}$ to $C_c(Z) \subset \mathsf{X}_r$ induces an isomorphism of the quotient imprimitivity bimodule $\mathsf{X}/\mathsf{X}\cdot I_{C^*_r(H)}$ onto X_r.*

Proof. If $\phi \in C_c(Z)$, then

$$\|\phi\|^2_{\mathsf{X}} = \|\langle \phi\,,\,\phi\rangle_\star\|_{C^*(H)} \geq \|\langle \phi\,,\,\phi\rangle_\star\|_{C^*_r(H)} = \|\phi\|_{\mathsf{X}_r}.$$

Therefore the identity map from $C_c(Z) \subset \mathsf{X}_r$ to $C_c(Z) \subset \mathsf{X}$ induces a surjection of X onto X_r. Let Y denote the kernel of this surjection. Then Y is a closed sub-bimodule of X such that X_r is isomorphic to X/Y as imprimitivity bimodules.

The Rieffel correspondence (in the form of [13, Theorem 3.22] and [13, Lemma 3.23]) implies that

$$\mathsf{Y} = \mathsf{X}\cdot I = J\cdot \mathsf{X},$$

where I and J are ideals in $C^*(H)$ and $C^*(G)$, respectively, such that $\mathsf{X}\text{-}\operatorname{Ind}(I) = J$, and where

$$I = \overline{\operatorname{span}}\{\,\langle x\,,\,y\rangle_\star : x \in \mathsf{X} \text{ and } y \in \mathsf{Y}\,\} = \overline{\operatorname{span}}\{\,\langle y\,,\,y\rangle_\star : y \in Y\,\}.$$

Thus $I \subset I_{C^*_r(H)}$. On the other hand, if $b \in I_{C^*_r(H)}$, then for all x and y in X, we have $\langle x\,,\,y\rangle_\star b = \langle x\,,\,y\cdot b\rangle_\star \in I$. Since $\langle \cdot\,,\,\cdot\rangle_\star$ is full, it follows that $b \in I$. Therefore $I = I_{C^*_r(H)}$. Similarly, we also must have $J = I_{C^*_r(G)}$. This completes the proof. □

Corollary 18. *Suppose that G, H and Z are as in Theorem 17. If π is a representation of $C^*(H)$ that factors through $C^*_r(H)$, then $\mathsf{X}\text{-}\operatorname{Ind}\pi$ factors through $C^*_r(G)$.*

Proof. By assumption, $I_{C^*_r(H)} \subset \ker\pi$. But then by [13, Proposition 3.24],

$$I_{C^*_r(G)} = \mathsf{X}\text{-}\operatorname{Ind}(I_{C^*_r(H)}) \subset \mathsf{X}\text{-}\operatorname{Ind}(\ker\pi) = \ker(\mathsf{X}\text{-}\operatorname{Ind}\pi).$$ □

Appendix A. Separability Hypotheses in the Disintegration Theorem

Let G be a second countable locally compact Hausdorff[2] groupoid. A *pre-representation* of $C_c(G)$ on a dense subspace $\mathcal{H}_0$ of a Hilbert space $\mathcal{H}$ is a homomorphism $L : C_c(G) \to \operatorname{Lin}(\mathcal{H}_0)$ with the following properties.

(a) For $f \in C_c(G)$ and $h, k \in \mathcal{H}_0$, $\big(L(f)h \mid k\big) = \big(h \mid L(f^*)k\big)$.

(b) For each $h, k \in \mathcal{H}_0$, $f \mapsto \big(L(f)h \mid k\big)$ is continuous in the inductive-limit topology on $C_c(G)$.

(c) The subspace $\operatorname{span}\{\, L(f) : f \in C_c(G) \text{ and } h \in \mathcal{H}_0 \,\}$ is dense in $\mathcal{H}$.

Renault's Disintegration Theorem implies that *if $\mathcal{H}$ is separable*, then L is the restriction of a representation $\bar{L}$ on $\mathcal{H}$ which is equivalent to the integrated form of a unitary representation of G. In particular, L is bounded in the $\|\cdot\|_I$-norm; indeed, $\|L(f)\| \le \|f\|_I$ for all $f \in C_c(G)$.

Conversely, if L is $\|\cdot\|_I$-bounded, L extends to a representation $\bar{L}$ via standard arguments.

Unfortunately, the hypothesis that $\mathcal{H}$ (or equivalently, $\mathcal{H}_0$) have a countable dense subset was omitted from the statement of the Disintegration Theorem in [8, Theorem 7.8] as well as in its generalizations in [8, Theorem 7.12] and [7, Theorem 4.13]. Although separability was a standing assumption in both [8] and [7], the omission of this hypothesis in the statements of the Disintegration results was, well, misleading at best. (Note that $\mathcal{H}$ must be separable if $\bar{L}$ is to be equivalent to the integrated form of some unitary representation. The later acts on a direct integral of Hilbert spaces, and that theory only makes sense in the presence of separability.)

Remark 19 (Arbitrary $\mathcal{H}_0$). Fortunately, in most applications, and in particular in the applications in this paper, we only want to invoke the Disintegration Theorem to show that L is bounded and therefore extends to a bona fide representation of $C^*(L)$ on $\mathcal{H}$. (That is, it is not necessary to show that L is the integrated form of a unitary representation.) *When this is the case, we do not need the hypothesis that $\mathcal{H}_0$ is separable.* To see that L is bounded, we just need to establish that for each $h_0 \in \mathcal{H}_0$ of norm one, $\|L(f)h_0\| \le \|f\|_I$. For this, it suffices to consider the restriction of L to the cyclic subspace

$$\mathcal{H}_{00} := \{\, L(f)h_0 : f \in C_c(G) \,\}.$$

Then L defines a pre-representation $L_0 : C_c(G) \to \operatorname{Lin}(\mathcal{H}_{00})$. Since G is second countable, $C_c(G)$ has a countable dense set $\{f_i\}$ in the inductive-limit topology, and the continuity condition of a pre-representation implies that $\{L(f_i)h_0\}$ is dense in $\mathcal{H}_{00}$. Then the Disintegration Theorem applies to L_0, and

$$\|L(f)h_0\| = \|L_0(f)h_0\| \le \|f\|_I.$$

Therefore L is bounded on $\mathcal{H}_0$ and extends as claimed.

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[2] After replacing $C_c(G)$ with the vector space $\mathscr{C}(G)$ of functions generated by the functions in $C_c(V)$ for Hausdorff open sets $V \subset G$, the remarks in this appendix apply equally well to second countable locally compact, locally Hausdorff groupoids as studied in [8].

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School of Mathematics and Applied Statistics, University of Wollongong, NSW 2522, Australia
E-mail address: asims@uow.edu.au

Department of Mathematics, Dartmouth College, Hanover, NH 03755-3551
E-mail address: dana.williams@Dartmouth.edu