PRODUCT SYSTEMS OF GRAPHS AND THE TOEPLITZ ALGEBRAS OF HIGHER-RANK GRAPHS

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Abstract. There has recently been much interest in the C^*-algebras of directed graphs. Here we consider product systems E of directed graphs over semigroups and associated C^*-algebras $C^*(E)$ and $\mathcal{T}C^*(E)$ which generalise the higher-rank graph algebras of Kumjian-Pask and their Toeplitz analogues. We study these algebras by constructing from E a product system $X(E)$ of Hilbert bimodules, and applying recent results of Fowler about the Toeplitz algebras of such systems. Fowler's hypotheses turn out to be very interesting graph-theoretically, and indicate new relations which will have to be added to the usual Cuntz-Krieger relations to obtain a satisfactory theory of Cuntz-Krieger algebras for product systems of graphs; our algebras $C^*(E)$ and $\mathcal{T}C^*(E)$ are universal for families of partial isometries satisfying these relations.

Our main result is a uniqueness theorem for $\mathcal{T}C^*(E)$ which has particularly interesting implications for the C^*-algebras of non-row-finite higher-rank graphs. This theorem is apparently beyond the reach of Fowler's theory, and our proof requires a detailed analysis of the expectation onto the diagonal in $\mathcal{T}C^*(E)$.

1. Introduction

The C^*-algebras $C^*(E)$ of infinite directed graphs E are generalisations of the Cuntz-Krieger algebras which include many interesting C^*-algebras and provide a rich supply of models for simple purely infinite algebras (see, for example, [13, 3, 9, 19]). In the first papers, it was assumed for technical reasons that the graphs were locally finite. However, after $C^*(E)$ had been realised as the Cuntz-Pimsner algebra $\mathcal{O}_{X(E)}$ of a Hilbert bimodule $X(E)$ in [7], it was noticed that $\mathcal{O}_{X(E)}$ made sense for arbitrary infinite graphs. The analysis in [7] applied to the Toeplitz algebra $\mathcal{T}_{X(E)}$ rather than $\mathcal{O}_{X(E)}$, but the two coincide for some infinite graphs E, and hence the results of [7] gave information about $\mathcal{O}_{X(E)}$ for these graphs. The results of [7] therefore suggested an appropriate definition of $C^*(E)$ for arbitrary E, which was implemented in [6].

Higher-rank analogues of Cuntz-Krieger algebras and of the C^*-algebras of row-finite graphs have been studied by Robertson-Steger [18] and Kumjian-Pask [11], respectively. It was observed in [8] that the higher-rank graphs of Kumjian and Pask could be viewed as product systems of graphs over the semigroup $\mathbb{N}^k$. The main object of this paper is to extend the construction $E \mapsto X(E)$ to product systems of graphs over $\mathbb{N}^k$ and other semigroups, to apply the results of [5] to the resulting product systems of Hilbert bimodules, and to see what insight might be gained into the C^*-algebras of arbitrary higher-rank graphs.

It is relatively easy to extend the construction of $X(E)$ to product systems, and to identify *Toeplitz E-families* which correspond to the Toeplitz representations of $X(E)$

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studied in [5]. The story becomes interesting when we investigate the conditions on E and on Toeplitz E-families which ensure that we can apply [5, Theorem 7.2] to the corresponding representation of $X(E)$. To understand the issues, we digress briefly.

The isometric representation theory of semigroups suggests that in general $\mathcal{T}_{X(E)}$ will be too big to behave like a Cuntz-Krieger algebra, and that we should restrict attention to the Nica-covariant representations of [15, 14, 4, 5]. However, Nica covariance is in general a spatial phenomenon, and to talk about the universal C^*-algebra $\mathcal{T}_{\text{cov}}(X)$ generated by a Nica-covariant Toeplitz representation of a product system X of bimodules, we need to assume that X is compactly aligned in the sense of [4, 5].

We identify the *finitely aligned* product systems E of graphs for which $X(E)$ is compactly aligned, and the *Toeplitz-Cuntz-Krieger E-families* $\{S_\lambda\}$ which correspond to Nica-covariant Toeplitz representations of $X(E)$. The C^*-algebra generated by $\{S_\lambda\}$ is then spanned by the products $S_\lambda S_\mu^*$, as Cuntz-Krieger algebras and their Toeplitz analogues are. We therefore define the Toeplitz algebra $\mathcal{T}C^*(E)$ of a finitely aligned product system E to be the universal C^*-algebra generated by a Toeplitz-Cuntz-Krieger E-family; for technical reasons, we only define the Cuntz-Krieger algebra $C^*(E)$ to be the appropriate quotient of $\mathcal{T}C^*(E)$ when E has no sinks.

Fowler's [5, Theorem 7.2] gives a spatial condition under which a Nica-covariant Toeplitz representation of a compactly aligned product system X of Hilbert bimodules is faithful on $\mathcal{T}_{\text{cov}}(X)$. Since $\mathcal{T}C^*(E)$ has essentially the same representation theory as $\mathcal{T}_{\text{cov}}(X(E))$, Fowler's theorem describes some faithful representations of $\mathcal{T}C^*(E)$. However, the resulting theorem about Toeplitz-Cuntz-Krieger E-families is not as sharp as we would like, for the same reasons that [7, Theorem 2.1] is not: applied to the single graph E with $\mathcal{T}C^*(E) = \mathcal{O}_\infty$, it says that isometries $\{S_i\}$ satisfying $1 > \sum_{i=1}^{\infty} S_i S_i^*$ generate an isomorphic copy of $\mathcal{O}_\infty$, whereas we know from [1] that $1 \geq \sum_{i=1}^{\infty} S_i S_i^*$ suffices. Our main theorem is sharp in this sense: it is an analogue of [7, Theorem 3.1] rather than [7, Theorem 2.1]. It suggests an appropriate set of Cuntz-Krieger relations for product systems of not-necessarily-row-finite graphs, and gives a uniqueness theorem of Cuntz-Krieger type for k-graphs in which each vertex receives infinitely many edges of each degree.

We start with a short review of the basic facts about graphs and the Cuntz-Krieger bimodule $X(E)$ of a single graph E. In §3, we associate to each product system E of graphs a product system $X(E)$ of Cuntz-Krieger bimodules (Proposition 3.2). In §4, we define Toeplitz E-families, and show that there is a one-to-one correspondence between such families and Toeplitz representations of $X(E)$ (Theorem 4.2). We then restrict attention to product systems over the quasi-lattice ordered semigroups of Nica, and identify the finitely aligned product systems E of graphs for which $X(E)$ is compactly aligned (Theorem 5.4). In §6, we discuss Nica covariance, and show that for finitely aligned systems, it becomes a familiar relation which is automatically satisfied by Cuntz-Krieger families of a single graph. By adding this relation to those of a Toeplitz family, we obtain an appropriate definition of Toeplitz-Cuntz-Krieger E-families for more general E, and then $\mathcal{T}C^*(E)$ is the universal C^*-algebra generated by such a family. We can now apply Fowler's theorem to $X(E)$ (Proposition 7.6), and deduce that the Fock representation of $\mathcal{T}C^*(E)$ is faithful (Corollary 7.7).

Our main Theorem 8.1 is a C^*-algebraic uniqueness theorem. It does not appear to follow from Fowler's results: its proof requires a detailed analysis of the expectation onto the diagonal in $\mathcal{T}C^*(E)$ and its spatial implementation, as well as an application of Corollary 7.7. In the last section, we apply Theorem 8.1 to the k-graphs of [11]. Our results are all interesting in this case, and those interested primarily in k-graphs could assume $P = \mathbb{N}^k$ throughout the paper without losing the main points.

2. Preliminaries

2.1. **Graphs and Cuntz-Krieger families.** A *directed graph* $E = (E^0, E^1, r, s)$ consists of a countable vertex set E^0, a countable edge set E^1, and range and source maps $r, s : E^1 \to E^0$. All graphs in this paper are directed.

A *Toeplitz-Cuntz-Krieger E-family* in a C^*-algebra B consists of mutually orthogonal projections $\{p_v : v \in E^0\}$ in B and partial isometries $\{s_\lambda : \lambda \in E^1\}$ in B satisfying $s_\lambda^* s_\lambda = p_{r(\lambda)}$ for $\lambda \in E^1$ and

$$p_v \geq \sum_{\lambda \in F} s_\lambda s_\lambda^* \quad \text{for every } v \in E^0 \text{ and every finite set } F \subset s^{-1}(v).$$

It is a *Cuntz-Krieger E-family* if

$$p_v = \sum_{\lambda \in s^{-1}(v)} s_\lambda s_\lambda^* \quad \text{whenever } s^{-1}(v) \text{ is finite and nonempty.}$$

2.2. **Hilbert bimodules.** Let A be a C^*-algebra. A *right-Hilbert $A - A$ bimodule* (or *Hilbert bimodule over A*) is a right Hilbert A-module X together with a left action $(a, x) \mapsto a \cdot x$ of A by adjointable operators on X; we denote by ϕ the homomorphism of A into $\mathcal{L}(X)$ given by the left action. We say X is *essential* if

$$\overline{\operatorname{span}}\{a \cdot x : a \in A,\ x \in X\} = X.$$

A *Toeplitz representation* (ψ, π) of a Hilbert bimodule X in a C^*-algebra B consists of a linear map $\psi : X \to B$ and a homomorphism $\pi : A \to B$ such that

$$\psi(x \cdot a) = \psi(x)\pi(a), \quad \psi(a \cdot x) = \pi(a)\psi(x), \quad \text{and} \quad \psi(x)^*\psi(y) = \pi(\langle x, y\rangle_A)$$

for $x, y \in X$ and $a \in A$. There is then a unique homomorphism $\psi^{(1)} : \mathcal{K}(X) \to B$ such that

$$\psi^{(1)}(\Theta_{x,y}) = \psi(x)\psi(y)^* \quad \text{for } x, y \in X;$$

see [16, page 202], [10, Lemma 2.2], or [7, Remark 1.7] for details. The representation (ψ, π) is *Cuntz-Pimsner covariant* if

$$\psi^{(1)}(\phi(a)) = \pi(a) \quad \text{whenever } \phi(a) \in \mathcal{K}(X).$$

Pimsner associated to each Hilbert bimodule X a C^*-algebra $\mathcal{T}_X$ which is universal for Toeplitz representations of X, and a quotient $\mathcal{O}_X$ which is universal for Cuntz-Pimsner covariant Toeplitz representations of X ([16]; see also [7, §1]).

2.3. **Cuntz-Krieger bimodules.** The Cuntz-Krieger bimodule $X(E)$ of a graph E, as in [7, Example 1.2], consists of the functions $x : E^1 \to \mathbb{C}$ such that

$$\rho_x : v \mapsto \sum_{\lambda \in E^1, r(\lambda)=v} |x(\lambda)|^2 \tag{2.1}$$

vanishes at infinity on E^0. With

$$(x \cdot a)(\lambda) := x(\lambda)a(r(\lambda)) \text{ and } (a \cdot x)(\lambda) := a(s(\lambda))x(\lambda) \quad \text{for } \lambda \in E^1\text{, and}$$

$$\langle x, y\rangle_{C_0(E^0)}(v) := \sum_{\lambda \in E^1, r(\lambda)=v} \overline{x(\lambda)}y(\lambda) \quad \text{for } v \in E^0,$$

$X(E)$ is a Hilbert bimodule over $C_0(E^0)$. The Toeplitz representations of $X(E)$ are in one-to-one correspondence with the Toeplitz-Cuntz-Krieger E-families via $(\psi, \pi) \leftrightarrow \{\psi(\delta_\lambda), \pi(\delta_v)\}$ [7, Example 1.2]. Hence $\mathcal{T}_{X(E)}$ is universal for Toeplitz-Cuntz-Krieger E-families. When E has no sinks, the left action of $C_0(E^0)$ on $X(E)$ is faithful, the Cuntz-Pimsner covariant representations correspond to Cuntz-Krieger E-families, and the quotient $\mathcal{O}_{X(E)}$ is the usual graph C^*-algebra $C^*(E)$.

Because of the correspondence $(\psi, \pi) \leftrightarrow \{\psi(\delta_\lambda), \pi(\delta_v)\}$, it is convenient in calculations to work with the point masses $\delta_\lambda \in X(E)$. The following lemma explains why this suffices.

Lemma 2.1. *The space $X_c(E) := C_c(E^1)$ is a dense submodule of $X(E)$, and the point masses $\{\delta_\lambda : \lambda \in E^1\}$ are a vector-space basis for $X_c(E^1)$.*

Proof. As a Banach space, $X(E)$ is the c_0-direct sum $\bigoplus_{v\in E^0} \ell^2(r^{-1}(v))$, and $X_c(E)$ is the algebraic direct sum of the subspaces $C_c(r^{-1}(v))$. So it is standard that $X_c(E)$ is dense. For $x \in X_c(E)$, we have $x = \sum_{\lambda\in E^1} x(\lambda)\delta_\lambda$. □

3. Product systems of graphs and of Hilbert bimodules

Throughout the next two sections, P denotes an arbitrary countable semigroup with identity e. If $E = (E^0, E^1, r_E, s_E)$ and $F = (E^0, F^1, r_F, s_F)$ are two graphs with the same vertex set E^0, then $E \times_{E^0} F$ denotes the graph with $(E \times_{E^0} F)^0 := E^0$,

$$(E \times_{E^0} F)^1 := \{(\lambda, \mu) : \lambda \in E^1, \mu \in F^1, r_E(\lambda) = s_F(\mu)\},$$

and $s(\lambda, \mu) := s_E(\lambda)$, $r(\lambda, \mu) := r_F(\mu)$.

We recall from [8] that a *product system* (E, φ) *of graphs over* P consists of graphs $\{(E^0, E^1_p, r_p, s_p) : p \in P\}$ with common vertex set E^0 and disjoint edge sets E^1_p, and isomorphisms $\varphi_{p,q} : E_p \times_{E^0} E_q \to E_{pq}$ for $p, q \in P$ satisfying the associativity condition

$$\varphi_{pq,r}(\varphi_{p,q}(\lambda, \mu)), \nu) = \varphi_{p,qr}(\lambda, \varphi_{q,r}(\mu, \nu)) \tag{3.1}$$

for all $p, q, r \in P$, $(\lambda, \mu) \in (E_p \times_{E^0} E_q)^1$, and $(\mu, \nu) \in (E_q \times_{E^0} E_r)^1$; we require that

$$E_e = (E^0, E^0, \mathrm{id}_{E^0}, \mathrm{id}_{E^0}).$$

We write $d(\lambda) = p$ to mean $\lambda \in E^1_p$; because the E^1_p are disjoint, this gives a well-defined *degree map* $d : E^1 := \bigcup_{p\in P} E^1_p \to P$, which gives the vertices $E^0 = E^1_e$ degree e. The range and source maps combine to give maps $r, s : E^1 \to E^0$.

The isomorphisms $\varphi_{p,q}$ in a product system (E, φ) combine to give a partial multiplication on E^1: for $(\lambda, \mu) \in E^1_p \times_{E^0} E^1_q$, we define $\lambda\mu = \varphi_{p,q}(\lambda, \mu) \in E^1_{pq}$. This multiplication is associative by (3.1). Since each $\varphi_{p,q}$ is an isomorphism, the multiplication has the following *factorisation property*: for each $\gamma \in E^1_{pq}$, there is a unique $(\lambda, \mu) \in (E_p \times_{E^0} E_q)^1$ such that $\gamma = \lambda\mu$. It follows that if $\lambda \in E^1_{pqr}$, then there is a unique $\lambda(p, pq) \in E^1_q$ such that $\lambda = \lambda'\lambda(p, pq)\lambda''$ with $d(\lambda') = p$ and $d(\lambda'') = r$. By (3.1) and the factorisation property, $s(\lambda)\lambda = \lambda = \lambda r(\lambda)$ for all λ.

A single graph E gives a product system over $\mathbb{N}$ in which E^1_n consists of the paths of length n in E. More generally:

Example 3.1 (k-graphs). It is shown in [8, Examples 1.5, (4)] that the product systems of graphs over $\mathbb{N}^k$ are essentially the same as the k-graphs of [11, Definitions 1.1]:

- Given a product system (E, φ) of graphs over $\mathbb{N}^k$, let Λ_E be the category with objects E^0 and morphisms E^1, with $\operatorname{dom}(\lambda) := r(\lambda)$ and $\operatorname{cod}(\lambda) := s(\lambda)$. The degree map is that of E, the morphism $\lambda \circ \mu$ is by definition the morphism associated to the edge $\lambda\mu$, and the factorisation property for Λ_E reduces to that of E.
- Given a k-graph (Λ, d), let $(E_\Lambda)^0 := \Lambda^0$, $(E_\Lambda)^1_n := \Lambda^n$ for $n \in \mathbb{N}^k$, $\lambda\mu := \lambda \circ \mu \in \Lambda^{m+n}$ whenever $(\lambda, \mu) \in (E_m \times_{E^0} E_n)^1$, and define $r := \operatorname{dom}$ and $s := \operatorname{cod}$.

The direction of the edges is reversed in going from (Λ, d) to $(E_\Lambda, \varphi_\Lambda)$ to ensure that the representations of the two coincide (compare Definition 4.1 with [11, Definitions 1.5]).

Proposition 3.2. *If (E, φ) is a product system of graphs over P, then there is a unique associative multiplication on $X(E) := \bigcup_{p\in P} X(E_p)$ such that*

$$\delta_\lambda \delta_\mu := \begin{cases} \delta_{\lambda\mu} & \textit{if } (\lambda, \mu) \in (E_{d(\lambda)} \times_{E^0} E_{d(\mu)})^1 \\ 0 & \textit{otherwise,} \end{cases} \tag{3.2}$$

and $X(E)$ thus becomes a product system of Hilbert bimodules over $C_0(E^0)$ as in [5, Definition 2.1].

Remark 3.3. We have described the multiplication using point masses because we want to use them in calculations. However, we also write it out explicitly in Corollary 3.4.

Proof of Proposition 3.2. It follows from Lemma 2.1 that the elements $\delta_\lambda \otimes \delta_\mu$ are a basis for the algebraic tensor product $X_c(E_p) \odot X_c(E_q)$, and hence there is a well-defined linear map $\pi : X_c(E_p) \odot X_c(E_q) \to X_c(E_{pq})$ such that

$$\pi(\delta_\lambda \otimes \delta_\mu) = \begin{cases} \delta_{\lambda\mu} & \text{if } (\lambda, \mu) \in (E_{d(\lambda)} \times_{E^0} E_{d(\mu)})^1 \\ 0 & \text{otherwise.} \end{cases}$$

Let $\lambda, \mu, \eta, \xi \in E^1$. Then

$$\begin{aligned} \big\langle \delta_\lambda \otimes \delta_\mu, \delta_\eta \otimes \delta_\xi \big\rangle_{C_0(E^0)}(v) &= \big\langle \langle \delta_\eta, \delta_\lambda \rangle_{C_0(E^0)} \cdot \delta_\mu, \delta_\xi \big\rangle_{C_0(E^0)}(v) \\ &= \begin{cases} 1 & \text{if } \eta = \lambda,\ \xi = \mu,\ r(\lambda) = s(\mu) \text{ and } r(\mu) = v \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{3.3}$$

On the other hand,

$$\begin{aligned}\big\langle \pi(\delta_\lambda \otimes \delta_\mu), \pi(\delta_\eta \otimes \delta_\xi)\big\rangle_{C_0(E^0)}(v) \\ &= \begin{cases} \langle \delta_{\lambda\mu}, \delta_{\eta\xi}\rangle_{C_0(E^0)}(v) & \text{if } r(\lambda) = s(\mu) \text{ and } r(\eta) = s(\xi) \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} 1 & \text{if } r(\lambda) = s(\mu),\ r(\eta) = s(\xi),\ \lambda\mu = \eta\xi \text{ and } r(\mu) = v \\ 0 & \text{otherwise,} \end{cases}\end{aligned}$$

which by the factorisation property is (3.3). Since $X_c(E_p)$ is dense in $X(E_p)$ (Lemma 2.1), it follows that π extends to an isometric linear isomorphism of $X(E_p) \otimes_{C_0(E^0)} X(E_q)$ onto $X(E_{pq})$. It is easy to check on dense subspaces $X_c(E_p)$ and $\operatorname{span}\{\delta_v\} \subset C_0(E^0)$ that π is an isomorphism of Hilbert $C_0(E^0)$-bimodules. We now define $xy := \pi(x \otimes y)$, and associativity of this multiplication follows from (3.1). More calculations on dense subspaces show that $xa = x \cdot a$ and $ax = a \cdot x$ for $a \in C_0(E^0) = X(E_e)$ and $x \in X(E_p)$. □

Corollary 3.4. *For $x \in X(E_p)$ and $y \in X(E_q)$, we have*

$$(xy)(\lambda\mu) = x(\lambda)y(\mu) \quad \textit{for } (\lambda, \mu) \in (E_p \times_{E^0} E_q)^1. \tag{3.4}$$

Proof. The multiplication extends to an isomorphism of $X(E_p) \otimes_{C_0(E^0)} X(E_q)$ onto $X(E_{pq})$, $(x, y) \mapsto x \otimes y$ is continuous, and the various evaluation maps $z \mapsto z(\lambda)$ are continuous, so Lemma 2.1 implies that it is enough to prove (3.4) for $x \in X_c(E_p)$ and $y \in X_c(E_q)$. For such x, y we have

$$(xy)(\lambda\mu) = \sum_{\alpha \in E_p^1, \beta \in E_q^1} x(\alpha)y(\beta)(\delta_\alpha\delta_\beta)(\lambda\mu),$$

which collapses to $x(\lambda)y(\mu)$ by the factorisation property. □

4. Representations of product systems

Throughout this section, (E, φ) is a product system of graphs over P.

Definition 4.1. Partial isometries $\{s_\lambda : \lambda \in E^1\}$ in a C^*-algebra B form a *Toeplitz E-family* if:

1. $\{s_v : v \in E^0\}$ are mutually orthogonal projections,
2. $s_\lambda s_\mu = s_{\lambda\mu}$ for all $\lambda, \mu \in E^1$ such that $r(\lambda) = s(\mu)$,
3. $s_\lambda^* s_\lambda = s_{r(\lambda)}$ for all $\lambda \in E^1$, and
4. for all $p \in P \setminus \{e\}, v \in E^0$ and every finite $F \subset s_p^{-1}(v)$, $s_v \geq \sum_{\lambda \in F} s_\lambda s_\lambda^*$.

We recall from [5] that a Toeplitz representation ψ of a product system X of bimodules consists of linear maps $\psi_p : X_p \to B$ such that each (ψ_p, ψ_e) is a Toeplitz representation of X_p, and $\psi_p(x)\psi_q(y) = \psi_{pq}(xy)$. It is Cuntz-Pimsner covariant if each (ψ_p, ψ_e) is Cuntz-Pimsner covariant. Fowler proves that there is a C^*-algebra $\mathcal{T}_X$ generated by a universal Toeplitz representation i_X, and a quotient $\mathcal{O}_X$ generated by a universal Cuntz-Pimsner covariant representation j_X [5, §2].

Theorem 4.2. *Let (E, φ) be a product system of graphs over a semigroup P, and let $X(E)$ be the corresponding product system of Cuntz-Krieger bimodules. If ψ is a Toeplitz representation of $X(E)$, then*

$$\{s_\lambda := \psi_{d(\lambda)}(\delta_\lambda) : \lambda \in E^1\} \tag{4.1}$$

is a Toeplitz E-family; conversely, if $\{s_\lambda : \lambda \in E^1\}$ is a Toeplitz E-family, then the map

$$x \in C_c(E^1_p) \mapsto \sum_{\lambda \in E^1_p} x(\lambda) s_\lambda \tag{4.2}$$

extends to a Toeplitz representation of $X(E)$ from which we can recover $s_\lambda = \psi_{d(\lambda)}(\delta_\lambda)$. The representation ψ is Cuntz-Pimsner covariant if and only if $\{s_\lambda\}$ satisfies

$$s_v = \sum_{\lambda \in s_p^{-1}(v)} s_\lambda s_\lambda^* \text{ whenever } s_p^{-1}(v) \text{ is finite (possibly empty).} \tag{4.3}$$

Proof. If ψ is a Toeplitz representation of $X(E)$, then [7, Example 1.2] shows that

$$\{\psi_e(\delta_v), \psi_p(\delta_\lambda) : v \in E^0, \lambda \in E^1_p\}$$

is a Toeplitz-Cuntz-Krieger family for E_p as in [7], and this gives (1), (3), and (4) of Definition 4.1. Definition 4.1(2) follows from (3.2) because ψ is a homomorphism.

Now suppose that ψ is Cuntz-Pimsner covariant and $s_p^{-1}(v)$ is finite. Write $\phi_p : C_0(E^0) \to \mathcal{L}(X_p)$ for the homomorphism that implements the left action on X_p. Then

$$\sum_{\lambda \in s_p^{-1}(v)} \psi_p(\delta_\lambda)\psi_p(\delta_\lambda)^* = \sum_{\lambda \in s_p^{-1}(v)} \psi_p^{(1)}(\Theta_{\delta_\lambda,\delta_\lambda}) = \psi_p^{(1)}\Big(\sum_{\lambda \in s_p^{-1}(v)} \Theta_{\delta_\lambda,\delta_\lambda}\Big). \tag{4.4}$$

For $x \in X_p$, $w \in E^0$ and $\mu \in E^1_p$,

$$\Big(\sum_{\lambda \in s_p^{-1}(w)} \Theta_{\delta_\lambda,\delta_\lambda}(x)\Big)(\mu) = \begin{Bmatrix} x(\mu) & \text{if } \mu \in s_p^{-1}(w) \\ 0 & \text{otherwise} \end{Bmatrix} = (\delta_w \cdot x)(\mu).$$

Hence the right hand side of (4.4) is just $\psi_p^{(1)}(\phi_p(\delta_v))$. Since $\phi_p(\delta_v)$ belongs to $\mathcal{K}(X_p)$ [7, Proposition 4.4], Cuntz-Pimsner covariance gives $\psi_p^{(1)}(\phi_p(\delta_v)) = \psi_e(\delta_v)$. Thus

$$\sum_{\lambda \in s_p^{-1}(v)} s_\lambda s_\lambda^* = \sum_{\lambda \in s_p^{-1}(v)} \psi_p(\delta_\lambda)\psi_p(\delta_\lambda)^* = \psi_e(\delta_v) = s_v.$$

If $\{s_\lambda : \lambda \in E^1\}$ is a Toeplitz E-family, [7, Example 1.2] implies that $\psi_p(\delta_\lambda) := s_\lambda$ extend to Toeplitz representations (ψ_p, ψ_e) of X_p for $p \in P$; since

$$\psi_{pq}(\delta_\lambda \delta_\mu) = \psi_{pq}(\delta_{\lambda\mu}) = s_{\lambda\ mu} = s_\lambda s_\mu = \psi_p(\delta_\lambda)\psi_q(\delta_\mu),$$

it follows that ψ is a Toeplitz representation of $X(E)$. We trivially have $s_\lambda = \psi_{d(\lambda)}(\delta_\lambda)$.

If $\{s_\lambda : \lambda \in E^1\}$ satisfies (4.3), then for $p \in P$ and $v \in E^0$ with $s_p^{-1}(v)$ finite,

$$\psi_p^{(1)}(\phi_p(\delta_v)) = \psi_p^{(1)}\Big(\sum_{\lambda \in s_p^{-1}(v)} \Theta_{\delta_\lambda,\delta_\lambda}\Big) = \sum_{\lambda \in s_p^{-1}(v)} \psi_p(\delta_\lambda)\psi_p(\delta_\lambda)^*,$$

which is $\psi_e(\delta_v)$ by (4.3). Proposition 4.4 of [7] ensures that $\{\delta_v : |s_p^{-1}(v)| < \infty\}$ spans a dense subspace of $\{a \in C_0(E^0) : \phi(a) \in \mathcal{K}(X_p)\}$, so ψ is Cuntz-Pimsner covariant. □

Corollary 4.3. *Let (E, φ) be a product system of graphs over a semigroup P. Then $(\mathcal{T}_{X(E)}, i_{X(E)})$ is universal for Toeplitz E-families in the sense that*

(1) *$\{s_\lambda\} := \{i_{X(E)}(\delta_\lambda)\}$ is a Toeplitz E-family which generates $\mathcal{T}_{X(E)}$; and*
(2) *for every Toeplitz E-family $\{s_\lambda\}$, there is a representation ψ_* of $\mathcal{T}_{X(E)}$ such that $(\psi_* \circ i_{X(E)})(\delta_\lambda) = s_\lambda$ for every $\lambda \in E^1$.*

Similarly, $(\mathcal{O}_{X(E)}, j_{X(E)})$ is universal for Toeplitz E-families satisfying (4.3).

Proof. This follows from Theorem 4.2 and the universal properties of $\mathcal{T}_{X(E)}$ and $\mathcal{O}_{X(E)}$ described in [5, Propositions 2.8 and 2.9]. □

If (E, φ) is a product system of row-finite graphs without sinks over $\mathbb{N}^k$, then Λ_E is row-finite and has no sources as in [11], and the Toeplitz E-families which satisfy (4.3) are precisely the $*$-representations of Λ_E. Hence:

Corollary 4.4. *Let Λ be a row-finite k-graph with no sources as in* [11]*, define E_Λ as in Example 3.1, and let $X = X(E_\Lambda)$. Then there is an isomorphism of $C^*(\Lambda)$ onto $\mathcal{O}_X$ carrying s_λ to $i_X(\delta_\lambda)$.*

Remark 4.5. If there are vertices which are sinks in one or more E_p, then some subtle issues arise, and the Toeplitz E-families satisfying (4.3) are not necessarily the Cuntz-Krieger Λ_E-families studied in [17]. Here, though, we care primarily about Toeplitz familes, and the presence of sinks does not cause problems.

5. Compactly aligned product systems of Cuntz-Krieger bimodules

The compactly aligned product systems are a large class of product systems whose Toeplitz algebras have been analysed in [4] and [5]. To apply the results of [5], we need to identify the product systems E of graphs for which $X(E)$ is compactly aligned.

In compactly aligned product systems, the underlying semigroup P has to be quasi-lattice ordered in the sense of Nica [15, 14]. Suppose P is a subsemigroup of a group G such that $P \cap P^{-1} = \{e\}$. Then $g \leq h \iff g^{-1}h \in P$ defines a partial order on G, and P is *quasi-lattice ordered* if every finite subset of G with an upper bound in P has a least upper bound in P. (Strictly speaking, it is the pair (G, P) which is quasi-lattice ordered.) If two elements p and q have a common upper bound in P, $p \vee q$ denotes their least upper bound; otherwise, we write $p \vee q = \infty$.

Totally ordered groups, free groups, and products of these groups are all quasi-lattice ordered. The main example of interest to us is $(G, P) = (\mathbb{Z}^k, \mathbb{N}^k)$, which is actually *lattice-ordered*: each pair $m, n \in \mathbb{N}^k$ has a least upper bound $m \vee n$ with ith coordinate $(m \vee n)_i := \max\{m_i, n_i\}$.

Let X be a product system of bimodules over a quasi-lattice ordered semigroup P, and suppose $p, q \in P$ have $p \vee q < \infty$. Since $S \in \mathcal{L}(X_p)$ acts as an adjointable operator $S \otimes 1$ on $X_p \otimes_A X_{p^{-1}(p \vee q)}$, the isomorphism of $X_p \otimes_A X_{p^{-1}(p \vee q)}$ onto $X_{p \vee q}$ induced by the multiplication gives an action of $\mathcal{L}(X_p)$ on $X_{p \vee q}$; we write $S_p^{p \vee q}$ for the image of $S \in \mathcal{L}(X_p)$, so that $S_p^{p \vee q}$ is characterised by

$$S_p^{p \vee q}(xy) := (Sx)y \quad \text{for } x \in X_p,\ y \in X_{p^{-1}(p \vee q)}. \tag{5.1}$$

The product system X is *compactly aligned* [5, Definition 5.7] if

$$S \in \mathcal{K}(X_p) \text{ and } T \in \mathcal{K}(X_q) \text{ imply } (S_p^{p \vee q})(T_q^{p \vee q}) \in \mathcal{K}(X_{p \vee q}).$$

When $X = X(E)$ is a product system of Cuntz-Krieger bimodules, Lemma 2.1 implies that the point masses span dense subspaces of $X(E_p)$, and the rank-one operators $\Theta_{x,y}$ span dense subspaces of $\mathcal{K}(X)$; thus to prove that $X(E)$ is compactly aligned, it suffices to check that every

$$(\Theta_{\delta_{\mu_1},\delta_{\mu_2}})_p^{p\vee q}(\Theta_{\delta_{\nu_1},\delta_{\nu_2}})_q^{p\vee q} \quad \text{belongs to } \mathcal{K}(X(E_{p\vee q})). \tag{5.2}$$

To prove that a given $X(E)$ is not compactly aligned, we need to be able to recognise non-compact operators on $X(E)$.

Lemma 5.1. *Let $X(E)$ be the Cuntz-Krieger bimodule of a graph, and let $S \in \mathcal{K}(X(E))$. Then the function $x_S : E^1 \to \mathbb{R}$ defined by $x_S(\lambda) := \|S(\delta_\lambda)\|_{C_0(E^0)}$ vanishes at infinity on E^1.*

Proof. First suppose $S = \Theta_{x,y}$ for some $x, y \in X(E)$. Then for $\lambda \in E^1$, we have

$$\|\Theta_{x,y}(\delta_\lambda)\|^2 = \sum_{r(\mu)=r(\lambda)} |x(\mu)y(\lambda)|^2 \leq |y(\lambda)|^2\|x\|^2;$$

since $y \in X(E) \subset C_0(E^1)$, so is $\lambda \mapsto \|\Theta_{x,y}(\delta_\lambda)\|$. Easy calculations show that $|x_{wS+zT}(\lambda)| \leq |w|\,|x_S(\lambda)| + |z|\,|x_T(\lambda)|$ and $|x_S(\lambda)| \leq \|S\|_{\mathcal{L}(X(E))}$, so the result for arbitrary $S \in \mathcal{K}(X(E))$ follows by linearity and continuity. □

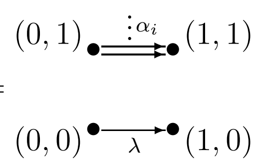
Example 5.2. (A Cuntz-Krieger bimodule which is not compactly aligned.) Let $(G, P) = (\mathbb{Z}^2, \mathbb{N}^2)$. Let $E^0 := \{(0,0), (0,1), (1,0), (1,1)\}$,

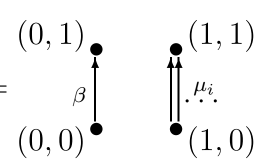
$$E^1_{(1,0)} := \{\lambda\} \cup \{\alpha_i : i \in \mathbb{N}\}, \quad E^1_{(0,1)} := \{\mu_i : i \in \mathbb{N}\} \cup \{\beta\},$$

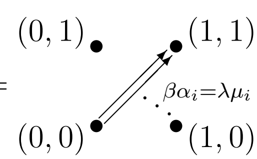
and define

$$r(\lambda) = (1,0), \quad s(\lambda) = (0,0), \quad r(\alpha_i) = (1,1), \quad s(\alpha_i) = (0,1), \quad \text{and}$$
$$r(\mu_i) = (1,1), \quad s(\mu_i) = (1,0), \quad r(\beta) = (0,1), \quad s(\beta) = (0,0).$$

By [8, Theorem 2.1], there is a unique product system E over $\mathbb{N}^2$ in which $\beta\alpha_i = \lambda\mu_i$. In pictures:

$E_{(1,0)} =$ 

$E_{(0,1)} =$ 

$E_{(1,1)} =$ 

For $S := \Theta_{\delta_\lambda,\delta_\lambda}$ and $T := \Theta_{\delta_\beta,\delta_\beta}$, we can compute $S^{(1,1)}_{(1,0)} \circ T^{(1,1)}_{(0,1)}(\delta_{\lambda\mu_i})$ using (5.1). To evaluate $T^{(1,1)}_{(0,1)}(\delta_{\lambda\mu_i})$ we need to factor $\lambda\mu_i$ as $\beta\alpha_i$, so that $\delta_{\lambda\mu_i} = \delta_\beta\delta_{\alpha_i}$. Then

$$\begin{aligned} S^{(1,1)}_{(1,0)} \circ T^{(1,1)}_{(0,1)}(\delta_{\lambda\mu_i}) &= S^{(1,1)}_{(1,0)}(T(\delta_\beta)\delta_{\alpha_i}) = S^{(1,1)}_{(1,0)}(\delta_\beta\delta_{\alpha_i}) \\ &= S^{(1,1)}_{(1,0)}(\delta_\lambda\delta_{\mu_i}) = S(\delta_\lambda)\delta_{\mu_i} = \delta_{\lambda\mu_i}. \end{aligned} \tag{5.3}$$

Thus $\lambda\mu_i \mapsto \big\|S^{(1,1)}_{(1,0)} \circ T^{(1,1)}_{(0,1)}(\delta_{\lambda\mu_i})\big\|$ does not vanish at infinity on $E^1_{(1,1)}$. Lemma 5.1 therefore implies that $S^{(1,1)}_{(1,0)} \circ T^{(1,1)}_{(0,1)}$ is not compact, and E is not compactly aligned.

To identify the E for which $X(E)$ is compactly aligned, we legislate out the behaviour which makes Example 5.2 work. More precisely:

Definition 5.3. Suppose (E,φ) is a product system of graphs over a quasi-lattice ordered semigroup P, and let $\mu \in E^1_p$ and $\nu \in E^1_q$. A *common extension* of μ and ν is a path γ such that $\gamma(0,p) = \mu$ and $\gamma(0,q) = \nu$. Notice that $d(\gamma)$ is then an upper bound for p and q, so $p \vee q < \infty$; we say that γ is a *minimal common extension* if $d(\gamma) = p \vee q$. We denote by $\mathrm{MCE}(\mu,\nu)$ the set of minimal common extensions of μ and ν, and say that (E,φ) is *finitely aligned* if $\mathrm{MCE}(\mu,\nu)$ is finite (possibly empty) for all $\mu,\nu \in E^1$.

Theorem 5.4. *Let (E,φ) be a product system of graphs over a quasi-lattice ordered semigroup P. Then $X(E)$ is compactly aligned if and only if (E,φ) is finitely aligned.*

Proof. If $\mathrm{MCE}(\lambda,\beta)$ is infinite for some α and β, there are infinitely many paths μ_i and α_i such that $\lambda\mu_i = \beta\alpha_i$, and the argument of Example 5.2 shows that $X(E)$ is not compactly aligned. Suppose that (E,φ) is finitely aligned, $p,q \in P$ satisfy $p\vee q < \infty$, and $\mu_1,\mu_2 \in E^1_p$, $\nu_1,\nu_2 \in E^1_q$. Then computations like (5.3) show that $(\Theta_{\delta_{\nu_1},\delta_{\nu_2}})^{p\vee q}_q(\delta_\lambda) = 0$ unless $\lambda(e,q) = \nu_2$, and then with $\sigma := \nu_1\lambda(q,p\vee q)$ we have

$$\begin{aligned}(\Theta_{\delta_{\mu_1},\delta_{\mu_2}})^{p\vee q}_p(\Theta_{\delta_{\nu_1},\delta_{\nu_2}})^{p\vee q}_q(\delta_\lambda) &= \delta_{\nu_2}(\lambda(0,q))\delta_{\mu_2}(\sigma(0,p))\delta_{\mu_1\sigma(p,p\vee q)}\\ &= \begin{cases}\delta_{\mu_1\sigma(p,p\vee q)} & \text{if } \sigma(0,p) = \mu_2\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

Thus

$$(\Theta_{\delta_{\mu_1},\delta_{\mu_2}})^{p\vee q}_p(\Theta_{\delta_{\nu_1},\delta_{\nu_2}})^{p\vee q}_q = \sum_{\sigma\in\mathrm{MCE}(\mu_2,\nu_1)} \Theta_{\delta_{\mu_1\sigma(p,p\vee q)},\delta_{\nu_2\sigma(q,p\vee q)}},$$

which belongs to $\mathcal{K}(X(E))$ because $\mathrm{MCE}(\mu_2,\nu_1)$ is finite. □

6. Nica covariance

In this section, we show that when $X = X(E)$, Fowler's Nica-covariance condition reduces to an extra relation for Toeplitz E-families, which will look familiar to anyone who has studied any generalisation of Cuntz-Krieger algebras. This relation automatically holds for Toeplitz-Cuntz-Krieger families of single graphs, but is not automatic for the Toeplitz families of product systems.

Suppose X is a product system of $A-A$ bimodules over a quasi-lattice ordered semigroup P, and ψ is a nondegenerate Toeplitz representation of X on $\mathcal{H}$. Fowler shows in [5, Proposition 4.1] that there is an action $\alpha^\psi : P \to \operatorname{End}\psi_e(A)'$ such that

$$\alpha^\psi_p(T)\psi_p(x) = \psi_p(x)T \text{ for } T \in \psi_e(A)' \text{ and } \alpha^\psi_p(1)h = 0 \text{ for } h \in \psi_p(X_p)^\perp. \tag{6.1}$$

The representation ψ is *Nica covariant* if

$$\alpha^\psi_p(1_p)\alpha^\psi_q(1_q) = \begin{cases}\alpha^\psi_{p\vee q}(1_{p\vee q}) & \text{if } p\vee q < \infty\\ 0 & \text{otherwise.}\end{cases} \tag{6.2}$$

We denote by $(\mathcal{T}_{\mathrm{cov}}(X), i_X)$ the pair which is universal for Nica-covariant Toeplitz representations of X in the sense of [5, Theorem 6.3]. When X is compactly aligned, it follows from [5, Lemma 5.5 and Proposition 5.6] that the Nica covariance condition (6.2) makes sense for a representation taking values in a C^*-algebra, and then $(\mathcal{T}_{\mathrm{cov}}(X), i_X)$ is universal in the usual sense of the word.

When P is the positive cone in a totally ordered group, $p \vee q$ is either p or q, and Nica covariance is automatic. Thus Toeplitz representations of a single Cuntz-Krieger bimodule $X(E)$ are always Nica covariant. For product systems of row-finite graphs over lattice-ordered semigroups such as $\mathbb{N}^k$, Nica covariance is a consequence of Cuntz-Pimsner covariance:

Lemma 6.1. *Let (E, φ) be a product system of graphs over a lattice-ordered semigroup P. If every E_p is row-finite, then every Toeplitz representation of $X(E)$ which is Cuntz-Pimsner covariant is also Nica covariant. In particular, if Λ is a row-finite k-graph, every Cuntz-Pimsner covariant representation of $X(E_\Lambda)$ is Nica covariant.*

Proof. Since each E_p is row-finite, $C_0(E^0)$ acts by compact operators on the left of each $X(E_p)$ [7, Proposition 4.4], and the result follows from [5, Proposition 5.4]. □

Corollary 6.2. *Let (E, φ) be a product system of row-finite graphs over a lattice-ordered semigroup P. Then $\mathcal{O}_{X(E)}$ is isomorphic to a quotient of $\mathcal{T}_{\mathrm{cov}}(X(E))$.*

Proposition 6.3. *Let (E, φ) be a product system of graphs over a quasi-lattice ordered semigroup P, and let ψ be a nondegenerate Toeplitz representation of $X(E)$ on $\mathcal{H}$. For $p \in P$, $T \in B(\mathcal{H})$ and $h \in \mathcal{H}$, the sum*

$$\sum_{\lambda \in E_p^1} \psi_p(\delta_\lambda) T \psi_p(\delta_\lambda)^* h$$

converges in $\mathcal{H}$; if $T \in \psi_e(C_0(E^0))'$, it converges to $\alpha_p^\psi(T)h$.

Proof. By [5, Proposition 4.1(1)], it suffices to work with a representation (ψ, π) of a single graph E, and show

(1) that the sum $\alpha(T)h := \sum_{\lambda \in E^1} \psi(\delta_\lambda) T \psi(\delta_\lambda)^* h$ converges for all $h \in \mathcal{H}$;
(2) that $\alpha(T) \in B(\mathcal{H})$ for each $T \in B(\mathcal{H})$;
(3) that α is an endomorphism of $\pi(C_0(E^0))'$; and
(4) that α satisfies $\alpha(T)\psi(x) = \psi(x)T$ for $T \in \psi_e(C_0(E^0))'$, and $\alpha(1)|_{(\psi(X)\mathcal{H})^\perp} = 0$.

Because the $\psi(\delta_\lambda)$ are partial isometries with orthogonal ranges, we have

$$\sum_{\lambda \in E^1} \|\psi(\delta_\lambda) T \psi(\delta_\lambda)^* h\|^2 \le \sum_{\lambda \in E^1} \|T\|^2 \|\psi(\delta_\lambda)^* h\|^2 \le \|T\|^2 \|h\|^2.$$

Thus $\sum_{\lambda \in E^1} \psi(\delta_\lambda) T \psi(\delta_\lambda)^* h$ is a sum of orthogonal vectors which converges in $\mathcal{H}$, and the sum satisfies

$$\|\alpha(T)h\|^2 = \Big\| \sum_{\lambda \in E^1} \psi(\delta_\lambda) T \psi(\delta_\lambda)^* h \Big\|^2 = \sum_{\lambda \in E^1} \|\psi(\delta_\lambda) T \psi(\delta_\lambda)^* h\|^2 \le \|T\|^2 \|h\|^2.$$

This gives (1) and (2).

Multiplying $\psi(\delta_\lambda)T\psi(\delta_\lambda)^*$ on either side by $\psi(\delta_v)$ gives 0 unless $v = s(\lambda)$, and leaves it alone if $v = s(\lambda)$. Thus each $\psi(\delta_\lambda)T\psi(\delta_\lambda)^*$ belongs to $\pi(C_0(E^0))'$, and so does the strong sum $\alpha(T)$. If S and T belong to $\pi(C_0(E^0))'$, then

$$\begin{aligned}\psi(\delta_\lambda)S\psi(\delta_\lambda)^*\psi(\delta_\mu)T\psi(\delta_\mu)^* &= \psi(\delta_\lambda)S\psi(\langle\delta_\lambda,\delta_\mu\rangle_{C_0(E^0)})T\psi(\delta_\mu)^*\\ &= \begin{cases}\psi(\delta_\lambda)ST\psi(\langle\delta_\lambda,\delta_\mu\rangle_{C_0(E^0)})\psi(\delta_\mu)^* & \text{if } \mu=\lambda\\ 0 & \text{otherwise}\end{cases}\\ &= \begin{cases}\psi(\delta_\lambda)ST\psi(\delta_\lambda)^* & \text{if } \mu=\lambda\\ 0 & \text{otherwise,}\end{cases}\end{aligned}$$

and it follows by taking sums and limits that α is multiplicative on $\pi(C_0(E^0))'$. It is clearly $*$-preserving.

For (4), we let $T \in \psi_e(C_0(E^0))'$ and calculate:

$$\alpha(T)\psi(\delta_\lambda) = \sum_{\mu\in E^1}\psi(\delta_\mu)T\psi(\delta_\mu)^*\psi(\delta_\lambda) = \psi(\delta_\lambda)T\pi(\delta_{r(\lambda)}) = \psi(\delta_\lambda)\pi(\delta_{r(\lambda)})T = \psi(\delta_\lambda)T.$$

Extending by linearity gives $\alpha(T)\psi(x) = \psi(x)T$ for $x \in X_c(E)$, which suffices by continuity. If $h \perp \psi(X)\mathcal{H}$, then $\psi(\delta_\lambda)^*h = 0$ for all λ, and $\alpha(T)h = 0$. □

Suppose that $\{S_\lambda\} \subset B(\mathcal{H})$ is a Toeplitz E-family for a product system (E,φ) of graphs over a quasi-lattice ordered semigroup P. Proposition 6.3 implies that the corresponding Toeplitz representation ψ of $X(E)$ is Nica covariant if and only if

$$\Big(\textstyle\sum_{\mu\in E_p^1}S_\mu S_\mu^*\Big)\Big(\sum_{\nu\in E_q^1}S_\nu S_\nu^*\Big) = \begin{cases}\sum_{\lambda\in E^1_{p\vee q}}S_\lambda S_\lambda^* & \text{if } p\vee q<\infty\\ 0 & \text{otherwise.}\end{cases} \tag{6.3}$$

The sums in (6.3) may be infinite, and then only converge in the strong operator topology, so this is a spatial criterion rather than a C^*-algebraic one. When E is finitely aligned, however, there is an equivalent condition which only uses finite sums.

Proposition 6.4. *Let (E,φ) be a finitely aligned product system of graphs over a quasi-lattice ordered semigroup P, and let $\{S_\lambda\} \subset B(\mathcal{H})$ be a Toeplitz E-family. The corresponding Toeplitz representation ψ of $X(E)$ is Nica covariant if and only if, for all $p,q \in P$, $\mu \in E_p^1$ and $\nu \in E_q^1$, we have*

$$S_\mu^*S_\nu = \sum_{\mu\alpha=\nu\beta\in \mathrm{MCE}(\mu,\nu)} S_\alpha S_\beta^* \quad \textit{(which is 0 if } p\vee q=\infty\textit{).} \tag{6.4}$$

Proof. First suppose ψ is Nica covariant, and let $\mu \in E_p^1$ and $\nu \in E_q^1$. Then because the S_λ corresponding to λ of the same degree have mutually orthogonal ranges, we have

$$\begin{aligned}S_\mu^*S_\nu &= S_\mu^*\Big(\textstyle\sum_{\gamma\in E_p^1}S_\gamma S_\gamma^*\Big)\Big(\sum_{\sigma\in E_q^1}S_\sigma S_\sigma^*\Big)S_\nu\\ &= \begin{cases}S_\mu^*\big(\sum_{\lambda\in E^1_{p\vee q}}S_\lambda S_\lambda^*\big)S_\nu & \text{if } p\vee q<\infty\\ 0 & \text{if } p\vee q=\infty\end{cases}\\ &= \sum_{\mu\alpha=\nu\beta\,\in\,\mathrm{MCE}(\mu,\nu)} S_\alpha S_\beta^*,\end{aligned}$$

because $(S_\mu^* S_\lambda)(S_\lambda^* S_\nu) = 0$ unless $\lambda = \mu\alpha = \nu\beta$, and $\mathrm{MCE}(\mu,\nu)$ is empty if $p \vee q = \infty$.

On the other hand, let $p, q \in P$ and suppose that (6.4) holds. Then

$$\Big(\textstyle\sum_{\mu\in E_p^1} S_\mu S_\mu^*\Big)\Big(\textstyle\sum_{\nu\in E_q^1} S_\nu S_\nu^*\Big) = \sum_{\mu\in E_p^1,\nu\in E_q^1} S_\mu \Big(\textstyle\sum_{\mu\alpha=\nu\beta\in \mathrm{MCE}(\mu,\nu)} S_\alpha S_\beta^*\Big) S_\nu^*$$

which is $\sum\{S_\lambda S_\lambda^* : \lambda \in E^1_{p\vee q}\}$ if $p \vee q < \infty$ because the factorisation property implies that each λ appears exactly once as a $\mu\alpha$ and as a $\nu\beta$, and 0 if $p \vee q = \infty$ because then each $\mathrm{MCE}(\mu,\nu)$ is empty. □

7. Toeplitz-Cuntz-Krieger families

Relation (6.4) is familiar: some version of it is used in every theory of Cuntz-Krieger algebras to ensure that $\mathrm{span}\{S_\mu S_\nu^*\}$ is a dense $*$-subalgebra of $C^*(\{S_\mu\})$ (see, for example, [2, Lemma 2.2], [12, Lemma 1.1], [17, Proposition 3.5]). As Lemma 6.1 shows, it is often automatic when the graphs are row-finite, but otherwise it will have to be assumed if we want $C^*(\{S_\mu\})$ to behave like a Cuntz-Krieger algebra.

We therefore make the following definition:

Definition 7.1. Let E be a finitely aligned product system of graphs over a quasi-lattice ordered semigroup P. Partial isometries $\{s_\lambda : \lambda \in E^1\}$ in a C^*-algebra B form a *Toeplitz-Cuntz-Krieger E-family* if:

(1) $\{s_v : v \in E^0\}$ are mutually orthogonal projections,
(2) $s_\lambda s_\mu = s_{\lambda\mu}$ for all $\lambda, \mu \in E^1$ such that $r(\lambda) = s(\mu)$,
(3) $s_\lambda^* s_\lambda = s_{r(\lambda)}$ for all $\lambda \in E^1$,
(4) for all $p \in P \setminus \{e\}, v \in E^0$ and every finite $F \subset s_p^{-1}(v)$, $s_v \geq \sum_{\lambda\in F} s_\lambda s_\lambda^*$,
(5) $s_\mu^* s_\nu = \sum_{\mu\alpha=\nu\beta\,\in\,\mathrm{MCE}(\mu,\nu)} s_\alpha s_\beta^*$ for all $\mu, \nu \in E^1$.

They form a *Cuntz-Pimsner E-family* if they also satisfy

(6) $s_v = \sum_{\lambda\in s_p^{-1}(v)} s_\lambda s_\lambda^*$ whenever $s_p^{-1}(v)$ is finite.

Remark 7.2. Multiplying both sides of (5) on the left by s_μ and on the right by s_ν^* gives

$$(s_\mu s_\mu^*)(s_\nu s_\nu^*) = \sum_{\gamma\in\mathrm{MCE}(\mu,\nu)} s_\gamma s_\gamma^*, \tag{7.1}$$

and this is equivalent to (5) because we can get back by multiplying on the left by s_μ^* and on the right by s_ν.

Remark 7.3. We have called families satisfying (6) Cuntz-Pimsner families rather than Cuntz-Krieger families because of the problems with sinks mentioned in Remark 4.5: if v is a sink in a single graph E, then (6) implies that $s_v = 0$, whereas the generally acccepted Cuntz-Krieger relations impose no relation at v. The Cuntz-Pimsner families are the ones which correspond to Cuntz-Pimsner covariant representations of $X(E)$.

Example 7.4 (The Fock representation). For $\lambda \in E^1$, let S_λ be the partial isometry on $\ell^2(E^1)$ such that

$$S_\lambda e_\mu := \begin{cases} e_{\lambda\mu} & \text{if } r(\lambda) = s(\mu) \\ 0 & \text{otherwise.}\end{cases}$$

We claim that $\{S_\lambda : \lambda \in E^1\}$ is a Toeplitz-Cuntz-Krieger E-family. Conditions (1)–(3) of Definition 7.1 are obvious, and (4) holds because

$$\Big(S_v - \textstyle\sum_{\lambda \in s_p^{-1}(v)} S_\lambda S_\lambda^*\Big) e_v = e_v \tag{7.2}$$

for all $v \in E^0$ and $p \in P \setminus \{e\}$. To verify (5), we compute on the one hand

$$\big(S_\lambda^* S_\mu e_\nu | e_\sigma\big) = \big(S_\mu e_\nu | S_\lambda e_\sigma\big) = \begin{cases} 1 & \text{if } \mu\nu = \lambda\sigma \\ 0 & \text{otherwise,} \end{cases}$$

and on the other hand,

$$\begin{aligned}\Big(\sum_{\lambda\alpha=\mu\beta\in \mathrm{MCE}(\lambda,\mu)} S_\alpha S_\beta^* e_\nu \Big| e_\sigma\Big) &= \sum_{\lambda\alpha=\mu\beta\in \mathrm{MCE}(\lambda,\mu)} \big(S_\beta^* e_\nu | S_\alpha^* e_\sigma\big) \\ &= \sum_{\lambda\alpha=\mu\beta\in \mathrm{MCE}(\lambda,\mu)} \begin{cases} 1 & \text{if } \nu = \beta\tau \text{ and } \sigma = \alpha\tau \text{ for some } \tau \\ 0 & \text{otherwise.} \end{cases}\end{aligned}$$

By the factorisation property, at most one term in this last sum can be nonzero, and there is one precisely when $\lambda\alpha\tau = \mu\beta\tau$ for some $\lambda\alpha = \mu\beta \in \mathrm{MCE}(\lambda, \mu)$, giving (5).

If there is a vertex v which emits just finitely many edges in some E_p, then (7.2) implies that (6) does not hold, and hence $\{S_\lambda\}$ is not a Cuntz-Pimsner family.

If (E, φ) is finitely aligned, then Theorem 4.2 and Proposition 6.4 imply that the Toeplitz E-family $\{i_{X(E)}(\delta_\lambda) : \lambda \in E^1\}$ in $\mathcal{T}_{\mathrm{cov}}(X(E))$ is a Toeplitz-Cuntz-Krieger E-family. It then follows from Lemma 2.1 that $\mathcal{T}_{\mathrm{cov}}(X(E))$ is generated by $\{i_{X(E)}(\delta_\lambda)\}$. We can now apply the other direction of Theorem 4.2 to see that $\mathcal{T}_{\mathrm{cov}}(X(E))$ is universal for Toeplitz-Cuntz-Krieger E-families. Thus:

Corollary 7.5. *Let (E, φ) be a finitely aligned product system of graphs over a quasi-lattice ordered semigroup P. Then $(\mathcal{T}_{\mathrm{cov}}(X(E)), \{i_{X(E)}(\delta_\lambda)\})$ is universal for Toeplitz-Cuntz-Krieger E-families.*

In view of Corollary 7.5, we define $\mathcal{T}C^*(E)$ to be the universal algebra $\mathcal{T}_{\mathrm{cov}}(X(E))$. If there are no sinks, we define $C^*(E)$ to be the quotient of $\mathcal{T}C^*(E)$ which is universal for Cuntz-Pimsner E-families. If Λ is a row-finite k-graph with no sources, it follows from Lemma 6.1 that $C^*(E_\Lambda)$ is the C^*-algebra $C^*(\Lambda)$ studied in [11].

From now on, we denote by $\{s_\lambda : \lambda \in E^1\}$ the canonical generating family in $\mathcal{T}C^*(E)$, and if $\{t_\lambda : \lambda \in E^1\}$ is a Toeplitz-Cuntz-Krieger E-family in a C^*-algebra B, then we write π_t for the homomorphism of $\mathcal{T}C^*(E)$ into B such that $\pi_t(s_\lambda) = t_\lambda$.

We now see what Fowler's theory tells us about faithful representations.

Proposition 7.6. *Let (G, P) be quasi-lattice ordered with G amenable, and let (E, φ) be a finitely aligned product system of graphs over P. Let $\{S_\lambda : \lambda \in E^1\}$ be a Toeplitz-Cuntz-Krieger E-family in $B(\mathcal{H})$, and suppose that for every finite subset R of $P \setminus \{e\}$ and every $v \in E^0$, we have*

$$\prod_{p \in R} \Big(S_v - \textstyle\sum_{\lambda \in s_p^{-1}(v)} S_\lambda S_\lambda^*\Big) > 0. \tag{7.3}$$

Then the corresponding representation $\pi_S : \mathcal{T}C^*(E) \to B(\mathcal{H})$ is faithful.

Proof. We consider the representation ψ of $X(E)$ associated to $\{S_\lambda\}$. Theorem 5.4 says that $X(E)$ is compactly aligned, and Proposition 6.4 that ψ is Nica covariant. Since the δ_v span a dense subspace of $C_0(E^0)$ and the $\psi_e(\delta_v) = S_v$ are mutually orthogonal, Proposition 6.3 implies that (7.3) is equivalent to the displayed hypothesis in [5, Theorem 7.2]. Thus [5, Theorem 7.2] implies that ψ_* is faithful on $\mathcal{T}_{\mathrm{cov}}(X(E))$. But π_S is by definition the representation ψ_* of $\mathcal{T}C^*(E) := \mathcal{T}_{\mathrm{cov}}(X(E))$. □

Corollary 7.7. *Let (G, P) be a quasi-lattice ordered group such that G is amenable, and let (E, φ) be a finitely aligned product system of graphs over P. Then the representation π_S of $\mathcal{T}C^*(E)$ associated to the Fock representation of Example 7.4 is faithful.*

Proof. Equation (7.3) follows from (7.2). □

8. A C^*-algebraic uniqueness theorem

Theorem 8.1. *Let (G, P) be a quasi-lattice ordered group such that G is amenable, and let (E, φ) be a finitely aligned product system of graphs over P. Let $\{t_\lambda : \lambda \in E^1\}$ be a Toeplitz-Cuntz-Krieger E-family in a C^*-algebra B. Suppose that for every finite subset R of $P \setminus \{e\}$, every $v \in E^0$, and every collection of finite sets $F_p \subset s_p^{-1}(v)$, we have*

$$\prod_{p\in R}\Big(t_v - \textstyle\sum_{\lambda\in F_p} t_\lambda t_\lambda^*\Big) > 0. \tag{8.1}$$

Then the associated homomorphism $\pi_t : \mathcal{T}C^*(E) \to B$ is injective.

To prove Theorem 8.1, we first establish that there is a linear map Φ^E onto the diagonal in $\mathcal{T}C^*(E)$ which is faithful on positive elements, and show that there is a norm-decreasing linear map Φ^B on $\pi_t(\mathcal{T}C^*(E))$ such that $\pi_t \circ \Phi^E = \Phi^B \circ \pi_t$.

Proposition 8.2. *There is a linear map $\Phi^E : \mathcal{T}C^*(E) \to \mathcal{T}C^*(E)$ such that*

$$\Phi^E(s_\lambda s_\mu^*) = \begin{cases} s_\lambda s_\lambda^* & \text{if } \lambda = \mu \\ 0 & \text{otherwise,} \end{cases}$$

and Φ^E is faithful on positive elements.

Proof. Let $\{e_i : i \in I\}$ be an orthonormal basis for $\mathcal{H}$, and for $i \in I$, let P_i be the projection onto $\mathbb{C}e_i$. Then for $T \in B(\mathcal{H})$, $\sum_{i\in I} P_i T P_i$ converges in the strong operator topology, and $T \mapsto \sum_{i\in I} P_i T P_i$ is the diagonal map on $B(\mathcal{H})$ which takes the rank-one operator Θ_{e_i,e_j} to Θ_{e_i,e_i} if $i = j$ and to 0 otherwise. It follows that this diagonal map is linear and norm-decreasing, and it is faithful on positive elements: $\Phi(T^*T) = 0$ implies $(T^*Te_i|e_i) = 0$ for all i, and hence $T = 0$.

Let $\mathcal{H} := \ell^2(E^1)$ and let $\{S_\lambda : \lambda \in E^1\}$ be the Toeplitz-Cuntz-Krieger family of Example 7.4. Then a calculation using the basis elements $\{e_\nu : \nu \in E^1\}$ shows that

$$P_\gamma S_\lambda S_\mu^* P_\gamma = \begin{cases} P_\gamma & \text{if } \lambda = \mu = \gamma(e, d(\mu)) \\ 0 & \text{otherwise.} \end{cases}$$

Thus if Φ denotes the diagonal map on $\ell^2(E^1)$, then

$$\Phi(S_\lambda S_\mu^*) = P_{\overline{\mathrm{span}}\{e_\gamma : \lambda=\mu=\gamma(e,d(\mu))\}} = \begin{cases} S_\lambda S_\lambda^* & \text{if } \lambda = \mu \\ 0 & \text{otherwise.} \end{cases}$$

Because the representation π_S associated to the Fock representation is faithful by Corollary 7.7, and because Φ has the required properties, we can pull Φ back to $\mathcal{T}C^*(E)$ to get the required map Φ^E. $\square$

We must now establish the existence of $\Phi^B : \pi_t(\mathcal{T}C^*(E)) \to \pi_t(\mathcal{T}C^*(E))$ and show that π_t is faithful on $\Phi^E(\mathcal{T}C^*(E))$. To do this, we analyse the structure of the diagonal $\Phi^E(\mathcal{T}C^*(E))$. Since $\mathcal{T}C^*(E)$ is spanned by elements of the form $s_\lambda s_\mu^*$, we consider the image of $\operatorname{span}\{s_\lambda s_\mu^* : \lambda, \mu \in E^1\}$ in the diagonal. We show that for a finite subset F of E^1, $C^*(\{t_\lambda t_\lambda^* : \lambda \in F\})$ sits inside a finite-dimensional diagonal subalgebra of B, and use the matrix units in this diagonal subalgebra to show that Φ^B exists and is norm-decreasing. We can then show that π_t is faithful on $\operatorname{span}\{s_\lambda s_\lambda^* : \lambda \in E^1\}$ just by checking that the matrix units are nonzero.

Condition (5) of Definition 7.1 shows that $C^*(\{t_\lambda t_\lambda^* : \lambda \in F\})$ is typically bigger than $\operatorname{span}\{t_\lambda t_\lambda^* : \lambda \in F\}$; the two can only be equal if $\lambda, \mu \in F$ implies $\operatorname{MCE}(\lambda, \mu) \subset F$. Thus we need to pass to a larger finite set H such that $\lambda, \mu \in H$ imply $\operatorname{MCE}(\lambda, \mu) \subset H$.

Definition 8.3. For each finite subset F of E^1, let

$$\operatorname{MCE}(F) := \{\lambda \in E^1 : d(\lambda) = \textstyle\bigvee_{\alpha \in F} d(\alpha) \text{ and } \lambda(e, d(\alpha)) = \alpha \text{ for all } \alpha \in F\},$$

and let $\vee F := \bigcup_{G \subset F} \operatorname{MCE}(G)$.

Definition 8.3 is consistent with Definition 5.3, since $\operatorname{MCE}(\{\lambda, \mu\}) = \operatorname{MCE}(\lambda, \mu)$.

Lemma 8.4. *Let F be a finite subset of E^1. Then*

(1) *$F \subset \vee F$;*
(2) *$\vee F$ is the union of the disjoint sets $\vee\{\lambda \in F : s(\lambda) = v\}$ over $v \in s(F)$;*
(3) *$\vee F$ is finite; and*
(4) *$G \subset \vee F$ implies $\operatorname{MCE}(G) \subset \vee F$.*

Proof. (1) For $\lambda \in F$, $\{\lambda\} \subset F$ and $\lambda \in \operatorname{MCE}(\{\lambda\})$.

(2) If $\lambda, \mu \in G$ and $s(\lambda) \neq s(\mu)$, then $\operatorname{MCE}(G)$ is empty.

(3) It suffices to show that if $F \subset E^1$ is finite, then $\operatorname{MCE}(F)$ is finite. When $|F| = 1$, this assertion is trivial. Suppose as an inductive hypothesis that $\operatorname{MCE}(F)$ is finite whenever $|F| \le k$ for some $k \ge 1$, and suppose that $|F| = k + 1$. Let $\lambda \in F$, and let $F' := F \setminus \{\lambda\}$. Suppose that $\gamma \in \operatorname{MCE}(F)$. Since $\gamma(e, \bigvee_{\alpha \in F'} d(\alpha)) \in \operatorname{MCE}(F')$, we have $\gamma \in \operatorname{MCE}(\lambda, \mu)$ for some $\mu \in \operatorname{MCE}(F')$. Hence $|\operatorname{MCE}(F)| \le \sum_{\mu \in \operatorname{MCE}(F')} |\operatorname{MCE}(\lambda, \mu)|$. Each term in this sum is finite because (E, φ) is finitely aligned, and the sum has only finitely many terms by the inductive hypothesis. Hence $\operatorname{MCE}(F)$ is finite.

(4) Let $G \subset \vee F$ and for $\alpha \in G$ choose $G_\alpha \subset F$ such that $\alpha \in \operatorname{MCE}(G_\alpha)$. Let $H := \bigcup_{\alpha \in G} G_\alpha$. We will show that $\operatorname{MCE}(G) \subset \operatorname{MCE}(H) \subset \vee F$. Suppose $\lambda \in \operatorname{MCE}(G)$. Then $d(\lambda) = \bigvee_{\alpha \in G} d(\alpha) = \bigvee_{\alpha \in G} \big(\bigvee_{\beta \in G_\alpha} d(\beta)\big) = \bigvee_{\beta \in H} d(\beta)$. For $\beta \in H$, choose $\alpha \in G$ such that $\beta \in G_\alpha$. Then $\lambda(e, d(\beta)) = \alpha(e, d(\beta)) = \beta$. Thus $\lambda \in \operatorname{MCE}(H)$. $\square$

It follows from Lemma 8.4(4) that $\lambda, \mu \in \vee F$ implies that $\operatorname{MCE}(\lambda, \mu) \subset \vee F$. Consequently, Lemma 8.4(1) and (7.1) imply that

$$C^*(\{t_\lambda t_\lambda^* : \lambda \in F\}) \subset C^*(\{t_\lambda t_\lambda^* : \lambda \in \vee F\}) = \operatorname{span}\{t_\lambda t_\lambda^* : \lambda \in \vee F\}.$$

To write this as a diagonal matrix algebra, we need to be able to orthogonalise the range projections associated to the edges in $\vee F$.

Lemma 8.5. *Let $\lambda \in E^1$. If $F \subset s^{-1}(r(\lambda))$ is finite and $r(\lambda) \notin F$, then*
$$t_\lambda t_\lambda^*\Big(\prod_{\mu\in F}(t_{s(\lambda)} - t_{\lambda\mu}t_{\lambda\mu}^*)\Big) > 0.$$

Proof. We have
$$\Big\|t_\lambda t_\lambda^*\Big(\prod_{\mu\in F}(t_{s(\lambda)} - t_{\lambda\mu}t_{\lambda\mu}^*)\Big)\Big\| = \Big\|\prod_{\mu\in F}(t_\lambda t_\lambda^* - t_{\lambda\mu}t_{\lambda\mu}^*)\Big\| = \Big\|t_\lambda\Big(\prod_{\mu\in F}(t_{r(\lambda)} - t_\mu t_\mu^*)\Big)t_\lambda^*\Big\|,$$
which is nonzero by (8.1). □

We now define our matrix units. First note that (7.1) for the Toeplitz-Cuntz-Krieger family $\{t_\lambda\}$ implies that the range projections $t_\lambda t_\lambda^*$ commute with each other. Thus for every finite subset F of E^1 and every $\lambda \in \vee F$, the operator $Q_\lambda^{\vee F}$ defined by
$$Q_\lambda^{\vee F} := t_\lambda t_\lambda^*\Big(\prod_{\lambda\alpha\in\vee F,\, d(\alpha)\neq e}(t_{s(\lambda)} - t_{\lambda\alpha}t_{\lambda\alpha}^*)\Big)$$
is a projection which commutes with every $t_\mu t_\mu^*$.

Proposition 8.6. *Let F be a finite subset of E^1 such that $\lambda \in F$ implies $s(\lambda) \in F$. Then $\{Q_\lambda^{\vee F} : \lambda \in \vee F\}$ is a collection of nonzero mutually orthogonal projections in B such that* $\operatorname{span}\{Q_\lambda^{\vee F} : \lambda \in \vee F\} = \operatorname{span}\{t_\lambda t_\lambda^* : \lambda \in \vee F\}$. *In particular,*
$$\sum_{\lambda\in\vee F} Q_\lambda^{\vee F} = \sum_{v\in s(F)} t_v. \tag{8.2}$$

The key to proving Proposition 8.6 is establishing (8.2), which we do by induction on $|F|$. This requires two technical lemmas.

Lemma 8.7. *Let F be as in Proposition 8.6, suppose $\lambda \in F \setminus E^0$ and let $G := F \setminus \{\lambda\}$. Then for every $\gamma \in \vee F \setminus \vee G$ there is a unique $\mu_\gamma \in \vee G$ such that*
$$\text{if } \mu \in \vee G \text{ and } \gamma(e, d(\mu)) = \mu \text{ then } d(\mu) \le d(\mu_\gamma). \tag{8.3}$$
We then have $\gamma \in \operatorname{MCE}(\mu_\gamma, \lambda)$; in particular, $d(\gamma) = d(\mu_\gamma) \vee d(\lambda)$.

Proof. For $\gamma \in \vee F \setminus \vee G$, let $(\vee G)_\gamma := \{\mu \in \vee G : \gamma(e, d(\mu)) = \mu\}$, which is nonempty because $s(\gamma) \in (\vee G)_\gamma$. For every $\mu \in (\vee G)_\gamma$, $d(\mu) \le d(\gamma)$, so $d := \bigvee_{\mu\in(\vee G)_\gamma} d(\mu)$ satisfies $d \le d(\gamma)$. Lemma 8.4(4) shows that $\gamma(e, d) \in \vee G$, and then $\mu_\gamma := \gamma(e, d)$ has the required property. To see that $\gamma \in \operatorname{MCE}(\mu_\gamma, \lambda)$, notice that $\gamma \in \vee F \setminus \vee G$ implies $\gamma \in \operatorname{MCE}(\mu, \lambda)$ for some $\mu \in \vee G$. Thus $\mu \in (\vee G)_\gamma$, $d(\mu) \le d(\mu_\gamma)$, and
$$d(\gamma) = d(\mu) \vee d(\lambda) \le d(\mu_\gamma) \vee d(\lambda).$$
On the other hand, we have $d(\gamma) \ge d(\mu_\gamma)$ by definition, and $d(\gamma) \ge d(\lambda)$ since $\gamma \in \operatorname{MCE}(\lambda, \mu)$. Hence $d(\gamma) = d(\mu_\gamma) \vee d(\lambda)$, and $\gamma \in \operatorname{MCE}(\mu_\gamma, \lambda)$. □

Lemma 8.8. *Let F be as in Proposition 8.6, suppose $\lambda \in F \setminus E^0$ and let $G := F \setminus \{\lambda\}$. Then for each $\delta \in \vee F \setminus \vee G$,*
$$Q_\delta^{\vee F} = Q_{\mu_\delta}^{\vee G} t_\delta t_\delta^*. \tag{8.4}$$

Proof. We shall show that

(1) $Q_\delta^{\vee F} = Q_{\mu_\delta}^{\vee G} Q_\delta^{\vee F}$, and

(2) $Q^{\vee G}_{\mu_\delta} t_{\delta\varepsilon} t^*_{\delta\varepsilon} = 0$ whenever $\delta\varepsilon \in \vee F$ and $d(\varepsilon) \neq e$,

and then use these to prove (8.4).

To prove (1), let $\delta \in \vee F \setminus \vee G$. Since $t_{\mu_\delta} t^*_{\mu_\delta} \geq t_\delta t^*_\delta$,

$$Q^{\vee G}_{\mu_\delta} Q^{\vee F}_\delta = t_\delta t^*_\delta \Big(\prod_{\mu_\delta \nu \in \vee G,\, d(\nu) \neq e} (t_{s(\delta)} - t_{\mu_\delta \nu} t^*_{\mu_\delta \nu}) \Big) Q^{\vee F}_\delta.$$

Suppose $\mu_\delta \nu \in \vee G$ and $d(\nu) \neq e$. Then

$$t_\delta t^*_\delta (t_{s(\delta)} - t_{\mu_\delta \nu} t^*_{\mu_\delta \nu}) = t_\delta t^*_\delta - \sum_{\gamma \in \mathrm{MCE}(\delta, \mu_\delta \nu)} t_\gamma t^*_\gamma \quad \text{by (7.1)}.$$

Now suppose $\gamma \in \mathrm{MCE}(\delta, \mu_\delta \nu)$. Then $d(\mu_\gamma) \geq d(\mu_\delta \nu)$ because $\mu_\delta \nu \in \vee G$, and $d(\mu_\delta \nu) > d(\mu_\delta)$ because $d(\nu) \neq e$. In particular $\gamma \neq \delta$. But $\gamma(e, d(\delta)) = \delta$ because $\gamma \in \mathrm{MCE}(\delta, \mu_\delta \nu)$. Hence there exists $\varepsilon \in E^1$ such that $d(\varepsilon) \neq e$ and $\gamma = \delta\varepsilon$. Since δ and $\mu_\delta \nu$ are in $\vee F$, Lemma 8.4(4) ensures that $\gamma \in \vee F$, so $t_{s(\delta)} - t_\gamma t^*_\gamma$ is a factor in $Q^{\vee F}_\delta$, and $t_\gamma t^*_\gamma Q^{\vee F}_\delta = 0$. Thus

$$t_\delta t^*_\delta (t_{s(\delta)} - t_{\mu_\delta \nu} t^*_{\mu_\delta \nu}) Q^{\vee F}_\delta = t_\delta t^*_\delta Q^{\vee F}_\delta - \Big(\sum_{\gamma \in \mathrm{MCE}(\delta, \mu_\delta \nu)} t_\gamma t^*_\gamma \Big) Q^{\vee F}_\delta = Q^{\vee F}_\delta.$$

Applying this equation to each $\mu_\delta \nu \in \vee G$ with $d(\nu) \neq e$ establishes (1).

To prove (2), suppose that $\delta\varepsilon \in \vee F$ with $d(\varepsilon) \neq e$. Then $\mu_{\delta\varepsilon} \in \vee G$, and $\mu_{\delta\varepsilon} \neq \mu_\delta$: if $\mu_{\delta\varepsilon} = \mu_\delta$, then $d(\delta\varepsilon) = d(\lambda) \vee d(\mu_{\delta\varepsilon}) = d(\lambda) \vee d(\mu_\delta) = d(\delta)$, contradicting $d(\varepsilon) \neq e$. However, $(\delta\varepsilon)(e, d(\mu_\delta)) = \delta(e, d(\mu_\delta)) = \mu_\delta$, so Lemma 8.7 implies that $d(\mu_\delta) < d(\mu_{\delta\varepsilon})$, and $\mu_{\delta\varepsilon} = \mu_\delta \alpha$ for some α with $d(\alpha) \neq e$. Since $\mu_{\delta\varepsilon} \in \vee G$, it follows that

$$Q^{\vee G}_{\mu_\delta} t_{\delta\varepsilon} t^*_{\delta\varepsilon} \leq (t_{s(\mu_\delta)} - t_{\mu_\delta \alpha} t^*_{\mu_\delta \alpha}) t_{\delta\varepsilon} t^*_{\delta\varepsilon},$$

which vanishes because $\mu_\delta \alpha = (\delta\varepsilon)(e, d(\mu_{\delta\varepsilon}))$. This gives (2).

To finish off, we compute:

$$\begin{aligned} Q^{\vee F}_\delta &= Q^{\vee G}_{\mu_\delta} Q^{\vee F}_\delta \quad \text{by (1)} \\ &= Q^{\vee G}_{\mu_\delta} \Big(\prod_{\delta\varepsilon \in \vee F,\, d(\varepsilon) \neq e} (t_{s(\mu_\delta)} - t_{\delta\varepsilon} t^*_{\delta\varepsilon}) \Big) t_\delta t^*_\delta \\ &= Q^{\vee G}_{\mu_\delta} t_\delta t^*_\delta \quad \text{by (2)}. \end{aligned}$$

□

Proof of Proposition 8.6. The $Q^{\vee F}_\lambda$ are nonzero by Lemma 8.5. To see that the $Q^{\vee F}_\lambda$ are orthogonal, suppose that $\lambda \neq \mu \in \vee F$. If $d(\lambda) = d(\mu)$ then $Q^{\vee F}_\lambda Q^{\vee F}_\mu \leq t_\lambda t^*_\lambda t_\mu t^*_\mu = 0$ by (4) of Definition 7.1. So suppose that $d(\lambda) \neq d(\mu)$. We can assume without loss of generality that $d(\lambda) \vee d(\mu) > d(\lambda)$. Then $\gamma \in \mathrm{MCE}(\lambda, \mu)$ implies $\gamma = \lambda\alpha$ where $d(\alpha) \neq e$, and $\gamma \in \vee F$ by Lemma 8.4(4). Thus (7.1) shows that

$$Q^{\vee F}_\lambda Q^{\vee F}_\mu \leq \Big(\sum_{\gamma \in \mathrm{MCE}(\lambda, \mu)} t_\gamma t^*_\gamma \Big) Q^{\vee F}_\lambda = 0.$$

Assuming that (8.2) has been established, let $\lambda \in \vee F$ and calculate:

$$\begin{aligned}
t_\lambda t_\lambda^* &= t_\lambda t_\lambda^* \Big(\sum_{\mu \in \vee F} Q_\mu^{\vee F} \Big) \quad \text{by (8.2)} \\
&= \sum_{\mu \in \vee F} \Big(t_\lambda t_\lambda^* t_\mu t_\mu^* \Big(\prod_{\mu\alpha \in \vee F,\, d(\alpha) \neq e} (t_{s(\mu)} - t_{\mu\alpha} t_{\mu\alpha}^*) \Big) \Big) \\
&= \sum_{\mu \in \vee F} \Big(\Big(\sum_{\gamma \in \mathrm{MCE}(\lambda,\mu)} t_\gamma t_\gamma^* \Big) \Big(\prod_{\mu\alpha \in \vee F,\, d(\alpha) \neq e} (t_{s(\mu)} - t_{\mu\alpha} t_{\mu\alpha}^*) \Big) \Big). \qquad (8.5)
\end{aligned}$$

Suppose $\mu \in \vee F$ and $\mu \neq \lambda\lambda'$ for any path λ', and that $\gamma \in \mathrm{MCE}(\lambda, \mu)$. Lemma 8.4(4) ensures that $\gamma \in \vee F$, and $\gamma \neq \mu$ because $\mu \neq \lambda\lambda'$. Thus $\gamma = \mu\alpha$ for some path α such that $d(\alpha) \neq e$. Hence the product in (8.5) vanishes for such μ, and (8.5) collapses to

$$t_\lambda t_\lambda^* = \sum_{\lambda\lambda' \in \vee F} Q_{\lambda\lambda'}^{\vee F}.$$

It therefore suffices to establish (8.2). Indeed, $Q_\lambda^{\vee F} \leq s(\lambda)$ for all λ, so Lemma 8.4(2) shows that it suffices to establish (8.2) when $F \subset s^{-1}(v)$ for some $v \in E^0$. We do this by induction on $|F|$. Recall that $\lambda \in F$ implies $s(\lambda) \in F$, so if $|F| = 1$ then $F = \vee F = \{v\}$ and $Q_v^{\vee F} = t_v$.

Suppose that $|F| = k + 1 \geq 2$, and that the proposition holds for all subsets of $s^{-1}(v)$ containing v and having at most k elements. Since $|F| > 1$ there exists $\lambda \neq v$ in F. Let $G := F \setminus \{\lambda\}$. For $\mu \in \vee G$, we have

$$Q_\mu^{\vee F} = t_\mu t_\mu^* \Big(\prod_{\mu\alpha \in \vee G,\, d(\alpha) \neq e} (t_v - t_{\mu\alpha} t_{\mu\alpha}^*) \Big) \Big(\prod_{\gamma = \mu\beta \in \vee F \setminus \vee G} (t_v - t_\gamma t_\gamma^*) \Big).$$

Suppose that $t_v - t_\gamma t_\gamma^*$ is a factor in the second product and $\mu_\gamma \neq \mu$. Then $\mu_\gamma = \mu\alpha$ for some α such that $d(\alpha) \neq e$ because μ_γ is the maximal subpath of γ in $\vee G$. Thus $t_v - t_\gamma t_\gamma^*$ is larger than the factor $t_v - t_{\mu_\gamma} t_{\mu_\gamma}^*$ from the first product. So such terms in the second product can be deleted without changing the product, and we have

$$Q_\mu^{\vee F} = Q_\mu^{\vee G} \Big(\prod_{\gamma \in \vee F \setminus \vee G,\, \mu_\gamma = \mu} (t_v - t_\gamma t_\gamma^*) \Big).$$

Thus

$$\begin{aligned}
\sum_{\lambda \in \vee F} Q_\lambda^{\vee F} &= \sum_{\mu \in \vee G} Q_\mu^{\vee G} \Big(\prod_{\gamma \in \vee F \setminus \vee G,\, \mu_\gamma = \mu} (t_v - t_\gamma t_\gamma^*) \Big) + \sum_{\delta \in \vee F \setminus \vee G} Q_\delta^{\vee F} \\
&= \sum_{\mu \in \vee G} \Big(Q_\mu^{\vee G} \Big(\prod_{\gamma \in \vee F \setminus \vee G,\, \mu_\gamma = \mu} (t_v - t_\gamma t_\gamma^*) \Big) + \sum_{\delta \in \vee F \setminus \vee G,\, \mu_\delta = \mu} Q_\delta^{\vee F} \Big)
\end{aligned}$$

by Lemma 8.7, and Lemma 8.8 gives

$$\begin{aligned}
\sum_{\lambda \in \vee F} Q_\lambda^{\vee F} &= \sum_{\mu \in \vee G} \Big(Q_\mu^{\vee G} \Big(\prod_{\gamma \in \vee F \setminus \vee G,\, \mu_\gamma = \mu} (t_v - t_\gamma t_\gamma^*) \Big) + \sum_{\delta \in \vee F \setminus \vee G,\, \mu_\delta = \mu} Q_{\mu_\delta}^{\vee G} t_\delta t_\delta^* \Big) \\
&= \sum_{\mu \in \vee G} Q_\mu^{\vee G} \Big(\Big(\prod_{\gamma \in \vee F \setminus \vee G,\, \mu_\gamma = \mu} (t_v - t_\gamma t_\gamma^*) \Big) + \sum_{\delta \in \vee F \setminus \vee G,\, \mu_\delta = \mu} t_\delta t_\delta^* \Big). \qquad (8.6)
\end{aligned}$$

If $\mu \in \vee G$ and $\delta \in \vee F \setminus \vee G$ satisfies $\mu_\delta = \mu$, then Lemma 8.7 implies that $d(\delta) = d(\mu) \vee d(\lambda)$. Thus $\{t_\delta t_\delta^* : \mu_\delta = \mu\}$ are mutually orthogonal, and (8.6) is just $\sum_{\mu \in \vee G} Q_\mu^{\vee G}$. Applying the inductive hypothesis to G now establishes (8.2) for the given F. □

Proposition 8.9. *There is a norm-decreasing linear map*

$$\Phi^B : C^*(\{t_\lambda : \lambda \in E^1\}) \to \overline{\operatorname{span}}\{t_\lambda t_\lambda^* : \lambda \in E^1\}$$

such that $\Phi^B \circ \pi_t = \pi_t \circ \Phi^E$.

Proof. It suffices to show that if $F \subset E^1$ is finite and $\{\alpha_{\lambda,\mu} : \lambda, \mu \in F\} \subset \mathbb{C}$, then $\big\| \sum_{\lambda,\mu \in F} \alpha_{\lambda,\mu} t_\lambda t_\mu^* \big\| \geq \big\| \sum_{\lambda \in F} \alpha_{\lambda,\lambda} t_\lambda t_\lambda^* \big\|$.

Since $\sum_{\gamma \in F} Q_\gamma^{\vee F} = \sum_{v \in s(F)} t_v$ and the $Q_\gamma^{\vee F}$ commute with the $t_\lambda t_\lambda^*$, there exists $\gamma \in \vee F$ such that

$$\Big\| Q_\gamma^{\vee F} \Big(\sum_{\lambda \in F} \alpha_{\lambda,\lambda} t_\lambda t_\lambda^* \Big) \Big\| = \Big\| \sum_{\lambda \in F} \alpha_{\lambda,\lambda} t_\lambda t_\lambda^* \Big\|. \tag{8.7}$$

If $\lambda \in F$ and $\gamma \neq \lambda\beta$ for any β, then $\delta \in \operatorname{MCE}(\lambda, \gamma)$ implies $d(\delta) > d(\gamma)$, giving

$$Q_\gamma^{\vee F} t_\lambda = Q_\gamma^{\vee F} t_\lambda t_\lambda^* t_\lambda = \Big(\prod_{\gamma\beta \in \vee F,\, d(\beta) \neq e} \big(t_\gamma t_\gamma^* - t_{\gamma\beta} t_{\gamma\beta}^*\big) \Big) \Big(\sum_{\delta \in \operatorname{MCE}(\gamma,\lambda)} t_\delta t_\delta^* \Big) t_\lambda = 0.$$

Thus

$$Q_\gamma^{\vee F} \Big(\sum_{\lambda,\mu \in F} \alpha_{\lambda,\mu} t_\lambda t_\mu^* \Big) Q_\gamma^{\vee F} = Q_\gamma^{\vee F} \Big(\sum_{\substack{\lambda,\mu \in F \\ \gamma(e,d(\lambda)) = \lambda \\ \gamma(e,d(\mu)) = \mu}} \alpha_{\lambda,\mu} t_\lambda t_\mu^* \Big) Q_\gamma^{\vee F}.$$

In particular, notice that for $\lambda \in \vee F$,

$$Q_\gamma^{\vee F} t_\lambda t_\lambda^* = \begin{cases} Q_\gamma^{\vee F} & \text{if } d(\gamma) \geq d(\lambda) \text{ and } \gamma(e, d(\lambda)) = \lambda \\ 0 & \text{otherwise.} \end{cases} \tag{8.8}$$

We will replace $Q_\gamma^{\vee F}$ with a smaller nonzero projection Q_γ so that the remaining off-diagonal terms are eliminated. Since $0 < Q_\gamma \leq Q_\gamma^{\vee F}$, we will then have

$$Q_\gamma t_\lambda t_\lambda^* = \begin{cases} Q_\gamma & \text{if } d(\gamma) \geq d(\lambda) \text{ and } \gamma(e, d(\lambda)) = \lambda \\ 0 & \text{otherwise,} \end{cases} \tag{8.9}$$

which, in conjunction with (8.8), will imply that

$$\Big\| Q_\gamma \Big(\sum_{\lambda \in F} \alpha_{\lambda,\lambda} t_\lambda t_\lambda^* \Big) \Big\| = \Big| \sum_{\substack{\lambda \in F,\, d(\lambda) \leq d(\gamma), \\ \gamma(e,d(\lambda)) = \lambda}} \alpha_{\lambda,\lambda} \Big| = \Big\| Q_\gamma^{\vee F} \Big(\sum_{\lambda \in F} \alpha_{\lambda,\lambda} t_\lambda t_\lambda^* \Big) \Big\|. \tag{8.10}$$

To produce Q_γ, we consider pairs $\lambda, \mu \in \vee F$ such that $\gamma(e, d(\lambda)) = \lambda$ and $\gamma(e, d(\mu)) = \mu$. For each such (λ, μ), factorise γ as $\lambda\lambda' = \gamma = \mu\mu'$, and define

$$d_\gamma(\lambda, \mu) := \big\{\sigma : \sigma = \delta(d(\lambda'), d(\delta)) \text{ or } \sigma = \delta(d(\mu'), d(\delta)) \text{ for some } \delta \in \operatorname{MCE}(\lambda', \mu')\big\}.$$

Now λ' and μ' are uniquely determined by λ, μ and γ, each $\operatorname{MCE}(\lambda', \mu')$ is finite, and $\delta(d(\lambda'), d(\delta))$ and $\delta(d(\mu'), d(\delta))$ are uniquely determined by $\delta \in \operatorname{MCE}(\lambda', \mu')$, so each

$d_\gamma(\lambda,\mu)$ is finite. Let

$$Q_\gamma := Q_\gamma^{\vee F} \prod_{\substack{\lambda\neq\mu\in\vee F,\,\gamma(e,d(\lambda))=\lambda,\\ \gamma(e,d(\mu))=\mu,\,\sigma\in d_\gamma(\lambda,\mu)}} \big(t_\gamma t_\gamma^* - t_{\gamma\sigma}t_{\gamma\sigma}^*\big).$$

Lemma 8.5 implies $Q_\gamma > 0$, and $Q_\gamma \leq Q_\gamma^{\vee F}$ by definition, so we have (8.9) and (8.10). For $\lambda,\mu\in\vee F$ with $\lambda\lambda' = \gamma = \mu\mu'$ and $\lambda\neq\mu$, we calculate:

$$\begin{aligned} Q_\gamma t_\lambda t_\mu^* Q_\gamma &= Q_\gamma(t_\lambda(t_{\lambda'}t_{\lambda'}^*t_{\mu'}t_{\mu'}^*)t_\mu^*)Q_\gamma \\ &= Q_\gamma\Big(t_\lambda\Big(\sum_{\nu\in\mathrm{MCE}(\lambda',\mu')} t_\nu t_\nu^*\Big)t_\mu^*\Big)Q_\gamma, \end{aligned}$$

which vanishes because $\nu\in\mathrm{MCE}(\lambda',\mu')$ implies that $\lambda\nu = \gamma\sigma$ for some $\sigma\in d_\gamma(\lambda,\mu)$. Thus

$$\begin{aligned} \Big\|\sum_{\lambda,\mu\in F}\alpha_{\lambda,\mu}t_\lambda t_\mu^*\Big\| &\geq \Big\|Q_\gamma\Big(\sum_{\lambda,\mu\in F}\alpha_{\lambda,\mu}t_\lambda t_\mu^*\Big)Q_\gamma\Big\| \\ &= \Big\|Q_\gamma\Big(\sum_{\lambda\in F}\alpha_{\lambda,\lambda}t_\lambda t_\lambda^*\Big)Q_\gamma\Big\| \\ &= \Big\|Q_\gamma^{\vee F}\Big(\sum_{\lambda\in F}\alpha_{\lambda,\lambda}t_\lambda t_\lambda^*\Big)\Big\| \quad \text{by (8.10)} \\ &= \Big\|\sum_{\lambda\in F}\alpha_{\lambda,\lambda}t_\lambda t_\lambda^*\Big\| \qquad \text{by (8.7).} \end{aligned}$$

□

Proof of Theorem 8.1. It suffices to show that if F is a finite subset of E^1 and

$$a = \sum_{\lambda,\mu\in F}\alpha_{\lambda,\mu}s_\lambda s_\mu^*,$$

then $\pi_t(a) = 0$ implies $a = 0$. Suppose $\pi_t(a) = 0$. Then $\pi_t(a^*a) = 0$, $\Phi^B(\pi_t(a^*a)) = 0$, and Proposition 8.9 implies that $\pi_t(\Phi^E(a^*a)) = 0$. Now $\Phi^E(a^*a)$ belongs to $D := \operatorname{span}\{s_\lambda s_\lambda^* : \lambda\in\vee F\}$, and applying Proposition 8.6 to the universal Toeplitz-Cuntz-Krieger E-family $\{s_\lambda\}$ shows that D is a finite-dimensional diagonal matrix algebra with matrix units

$$\Big\{e_{\lambda,\lambda} := s_\lambda s_\lambda^*\Big(\prod_{\lambda\alpha\in\vee F,\,d(\alpha)\neq e}(s_{s(\lambda)} - s_{\lambda\alpha}s_{\lambda\alpha}^*)\Big) : \lambda\in\vee F\Big\}.$$

Lemma 8.5 implies that $\pi_t(e_{\lambda,\lambda}) \neq 0$ for $\lambda\in\vee F$, so π_t is faithful on D. In particular $\|\Phi^E(a^*a)\| = \|\pi_t(\Phi^E(a^*a))\| = 0$. Proposition 8.2 now shows that $a^*a = 0$, and hence $a = 0$. □

9. The C^*-algebra of an infinite k-graph

We show how the finitely-aligned hypothesis, relation (5) of Definition 7.1, and the hypothesis (8.1) in Theorem 8.1 all simplify when the underlying semigroup is $\mathbb{N}^k$. We then prove a uniqueness theorem for the C^*-algebras of k-graphs in which every vertex receives infinitely many paths of every degree.

9.1. **Product systems of graphs over $\mathbb{N}^k$.**

Lemma 9.1. *Let (E,φ) be a product system of graphs over $\mathbb{N}^k$. Then (E,φ) is finitely aligned if and only if*

$$\mathrm{MCE}(\mu,\nu) \text{ is finite for every pair } \mu\in E^1_{e_i} \text{ and } \nu\in E^1_{e_j} \text{ with } i\neq j. \tag{9.1}$$

Proof. Every finitely aligned system trivially satisfies (9.1). For the reverse implication, suppose E satisfies (9.1). Then $\mathrm{MCE}(\mu,\nu)$ is finite whenever $|d(\mu)\vee d(\nu)|\leq 2$. Suppose as an inductive hypothesis that $\mathrm{MCE}(\mu,\nu)$ is finite whenever $|d(\mu)\vee d(\nu)|\leq n$, and consider $\mu\in E^1_p$, $\nu\in E^1_q$ with $|p\vee q|=n+1$.

If the coordinate-wise minimum $p\wedge q$ of p and q is nonzero, then either $\mu(0,p\wedge q)\neq \nu(0,p\wedge q)$, in which case the factorisation property implies $\mathrm{MCE}(\mu,\nu)=\emptyset$, or

$$\mathrm{MCE}(\mu,\nu)=\big\{\mu(0,p\wedge q)\gamma:\gamma\in\mathrm{MCE}(\mu(p\wedge q,p),\nu(p\wedge q,q))\big\}$$

is finite by the inductive hypothesis. Thus we may assume that $p\wedge q=0$, and hence that $p\vee q=p+q$. If $p\geq q$ or $q\geq p$ then $\mathrm{MCE}(\mu,\nu)$ has at most one element. So we may further assume that there exist $i\neq j$ such that $p_i>q_i$ and $q_j>p_j$. Since $p\wedge q=0$, this implies that $p_j=q_i=0$.

Now let $\gamma\in\mathrm{MCE}(\mu,\nu)$. Then $d(\gamma)-e_i=p+q-e_i=(p-e_i)\vee q$ since $q_i=0$. Thus $\gamma_i:=\gamma(0,d(\gamma)-e_i)$ satisfies

$$\gamma_i(0,p-e_i)=\gamma(0,p-e_i)=\mu(0,p-e_i)\ \text{ and }\ \gamma_i(0,q)=\gamma(0,q)=\nu,$$

so $\gamma_i\in\mathrm{MCE}(\mu(0,p-e_i),\nu)$. Similarly, $\gamma_j:=\gamma(0,d(\gamma)-e_j)\in\mathrm{MCE}(\mu,\nu(0,q-e_j))$. But now $p\vee q=d(\gamma_i)+e_i=d(\gamma_j)+e_j$, and since $i\neq j$, it follows that $d(\gamma)=d(\gamma_i)\vee d(\gamma_j)$. Furthermore, $\gamma(0,d(\gamma_i))=\gamma_i$ and $\gamma(0,d(\gamma_j))=\gamma_j$, so $\gamma\in\mathrm{MCE}(\gamma_i,\gamma_j)$. Hence

$$|\mathrm{MCE}(\mu,\nu)|\leq\sum_{\substack{\gamma_i\in\mathrm{MCE}(\mu(0,p-e_i),\nu)\\ \gamma_j\in\mathrm{MCE}(\mu,\nu(0,q-e_j))}}|\mathrm{MCE}(\gamma_i,\gamma_j)|.$$

By the inductive hypothesis, $\mathrm{MCE}(\mu(0,p-e_i),\nu)$ and $\mathrm{MCE}(\mu,\nu(0,q-e_j))$ are finite, so the sum has only finitely many terms. Thus we take $\gamma_i\in\mathrm{MCE}(\mu(0,p-e_i),\nu)$ and $\gamma_j\in\mathrm{MCE}(\mu,\nu(0,q-e_j))$, and show that $\mathrm{MCE}(\gamma_i,\gamma_j)$ is finite. If it is nonempty, then the initial segments of degree $(p\vee q)-e_i-e_j$ of γ_i and γ_j are the same; call it β, and write $\gamma_i=\beta\gamma_i'$, $\gamma_j=\beta\gamma_j'$. Then $d(\gamma_i')=e_i$ and $d(\gamma_j')=e_j$, so $|\,\mathrm{MCE}(\gamma_i,\gamma_j)|=|\,\mathrm{MCE}(\gamma_i',\gamma_j')|$ is finite by (9.1). □

Lemma 9.2. *Let (E,φ) be a finitely aligned product system of graphs over $\mathbb{N}^k$. Then a Toeplitz E-family $\{t_\lambda\}$ is a Toeplitz-Cuntz-Krieger E-family if and only if*

$$t_\mu^*t_\nu=\sum_{\mu\alpha=\nu\beta\in\mathrm{MCE}(\mu,\nu)}t_\alpha t_\beta^* \tag{9.2}$$

for every $\mu\in E^1_{e_i}$ and $\nu\in E^1_{e_j}$ with $s(\mu)=s(\nu)$ and $i\neq j$.

Proof. Since (9.2) is a special case of Definition 7.1(5), we have to show that (9.2) implies Definition 7.1(5). If $|d(\mu)\vee d(\nu)|\leq 2$, this is trivially true. Suppose as an inductive hypothesis that (6.4) holds whenever $|d(\mu)\vee d(\nu)|\leq n$ for some $n\geq 2$. Suppose $\mu\in E^1_p$ and $\nu\in E^1_q$ where p and q satisfy $|p\vee q|=n+1$. We give separate arguments for $p\wedge q\neq 0$ and $p\wedge q=0$.

If $p \wedge q \neq 0$, then

$$t^*_\mu t_\nu = t^*_{\mu(p\wedge q,p)} t^*_{\mu(0,p\wedge q)} t_{\nu(0,p\wedge q)} t_{\nu(p\wedge q,q)}$$

$$= \begin{cases} t^*_{\mu(p\wedge q,p)} t_{\nu(p\wedge q,q)} & \text{if } \mu(0,p\wedge q) = \nu(0,p\wedge q) \\ 0 & \text{otherwise.} \end{cases} \tag{9.3}$$

The set $\operatorname{MCE}(\mu,\nu)$ is empty unless $\mu(0,p\wedge q) = \nu(0,p\wedge q)$, and if so we have

$$\operatorname{MCE}(\mu,\nu) = \big\{\mu(0,p\wedge q)\gamma : \gamma \in \operatorname{MCE}(\mu(p\wedge q,p),\nu(p\wedge q,q))\big\}.$$

Applying the inductive hypothesis to (9.3) gives Definition 7.1(5).

Now suppose $p \wedge q = 0$, or equivalently that $p \vee q = p + q$. Since $|p \vee q| \geq 3$, we can assume that $|q| \geq 2$. If $p \geq q$ then (6.4) is trivial, so we may further assume that there exists i such that $q_i > p_i$, and then $p\wedge q = 0$ forces $p_i = 0$. In particular, $|p\vee(q-e_i)| = n$, and the inductive hypothesis gives

$$t^*_\mu t_\nu = t^*_\mu t_{\nu(0,q-e_i)} t_{\nu(q-e_i,q)} = \Big(\sum_{\mu\delta=\nu(0,q-e_i)\varepsilon\in\operatorname{MCE}(\mu,\nu(0,q-e_i))} t_\delta t^*_\varepsilon\Big) t_{\nu(q-e_i,q)}.$$

Each ε appearing in this sum has $d(\varepsilon) = p$, so $d(\varepsilon) \vee d(\nu(q-e_i,q) = p + e_i$, which has length at most n because $|q| \geq 2$. Thus we can apply the inductive hypothesis to each summand to get

$$t^*_\mu t_\nu = \sum_{\substack{\mu\delta=\nu(0,q-e_i)\varepsilon\in\operatorname{MCE}(\mu,\nu(0,q-e_i))\\ \varepsilon\sigma=\nu(q-e_i,q)\tau\in\operatorname{MCE}(\varepsilon,\nu(q-e_i,q))}} t_{\delta\sigma} t^*_\tau. \tag{9.4}$$

It remains to show that the pairs $(\delta\sigma,\tau)$ arising in this sum are precisely the pairs (α,β) arising in the right-hand side of (6.4). Given $(\delta\sigma,\tau)$, we certainly have

$$\mu\delta\sigma = \nu(0,q-e_i)\varepsilon\sigma = \nu(0,q-e_i)\nu(q-e_i,q)\tau = \nu\tau,$$

and $d(\delta\sigma) = d(\delta) + d(\sigma) = q - e_i + e_i = q$, so $\mu\delta\sigma \in \operatorname{MCE}(\mu,\nu)$. Conversely, given (α,β), we take $\delta := \alpha(0,q-e_i)$, $\sigma := \alpha(q-e_i,q)$ and $\tau := \beta$. □

Lemma 9.3. *Let E be a finitely aligned product system of graphs over $\mathbb{N}^k$. Then a Toeplitz E-family $\{t_\lambda\}$ satisfies* (8.1) *if and only if*

$$\prod_{m=1}^{k}\Big(t_v - \sum_{\lambda\in G_m} t_\lambda t^*_\lambda\Big) > 0 \tag{9.5}$$

for every choice of finite sets $G_m \subset s^{-1}_{e_m}(v)$.

Proof. The necessity of (9.5) is obvious. Suppose (9.5) holds, and R, v and F_p are as in Theorem 8.1. For $p \in R$, choose i_p such that $p_{i_p} > 0$, and for each m, set

$$G_m := \bigcup_{\{p\in R: i_p=m\}} \{\lambda(0,e_m) : \lambda \in F_p\}.$$

Then each G_m is a finite subset of $s_{e_m}^{-1}(v)$, and

$$\prod_{p\in R}\Big(t_v-\sum_{\lambda\in F_p}t_\lambda t_\lambda^*\Big)\geq\prod_{p\in R}\Big(t_v-\sum_{\lambda\in F_p}t_{\lambda(0,e_{i_p})}t_{\lambda(0,e_{i_p})}^*\Big)$$
$$=\prod_{m=1}^{k}\Big(t_v-\sum_{\mu\in G_m}t_\mu t_\mu^*\Big),$$

which is nonzero by (9.5). □

9.2. **The C^*-algebra of an infinite k-graph.** If (Λ,d) is a k-graph, and $\lambda,\mu\in\Lambda$, we regard $\mathrm{MCE}(\lambda,\mu)\subset(E_\Lambda)^1$ as a subset of Λ. In view of Lemma 9.2, we say that Λ is finitely aligned if $\mathrm{MCE}(\lambda,\mu)$ is finite whenever $d(\lambda)=e_i$ and $d(\mu)=e_j$. By a Toeplitz-Cuntz-Krieger Λ-family we mean a Toeplitz-Cuntz-Krieger E_Λ-family. If Λ has no sources, so that the graphs in E_Λ have no sinks, then we define a Cuntz-Krieger Λ-family to be a Cuntz-Pimsner E_Λ-family. We have only made this last definition for k-graphs without sources to avoid clashing with the definitions given for row-finite graphs in [17]; for row-finite k-graphs without sources, therefore, our $C^*(E_\Lambda)$ coincides with the graph algebra $C^*(\Lambda)$ used in [11] and [17].

Recall that $\Lambda^n(v):=\{\lambda\in\Lambda: d(\lambda)=n \text{ and } \mathrm{cod}(\lambda)=v\}$. If $|\Lambda^{e_i}(v)|=\infty$ for every $v\in\Lambda^0$, and every $1\leq i\leq k$, then conditions (6) and (4) of Definition 7.1 are equivalent, so Theorem 8.1 gives a uniqueness theorem for $C^*(\Lambda)$.

Corollary 9.4. *Let (Λ,d) be a finitely aligned k-graph such that $|\Lambda^{e_i}(v)|=\infty$ for every $v\in\Lambda^0$ and $1\leq i\leq k$. Let $\{t_\lambda:\lambda\in\Lambda\}$ be a Cuntz-Krieger Λ-family such that $t_v\neq 0$ for all $v\in\Lambda^0$. Then the representation π_t of $C^*(\Lambda):=C^*(E_\Lambda)$ is faithful.*

Proof. That each $|\Lambda^{e_i}(v)|=\infty$ implies both that $C^*(E_\Lambda)=\mathcal{T}C^*(E_\Lambda)$, and that Λ has no sources, so that $C^*(\Lambda):=C^*(E_\Lambda)$. Lemma 9.1 implies that $(E_\Lambda,\varphi_\Lambda)$ is finitely aligned. To establish (8.1), we fix $v\in\Lambda^0$ and finite sets $G_m\subset\Lambda^{e_m}(v)$ for $1\leq m\leq k$. By Lemma 9.3, it suffices to show that

$$\prod_{m=1}^{k}\Big(t_v-\sum_{\lambda\in G_m}t_\lambda t_\lambda^*\Big)>0.$$

We shall construct paths $\mu_m\in\Lambda(v)$ of degree $\sum_{i=1}^m e_i$ for $m\leq k$ such that $\mu_m(0,e_i)$ does not belong to G_i for $1\leq i\leq m$. We take μ_1 to be any edge of degree e_1 which is not in G_1. If we have μ_m, then because the set $\Lambda^{e_{m+1}}(\mathrm{dom}(\mu_m))$ is infinite, there is a path $\mu_{m+1}=\mu_m\alpha$ of degree $\sum_{i=1}^{m+1}e_i$ which is not in the finite set $\bigcup_{\lambda\in G_{m+1}}\mathrm{MCE}(\mu_m,\lambda)$. Then $\mu_{m+1}(0,e_i)=\mu_m(0,e_i)$ is not in G_i for $i\leq m$, and $\mu_{m+1}(0,e_{m+1})$ cannot be in G_{m+1} because $\mu_{m+1}\in\mathrm{MCE}(\mu_m,\mu(0,e_{m+1}))$.

Now for $\lambda\in G_i$, we have $\mathrm{MCE}(\lambda,\mu_k)=\emptyset$, and relation (5) of Definition 7.1 in the form (7.1) gives $t_\lambda t_\lambda^* t_{\mu_k}t_{\mu_k}^*=0$. Thus

$$\prod_{m=1}^{k}\Big(t_v-\sum_{\lambda\in G_m}t_\lambda t_\lambda^*\Big)t_{\mu_k}t_{\mu_k}^*=t_{\mu_k}t_{\mu_k}^*,$$

which is nonzero because $t_{\mu_k}^*t_{\mu_k}=t_{s(\mu_k)}$ is nonzero. Since $\mathbb{Z}^k$ is amenable, the result now follows from Theorem 8.1. □

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