

# SELF-SIMILAR GROUPOID ACTIONS ON $k$ -GRAPHS, AND INVARIANCE OF $K$ -THEORY FOR COCYCLE HOMOTOPIES

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ABSTRACT. We establish conditions under which an inclusion of finitely aligned left-cancellative small categories induces inclusions of twisted  $C^*$ -algebras. We also present an example of an inclusion of finitely aligned left-cancellative monoids that does not induce a homomorphism even between (untwisted) Toeplitz algebras. We prove that the twisted  $C^*$ -algebras of a jointly faithful self-similar action of a countable discrete amenable groupoid on a row-finite  $k$ -graph with no sources, with respect to homotopic cocycles, have isomorphic  $K$ -theory.

## 1. INTRODUCTION

In this article, we establish that for self-similar actions of groupoids on  $k$ -graphs, the  $K$ -theory of the twisted  $C^*$ -algebras is invariant under 2-cocycle homotopy.

It is a recurring theme that the  $K$ -theory of twisted  $C^*$ -algebras is invariant under homotopies of cocycles (see for example [Ell84, ELPW10, Gil15, KPS15]). An early result of this nature is due to Elliott [Ell84], who showed that the  $K$ -theory of each rank- $n$  noncommutative torus—which can be regarded as a twisted group  $C^*$ -algebra of  $\mathbb{Z}^n$ —is isomorphic to the  $K$ -theory of the (untwisted) group  $C^*$ -algebra  $C^*(\mathbb{Z}^n) \cong C(\mathbb{T}^n)$ . Elliott’s proof involves inducting on the dimension  $n$ , using a five-lemma argument based on naturality of the Pimsner–Voiculescu exact sequence. We employ a technique based on Elliott’s argument to prove our main theorem.

In [MS24] we introduced twisted  $C^*$ -algebras for self-similar groupoid actions on  $k$ -graphs. A 2-cocycle  $\sigma$  for a self-similar action of a groupoid  $\mathcal{G}$  on  $k$ -graph  $\Lambda$  is, by definition, a categorical 2-cocycle on the Zappa–Szépe product category  $\mathcal{G} \bowtie \Lambda$ . The corresponding twisted  $C^*$ -algebra  $C^*(\mathcal{G} \bowtie \Lambda, \sigma)$  agrees, in the untwisted case, with the algebras of [Nek04, LRRW18, ABRW19]. It is natural to ask about invariance of  $K$ -theory as  $\sigma$  varies continuously. Our main theorem can be summarised as follows.

**Theorem (Theorem 4.9).** *Fix  $k \geq 0$ . Suppose that  $\mathcal{G}$  is a countable discrete amenable groupoid acting self-similarly and jointly-faithfully on a row-finite  $k$ -graph  $\Lambda$  with no sources. If  $\sigma_0$  and  $\sigma_1$  are homotopic 2-cocycles on  $\mathcal{G} \bowtie \Lambda$ , then*

$$K_*(C^*(\mathcal{G} \bowtie \Lambda, \sigma_0)) \cong K_*(C^*(\mathcal{G} \bowtie \Lambda, \sigma_1)).$$

Indeed, like Elliott, we prove the stronger statement that the twisted  $C^*$ -algebras for the cocycles along the homotopy assemble into a  $C([0, 1])$ -algebra for which the point-evaluation  $C^*$ -homomorphisms all induce isomorphisms in  $K$ -theory. As in Elliott’s argument, we induct on  $k$ . However, we use the Pimsner exact sequence for Cuntz–Pimsner algebras [Pim97] rather than the Pimsner–Voiculescu sequence for crossed products [PV80]. In particular, we show that the twisted  $C^*$ -algebra  $C^*(\mathcal{G} \bowtie \Lambda, \sigma)$ , with  $\Lambda$  a  $(k + 1)$ -graph,

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can be realised as the Cuntz–Pimsner algebra of a  $C^*$ -correspondence over  $C^*(\mathcal{G} \rtimes \Gamma, \sigma)$ , where  $\Gamma \subseteq \Lambda$  is a sub- $k$ -graph.

To do so, we investigate how inclusions of finitely aligned left-cancellative small categories correspond to inclusions of twisted versions of the associated  $C^*$ -algebras introduced by Spielberg [Spi20]. This turns out to be more complicated than we expected. We give an example (Example 3.17) of an inclusion of finitely aligned left-cancellative monoids that does not induce a  $*$ -homomorphism between the associated Toeplitz algebras.

The paper is structured as follows. Section 2 contains preliminary material on self-similar actions and Zappa–Szépe products. In Section 3 we introduce twisted  $C^*$ -algebras of finitely aligned left-cancellative small categories, generalising the untwisted algebras of Spielberg [Spi20]. In particular, we identify sufficient conditions for an inclusion of categories to induce inclusions of  $C^*$ -algebras. In Section 4 we introduce the  $C^*$ -algebra of a finitely aligned left-cancellative small category twisted by a homotopy of 2-cocycles, and use it to prove our main result, Theorem 4.9.

## 2. PRELIMINARIES

**2.1. Categories, groupoids, and  $k$ -graphs.** Throughout this article,  $\mathcal{C}$  denotes a countable discrete small category. We identify  $\mathcal{C}$  with its set of morphisms and denote its set of objects (identified with the corresponding identity morphisms) by  $\mathcal{C}^0$ . The domain and codomain maps become maps  $r: \mathcal{C} \rightarrow \mathcal{C}^0$  and  $s: \mathcal{C} \rightarrow \mathcal{C}^0$  taking a morphism to its *range* and *source*. We write  $\mathcal{C}^n$  for the collection of composable  $n$ -tuples in  $\mathcal{C}$ . That is  $(c_1, \dots, c_n) \in \mathcal{C}^n$  if  $s(c_i) = r(c_{i+1})$  for all  $i$ . We extend  $r$  and  $s$  to  $\mathcal{C}^n$  by  $r(c_1, \dots, c_n) = r(c_1)$  and  $s(c_1, \dots, c_n) = s(c_n)$ . For  $c_1, c_2 \in \mathcal{C}$  we define

$$c_1\mathcal{C} := \{c_1c: (c_1, c) \in \mathcal{C}^2\}, \quad \mathcal{C}c_2 := \{cc_2: (c, c_2) \in \mathcal{C}^2\}, \quad \text{and} \\ c_1\mathcal{C}c_2 := \{c_1cc_2: (c_1, c, c_2) \in \mathcal{C}^3\};$$

if  $c_1, c_2 \in \mathcal{C}^0$  then  $c_1\mathcal{C}c_2 = c_1\mathcal{C} \cap \mathcal{C}c_2$ .

A *groupoid*  $\mathcal{G}$  is a small category such that every  $g \in \mathcal{G}$  has an inverse  $g^{-1} \in \mathcal{G}$  such that  $gg^{-1} = r(g)$  and  $g^{-1}g = s(g)$ .

Let  $k \in \mathbb{N}$ . A  *$k$ -graph* is a countable small category  $\Lambda$  together with a functor  $d: \Lambda \rightarrow \mathbb{N}^k$ , called the *degree functor*, such that composition in  $\Lambda$  satisfies the *unique factorisation property*: for each  $\lambda \in \Lambda$  and  $m, n \in \mathbb{N}^k$  such that  $d(\lambda) = m+n$  there exist unique  $\mu, \nu \in \Lambda$  such that  $\lambda = \mu\nu$ ,  $d(\mu) = m$ , and  $d(\nu) = n$ . If  $d(\lambda) = n$  we say that  $\lambda$  has *degree*  $n$ . We write  $\Lambda^n := d^{-1}(n)$ . We call elements of  $\Lambda^0$  *vertices*. We denote the element of  $\mathbb{N}^k$  with a 1 in the  $i$ -th component and 0 elsewhere by  $e_i$ . For  $m, n \in \mathbb{N}^k$  we write  $m \leq n$  if  $m_i \leq n_i$  for all  $i$ .

A  $k$ -graph  $\Lambda$  is *row-finite* if  $|v\Lambda^n| < \infty$  for all  $n \in \mathbb{N}^k$  and  $v \in \Lambda^0$ . It has *no sources* if  $v\Lambda^n \neq \emptyset$  for all  $n \in \mathbb{N}^k$  and  $v \in \Lambda^0$ . It is *locally convex* if for every  $1 \leq i, j \leq k$  with  $i \neq j$ , whenever  $e \in \Lambda^{e_i}$  and  $r(e)\Lambda^{e_j} \neq \emptyset$ , we have  $s(e)\Lambda^{e_j} \neq \emptyset$ . Every  $k$ -graph with no sources is locally convex.

For each  $n \in \mathbb{N}^k$  we define, as in [RSY03],

$$\Lambda^{\leq n} := \{\lambda \in \Lambda \mid d(\lambda) \leq n \text{ and if } d(\lambda)_i < n_i \text{ then } s(\lambda)\Lambda^{e_i} = \emptyset\}.$$

It is potentially confusing that  $\Lambda^{\leq n} \neq \bigcup_{m \leq n} \Lambda^m$ ; however the notation is, by now, standard.

**2.2. Zappa–Szépe products and self-similar actions.** For details of the following, see [MS24, Section 3]. A *left action* of a small category  $\mathcal{C}$  on a small category  $\mathcal{D}$  with  $\mathcal{C}^0 = \mathcal{D}^0$

consists of a map

$$\triangleright: \mathcal{C} \ast_r \mathcal{D} := \{(c, d) \in \mathcal{C} \times \mathcal{D} \mid s(c) = r(d)\} \rightarrow \mathcal{D}$$

such that  $r(d) \triangleright d = d$ ,  $c \triangleright s(c) = r(c)$ ,  $r(c \triangleright d) = r(c)$ , and  $(cc') \triangleright d = c \triangleright (c' \triangleright d)$  for all  $c, c' \in \mathcal{C}$  and  $d \in \mathcal{D}$  for which the formulas make sense. A *right action*  $\triangleleft: \mathcal{C} \ast_r \mathcal{D} \rightarrow \mathcal{C}$  is defined symmetrically.

Following [MS24, Definition 3.1], a *matched pair*  $(\mathcal{C}, \mathcal{D})$  consists of a pair of small categories  $\mathcal{C}$  and  $\mathcal{D}$  with  $\mathcal{C}^0 = \mathcal{D}^0$ , together with a left action  $\triangleright: \mathcal{C} \ast_r \mathcal{D} \rightarrow \mathcal{D}$  of  $\mathcal{C}$  on  $\mathcal{D}$  and a right action  $\triangleleft: \mathcal{C} \ast_r \mathcal{D} \rightarrow \mathcal{C}$  of  $\mathcal{D}$  on  $\mathcal{C}$  such that  $s(c \triangleright d) = r(c \triangleleft d)$  for all  $(c, d) \in \mathcal{C} \ast_r \mathcal{D}$  and such that for all  $(c_1, c_2, d_1, d_2) \in \mathcal{C}^2 \ast_r \mathcal{D}^2$  we have

$$c_2 \triangleright (d_1 d_2) = (c_2 \triangleright d_1)((c_2 \triangleleft d_1) \triangleright d_2) \quad \text{and} \quad (c_1 c_2) \triangleleft d_1 = (c_1 \triangleleft (c_2 \triangleright d_1))(c_2 \triangleleft d_1).$$

**Definition 2.1** ([MS24, Definition 3.6]). Let  $(\mathcal{C}, \mathcal{D})$  be a matched pair of small categories. The *Zappa–Szépe* product  $\mathcal{C} \bowtie \mathcal{D}$  is the small category with objects  $\mathcal{C}^0 = \mathcal{D}^0$  and morphisms  $\{dc \mid (d, c) \in \mathcal{D} \ast_r \mathcal{C}\}$ , in which the range and source maps are given by  $r(dc) = r(d)$  and  $s(dc) = s(c)$ , and composition is defined by the formula

$$d_1 c_1 d_2 c_2 = d_1 (c_1 \triangleright d_2) (c_1 \triangleleft d_2) c_2$$

whenever  $d_1 c_1, d_2 c_2 \in \mathcal{C} \bowtie \mathcal{D}$  with  $s(c_1) = r(d_2)$ .

By [MS24, Lemma 3.5] the Zappa–Szépe product  $\mathcal{C} \bowtie \mathcal{D}$  is indeed a small category. By [MS24, Proposition 3.13] it is characterised by a unique-factorisation property: if  $\mathcal{E}$  is a small category and  $\mathcal{C}$  and  $\mathcal{D}$  are wide subcategories of  $\mathcal{E}$  such that for any  $e \in \mathcal{E}$  there are unique  $d \in \mathcal{D}$  and  $c \in \mathcal{C}$  such that  $e = dc$ , then  $\mathcal{E}$  is isomorphic to a Zappa–Szépe product  $\mathcal{C} \bowtie \mathcal{D}$ .

In [MS24, Proposition 3.29] it is shown that  $k$ -graphs can be described as Zappa–Szépe products. We recap this briefly in the following example.

*Example 2.2.* Let  $\Lambda$  be a  $k$ -graph and fix  $p, q > 0$  such that  $p + q = k$ . Then  $\Lambda_p := \Lambda^{\mathbb{N}^p \times \{0\}}$  is a  $p$ -graph and  $\Lambda_q := \Lambda^{\{0\} \times \mathbb{N}^q}$  is a  $q$ -graph.

If  $\lambda \in \Lambda_p$  and  $\mu \in s(\lambda)\Lambda_q$ , then uniqueness of factorisation in  $\Lambda$  gives  $\mu' \in \Lambda_q$  and  $\lambda' \in \Lambda_p$  such that  $\lambda\mu = \mu'\lambda'$ . Setting  $\lambda \triangleright \mu = \mu'$  defines a left action of  $\Lambda_p$  on  $\Lambda_q$  and setting  $\lambda \triangleleft \mu = \lambda'$  defines a right action of  $\Lambda_q$  on  $\Lambda_p$ . Uniqueness of factorisation in  $\Lambda$  implies that  $\Lambda \cong \Lambda_p \bowtie \Lambda_q$ . In particular, if  $\Lambda$  is a  $(k + 1)$ -graph, then  $\Gamma := \Lambda^{\mathbb{N}^k}$  is a  $k$ -graph and  $\Lambda^{\mathbb{N}^{k+1}}$  is the path category  $E^*$  of a directed graph  $E$  such that  $\Lambda \cong \Gamma \bowtie E^*$ .

Zappa–Szépe products also capture self-similar actions of groupoids on  $k$ -graphs (see [MS24, Proposition 3.32]). We use the following definition of a self-similar action.

**Definition 2.3** ([MS24, Definition 3.33]). Let  $\Lambda$  be a  $k$ -graph and let  $\mathcal{G}$  be a discrete groupoid with  $\mathcal{G}^0 = \Lambda^0$ . A *self-similar action* of  $\mathcal{G}$  on  $\Lambda$  is a matched pair  $(\mathcal{G}, \Lambda)$  such that  $d(g \triangleright \lambda) = d(\lambda)$  for all  $g, \lambda$ . We say that  $\mathcal{G}$  is a *discrete groupoid acting self-similarly* on  $\Lambda$ .

### 3. TWISTED $C^*$ -ALGEBRAS OF FINITELY ALIGNED LEFT-CANCELLATIVE SMALL CATEGORIES

In this section we introduce twisted  $C^*$ -algebras for finitely aligned left-cancellative small categories as introduced by [Spi20]. We use the notation and setup of [BKQS18].

**3.1. Finitely aligned left-cancellative small categories.** A small category  $\mathcal{C}$  is *left-cancellative* if whenever  $a, b, c \in \mathcal{C}$  satisfy  $s(a) = r(b) = r(c)$ ,

$$ac = ab \quad \text{implies} \quad c = b.$$

Given a left-cancellative small category  $\mathcal{C}$ , we define an equivalence relation on  $\mathcal{C}$  by

$$a \sim b \quad \text{if and only if} \quad \text{there is an invertible } c \in \mathcal{C} \text{ such that } a = bc.$$

In other words,  $a \sim b$  if and only if  $a\mathcal{C} = b\mathcal{C}$ .

We extend the notion of equivalence to subsets of  $\mathcal{C}$ . If  $A, B \subseteq \mathcal{C}$  then we say that  $A \sim B$  if for every  $a \in A$  there exists  $b \in B$  such that  $a \sim b$  and for every  $b \in B$  there exists  $a \in A$  such that  $b \sim a$ . If  $A \sim B$ , then

$$\bigcup_{a \in A} a\mathcal{C} = \bigcup_{b \in B} b\mathcal{C}, \quad (3.1)$$

but the converse does not typically hold.

If  $a, a' \in A$  satisfy  $a \in a'\mathcal{C}$  then  $a\mathcal{C} \subseteq a'\mathcal{C}$ . We say that  $A \subseteq \mathcal{C}$  is *independent* if for all distinct  $a, a' \in A$  we have  $a \notin a'\mathcal{C}$ .

**Lemma 3.1** (cf. [BKQS18, p.1350]). *Let  $\mathcal{C}$  be a left-cancellative small category. If  $A, B \subseteq \mathcal{C}$  are independent and (3.1) holds, then  $A \sim B$ , and  $|A| = |B|$ .*

*Proof.* Fix  $a \in A$ . Since (3.1) holds, there exists  $b \in B$  such that  $a \in b\mathcal{C}$  and there exists an  $a' \in A$  such that  $b \in a'\mathcal{C}$ . Hence,  $a\mathcal{C} \subseteq b\mathcal{C} \subseteq a'\mathcal{C}$ . Independence of  $A$  implies that  $a = a'$  so  $a\mathcal{C} = b\mathcal{C}$  and therefore  $a \sim b$ . If  $b' \in B$  also satisfies  $a \sim b'$  then  $b \sim b'$  and independence of  $B$  implies that  $b = b'$ . So there is a unique function  $a \mapsto b_a$  from  $A$  to  $B$  such that  $a \sim b_a$  for all  $a \in A$ . A symmetric argument yields a function  $b \mapsto a_b$  from  $B \rightarrow A$ , which is inverse to  $a \mapsto b_a$ .  $\square$

We say that a left-cancellative small category  $\mathcal{C}$  is *finitely aligned* if for all  $a, b \in \mathcal{C}$  there is a finite subset  $F \subseteq \mathcal{C}$  such that

$$a\mathcal{C} \cap b\mathcal{C} = \bigcup_{c \in F} c\mathcal{C} =: F\mathcal{C}. \quad (3.2)$$

Since  $F$  is finite, by passing to a subset we can assume that  $F$  is independent, and therefore unique up to equivalence by Lemma 3.1. In this article, we work exclusively with finitely aligned left-cancellative small categories.

If  $\mathcal{C}$  is finitely aligned, an induction on  $|A|$  shows that for any finite set  $A \subseteq \mathcal{C}$  there is a finite independent set  $F \subseteq \mathcal{C}$ , unique up to equivalence, such that

$$\bigcap_{a \in A} a\mathcal{C} = \bigcup_{c \in F} c\mathcal{C}.$$

Following [BKQS18] we write  $\vee A$  for a choice of such a finite independent set  $F$ . If  $a, b \in \mathcal{C}$  then we write  $a \vee b$  instead of  $\vee\{a, b\}$ .

Let  $v \in \mathcal{C}^0$ . A subset  $A \subseteq v\mathcal{C}$  is *exhaustive* if for every  $c \in v\mathcal{C}$  there exists  $a \in A$  such that  $c\mathcal{C} \cap a\mathcal{C} \neq \emptyset$ .

**Lemma 3.2** ([BKQS18, Lemma 2.3]). *Let  $\mathcal{C}$  be a left-cancellative small category. For  $v \in \mathcal{C}^0$ , if  $A \subseteq v\mathcal{C}$  is exhaustive and  $A \sim B$ , then  $B \subseteq v\mathcal{C}$  is exhaustive.*

*Example 3.3.* Every groupoid  $\mathcal{G}$  is a finitely aligned left-cancellative small category since  $g\mathcal{G} = r(g)\mathcal{G}$  for all  $g \in \mathcal{G}$ .

*Example 3.4.* The path category  $E^*$  of a directed graph  $E$  is a finitely aligned left-cancellative small category. For  $k \geq 2$  some  $k$ -graphs are *not* finitely aligned (see [RS05, Examples 3.1 and 5.2] for example).

**Lemma 3.5.** Fix  $S \subseteq \{1, \dots, k\}$  and let  $\mathbb{N}^S := \{n \in \mathbb{N}^k \mid n_i = 0 \text{ for } i \notin S\}$ . Let  $\Lambda$  be a finitely aligned  $k$ -graph. Then  $\Lambda^{\mathbb{N}^S}$  is finitely aligned.

*Proof.* Fix  $\alpha, \beta \in \Lambda^{\mathbb{N}^S}$ . Then  $\alpha \vee \beta$ , as calculated in  $\Lambda$ , is a subset of  $\Lambda^{\mathbb{N}^S}$  and hence is equal to  $\alpha \vee \beta$  as calculated in  $\Lambda^{\mathbb{N}^S}$ .  $\square$

**Lemma 3.6.** Let  $\mathcal{G}$  be a groupoid and let  $\mathcal{C}$  be a finitely aligned left-cancellative small category such that  $(\mathcal{G}, \mathcal{C})$  is a matched pair. Then  $\mathcal{G} \rtimes \mathcal{C}$  is left cancellative and finitely aligned.

*Proof.* The left-cancellativity of  $\mathcal{G} \rtimes \mathcal{C}$  follows from [MS24, Lemma 3.15] since every element of  $\mathcal{G}$  is invertible.

For finite alignment let  $cg \in \mathcal{G} \rtimes \mathcal{C}$ . Since  $g$  is invertible we have  $cg(\mathcal{G} \rtimes \mathcal{C}) = c(\mathcal{G} \rtimes \mathcal{C})$ . Fix  $c_1, c_2 \in \mathcal{C}$  and suppose that  $c_1(\mathcal{G} \rtimes \mathcal{C}) \cap c_2(\mathcal{G} \rtimes \mathcal{C}) \neq \emptyset$ . Fix  $c_1c'_1g_1 = c_2c'_2g_2 \in \mathcal{G} \rtimes \mathcal{C}$ . Uniqueness of factorisation gives  $c_1c'_1 = c_2c'_2$ , so  $c_1\mathcal{C} \cap c_2\mathcal{C} \neq \emptyset$ . Since  $\mathcal{C}$  is finitely aligned, there is a finite subset  $F \subseteq \mathcal{C}$  and  $c \in F$  such that  $c_1c'_1 = c_2c'_2 = cc'$  for some  $c' \in \mathcal{C}$ . Hence,  $cc'g_1 = cc'g_2$  and it follows that  $c_1(\mathcal{G} \rtimes \mathcal{C}) \cap c_2(\mathcal{G} \rtimes \mathcal{C}) = \bigcup_{c \in F} c(\mathcal{G} \rtimes \mathcal{C})$ .  $\square$

**3.2. Twisted  $C^*$ -algebras.** To introduce twisted  $C^*$ -algebras of small categories we need to recall the notion of a  $\mathbb{T}$ -valued 2-cocycle.

**Definition 3.7.** A  $\mathbb{T}$ -valued 2-cocycle on a category  $\mathcal{C}$  is a map  $\sigma: \mathcal{C}^2 \rightarrow \mathbb{T}$  such that for every  $(c_1, c_2, c_3) \in \mathcal{C}^3$ ,

$$\sigma(c_2, c_3)\sigma(c_1, c_2c_3) = \sigma(c_1, c_2)\sigma(c_1c_2, c_3).$$

and such that  $\sigma(r(c), c) = 1 = \sigma(c, s(c))$  for all  $c \in \mathcal{C}$ .

*Remark 3.8.* The condition  $\sigma(r(c), c) = 1 = \sigma(c, s(c))$  for all  $c \in \mathcal{C}$  is often emphasised by saying that  $\sigma$  is a *normalised*, but in this paper all cocycles are normalised, so we drop the adjective.

Given a finite family  $\mathcal{P}$  of pairwise commuting projections in a  $C^*$ -algebra  $A$ , we write  $\bigvee \mathcal{P}$  for the smallest projection in  $A$  that dominates every element of  $\mathcal{P}$ . Explicitly,

$$\bigvee \mathcal{P} = \sum_{\emptyset \neq F \subseteq \mathcal{P}} (-1)^{|F|-1} \prod_{P \in F} P. \quad (3.3)$$

**Definition 3.9** (cf. [BKQS18, Definition 3.1]). Let  $\mathcal{C}$  be a finitely aligned left-cancellative small category and let  $\sigma$  be a  $\mathbb{T}$ -valued 2-cocycle on  $\mathcal{C}$ . A  $\sigma$ -twisted representation of  $\mathcal{C}$  in a  $C^*$ -algebra  $A$  is a map  $S: \mathcal{C} \rightarrow A$ ,  $c \mapsto S_c$  such that each  $S_c$  is a partial isometry and

$$(R1) \quad S_{c_1}S_{c_2} = \sigma(c_1, c_2)S_{c_1c_2} \text{ for all } (c_1, c_2) \in \mathcal{C}^2,$$

$$(R2) \quad S_c^*S_c = S_{s(c)} \text{ for all } c \in \mathcal{C}, \text{ and}$$

$$(R3) \quad S_{c_1}S_{c_1}^*S_{c_2}S_{c_2}^* = \bigvee_{c \in F} S_cS_c^* \text{ for all } c_1, c_2 \in \mathcal{C} \text{ and any finite independent set } F \text{ satisfying } c_1\mathcal{C} \cap c_2\mathcal{C} = F\mathcal{C}.$$

We say that  $S$  is *covariant* if, in addition, for all  $v \in \mathcal{C}^0$ ,

$$(R4) \quad S_v = \bigvee_{c \in F} S_cS_c^* \text{ for all finite exhaustive } F \subseteq v\mathcal{C}.$$

We consider two twisted  $C^*$ -algebras associated to each 2-cocycle on a finitely aligned left-cancellative small category.

**Definition 3.10.** Let  $\mathcal{C}$  be a finitely aligned left-cancellative small category, and let  $\sigma$  be a 2-cocycle on  $\mathcal{C}$ . The  $\sigma$ -twisted Toeplitz algebra of  $\mathcal{C}$  is the universal  $C^*$ -algebra  $\mathcal{TC}^*(\mathcal{C}, \sigma)$  generated by a  $\sigma$ -twisted representation  $t: \mathcal{C} \rightarrow \mathcal{TC}^*(\mathcal{C}, \sigma)$ ; that

is,  $\mathcal{TC}^*(\mathcal{C}, \sigma) = C^*(\{t_c \mid c \in \mathcal{C}\})$ , and for any  $\sigma$ -twisted representation  $S: \mathcal{C} \rightarrow A$  there is a unique  $*$ -homomorphism  $\Phi: \mathcal{TC}^*(\mathcal{C}, \sigma) \rightarrow A$  such that  $S = \Phi \circ t$ .

The  $\sigma$ -twisted  $C^*$ -algebra of  $\mathcal{C}$  is the universal  $C^*$ -algebra  $C^*(\mathcal{C}, \sigma)$  generated by a  $\sigma$ -twisted covariant representation  $s: \mathcal{C} \rightarrow C^*(\mathcal{C}, \sigma)$ . Let  $I$  be the ideal of  $\mathcal{TC}^*(\mathcal{C}, \sigma)$  generated by  $\{S_v - \bigvee_{c \in F} S_c S_c^* \mid v \in \mathcal{C}^0, F \subseteq v\mathcal{C} \text{ is finite exhaustive}\}$ . Then  $C^*(\mathcal{C}, \sigma) \cong \mathcal{TC}^*(\mathcal{C}, \sigma)/I$ .

The ‘‘untwisted’’ algebra  $C^*(\mathcal{C}, 1)$  of a countable finitely aligned left-cancellative small category coincides with the groupoid  $C^*$ -algebra  $C^*(G_2|_{\partial\Lambda})$  of [Spi20, Theorem 10.15].

The following is a generalisation of [BKQS18, Lemma 3.4].

**Lemma 3.11.** *Let  $\mathcal{C}$  be a finitely aligned left-cancellative small category, let  $\sigma$  be a 2-cocycle on  $\mathcal{C}$ , and let  $A$  be a  $C^*$ -algebra. If  $S: \mathcal{C} \rightarrow A$  satisfies (R1) and (R2), then*

- (i) *for every invertible  $c \in \mathcal{C}$  we have  $S_{c^{-1}} = \sigma(c^{-1}, c)S_c^*$  and  $S_c S_c^* = S_{r(c)}$ ;*
- (ii) *if  $a \sim b$  in  $\mathcal{C}$  then  $S_a S_a^* = S_b S_b^*$ ; and*
- (iii) *if  $A, B \subseteq \mathcal{C}$  satisfy  $A \sim B$ , then  $\bigvee_{a \in A} S_a S_a^* = \bigvee_{b \in B} S_b S_b^*$ .*

*Proof.* (i) Suppose that  $c \in \mathcal{C}$  is invertible. Then

$$\begin{aligned} S_{c^{-1}} &= S_{s(c)c^{-1}} = S_{s(c)} S_{c^{-1}} = S_c^* S_c S_{c^{-1}} \\ &= \sigma(c, c^{-1}) S_c^* S_{r(c)} = \sigma(c, c^{-1}) (S_{r(c)} S_c)^* = \sigma(c, c^{-1}) S_c^*. \end{aligned}$$

It follows that

$$S_c S_c^* = \overline{\sigma(c^{-1}, c)} S_c S_{c^{-1}} = \overline{\sigma(c^{-1}, c)} \sigma(c^{-1}, c) S_{r(c)} = S_{r(c)}.$$

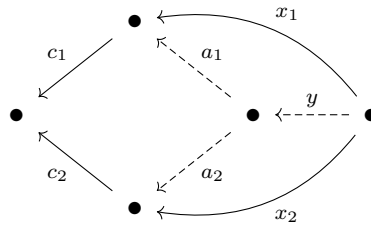
(ii) Suppose that  $a \sim b$ . Then there exists an invertible  $c \in \mathcal{C}$  such that  $a = bc$ . Now,

$$S_a S_a^* = S_{bc} S_{bc}^* = \overline{\sigma(b, c)} \sigma(b, c) S_b S_c S_c^* S_b^* = S_b S_{r(c)} S_b^* = S_b S_b^*.$$

(iii) follows immediately from (ii). □

**3.3. Maps induced by inclusions of subcategories.** We are interested in which inclusions of categories induce homomorphisms between twisted  $C^*$ -algebras. As Example 3.17 below demonstrates, it is not typically true that an inclusion of one category in another induces a  $*$ -homomorphism, even on the level of Toeplitz algebras. To identify the obstruction we introduce the following terminology.

**Definition 3.12.** Let  $\mathcal{X}$  be a finitely aligned left-cancellative small category. Let  $\mathcal{C}$  be a subcategory of  $\mathcal{X}$ . We say that  $\mathcal{C}$  is a *concordant* subcategory if for every  $c_1, c_2 \in \mathcal{C}$  with  $c_1 \mathcal{X} \cap c_2 \mathcal{X} \neq \emptyset$  there is a finite independent set  $F \subseteq \mathcal{C}$  generating  $c_1 \mathcal{C} \cap c_2 \mathcal{C}$  such that for all  $x_1, x_2 \in \mathcal{X}$  satisfying  $c_1 x_1 = c_2 x_2$  there exist  $c_1 a_1 = c_2 a_2 \in F$  and  $y \in \mathcal{X}$  such that the diagram



commutes.

*Remark 3.13.* A priori the  $a_1$  and  $a_2$  in Definition 3.12 need not belong to  $\mathcal{C}$  as long as  $c_1 a_1 = c_2 a_2$  does; but since  $c_1 a_1 = c_2 a_2 \in F \subseteq c_1 \mathcal{C} \cap c_2 \mathcal{C}$ , there exist  $a'_1, a'_2 \in \mathcal{C}$  such that  $c_1 a'_1 = c_1 a_1 = c_2 a_2 = c_2 a'_2$ , and then left cancellativity gives  $a_1 = a'_1 \in \mathcal{C}$  and  $a_2 = a'_2 \in \mathcal{C}$ .

*Remark 3.14.* If  $\mathcal{C}$  is a concordant subcategory of a finitely aligned left-cancellative small category  $\mathcal{X}$ , then  $\mathcal{C}$  is itself finitely aligned and left-cancellative.

**Notation 3.15.** If  $\mathcal{C}$  is a subcategory of  $\mathcal{X}$  and  $\sigma: \mathcal{X}^2 \rightarrow \mathbb{T}$  is a 2-cocycle then we abuse notation and also write  $\sigma$  for the restriction  $\sigma|_{\mathcal{C}^2}: \mathcal{C}^2 \rightarrow \mathbb{T}$ .

**Proposition 3.16.** *Let  $\mathcal{X}$  be a finitely aligned left-cancellative small category and suppose that  $\mathcal{C}$  is a concordant subcategory of  $\mathcal{X}$ . Let  $t^{\mathcal{C}}: \mathcal{C} \rightarrow \mathcal{TC}^*(\mathcal{C}, \sigma)$  and  $t^{\mathcal{X}}: \mathcal{X} \rightarrow \mathcal{TC}^*(\mathcal{X}, \sigma)$  be universal representations. For any 2-cocycle  $\sigma: \mathcal{X}^2 \rightarrow \mathbb{T}$  there is a unique  $*$ -homomorphism  $\Phi: \mathcal{TC}^*(\mathcal{C}, \sigma) \rightarrow \mathcal{TC}^*(\mathcal{X}, \sigma)$  such that  $\Phi(t_c^{\mathcal{C}}) = t_c^{\mathcal{X}}$  for all  $c \in \mathcal{C}$ . If, for every  $v \in \mathcal{C}^0$ , every finite exhaustive set  $F \subseteq v\mathcal{C}$  is also exhaustive in  $v\mathcal{X}$ , then  $\Phi$  descends to a  $*$ -homomorphism  $\overline{\Phi}: C^*(\mathcal{C}, \sigma) \rightarrow C^*(\mathcal{X}, \sigma)$  such that  $\overline{\Phi}(s_c) = s_c$  for all  $c \in \mathcal{C}$ .*

*Proof.* For the first statement it suffices to show that  $t^{\mathcal{X}}|_{\mathcal{C}}: \mathcal{C} \rightarrow \mathcal{TC}^*(\mathcal{X}, \sigma)$  satisfies (R1)–(R3) for  $(\mathcal{C}, \sigma)$ . That (R2) holds is immediate. Since the restriction of a 2-cocycle on  $\mathcal{X}$  to  $\mathcal{C}$  is a 2-cocycle, (R1) also holds.

For (R3) suppose that  $c_1, c_2 \in \mathcal{C}$ . First suppose that  $c_1\mathcal{X} \cap c_2\mathcal{X} = \emptyset$ . Then  $c_1\mathcal{C} \cap c_2\mathcal{C} = \emptyset$  and so (R3) holds since  $t_{c_1}^{\mathcal{C}} t_{c_1}^{\mathcal{C}*} t_{c_2}^{\mathcal{C}} t_{c_2}^{\mathcal{C}*} = 0$ . Now suppose that  $c_1\mathcal{X} \cap c_2\mathcal{X} \neq \emptyset$ . Take a finite independent set  $F \subseteq \mathcal{C}$  generating  $c_1\mathcal{C} \cap c_2\mathcal{C}$  satisfying the condition of Definition 3.12. We claim that  $F$  generates  $c_1\mathcal{X} \cap c_2\mathcal{X}$ . If  $x \in c_1\mathcal{X} \cap c_2\mathcal{X}$ , then there exist  $x_1, x_2 \in \mathcal{X}$  such that  $x = c_1x_1 = c_2x_2$ . By assumption, there exist  $c_1a_1 = c_2a_2 \in F$  and  $y \in \mathcal{X}$  such that  $x_1 = a_1y$  and  $x_2 = a_2y$ . Hence,

$$x = c_1x_1 = c_1a_1y \in F\mathcal{X}.$$

That is,  $F$  generates  $c_1\mathcal{X} \cap c_2\mathcal{X}$ . Relation (R3) gives  $t_{c_1}^{\mathcal{X}}(t_{c_1}^{\mathcal{X}})^* t_{c_1}^{\mathcal{X}}(t_{c_1}^{\mathcal{X}})^* = \bigvee_{c \in F} t_c^{\mathcal{X}}(t_c^{\mathcal{X}})^*$ , and since  $F \subseteq \mathcal{C}$ , this gives (R3) for the representation  $t^{\mathcal{X}}|_{\mathcal{C}}$  of  $(\mathcal{C}, \sigma)$ .

For the second statement let  $s: \mathcal{X} \rightarrow C^*(\mathcal{X}, \sigma)$  be a universal covariant representation. It suffices to show that  $s|_{\mathcal{C}}: \mathcal{C} \rightarrow C^*(\mathcal{X}, \sigma)$  satisfies (R4); but this follows immediately from the hypothesis that finite exhaustive sets in  $\mathcal{C}$  are also exhaustive in  $\mathcal{X}$ .  $\square$

Inclusions of the form  $\mathcal{C} \hookrightarrow \mathcal{C} \rtimes \mathcal{D}$  are not always concordant, even for monoids.

*Example 3.17.* Define an action  $\triangleright$  of  $\mathbb{F}_2^+ = \langle a, b \rangle$  on  $\mathbb{N}$  by  $w \triangleright n = n$  for all  $w \in \mathbb{F}_2^+$ , and  $n \in \mathbb{N}$  and a right action of  $\mathbb{N}$  on  $\mathbb{F}_2^+$  by  $w \triangleleft n = a^{|w|}$  for all  $w \in \mathbb{F}_2^+$  and  $n \in \mathbb{N} \setminus \{0\}$  and  $w \triangleleft 0 = w$ . Then  $(\mathbb{F}_2^+, \mathbb{N})$  is a matched pair; since  $w \triangleright \cdot: \mathbb{N} \rightarrow \mathbb{N}$  is injective for each  $w \in \mathbb{F}_2^+$ , [MS24, Lemma 3.15] implies that  $\mathcal{X} := \mathbb{F}_2^+ \rtimes \mathbb{N}$  is left cancellative.

Recall that for  $w, u \in \mathbb{F}_2^+$ , if  $w\mathbb{F}_2^+ \cap u\mathbb{F}_2^+$  is nonempty, then either  $w = uw'$  or  $u = wu'$ ; we then denote the unique minimal common extension by  $w \vee u$ .

To see that  $\mathcal{X}$  is finitely aligned fix  $n \in \mathbb{N}$  and  $w, u \in \mathbb{F}_2^+$ . We first claim that

$$nw\mathcal{X} \cap nu\mathcal{X} = \begin{cases} n(w \vee u)\mathcal{X} \cup (n+1)a^{\max\{|w|, |u|\}}\mathcal{X} & \text{if } w\mathbb{F}_2^+ \cap u\mathbb{F}_2^+ \neq \emptyset, \\ (n+1)a^{\max\{|w|, |u|\}}\mathcal{X} & \text{otherwise.} \end{cases} \quad (3.4)$$

For  $\supseteq$  note that if  $w\mathbb{F}_2^+ \cap u\mathbb{F}_2^+ \neq \emptyset$ , then  $n(w \vee u) \in nw\mathcal{X} \cap nu\mathcal{X}$ . For any  $v \in \mathbb{F}_2^+$  we have  $nv1 = n(v \triangleright 1)(v \triangleleft 1) = (n+1)a^{|v|}$ . In particular,  $(n+1)a^{\max\{|w|, |u|\}} \in nw\mathcal{X} \cap nu\mathcal{X}$ . Hence, the right-hand side of (3.4) is contained in the left-hand side.

Now suppose that  $x \in nw\mathcal{X} \cap nu\mathcal{X}$ . Then  $x = nwkv = nuk'v'$  for some  $k, k' \in \mathbb{N}$  and  $v, v' \in \mathbb{F}_2^+$ . So

$$(n+k)(w \triangleleft k)v = nwkv = nuk'v' = (n+k')(u \triangleleft k')v'.$$

By uniqueness of factorisation  $k = k'$ . If  $k = 0$ , then  $x = nwv = nuv'$  so  $wv = uv' \in w\mathbb{F}_2^+ \cap u\mathbb{F}_2^+$  and  $x \in n(w \vee u)\mathcal{X}$ . If  $k \geq 1$ , then  $x = (n+1)a^{|w|}(k-1)v = (n+1)a^{|u|}(k'-1)v' \in (n+1)a^{\max\{|w|, |u|\}}\mathcal{X}$ . So (3.4) holds as claimed.

Now suppose that  $nw\mathcal{X} \cap mu\mathcal{X}$  is nonempty, say  $x \in nw\mathcal{X} \cap mu\mathcal{X}$ . Without loss of generality,  $m \geq n$ , so  $m = n + n'$  for some  $n' > 0$ . If  $m = n$ , then (3.4) establishes that  $nw\mathcal{X} \cap mu\mathcal{X}$  is a finite union of principal right ideals. So suppose that  $m > n$ . Then  $x = nwkv = (n + n')uk'v'$  for some  $k, k' \in \mathbb{N}$  and  $v, v' \in \mathbb{F}_2^+$ . Uniqueness of factorisations implies that  $k \geq n'$ , so  $x = (n + n')a^{|w|}(k - n')v' = (n + n')uk'v'$ . It follows that  $nw\mathcal{X} \cap mu\mathcal{X} = ma^{|w|}\mathcal{X} \cap mu\mathcal{X}$  which is an intersection of the form covered by (3.4). Hence,  $\mathcal{X}$  is finitely aligned.

The inclusion  $\mathbb{F}_2^+ \hookrightarrow \mathcal{X}$  is not concordant since the ideal  $a\mathcal{X} \cap b\mathcal{X}$  contains  $b1 = 1a = a1$ , but  $a\mathbb{F}_2^+ \cap b\mathbb{F}_2^+ = \emptyset$ . On the level of Toeplitz algebras, in  $\mathcal{TC}^*(\mathbb{F}_2^+, 1)$  we have  $t_a t_a^* t_b t_b^* = 0$ , while in  $\mathcal{TC}^*(\mathcal{X}, 1)$  the element  $t_{1a} t_{1a}^*$  is a nonzero subprojection of  $t_a t_a^* t_b t_b^*$ . Since  $*$ -homomorphisms preserve orthogonality, there is no  $*$ -homomorphism  $\Phi: \mathcal{TC}^*(\mathbb{F}_2^+, 1) \rightarrow \mathcal{TC}^*(\mathcal{X}, 1)$  such that  $\Phi \circ t_{\mathbb{F}_2^+} = t_{\mathcal{X}}$ .

**Lemma 3.18.** *Fix  $k \geq 0$ . Suppose that  $\mathcal{G}$  is a discrete groupoid acting self-similarly on a  $(k + 1)$ -graph  $\Lambda$ . Let  $\Gamma = \Lambda^{\mathbb{N}^k}$  and  $E^* = \Lambda^{\mathbb{N}^{e_{k+1}}}$  so that  $\Lambda = \Gamma \bowtie E^*$  as in Example 2.2. Then  $\mathcal{G} \bowtie \Gamma \hookrightarrow \mathcal{G} \bowtie \Lambda$  is concordant.*

*Proof.* Fix  $\mu_1 g_1, \mu_2 g_2 \in \mathcal{G} \bowtie \Gamma$ . Suppose that  $x_1, x_2 \in \mathcal{G} \bowtie \Lambda$  satisfy  $\mu_1 g_1 x_1 = \mu_2 g_2 x_2$ . For  $i = 1, 2$  there exist unique  $\nu_i \in \Gamma$ ,  $\alpha_i \in E^*$  and  $h_i \in \mathcal{G}$  such that  $x_i = \nu_i \alpha_i h_i$ . We have

$$\mu_i g_i \nu_i (g \triangleleft \nu_i)^{-1} = \mu_i (g \triangleright \nu_i) (g \triangleleft \nu_i) (g \triangleleft \nu_i)^{-1} = \mu_i (g \triangleright \nu_i). \quad (3.5)$$

Since  $\mu_1 g_1 x_1 = \mu_2 g_2 x_2$ , we have

$$\mu_1 (g_1 \triangleright \nu_1) ((g_1 \triangleleft \nu_1) \triangleright \alpha_1) (g_1 \triangleleft \nu_1 \alpha_1) h_1 = \mu_2 (g_2 \triangleright \nu_2) ((g_2 \triangleleft \nu_2) \triangleright \alpha_2) (g_2 \triangleleft \nu_2 \alpha_2) h_2.$$

Uniqueness of factorisation in  $\Gamma \bowtie \Lambda$  implies that  $\mu_1 (g_1 \triangleright \nu_1) ((g_1 \triangleleft \nu_1) \triangleright \alpha_1) = \mu_2 (g_2 \triangleright \nu_2) ((g_2 \triangleleft \nu_2) \triangleright \alpha_2)$ , so uniqueness of factorisation in  $\Lambda = \Gamma \bowtie E^*$  implies that  $\mu_1 (g_1 \triangleright \nu_1) = \mu_2 (g_2 \triangleright \nu_2)$ . It follows from (3.5) that  $\mu_1 g_1 \nu_1 (g \triangleleft \nu_1)^{-1} = \mu_2 g_2 \nu_2 (g \triangleleft \nu_2)^{-1} \in \mu_1 g_1 (\mathcal{G} \bowtie \Gamma) \cap \mu_2 g_2 (\mathcal{G} \bowtie \Gamma)$ .

Since  $\mathcal{G} \bowtie \Gamma$  is finitely aligned, there is a finite set  $F \subseteq \mathcal{G} \bowtie \Gamma$  generating  $\mu_1 g_1 (\mathcal{G} \bowtie \Gamma) \cap \mu_2 g_2 (\mathcal{G} \bowtie \Gamma)$ . In particular, there exist  $a_1, a_2 \in \mathcal{G} \bowtie \Gamma$  with  $\mu_1 g_1 a_1 = \mu_2 g_2 a_2 \in F$  and  $y \in \mathcal{G} \bowtie \Gamma$  such that

$$\mu_1 g_1 \nu_1 (g \triangleleft \nu_1)^{-1} = \mu_1 g_1 a_1 y = \mu_2 g_2 a_2 y = \mu_2 g_2 \nu_2 (g \triangleleft \nu_2)^{-1}.$$

We have

$$\mu_i g_i x_i = \mu_i g_i \nu_i (g \triangleleft \nu_i)^{-1} (g_i \triangleleft \nu_i) \alpha_i h_i = \mu_i g_i a_i y (g_i \triangleleft \nu_i) \alpha_i h_i \in F (\mathcal{G} \bowtie \Lambda).$$

That is, the diagram

$$\begin{array}{ccccc}
 & & \bullet & \xleftarrow{\nu_1} & \bullet & & \\
 & \swarrow \mu_1 g_1 & \uparrow a_1 & & \downarrow g_1 \triangleleft \nu_1 & \swarrow \alpha_1 h_1 & \\
 \bullet & & \bullet & \xleftarrow{y} & \bullet & & \bullet \\
 & \searrow \mu_2 g_2 & \downarrow a_2 & & \uparrow g_2 \triangleleft \nu_2 & \searrow \alpha_2 h_2 & \\
 & & \bullet & \xleftarrow{\nu_2} & \bullet & & 
 \end{array}$$

commutes. So  $\mathcal{G} \bowtie \Gamma$  is concordant in  $\mathcal{G} \bowtie \Lambda$ .  $\square$

**Lemma 3.19.** *Fix  $k \geq 0$ . Let  $\Lambda$  be a locally convex  $(k + 1)$ -graph. Let  $\Gamma = \Lambda^{\mathbb{N}^k}$  and  $E^* = \Lambda^{\mathbb{N}^{e_{k+1}}}$ , so that  $\Lambda = \Gamma \bowtie E^*$  as in Example 2.2. Then every exhaustive set in  $\Gamma$  is exhaustive in  $\Lambda$ .*

*Proof.* Fix  $v \in \Lambda^0$  and a finite exhaustive set  $F \subseteq v\Gamma$ . Fix  $\lambda \in v\Lambda$ . Let  $n := (d(\lambda)_1, \dots, d(\lambda)_k) \in \mathbb{N}^k$  and  $n' := d(\lambda)_{k+1} e_{k+1}$ . Let  $m := \bigvee_{\mu \in F} d(\mu)$ .

Fix  $\tau \in s(\lambda) \Gamma^{\leq m}$ . Then  $\lambda \tau \in \Lambda^{\leq n+n'+m}$ . By [RSY03, Lemma 3.12],  $\lambda \tau \in \Lambda^{\leq m+n} \Lambda^{\leq n'}$ . By the factorisation property, there exists  $\tau' \in \Lambda^{\leq m+n} = \Gamma^{\leq m+n}$  and  $\alpha \in \Lambda^{n'}$  such that



$\lambda\tau = \tau'\alpha$ . Since  $F$  is exhaustive in  $\Gamma$ , there exists  $\mu \in F$  such that  $\mu\Gamma \cap \tau'\Gamma \neq \emptyset$ . By [RSY03, Lemma 3.12] again,  $\Gamma^{\leq m+n} = \Gamma^{\leq m}\Gamma^{\leq n}$ , and so  $\tau' = \eta\zeta$  for some  $\eta \in \Gamma^{\leq m}$  and  $\zeta \in \Gamma^{\leq n}$ . As  $\mu\Gamma \cap \tau'\Gamma \neq \emptyset$  we have  $\mu\Gamma \cap \eta\Gamma \neq \emptyset$ . Since  $d(\mu) \leq m$ , this forces  $\eta \in \mu\Gamma$ . Hence,  $\lambda\tau = \tau'\alpha = \eta\zeta\alpha \in \mu\Lambda$ , so  $\mu\Lambda \cap \lambda\Lambda \neq \emptyset$ . That is,  $F \subseteq v\Lambda$  is exhaustive.  $\square$

**Lemma 3.20.** *Fix  $k \geq 0$ . Suppose that  $\mathcal{G}$  is a discrete groupoid acting self-similarly on a locally convex  $(k+1)$ -graph  $\Lambda$ . Let  $\Gamma = \Lambda^{\mathbb{N}^k}$  and  $E^* = \Lambda^{\mathbb{N}^{e_{k+1}}}$  so that  $\Lambda = \Gamma \rtimes E^*$  as in Example 2.2. Then every exhaustive set in  $\mathcal{G} \rtimes \Gamma$  is exhaustive in  $\mathcal{G} \rtimes \Lambda$ .*

*Proof.* Fix an exhaustive set  $F \subseteq v(\mathcal{G} \rtimes \Lambda)$ . Then for every  $\mu' \in v\Gamma \subseteq \mathcal{G} \rtimes \Gamma$ , there exist  $\mu g \in F$  and  $\nu_1 h_1, \nu_2 h_2 \in \mathcal{G} \rtimes \Gamma$  such that  $\mu' \nu_1 h_1 = \mu g \nu_2 h_2 = \mu(g \triangleright \nu_2)(g \triangleleft \nu_2)h_2$ . By uniqueness of factorisation in  $\mathcal{G} \rtimes \Gamma$  we have  $\mu' \nu_1 = \mu(g \triangleright \nu_2)$  in  $\Gamma$ ; that is  $\mu'\Gamma \cap \mu\Gamma \neq \emptyset$ . In other words,  $F' := \{\mu \in \Gamma \mid \mu\mathcal{G} \cap F \neq \emptyset\}$  is exhaustive in  $\Gamma$ , so by Lemma 3.19  $F'$  is exhaustive in  $\Lambda$ .

Now fix  $\nu h \in \mathcal{G} \rtimes \Lambda$ . Since  $F'$  is exhaustive in  $\Lambda$ , there exist  $\mu g \in F$  and  $\mu', \nu' \in \Lambda$  such that  $\mu\mu' = \nu\nu'$ . We have

$$\nu h(h^{-1} \triangleright \nu')(h^{-1} \triangleleft \nu') = \nu h h^{-1} \nu' = \nu\nu' = \mu\mu' = \mu g g^{-1} \mu',$$

so  $\nu h(\mathcal{G} \rtimes \Lambda) \cap \mu g(\mathcal{G} \rtimes \Lambda) \neq \emptyset$ . That is,  $F$  is exhaustive in  $\mathcal{G} \rtimes \Lambda$ .  $\square$

**Theorem 3.21.** *Fix  $k \geq 0$ . Suppose that  $\mathcal{G}$  is a discrete groupoid acting self-similarly on a  $(k+1)$ -graph  $\Lambda$ . Let  $\Gamma = \Lambda^{\mathbb{N}^k}$  and  $E^* = \Lambda^{\mathbb{N}^{e_{k+1}}}$  so that  $\Lambda = \Gamma \rtimes E^*$  as in Example 2.2. Let  $\sigma$  be a  $\mathbb{T}$ -valued 2-cocycle on  $\mathcal{G} \rtimes \Lambda$ . There is a unique  $*$ -homomorphism  $\Phi: \mathcal{TC}^*(\mathcal{G} \rtimes \Gamma, \sigma) \rightarrow \mathcal{TC}^*(\mathcal{G} \rtimes \Lambda, \sigma)$  such that  $\Phi(t_{\gamma g}) = t_{\gamma g}$  for all  $\gamma g \in \mathcal{G} \rtimes \Gamma$ . If  $\Lambda$  is locally convex, then  $\Phi$  descends to a  $*$ -homomorphism  $\overline{\Phi}: C^*(\mathcal{G} \rtimes \Gamma, \sigma) \rightarrow C^*(\mathcal{G} \rtimes \Lambda, \sigma)$  such that  $\overline{\Phi}(s_{\gamma g}) = s_{\gamma g}$  for all  $\gamma g \in \mathcal{G} \rtimes \Gamma$ .*

*Proof.* By Lemma 3.5,  $\Gamma$  is finitely aligned. Hence, Lemma 3.6 implies that  $\mathcal{G} \rtimes \Lambda$  is left cancellative and finitely aligned. By Lemma 3.18 the inclusion  $\mathcal{G} \rtimes \Gamma \hookrightarrow \mathcal{G} \rtimes \Lambda$  is concordant, so Proposition 3.16 gives the desired homomorphism  $\Phi: \mathcal{TC}^*(\mathcal{G} \rtimes \Gamma, \sigma) \rightarrow \mathcal{TC}^*(\mathcal{G} \rtimes \Lambda, \sigma)$ . If  $\Lambda$  is locally convex, then Lemma 3.20 says that every finite exhaustive set in  $\mathcal{G} \rtimes \Gamma$  is also exhaustive in  $\mathcal{G} \rtimes \Lambda$ . So Proposition 3.16 implies that  $\Phi$  descends to a  $*$ -homomorphism  $\overline{\Phi}: C^*(\mathcal{G} \rtimes \Gamma, \sigma) \rightarrow C^*(\mathcal{G} \rtimes \Lambda, \sigma)$  such that  $\overline{\Phi}(s_{\gamma g}) = s_{\gamma g}$  for all  $\gamma g \in \mathcal{G} \rtimes \Gamma$ .  $\square$

We are interested in when the  $*$ -homomorphism  $\overline{\Phi}$  of Theorem 3.21 is injective. To this end, we extend the terminology of Yusnitha [Yus23b] to our setting.

**Definition 3.22.** A self-similar action of a discrete groupoid  $\mathcal{G}$  on a  $k$ -graph  $\Lambda$  is *jointly faithful* if for each  $v \in \Lambda^0$  and each  $n \in \mathbb{N}^{k+1}$  there exists  $\lambda \in \Lambda^n$  such that the map  $g \mapsto (g \triangleright \lambda, g \triangleleft \lambda)$  is injective on  $v\mathcal{G}v$ .

**Corollary 3.23.** *Consider a jointly faithful self-similar action of a discrete amenable groupoid  $\mathcal{G}$  on a row-finite  $(k+1)$ -graph  $\Lambda$  with no sources. Let  $\Gamma = \Lambda^{\mathbb{N}^k}$  and  $E^* = \Lambda^{\mathbb{N}^{e_{k+1}}}$ , so that  $\Lambda = \Gamma \rtimes E^*$  as in Example 2.2, and let  $\sigma$  be a  $\mathbb{T}$ -valued 2-cocycle on  $\mathcal{G} \rtimes \Lambda$ . Then the  $*$ -homomorphism  $\overline{\Phi}: C^*(\mathcal{G} \rtimes \Gamma, \sigma) \rightarrow C^*(\mathcal{G} \rtimes \Lambda, \sigma)$  of Theorem 3.21 is injective.*

*Proof.* By [MS24, Proposition 7.7] the  $*$ -homomorphism  $\iota_{\mathcal{G}}^{\Gamma}: C^*(\mathcal{G}, \sigma) \rightarrow C^*(\mathcal{G} \rtimes \Gamma, \sigma)$  satisfying  $\iota_{\mathcal{G}}^{\Gamma}(u_g) = s_g$  and the  $*$ -homomorphism  $\iota_{\mathcal{G}}^{\Lambda}: C^*(\mathcal{G}, \sigma) \rightarrow C^*(\mathcal{G} \rtimes \Lambda, \sigma)$  satisfying  $\iota_{\mathcal{G}}^{\Lambda}(u_g) = s_g$  are injective. Since  $\overline{\Phi} \circ \iota_{\mathcal{G}}^{\Gamma} = \iota_{\mathcal{G}}^{\Lambda}$ , the gauge-invariant uniqueness theorem [MS24, Corollary 7.10] implies that  $\overline{\Phi}$  is injective.  $\square$

## 4. HOMOTOPY OF COCYCLES

We are interested in the effect on  $C^*(\mathcal{C}, \sigma)$  of continuous variations in  $\sigma$ . To this end we introduce the following notion.

**Definition 4.1.** Let  $\mathbb{I} := [0, 1]$ . A *homotopy of 2-cocycles* on a small category  $\mathcal{C}$  is a continuous map  $\Sigma: \mathbb{I} \times \mathcal{C}^2 \rightarrow \mathbb{T}$  whose restriction  $\Sigma_t$  to  $\{t\} \times \mathcal{C}^2$  is a 2-cocycle on  $\mathcal{C}$  for each  $t \in \mathbb{I}$ . We say that the 2-cocycles  $\Sigma_0$  and  $\Sigma_1$  are *homotopic*. For each  $(c_1, c_2) \in \mathcal{C}^2$  we let  $\Sigma_\bullet(c_1, c_2)$  denote the continuous function  $t \mapsto \Sigma_t(c_1, c_2)$  on  $\mathbb{I}$ .

**4.1. Twisting representations by homotopies of 2-cocycles.** We consider representations of categories twisted by homotopies of cocycles. The idea is that such a representation should be thought of as a bundle of representations of the category twisted by continuously varying 2-cocycles.

**Definition 4.2.** Let  $\mathcal{C}$  be a finitely aligned left-cancellative small category and let  $\Sigma$  be a homotopy of 2-cocycles for  $\mathcal{C}$ . A  $\Sigma$ -*twisted representation* of  $\mathcal{C}$  in a  $C^*$ -algebra  $A$  is a pair  $(S, \tau)$  consisting of a map  $S: \mathcal{C} \rightarrow A$ ,  $c \mapsto S_c$  such that each  $S_c$  is a partial isometry, together with a  $*$ -homomorphism  $\tau_v: C(\mathbb{I}) \rightarrow A$  for each  $v \in \mathcal{C}^0$ , such that  $S$  satisfies (R2) and (R3), the  $\tau_v$  have mutually orthogonal images, and

$$(IR1) \quad \tau_v(1) = S_v \text{ for all } v \in \mathcal{C}^0,$$

$$(IR2) \quad \tau_{r(c)}(f)S_c = S_c\tau_{s(c)}(f) \text{ for all } c \in \mathcal{C} \text{ and } f \in C(\mathbb{I}), \text{ and}$$

$$(IR3) \quad S_{c_1}S_{c_2} = \tau_{r(c_1)}(\Sigma_\bullet(c_1, c_2))S_{c_1c_2} \text{ for all } (c_1, c_2) \in \mathcal{C}^2.$$

We say that  $(S, \tau)$  is *covariant* if  $S$  satisfies (R4).

Given a  $\Sigma$ -twisted representation  $(S, \tau)$  of  $\mathcal{C}$  in  $A$ , the  $C^*$ -algebra generated by  $(S, \tau)$ , denoted  $C^*(S, \tau)$ , is the  $C^*$ -subalgebra of  $A$  generated by  $\{S_c \mid c \in \mathcal{C}\} \cup \bigcup_{v \in \mathcal{C}^0} \tau_v(C(\mathbb{I}))$ . The  $C^*$ -algebra  $\mathcal{T}C_\mathbb{I}^*(\mathcal{C}; \Sigma)$  is the universal  $C^*$ -algebra generated by a  $\Sigma$ -twisted representation  $(t, \pi)$  in the sense that  $\mathcal{T}C_\mathbb{I}^*(\mathcal{C}; \Sigma) = C^*(t, \pi)$  and if  $(S, \tau)$  is a  $\Sigma$ -twisted representation in a  $C^*$ -algebra  $A$ , then there is a unique  $*$ -homomorphism  $\Phi: \mathcal{T}C_\mathbb{I}^*(\mathcal{C}; \Sigma) \rightarrow A$  such that  $S = \Phi \circ t$  and  $\tau_v = \Phi \circ \pi_v$  for all  $v \in \mathcal{C}^0$ . The  $C^*$ -algebra  $C_\mathbb{I}^*(\mathcal{C}; \Sigma)$  is the universal  $C^*$ -algebra generated by a covariant  $\Sigma$ -twisted representation  $(s, \pi)$ .

**Proposition 4.3** (cf. [Spi14, Proposition 6.7]). *Let  $(S, \tau)$  be a  $\Sigma$ -twisted representation of a finitely aligned left-cancellative small category  $\mathcal{C}$  in a  $C^*$ -algebra  $A$ . Let  $\mathcal{P} := \{\prod_{x \in F} S_x S_x^* \mid F \subseteq \mathcal{C} \text{ finite}\}$ . Then*

$$C^*(S, \tau) = \overline{\text{span}}\{\tau_{r(c)}(f)S_c S_d^* P \mid s(c) = s(d), f \in C(\mathbb{I}), P \in \mathcal{P}\}.$$

*Proof.* We follow the approach of [Spi14, Proposition 6.7]. Because the projections  $S_x S_x^*$  for  $x \in \mathcal{C}$  pairwise commute,  $\mathcal{P}$  consists of projections and is closed under multiplication. Let  $L := \text{span}\{\tau_{r(c)}(f)S_c S_d^* P \mid s(c) = s(d), f \in C(\mathbb{I}), P \in \mathcal{P}\}$ . Then  $L$  is a  $*$ -closed linear subspace of  $C^*(S, \tau)$  containing the generators of  $C^*(S, \tau)$  and their adjoints.

It suffices to show that  $L$  contains all monomials of the form  $S_{c_1} S_{d_1}^* \cdots S_{c_k} S_{d_k}^*$ . We will say that a monomial  $S_{c_1} S_{d_1}^* \cdots S_{c_k} S_{d_k}^*$  has length  $k$ .

Monomials of length one lie in  $L$  since each  $s_{r(c)} = \tau_{r(c)}(1_\mathbb{I})$  and each  $S_{r(d)} = \prod_{x \in \{r(d)\}} S_x S_x^*$ . Fix a nonzero monomial  $S_a S_b^* S_c S_d^* \in C^*(S, \tau)$  of length two. We claim that  $S_b^* S_c \in L$ . Let  $F$  be a finite independent set generating  $b\mathcal{C} \cap c\mathcal{C}$ , and write  $F = \{x_1, \dots, x_k\}$ . Then

$$S_b^* S_c = S_b^* S_b S_b^* S_c S_c^* S_c = S_b^* \left( \prod_{i=1}^k S_{x_i} S_{x_i}^* \right) S_c. \quad (4.1)$$

We have

$$\begin{aligned} \bigvee_{i=1}^k S_{x_i} S_{x_i}^* &= \sum_{i=1}^k S_{x_i} S_{x_i}^* \left( 1 - \bigvee_{j=1}^{i-1} S_{x_j} S_{x_j}^* \right) \\ &= \sum_{i=1}^k S_{x_i} S_{x_i}^* \left( 1 - \sum_{j=1}^{i-1} (-1)^j \sum_{1 \leq r_1 < \dots < r_j < i} S_{x_{r_1}} S_{x_{r_1}}^* \cdots S_{x_{r_j}} S_{x_{r_j}}^* \right). \end{aligned} \quad (4.2)$$

For each  $i$  there exist  $b_i, c_i \in \mathcal{C}$  such that  $x_i = bb_i = cc_i$ . So,

$$\begin{aligned} S_b^* S_{x_i} S_{x_i}^* S_c &= S_b^* S_b S_{b_i} S_{c_i}^* S_c^* S_c = S_{b_i} S_{c_i}^* \in L \quad \text{and} \\ S_c^* S_{x_i} S_{x_i}^* S_c &= S_c^* S_c S_{c_i} S_{b_i}^* S_c^* S_c = S_{c_i} S_{b_i}^* \in L. \end{aligned}$$

Since each  $x_{r_j} \in c\mathcal{C}$  we have  $S_{x_{r_j}} S_{x_{r_j}}^* \leq S_c S_c^*$ , so

$$\begin{aligned} S_b^* S_{x_{r_1}} S_{x_{r_1}}^* \cdots S_{x_{r_j}} S_{x_{r_j}}^* S_c &= S_b^* S_{x_{r_1}} S_{x_{r_1}}^* S_c \left( \prod_{l=2}^j S_c^* S_{x_{r_l}} S_{x_{r_l}}^* S_c \right) \\ &= S_{b_{r_1}} S_{c_{r_1}}^* \left( \prod_{l=2}^j S_{c_{r_l}} S_{c_{r_l}}^* \right) \in S_{b_{r_1}} S_{c_{r_1}}^* \mathcal{P} \subseteq L. \end{aligned}$$

Hence, (4.1) and (4.2) imply that  $S_b^* S_c \in L$ .

By the claim, there exist  $u_i, v_i \in \mathcal{C}$ , and  $P_i \in \mathcal{P}$  such that

$$S_a S_b^* S_c S_d^* = \sum_i S_a S_{u_i} S_{v_i}^* P_i S_d^*.$$

If  $r(x) = s(d)$  then  $S_d S_x S_x^* = S_d S_x S_x^* S_d^* S_d = S_{dx} S_{dx}^* S_d^*$ . In particular, for each  $i$  there exists  $P'_i \in \mathcal{P}$  such that  $P_i S_d^* = S_d^* P'_i$ . Hence,

$$S_a S_{u_i} S_{v_i}^* P_i S_d^* = \tau_{r(a)}(\Sigma_\bullet(a, u_i) \overline{\Sigma_\bullet(d, v_i)}) S_{au_i} S_{dv_i}^* P'_i \in L$$

for each  $i$  and so  $S_a S_b^* S_c S_d^* \in L$ .

Now, suppose inductively that every monomial of length  $k$  belongs to  $L$ . By the inductive hypothesis there exist  $a_i, b_i \in \mathcal{C}$ ,  $f_i \in C(\mathbb{I})$  and  $P_i \in \mathcal{P}$  such that

$$S_{c_1} S_{d_1}^* \cdots S_{c_{k+1}} S_{d_{k+1}}^* = S_{c_1} S_{d_1}^* \sum_i \tau_{r(a_i)}(f_i) S_{a_i} S_{b_i}^* P_i = \sum_i \tau_{r(c_1)}(f_i) S_{c_1} S_{d_1}^* S_{a_i} S_{b_i}^* P_i.$$

Since monomials of length two belong to  $L$  we are done.  $\square$

**Corollary 4.4.** *Let  $(S, \tau)$  be a  $\Sigma$ -twisted representation of a finitely aligned left-cancellative small category  $\mathcal{C}$  in a  $C^*$ -algebra  $A$ . Then there is a unique  $*$ -homomorphism  $\hat{\tau}: C(\mathbb{I}) \rightarrow \mathcal{ZM}(C^*(S, \tau))$  satisfying  $\hat{\tau}(f) S_v = \tau_v(f)$  for all  $v \in \mathcal{C}^0$  and  $f \in C(\mathbb{I})$ . This homomorphism is unital and makes  $C^*(S, \tau)$  into a  $C(\mathbb{I})$ -algebra.*

*Proof.* Fix  $a \in C^*(S, \tau)$ ,  $f \in C(\mathbb{I})$ , and  $\varepsilon > 0$ . We show that  $\sum_v \tau_v(f)$  converges strictly to a multiplier  $\hat{\tau}(f)$ . If  $f = 0$  this is trivial, so suppose  $f \neq 0$ . By Proposition 4.3 there exist a finite  $F \subseteq \mathcal{C}^0$  and  $a_0 \in \text{span}\{\tau_{r(c)}(g) S_c S_d^* P \mid r(c) \in F, s(c) = s(d), g \in C(\mathbb{I}), P \in \mathcal{P}\}$  such that  $\|a - a_0\| < \frac{\varepsilon}{2\|f\|}$ . If  $v \in \mathcal{C}^0 \setminus F$ , then  $\tau_v(f) a_0 = 0$ . Since the projections  $\tau_v(1) = S_v$  are mutually orthogonal, for any finite  $G \supseteq F$ ,

$$\begin{aligned} \left\| \sum_{v \in G} \tau_v(f) a - \sum_{v \in F} \tau_v(f) a \right\| &= \left\| \sum_{v \in G \setminus F} \tau_v(f) a \right\| \\ &\leq \left\| \sum_{v \in G \setminus F} \tau_v(f) (a - a_0) \right\| + \left\| \sum_{v \in G \setminus F} \tau_v(f) a_0 \right\| \\ &\leq \|f\| \|a - a_0\| + 0 < \frac{\varepsilon}{2}. \end{aligned}$$

Hence, the standard  $\varepsilon/2$  argument shows that the net  $(\sum_{v \in G} \tau_v(f)a)_{G \subseteq \mathcal{C}^0 \text{ finite}}$  is Cauchy. So  $\sum_v \tau_v(f)$  converges strictly in  $\mathcal{M}(C^*(S, \tau))$  to a multiplier  $\hat{\tau}(f)$ . Since each  $\tau_v$  is a  $*$ -homomorphism so is  $\hat{\tau}: f \mapsto \hat{\tau}(f)$ . Centrality of  $\tau_v(C(\mathbb{I}))$  follows from (IR2).

We claim that  $\sum_v S_v = \sum_v \tau_v(1)$  is an approximate identity for  $C^*(S, \tau)$ . Take  $a, \varepsilon, F$ , and  $a_0$  as above with  $f = 1$ . Since the  $S_v$  are mutually orthogonal,

$$\begin{aligned} \left\| \sum_{v \in G} S_v a - a \right\| &= \left\| \sum_{v \in G} S_v (a - a_0) \right\| + \left\| \sum_{v \in G} S_v a_0 - a \right\| \\ &\leq \left\| \sum_{v \in G} S_v \right\| \|a - a_0\| + \|a - a_0\| \leq 2\|a - a_0\| = \varepsilon. \end{aligned}$$

Hence,  $\sum_v \tau_v(1)$  converges strictly to the identity in  $\mathcal{M}(C^*(S, \tau))$ . That is,  $\hat{\tau}$  is unital.  $\square$

For the rest of this article we abuse notation and write  $\tau$  for the map  $\hat{\tau}$  of Corollary 4.4.

**Corollary 4.5.** *Fix  $k \geq 0$ . Suppose that  $\mathcal{G}$  is a countable discrete groupoid acting self-similarly on a finitely aligned  $k$ -graph  $\Lambda$ , and let  $\Sigma$  be a homotopy of cocycles on  $\mathcal{G} \rtimes \Lambda$ . Fix a  $\Sigma$ -twisted representation  $(S, \tau)$  of  $\mathcal{G} \rtimes \Lambda$ . Then*

$$C^*(S, \tau) = \overline{\text{span}}\{\tau(f)S_\lambda S_g S_\mu^* \mid \lambda, \mu \in \Lambda, g \in s(\lambda)\mathcal{G}s(\mu), f \in C(\mathbb{I})\}.$$

*Proof.* Let  $\mathcal{P}$  be the set of Proposition 4.3. We claim that  $\text{span } \mathcal{P} = \text{span}\{S_\lambda S_\lambda^* : \lambda \in \Lambda\}$ . We clearly have  $\supseteq$ . For the reverse, note that for each  $c \in \mathcal{G} \rtimes \Lambda$  there exist unique  $\lambda \in \Lambda$  and  $g \in \mathcal{G}$  such that  $S_c = \tau(\Sigma_\bullet(\lambda, g))S_\lambda S_g$ . By Lemma 3.11,  $S_c S_c^* = S_\lambda S_g S_g^* S_\lambda^* = S_\lambda S_\lambda^*$ . If  $\lambda, \mu \in \Lambda$ , then

$$S_\lambda S_\lambda^* S_\mu S_\mu^* = \sum_{\nu \in \lambda \vee \mu} S_\nu S_\nu^*.$$

Hence,  $\text{span}\{S_\lambda S_\lambda^* : \lambda \in \Lambda\}$  is closed under multiplication. Since it contains each  $S_x S_x^*$  it follows that it contains  $\text{span } \mathcal{P}$ . For  $\mu, \nu \in \Lambda$ ,

$$S_\mu^* S_\nu S_\nu^* = S_\mu^* S_\nu S_\nu^* S_\mu S_\mu^* = S_\mu^* \sum_{\alpha \in \mu \vee \nu} S_{\mu\alpha} S_{\mu\alpha}^* = \left( \sum_{\alpha \in \mu \vee \nu} S_\alpha S_\alpha^* \right) S_\mu^*. \quad (4.3)$$

Using Proposition 4.3 at the first equality, the claim above at the second, and (4.3) at the third, we obtain

$$\begin{aligned} C^*(S, \tau) &= \overline{\text{span}}\{\tau(f)S_c S_d^* P \mid c, d \in \mathcal{G} \rtimes \Lambda, s(c) = s(d), f \in C(\mathbb{I}), P \in \mathcal{P}\} \\ &= \overline{\text{span}}\{\tau(f)S_\lambda S_g S_\mu^* S_\nu S_\nu^* \mid \lambda, \mu, \nu \in \Lambda, g \in s(\lambda)\mathcal{G}s(\mu), f \in C(\mathbb{I})\} \\ &= \overline{\text{span}}\{\tau(f)S_\lambda S_g S_\mu^* \mid \lambda, \mu \in \Lambda, g \in s(\lambda)\mathcal{G}s(\mu), f \in C(\mathbb{I})\}. \quad \square \end{aligned}$$

**Lemma 4.6.** *Let  $\Sigma$  be a homotopy of cocycles on a finitely aligned left-cancellative small category  $\mathcal{C}$ . For each  $t \in \mathbb{I}$  there is a surjective homomorphism  $\varepsilon_t: C_{\mathbb{I}}^*(\mathcal{C}; \Sigma) \rightarrow C^*(\mathcal{C}; \Sigma_t)$  satisfying*

$$\varepsilon_t(s_c) = s_c \quad \text{and} \quad \varepsilon_t(\pi_v(f)) = f(t)s_v$$

for all  $\gamma \in \Gamma$ ,  $f \in C(\mathbb{I})$ , and  $v \in \mathcal{C}^0$ . This  $\varepsilon_t$  factors through an isomorphism of the fibre  $C_{\mathbb{I}}^*(\mathcal{C}; \Sigma)_t$  of  $C_{\mathbb{I}}^*(\mathcal{C}; \Sigma)$ —regarded as a  $C(\mathbb{I})$ -algebra as in Corollary 4.4—onto  $C^*(\mathcal{C}; \Sigma_t)$ .

*Proof.* Let  $s: \mathcal{C} \rightarrow C^*(\mathcal{C}; \Sigma_t)$  be a universal  $\Sigma_t$ -twisted representation of  $\mathcal{C}$ . For  $v \in \mathcal{C}^0$  define  $\tau_v: C(\mathbb{I}) \rightarrow C^*(\mathcal{C}; \Sigma_t)$  by  $\tau_v(f) = f(t)s_v$ . Then  $(s, \tau)$  is a  $\Sigma$ -twisted representation of  $\mathcal{C}$  and the universal property of  $C_{\mathbb{I}}^*(\mathcal{C}; \Sigma)$  induces a  $*$ -homomorphism  $\varepsilon_t: C_{\mathbb{I}}^*(\mathcal{C}; \Sigma) \rightarrow C^*(\mathcal{C}; \Sigma_t)$  satisfying  $\varepsilon_t(s_c) = s_c$  for all  $c \in \mathcal{C}$  and  $\varepsilon_t(\pi_v(f)) = f(t)s_v$  for all  $v \in \mathcal{C}^0$  and  $f \in C(\mathbb{I})$ . The image of  $\varepsilon_t$  contains the generators of  $C^*(\mathcal{C}; \Sigma_t)$  so  $\varepsilon_t$  is surjective.

For the final statement let  $A := C_{\mathbb{I}}^*(\mathcal{C}; \Sigma)$  and  $I_t = \pi(C_0(\mathbb{I} \setminus \{t\}))A$ . The fibre of  $A$  at  $t \in \mathbb{I}$  is  $A_t := A/I_t$ . By definition of  $\varepsilon_t$  and  $\pi$ , we have  $I_t \subseteq \ker(\varepsilon_t)$ . Hence,  $\varepsilon_t$  descends to a  $*$ -homomorphism  $\overline{\varepsilon}_t: A_t \rightarrow C^*(\mathcal{C}; \Sigma)$ .

We use the universal property of  $C^*(\mathcal{C}; \Sigma)$  to construct an inverse to  $\overline{\varepsilon}_t$ . Define  $S: \mathcal{C} \rightarrow A_t$  by  $S_c := s_c + I_t$  for  $c \in \mathcal{C}$ . Then  $S_c$  satisfies (R2)–(R4). For  $(c_1, c_2) \in \mathcal{C}^2$ ,

$$\pi(\Sigma_\bullet(c_1, c_2))s_{c_1c_2} - \Sigma_t(c_1, c_2)s_{c_1c_2} = \pi(\Sigma_\bullet(c_1, c_2) - \Sigma_t(c_1, c_2)1_{C(\mathbb{I})})s_{c_1c_2} \in I_t,$$

so

$$S_{c_1}S_{c_2} = \pi(\Sigma_\bullet(c_1, c_2))s_{c_1c_2} + I_t = \Sigma_t(c_1, c_2)s_{c_1c_2} + I_t = \Sigma_t(c_1, c_2)S_{c_1c_2}.$$

Hence,  $S$  is a  $\Sigma_t$ -twisted representation of  $\mathcal{C}$  in  $A_t$  and the universal property of  $C^*(\mathcal{C}; \Sigma_t)$  gives a  $*$ -homomorphism  $\psi: C^*(\mathcal{C}; \Sigma_t) \rightarrow A_t$  such that  $\psi(s_c) = S_c$  for all  $c \in \mathcal{C}$ . The maps  $\psi$  and  $\overline{\varepsilon}_t$  are mutually inverse on generators and hence are mutually inverse isomorphisms.  $\square$

## 4.2. Invariance of $K$ -theory under 2-cocycle homotopy.

**Definition 4.7.** Let  $\mathcal{C}$  be a finitely aligned left-cancellative small category. We say that  $K_*(C^*(\mathcal{C}, \bullet))$  is constant along homotopies if for any homotopy  $\Sigma$  of 2-cocycles and any  $t \in \mathbb{I}$ , the homomorphisms

$$K_*(\varepsilon_t): K_*(C^*(\mathcal{C}; \Sigma)) \rightarrow K_*(C^*(\mathcal{C}; \Sigma_t))$$

in  $K$ -theory, induced by the map  $\varepsilon_t$  of Lemma 4.6, are isomorphisms.

*Remark 4.8.* The property that  $K_*(C^*(\mathcal{C}, \bullet))$  is constant along homotopies is a property of  $\mathcal{C}$ , but we could not think of good suggestive terminology that better emphasises this. Note that while the condition is stated in terms of  $K_*(\varepsilon_t)$  for all  $t$ , in fact  $K_*(C^*(\mathcal{C}, \bullet))$  is constant along homotopies if and only if  $K_*(\varepsilon_0)$  and  $K_*(\varepsilon_1)$  are isomorphisms for every homotopy  $\Sigma$  of 2-cocycles.

Suppose that  $(\mathcal{C}, \mathcal{D})$  is a matched pair of categories. If  $\Sigma: \mathbb{I} \times (\mathcal{C} \bowtie \mathcal{D})^2 \rightarrow \mathbb{T}$  is a homotopy of 2-cocycles on  $\mathcal{C} \bowtie \mathcal{D}$ , then the restriction  $\Sigma|_{\mathcal{C}}: \mathbb{I} \times \mathcal{C}^2 \rightarrow \mathbb{T}$  defined by  $(\Sigma|_{\mathcal{C}})_t(c_1, c_2) := \Sigma_t(c_1, c_2)$  for  $(c_1, c_2) \in \mathcal{C}^2 \subseteq (\mathcal{C} \bowtie \mathcal{D})^2$  is a homotopy of cocycles on  $\mathcal{C}$ . We frequently just write  $\Sigma$  for this restriction. We can now state our main theorem.

**Theorem 4.9.** *Consider a jointly-faithful self-similar action of a countable discrete amenable groupoid  $\mathcal{G}$  on a row-finite  $k$ -graph  $\Lambda$  with no sources. Then  $K_*(C^*(\mathcal{G} \bowtie \Lambda, \bullet))$  is constant along homotopies.*

Our strategy for proving Theorem 4.9 is to apply Elliott’s inductive Five-Lemma argument [Ell84] to the Pimsner exact sequence in  $K$ -theory [Pim97, Theorem 4.8]. The following proposition provides the base case of the induction.

**Proposition 4.10.** *Let  $\mathcal{G}$  be a countable discrete amenable groupoid. Then  $K_*(C^*(\mathcal{G}, \bullet))$  is constant along homotopies.*

*Proof.* Fix a homotopy  $\Sigma$  of 2-cocycles on  $\mathcal{G}$  and a point  $t \in \mathbb{I}$ . Choose a subset  $X$  of  $\mathcal{G}^0$  that meets each  $\mathcal{G}$ -orbit exactly once. Since  $\mathcal{G}^0$  is discrete,  $P_X := \sum_{x \in X} S_x$  is a multiplier projection of  $C^*(\mathcal{G}, \Sigma)$ . Fix  $y \in \mathcal{G}^0$ . Since  $X$  meets each orbit there exists  $\gamma \in \mathcal{G}$  such that  $r(\gamma) = y$  and  $s(\gamma) \in X$ . So  $S_y = S_\gamma P_X S_\gamma^*$ . Hence,  $P_X$  is full.

Similarly, the characteristic function  $\mathbb{1}_X$  is a full multiplier projection of  $C^*(\mathcal{G}, \Sigma_t)$ , and the extension of  $\varepsilon_t$  to multiplier algebras carries  $P_X$  to  $\mathbb{1}_X$ . So it suffices to show that  $\varepsilon_t: P_X C^*(\mathcal{G}, \Sigma) P_X \rightarrow \mathbb{1}_X C^*(\mathcal{G}, \Sigma_t) \mathbb{1}_X$  induces an isomorphism in  $K$ -theory. We have

$$P_X C^*(\mathcal{G}, \Sigma) P_X = \bigoplus_{x \in X} S_x C^*(\mathcal{G}, \Sigma) S_x \cong \bigoplus_{x \in X} C^*(x\mathcal{G}x, \Sigma).$$

Since  $\mathcal{G}$  is amenable, each  $x\mathcal{G}x$  is amenable [Wil19, Corollary 9.78], and hence each  $C^*(x\mathcal{G}x, \Sigma)$  is canonically isomorphic to the  $C^*$ -algebra  $C_r^*(G, \Omega)$  described in the paragraph following [ELPW10, Theorem 0.3] for  $G = x\mathcal{G}x$  and  $\Omega = \Sigma|_{x\mathcal{G}x}$ . By [HK01, Corollary 9.2] each  $x\mathcal{G}x$  satisfies the Baum–Connes conjecture with coefficients. So [ELPW10,

Theorem 0.3] implies that the restriction of  $\varepsilon_t$  to  $C_{\mathbb{I}}^*(x\mathcal{G}x, \Sigma)$  induces an isomorphism  $K_*(\varepsilon_t)$  from  $K_*(C_{\mathbb{I}}^*(x\mathcal{G}x, \Sigma))$  to  $K_*(C^*(x\mathcal{G}x, \Sigma_t))$ . Since  $K$ -theory respects direct sums, it follows that

$$K_*(\varepsilon_t): K_*\left(\bigoplus_{x \in X} C_{\mathbb{I}}^*(x\mathcal{G}x, \Sigma)\right) \rightarrow K_*\left(\bigoplus_{x \in X} C^*(x\mathcal{G}x, \Sigma_t)\right)$$

is an isomorphism.  $\square$

*Remark 4.11.* **Proposition 4.10** does not imply that the  $K$ -theory of  $C^*(\mathcal{G}, \sigma)$  is independent of  $\sigma$ . There exist discrete amenable groups  $G$  and (non-homotopic) 2-cocycles  $\sigma_1, \sigma_2$  on  $G$  such that  $K_*(C^*(G, \sigma_1)) \not\cong K_*(C^*(G, \sigma_2))$ , [PR92, Proposition 3.11].

For the inductive step in the proof of **Theorem 4.9**, we need a Cuntz–Pimsner model for  $C_{\mathbb{I}}^*(\mathcal{G} \rtimes \Lambda; \Sigma)$  whose coefficient algebra is  $C_{\mathbb{I}}^*(\mathcal{G} \rtimes \Gamma; \Sigma)$  for a rank- $(k-1)$  subgraph  $\Gamma$  of  $\Lambda$ . We establish such a model in **Proposition 4.14** after proving two preliminary lemmas; the proof of **Theorem 4.9** then appears on page 15. From here until **Proposition 4.14**, we work in the following setup. Fix a jointly faithful self-similar action of a discrete amenable groupoid  $\mathcal{G}$  on a row-finite  $(k+1)$ -graph  $\Lambda$  with no sources. (Note:  $k$  could be 0.) Let  $\Gamma = \Lambda^{\text{N}^k}$  and  $E^* = \Lambda^{\text{N}^{e_{k+1}}}$  so that  $\Lambda = \Gamma \rtimes E^*$  as in **Example 2.2**. Let  $\Sigma$  be a homotopy of 2-cocycles on  $\mathcal{G} \rtimes \Lambda$ .

**Lemma 4.12.** *The inclusion  $\mathcal{G} \rtimes \Gamma \hookrightarrow \mathcal{G} \rtimes \Lambda$  induces an injective  $*$ -homomorphism  $\Pi: C_{\mathbb{I}}^*(\mathcal{G} \rtimes \Gamma; \Sigma) \rightarrow C_{\mathbb{I}}^*(\mathcal{G} \rtimes \Lambda; \Sigma)$ .*

*Proof.* Let  $(s^\Gamma, \tau^\Gamma)$  and  $(s^\Lambda, \tau^\Lambda)$  denote universal  $\Sigma$ -twisted representations of  $\mathcal{G} \rtimes \Gamma$  and  $\mathcal{G} \rtimes \Lambda$ , respectively. Then  $(s^\Lambda|_{\mathcal{G} \rtimes \Gamma}, \tau^\Lambda)$  satisfies (IR1), (IR2), (IR3), and (R2) as these relations for  $\mathcal{G} \rtimes \Lambda$  are identical to those for  $\mathcal{G} \rtimes \Gamma$  on elements of the latter. **Lemma 3.18** implies that  $(s^\Lambda|_{\mathcal{G} \rtimes \Gamma}, \tau^\Lambda)$  satisfies (R3), and **Lemma 3.20** implies that it satisfies (R4). Hence, the inclusion  $\mathcal{G} \rtimes \Gamma \hookrightarrow \mathcal{G} \rtimes \Lambda$  induces a  $*$ -homomorphism  $\Pi: C_{\mathbb{I}}^*(\mathcal{G} \rtimes \Gamma; \Sigma) \rightarrow C_{\mathbb{I}}^*(\mathcal{G} \rtimes \Lambda; \Sigma)$ .

To see that  $\Pi$  is injective, note that  $\Pi \circ \tau^\Gamma = \tau^\Lambda \circ \Pi$ , so  $\Pi$  is a homomorphism of  $C(\mathbb{I})$ -algebras. So it induces homomorphisms  $\Pi_t: C^*(\mathcal{G} \rtimes \Gamma; \Sigma_t) \rightarrow C^*(\mathcal{G} \rtimes \Lambda; \Sigma_t)$ . Since the norm on a  $C(\mathbb{I})$ -algebra is the supremum norm [Wil07, Proposition C.23 and Theorem C.26], it suffices to show that each  $\Pi_t$  is injective, which is **Corollary 3.23**.  $\square$

Let  $A := C_{\mathbb{I}}^*(\mathcal{G} \rtimes \Gamma; \Sigma)$ . We use **Lemma 4.12** to treat  $A$  as a subalgebra of  $C_{\mathbb{I}}^*(\mathcal{G} \rtimes \Lambda; \Sigma)$ .

**Lemma 4.13.** *The subspace*

$$X := \overline{\text{span}}\{s_e a : e \in E^1, a \in A\}$$

*of  $C_{\mathbb{I}}^*(\mathcal{G} \rtimes \Lambda; \Sigma)$  is a right Hilbert  $A$ -module with right action and inner product satisfying*

$$s_e a \cdot b = s_e ab \quad \text{and} \quad \langle s_e a \mid s_f b \rangle = \delta_{e,f} a^* b.$$

*The map  $\varphi: A \rightarrow \mathcal{L}_A(X)$  defined by  $\varphi(a)s_e b = as_e b$  makes  $(\varphi, X)$  a nondegenerate  $C^*$ -correspondence over  $A$ . The map  $\varphi$  is injective and takes values in the generalised compact operators  $\mathcal{K}_A(X)$ .*

*Proof.* Routine calculations show that  $X$  is a right pre-Hilbert  $A$ -module under the given right action and inner product. For each linear combination  $\sum_i s_{e_i} a_i \in X$  we have

$$\begin{aligned} \left\| \sum_i s_{e_i} a_i \right\|_X^2 &= \left\| \sum_{i,j} \langle s_{e_i} a_i \mid s_{e_j} a_j \rangle \right\| = \left\| \sum_i a_i^* s_{s(e_i)} a_i \right\| \\ &= \left\| \left( \sum_i s_{e_i} a_i \right)^* \left( \sum_j s_{e_j} a_j \right) \right\| = \left\| \sum_i s_{e_i} a_i \right\|_{C_{\mathbb{I}}^*(\mathcal{G} \rtimes \Lambda; \Sigma)}^2, \end{aligned}$$

so the norm on  $X$  agrees with the  $C^*$ -norm on  $C_{\mathbb{I}}^*(\mathcal{G} \rtimes \Lambda, \Sigma)$ . In particular,  $X$  is complete.

We claim that  $AX \subseteq X$ ; it then follows that  $X$  is a  $C^*$ -correspondence over  $A$  with respect to  $\varphi$  because  $\varphi(a^*)$  is an adjoint for  $\varphi(a)$ . It suffices to show that  $as_e \in X$  for each  $a$  in the spanning set described in [Corollary 4.5](#) and each  $e \in E^1$ . For this, fix  $e \in E^1$ ,  $f \in C(\mathbb{I})$ , and  $(\gamma_1, g, \gamma_2) \in \Gamma_{r^*s} \mathcal{G}_{s^*s} \Gamma$ . Then

$$\tau(f)s_{\gamma_1} s_g s_{\gamma_2} s_e^* = \sum_{\gamma_2 \alpha = e \beta \in \gamma_2 v e} \tau\left(f \overline{\Sigma_{\bullet}(\gamma_2, \alpha) \Sigma_{\bullet}(\beta, e)}\right) s_{\gamma_1} s_g s_{\alpha} s_{\beta}^*.$$

For each term in this sum,

$$\begin{aligned} s_{\gamma_1} s_g s_{\alpha} s_{\beta}^* &= \tau\left(\Sigma_{\bullet}(g, \alpha) \overline{\Sigma_{\bullet}(g \triangleright \alpha, g \triangleleft \alpha)}\right) s_{\gamma_1} s_{g \triangleright \alpha} s_{g \triangleleft \alpha} s_{\beta}^* \\ &= \tau\left(\Sigma_{\bullet}(g, \alpha) \overline{\Sigma_{\bullet}(g \triangleright \alpha, g \triangleleft \alpha)} \Sigma_{\bullet}(\gamma_1, g \triangleright \alpha)\right) s_{\gamma_1(g \triangleright \alpha)} s_{g \triangleleft \alpha} s_{\beta}^*. \end{aligned}$$

Since  $\alpha \in E^1$ , we have  $g \triangleright \alpha \in E^1$ . By uniqueness of factorisations in  $\Lambda$  there exist unique  $\alpha' \in E^1$  and  $\gamma'_1 \in \Gamma$  such that  $\gamma_1(g \triangleright \alpha) = \alpha' g'_1$ . So

$$s_{\gamma_1(g \triangleright \alpha)} s_{g \triangleleft \alpha} s_{\beta}^* = s_{\alpha' g'_1} s_{g \triangleleft \alpha} s_{\beta}^* = \tau\left(\overline{\Sigma_{\bullet}(\alpha', g'_1)}\right) s_{\alpha'} s_{g'_1} s_{g \triangleleft \alpha} s_{\beta}^* \in X.$$

Hence,  $\tau(f)s_{\gamma_1} s_g s_{\gamma_2} s_e^* \in X$ .

The  $(k+1)$ -graph  $\Lambda$  is row-finite with no sources, so for each  $v \in \Gamma^0$  the set  $\{e \in E^1 : r(e) = v\} \subseteq v\Lambda$  is finite and exhaustive. Since  $s_e = s_e s_{s(e)} \in X$ , we have  $s_e s_e^* \in \mathcal{K}(X)$  for each  $e \in E^1$ . Since  $\Gamma$  has no sources, for each  $v \in \Gamma^0$  we have,

$$s_v = \sum_{r(e)=v} s_e s_e^* \in \mathcal{K}(X).$$

Since  $(\sum_{v \in F} s_v)_{F \subseteq \text{finite } E^0}$  is an approximate unit for  $C_{\mathbb{I}}^*(\mathcal{G} \rtimes \Lambda; \Sigma)$ , we obtain  $\varphi(A) \subseteq \mathcal{K}(X)$ . This approximate unit belongs to  $A$ , so the left action is nondegenerate. If  $\varphi(a) = 0$  then

$$as_v = \sum_{r(e)=v} as_e s_e^* = \sum_{r(e)=v} \varphi(a) s_e s_e^* = 0,$$

so  $a = \lim_F \sum_{v \in F} as_v = 0$ . That is,  $\varphi$  is injective.  $\square$

**Proposition 4.14.** *Let  $A := C_{\mathbb{I}}^*(\mathcal{G} \rtimes \Gamma; \Sigma)$  and let  $X$  be as in [Lemma 4.13](#). The inclusions  $\iota_A : A \rightarrow C_{\mathbb{I}}^*(\mathcal{G} \rtimes \Lambda; \Sigma)$  and  $\iota_X : X \rightarrow C_{\mathbb{I}}^*(\mathcal{G} \rtimes \Lambda; \Sigma)$  comprise a covariant representation of  $X$  in  $C_{\mathbb{I}}^*(\mathcal{G} \rtimes \Lambda; \Sigma)$ . If  $(j_A, j_X)$  is a universal covariant representation of  $X$  in  $\mathcal{O}_X$ , then the  $*$ -homomorphism  $\Xi : \mathcal{O}_X \rightarrow C_{\mathbb{I}}^*(\mathcal{G} \rtimes \Lambda; \Sigma)$  satisfying  $\Xi(j_A(a)) = a$  and  $\Xi(j_X(x)) = x$  for all  $a \in A$  and  $x \in X$  is an isomorphism.*

*Proof.* By construction  $(\iota_A, \iota_X)$  is a representation of the correspondence  $(\varphi, X)$ . Since  $\iota_A(s_v) = \sum_{r(e)=v} s_e s_e^* = \sum_{r(e)=v} \iota_X(s_e) \iota_X(s_e)^*$  for all  $v \in \Lambda^0$ , and since  $(\sum_{v \in F} s_v)$  is approximate identity for  $A$ , the representation  $(\iota_A, \iota_X)$  is covariant [[Mun20](#), Lemma A.3.16]. Hence, there is a  $*$ -homomorphism  $\Xi : \mathcal{O}_X \rightarrow C_{\mathbb{I}}^*(\mathcal{G} \rtimes \Lambda; \Sigma)$  such that  $\Xi(a) = \iota_A(j_A(a))$  and  $\Xi(j_X(x)) = \iota_X(x)$  for all  $a \in A$  and  $x \in X$ .

The standard universal-property argument (see, for example, [[Rae05](#), Proposition 2.1]) yields a strongly continuous action  $\beta : \mathbb{T} \rightarrow \text{Aut}(C_{\mathbb{I}}^*(\mathcal{G} \rtimes \Lambda; \Sigma))$  such that  $\beta_z(\tau(f)s_{\lambda g}) = z^{d(\lambda)k+1} \tau(f)s_{\lambda g}$  for all  $f \in C(\mathbb{I})$  and  $\lambda g \in \mathcal{G} \rtimes \Lambda$ . Since  $\iota_A$  is injective and  $\Xi$  intertwines  $\beta$  with the gauge action on  $\mathcal{O}_X$ , [[Kat04](#), Theorem 6.4] implies that  $\Xi$  is injective. The subalgebra  $\Xi(\mathcal{O}_X)$  contains the generators of  $C_{\mathbb{I}}^*(\mathcal{G} \rtimes \Lambda; \Sigma)$ , so  $\Xi$  is a  $*$ -isomorphism.  $\square$

*Proof of [Theorem 4.9](#).* We induct on  $k$ . If  $k = 0$  the result is [Proposition 4.10](#).

Fix a jointly faithful self-similar action of a discrete amenable groupoid  $\mathcal{G}$  on a row-finite  $(k+1)$ -graph  $\Lambda$  with no sources and fix a homotopy  $\Sigma : \mathbb{I} \times \Lambda \rightarrow \mathbb{T}$  of 2-cocycles.

Let  $\Lambda = \Gamma \rtimes E^*$  as in [Example 2.2](#). Suppose as an inductive hypothesis that  $K_*(C^*(\mathcal{G} \rtimes \Gamma, \bullet))$  is constant along homotopies.

By [Proposition 4.14](#) the algebra  $B := C^*(\mathcal{G} \rtimes \Lambda; \Sigma)$  is the Cuntz–Pimsner algebra  $\mathcal{O}_X$  of a  $C^*$ -correspondence over  $A := C^*(\mathcal{G} \rtimes \Gamma; \Sigma)$  with injective left action by generalised compact operators. Hence, [\[Kat04, Theorem 8.6\]](#) gives an exact sequence

$$\begin{array}{ccccc} K_0(A) & \xrightarrow{\text{id}-[X]} & K_0(A) & \longrightarrow & K_0(B) \\ \partial \uparrow & & & & \downarrow \partial \\ K_1(B) & \longleftarrow & K_1(A) & \xleftarrow{\text{id}-[X]} & K_1(A) \end{array}$$

in  $K$ -theory.

Fix  $t \in \mathbb{I}$  and let  $A_t := C^*(\mathcal{G} \rtimes \Gamma; \Sigma_t)$  and  $B_t := C^*(\mathcal{G} \rtimes \Lambda; \Sigma_t)$ . Let  $\varepsilon_t^A: A \rightarrow A_t$  and  $\varepsilon_t^B: B \rightarrow B_t$  be as in [Lemma 4.6](#). Let  $I_t^A = \ker(\varepsilon_t^A) = A\tau(C_0(\mathbb{I} \setminus \{t\}))$  and  $I_t^B = \ker(\varepsilon_t^B) = B\tau(C_0(\mathbb{I} \setminus \{t\}))$ . By [\[Kat07, Corollary 1.4\]](#) the subspace  $XI_t^A = \overline{\text{span}}\{x \cdot a : x \in X, a \in I_t^A\}$  is an  $A$ -submodule of  $X$  and  $X_t := X/XI_t^A$  is a right  $A_t$ -module with respect to the obvious right  $A_t$ -action. Since  $I_t^A \subseteq I_t^B$  we can regard  $A_t$  as a  $C^*$ -subalgebra of  $B_t$  and  $X_t$  as a closed subspace of  $B_t$ . We also identify the quotient map  $X \rightarrow X_t$  with  $\varepsilon_t^X := \varepsilon_t^B|_X$ .

Since  $\tau(C_0(\mathbb{I} \setminus \{t\}))$  is central in  $\mathcal{MC}^*(\mathcal{G} \rtimes \Lambda; \Sigma)$  we have  $a\tau(f) \cdot s_e b = a s_e b \tau(f) \in XI_t^A$  for all  $f \in C_0(\mathbb{I} \setminus \{t\})$ ,  $a \in A$  and  $s_e b \in X$ . In particular, the left action of  $A$  on  $X$  descends to an adjointable left action of  $A_t$  on  $X_t$ , so  $X_t$  is a  $C^*$ -correspondence over  $A_t$ . Since  $s_v = \sum_{e \in vE^1} s_e s_e^*$  in  $B$ , the same is true in  $B_t$ , and since  $(\sum_{v \in F} s_v + I_t^B)_{F \subseteq E^0 \text{ finite}}$  is an approximate unit in  $B_t$ , the left action of  $A_t$  on  $X_t$  is by compact operators.

For injectivity of the left action suppose that  $a + I_t^A \in A_t$  satisfies  $(a + I_t^A)(s_e b + XI_t^A) = 0$  for all  $e \in E^1$  and  $b \in A$ . That is,  $a s_e b \in XI_t^A$  for all  $e \in E^1$  and  $b \in A$ . In particular,  $a s_e \in XI_t^A$  for all  $e \in E^1$ . This implies that  $a s_e s_e^* \in I_t^B$  for all  $e \in E^1$ , and so  $a s_v = \sum_{e \in vE^1} a s_e s_e^* \in I_t^B$  for all  $v \in E^0$ . Since  $(\sum_{v \in F} s_v)_{F \subseteq E^0 \text{ finite}}$  is an approximate unit we have  $a \in I_t^B$ . Fix an approximate unit  $(h_i)$  for  $C_0(\mathbb{I} \setminus \{t\})$ . Then  $(\tau(h_i))$  is an approximate unit for  $I_t^B$ . So,  $a = \lim \tau(h_i) a \in I_t^A$ . That is,  $a + I_t^A = 0$ .

Let  $(j_A, j_X)$  be a universal covariant representation of  $X$  in  $\mathcal{O}_X$  and let  $(j_{A_t}, j_{X_t})$  be a universal covariant representation of  $X_t$  in  $\mathcal{O}_{X_t}$ . The inclusions  $\iota_{A_t}: A_t \rightarrow B_t$  and  $\iota_{X_t}: X_t \rightarrow B_t$  define a covariant representation  $(\iota_{A_t}, \iota_{X_t})$  of  $X_t$  in  $B_t$ . By the universal property of  $\mathcal{O}_{X_t}$  there is a unique  $*$ -homomorphism  $\Psi: \mathcal{O}_{X_t} \rightarrow B_t$  such that  $\Psi(j_{A_t}(a + I_t^A)) = a + I_t^B$  and  $\Psi(j_{X_t}(x + XI_t^A)) = x + I_t^B$  for all  $a \in A$  and  $x \in X$ .

We construct an inverse to  $\Psi$ . The pair  $(\varepsilon_t^A, \varepsilon_t^X)$  is a coisometric correspondence morphism (in the sense of [\[Bre10, Definition 1.3\]](#)) from  $X$  to  $X_t$ , so by [\[Bre10, Proposition 1.4\]](#) there is a unique  $*$ -homomorphism  $\Phi: \mathcal{O}_X \rightarrow \mathcal{O}_{X_t}$  satisfying  $\Phi(j_A(a)) = j_{A_t}(a + I_t^A)$  and  $\Phi(j_X(x)) = j_{X_t}(x + XI_t^A)$  for all  $a \in A$  and  $x \in X$ . Let  $\Xi: \mathcal{O}_X \rightarrow B$  be the isomorphism of [Proposition 4.14](#). Then for each  $a \in A$  and  $x \in X$ ,

$$\Phi \circ \Xi^{-1}(a) = j_{A_t}(a + I_t^A) \quad \text{and} \quad \Phi \circ \Xi^{-1}(x) = j_{X_t}(x + XI_t^A).$$

Since  $\mathcal{O}_X$  is generated by  $j_A(A)$  and  $j_X(X)$ , [Proposition 4.14](#) shows that  $B$  is generated by  $\Xi(j_A(A)) = A$  and  $\Xi(j_X(X)) = X$ . So  $I_t^B$  is generated by  $A\tau(C_0(\mathbb{I} \setminus \{t\})) = I_t^A$  and  $X\tau(C_0(\mathbb{I} \setminus \{t\})) = XI_t^A$ . If  $a \in I_t^A$ , then  $\Phi \circ \Xi^{-1}(a) = j_{A_t}(a + I_t^A) = 0$ , and if  $x \in XI_t^A$ , then  $\Phi \circ \Xi^{-1}(x) = j_{X_t}(x + XI_t^A) = 0$ . Hence,  $I_t^B \subseteq \ker(\Phi \circ \Xi^{-1})$ , and so  $\Phi \circ \Xi^{-1}$  descends to a  $*$ -homomorphism  $\overline{\Phi}: B_t \rightarrow \mathcal{O}_{X_t}$  such that  $\overline{\Phi}(a + I_t^B) = j_{A_t}(a + I_t^A)$  and  $\overline{\Phi}(x + I_t^B) = j_{X_t}(x + XI_t^A)$  for all  $a \in A$  and  $x \in X$ . The maps  $\Psi$  and  $\overline{\Phi}$  are mutually inverse on generators, so are mutually inverse isomorphisms.



Another application of [Kat04, Theorem 8.6] together with naturality of the  $K$ -theory functor yields a commuting diagram

$$\begin{array}{ccccc}
 K_0(A) & \xrightarrow{\text{id}-[X]} & K_0(A) & \xrightarrow{j_A} & K_0(B) \\
 \uparrow & \searrow^{K_0(\varepsilon_t^A)} & \downarrow^{K_0(\varepsilon_t^A)} & & \swarrow^{K_0(\varepsilon_t^B)} \\
 & & K_0(A_t) & \xrightarrow{\text{id}-[X_t]} & K_0(A_t) & \xrightarrow{j_{A_t}} & K_0(B_t) \\
 & & \uparrow & & \downarrow & & \\
 & & K_1(B_t) & \xleftarrow{j_{A_t}} & K_1(A_t) & \xleftarrow{\text{id}-[X_t]} & K_1(A_t) \\
 & & \swarrow^{K_1(\varepsilon_t^B)} & & \uparrow^{K_1(\varepsilon_t^A)} & & \swarrow^{K_1(\varepsilon_t^A)} \\
 K_1(B) & \xleftarrow{j_A} & K_1(A) & \xleftarrow{\text{id}-[X]} & K_1(A), & & 
 \end{array}$$

of exact sequences. By the inductive hypothesis, the maps  $K_*(\varepsilon_t^A)$  are isomorphisms, so the Five Lemma implies that the  $K_*(\varepsilon_t^B)$  are isomorphisms.  $\square$

We recover a result of Gillaspy as a special case of [Theorem 4.9](#).

**Corollary 4.15** ([Gil15, Theorem 4.1]). *Let  $\Lambda$  be a row-finite  $k$ -graph with no sources. Then  $K_*(C^*(\Lambda, \bullet))$  is constant along homotopies.*

*Proof.* Let  $\mathcal{G} = \Lambda^0$ , the groupoid consisting solely of units. Then  $(\mathcal{G}, \Lambda)$  is a matched pair with  $\mathcal{G} \bowtie \Lambda \cong \Lambda$ . The result now follows immediately from [Theorem 4.9](#).  $\square$

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