

HOMOLOGY AND TWISTED C^* -ALGEBRAS FOR SELF-SIMILAR ACTIONS AND ZAPPA–SZÉP PRODUCTS

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ABSTRACT. We study the categorical homology of Zappa–Szép products of small categories, which include all self-similar actions. We prove that the categorical homology coincides with the homology of a double complex, and so can be computed via a spectral sequence involving homology groups of the constituent categories. We give explicit formulae for the isomorphisms involved, and compute the homology of a class of examples that generalise odometers. We define the C^* -algebras of self-similar groupoid actions on k -graphs twisted by 2-cocycles arising from this homology theory, and prove some fundamental results about their structure.

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1. INTRODUCTION

This paper achieves three main objectives:

- (1) to introduce a unifying framework, which we call matched pairs of categories, for self-similar actions, graphs of groups, Zappa–Szép products, and k -graphs;
- (2) to introduce homology and cohomology for matched pairs, and develop practical tools for computing them; and
- (3) to associate twisted C^* -algebras to self-similar groupoid actions on k -graphs, and establish fundamental structure theorems for these C^* -algebras.

Self-similar groups of automorphisms of trees were introduced in the early 1980s as models for new classes of groups. Grigorchuk used self-similar groups to describe the first example of a finitely generated group with intermediate growth [Gri80, Gri84], and Nekrashevych recently used them to produce the first simple groups of intermediate growth [Nek18]. Self-similar groups have been studied intensively ever since Grigorchuk’s work, including, since the seminal work of Nekrashevych [Nek05], via their C^* -algebras. Nekrashevych studies C^* -algebraic representations not just a self-similar group, but of the entire self-similar system: a unitary representation of the group and a Cuntz representation of the alphabet being acted upon. The resulting C^* -algebra encodes information about the self-similar system through both K -theory [Nek05] and KMS-data [LRRW14, EP17, LRRW18]. The former suggests that homological invariants of self-similar actions could be a profitable avenue of study.

Initially, self-similar actions were presented with an apparent asymmetry between the role of the group and the role of the alphabet. But recent generalisations [LRRW14, EP17, LRRW18, LY21, LawV22] make it increasingly clear that the roles of the two objects are symmetric, and that self-similar actions are closely related to Zappa–Szép products.

Introduced by Zappa [Zap42] and Szép [Sze50], Zappa–Szép products of groups are a generalisation of semidirect products in which each of the two constituent groups acts on the other; so both embed as (not necessarily normal) subgroups of the product. Subsequent generalisations include Zappa–Szép-style products of increasingly general pairs of algebraic objects: [Bri05, Law08, LRRW14, BPRRW17, BKQS18, LRRW18, LawV22, PO22, DL23].

Here, we start with a *matched pair of categories*: small categories \mathcal{C} and \mathcal{D} with common object set, a left action $(c, d) \mapsto c \triangleright d$ of \mathcal{C} on \mathcal{D} and a right action $(c, d) \mapsto c \triangleleft d$ of \mathcal{D} on \mathcal{C} satisfying the compatibility conditions of [Zap42, Sze50]. Each such pair determines a Zappa–Szép-product category $\mathcal{C} \bowtie \mathcal{D}$; this can be viewed either “externally” as the fibred product $\mathcal{D}_{s^*r} \mathcal{C}$ under a suitable multiplication, or “internally” as the universal category containing copies of \mathcal{C} and \mathcal{D} with a strict factorisation system as in [RW02] that implements \triangleright and \triangleleft . All of the algebraic product constructions mentioned above fit into this framework, as do graphs of groups [Bas93, Ser80] and k -graphs [KP00]. Their C^* -algebraic representations all boil down to representations, in the sense of Spielberg [Spe20], of the associated Zappa–Szép-product category.

In the study of C^* -algebras associated to algebraic or combinatorial objects, there is a well-established principle that interesting C^* -algebraic properties emerge when we twist the multiplication by a \mathbb{T} -valued 2-cocycle. The archetypal examples are the noncommutative tori A_θ , which are obtained by twisting the multiplication in unitary representations of \mathbb{Z}^k by \mathbb{T} -valued 2-cocycles—which themselves are simply computed in terms of characters (1-cocycles) on the constituent factors of \mathbb{Z} in \mathbb{Z}^k [OPT80]. To generalise this to matched pairs we need both a suitable definition of cohomology, and effective tools for computing it in terms of the cohomology of the constituent categories.

For topological spaces X and Y , the classical *Eilenberg–Zilber Theorem* gives an isomorphism between the (singular) homology of the chain complex $C_\bullet(X \times Y)$ and the total homology of the tensor product double complex $C_\bullet(X) \otimes_{\mathbb{Z}} C_\bullet(Y)$. We take this as our inspiration for analysing homology of Zappa–Szép products. We consider the classical categorical homology of $\mathcal{C} \bowtie \mathcal{D}$: n -chains are \mathbb{Z} -linear combinations of composable n -tuples, and boundary maps are alternating sums of the maps obtained by deleting the first or last entry in a composable tuple, or composing adjacent terms. We show that this homology can be computed in terms of a double complex, called the *matched complex*: its columns are chain complexes for the homology of \mathcal{D} with coefficients in modules spanned by composable tuples in \mathcal{C} ; and its rows are chain complexes for the homology of \mathcal{C} with coefficients in modules spanned by composable tuples in \mathcal{D} . The matched complex $C_{\bullet,\bullet}$ is not the tensor product $C_\bullet(\mathcal{C}) \otimes_{\mathbb{Z}} C_\bullet(\mathcal{D})$, but its terms are fibred products of a similar form.

The matched complex admits two natural homology theories—diagonal homology $H_\bullet^\Delta(\mathcal{C}, \mathcal{D})$ and total homology $H_\bullet^{\text{Tot}}(\mathcal{C}, \mathcal{D})$. These are isomorphic via explicit chain equivalences called the Eilenberg–Zilber map and the Alexander–Whitney map. The total homology $H_\bullet^{\text{Tot}}(\mathcal{C}, \mathcal{D})$, is defined in terms of the homology of the constituent categories \mathcal{C} and \mathcal{D} . So to see that the categorical homology $H_\bullet^\bowtie(\mathcal{C}, \mathcal{D})$ of $\mathcal{C} \bowtie \mathcal{D}$ suits our purposes, we use the method of acyclic models to construct explicit chain equivalences between the chain complex $C_\bullet^\bowtie(\mathcal{C}, \mathcal{D})$ defining $H_\bullet^\bowtie(\mathcal{C}, \mathcal{D})$ and the diagonal chain complex $C_\bullet^\Delta(\mathcal{C}, \mathcal{D})$. Combined with the Eilenberg–Zilber map, this gives a computable isomorphism $H_\bullet^{\text{Tot}}(\mathcal{C}, \mathcal{D}) \cong H_\bullet^\bowtie(\mathcal{C}, \mathcal{D})$. Dualising yields isomorphisms $H_{\text{Tot}}^\bullet(\mathcal{C}, \mathcal{D}; \mathbb{T}) \cong H_{\bowtie}^\bullet(\mathcal{C}, \mathcal{D}; \mathbb{T})$ in cohomology.

As an aside, this shows that if a category admits a strict factorisation system, then its categorical homology can be computed in terms of that of the embedded subcategories. This yields, for example, a potential iterative approach to computing homology for k -graphs.

We use our results to compute the homology of a class of self-similar groupoid actions on graphs that generalise the odometer. We calculate the homology in terms of the two nonzero homology groups of the underlying graph E , and the kernel and cokernel of an $E^0 \times E^1$ matrix encoding the orders of the odometers involved. *En passant*, we establish useful general results about homology for matched pairs in which one factor is the path category of a directed graph, or a bundle of monoids, with stronger results when the monoids are copies of \mathbb{Z} . These results would be well suited to computing the homology of Exel–Pardo systems [EP17].

The main motivation for our work on homology is to study twisted C^* -algebras of matched pairs. The point is that the natural definition of a C^* -algebraic representation of a matched pair, as made clear by Spielberg’s work [Spe20], is as a multiplicative map $\zeta \mapsto t_\zeta$ from its Zappa–Szép product category to a semigroup of partial isometries. Consequently, the natural definition of a twisted representation is in terms of a categorical 2-cocycle c on the Zappa–Szép product: we twist by the formula $s_\zeta s_\eta = c(\zeta, \eta) s_{\zeta\eta}$. However, the total homology (and cohomology) is a more computable theory, and clearly reflects the decomposition of the Zappa–Szép product category

into its constituent components. Our main homology theorem allows us to define and analyse the C^* -algebras in the natural way via categorical 2-cocycles, but pass to total cohomology when we wish to identify the possible twists for a given matched pair or produce nontrivial cocycles in concrete examples.

We explore this in the context of matched pairs consisting of a groupoid \mathcal{G} and a row-finite k -graph Λ with no sources (self-similar actions of groupoids on such k -graphs). This covers a fairly general class of examples with relatively complex cohomology, for which C^* -algebraic representations in the sense of Spielberg of the associated untwisted pair are well understood. Given a categorical 2-cocycle $c \in C_{\bowtie}^2(\mathcal{G}, \Lambda; \mathbb{T})$, we define a universal twisted Toeplitz algebra $\mathcal{TC}^*(\mathcal{G}, \Lambda; c)$ and a universal Cuntz–Krieger algebra $C^*(\mathcal{G}, \Lambda; c)$ in both of which all the generators are nonzero. We show that $\mathcal{TC}^*(\Lambda, c|_{\Lambda^2})$ embeds in $\mathcal{TC}^*(\mathcal{G}, \Lambda; c)$ and likewise that $C^*(\Lambda, c|_{\Lambda^2})$ embeds in $C^*(\mathcal{G}, \Lambda; c)$. We also show that if \mathcal{G} is amenable then $C^*(\mathcal{G}, c|_{\mathcal{G}^2})$ embeds in $\mathcal{TC}^*(\mathcal{G}, \Lambda; c)$, and that an addition condition developed by Yushman [Yus23] ensures that it also embeds in $C^*(\mathcal{G}, \Lambda; c)$. We establish a gauge-invariant uniqueness theorem for $C^*(\mathcal{G}, \Lambda; c)$ and prove that cohomologous 2-cocycles yield isomorphic twisted C^* -algebras.

We then construct twisted C^* -algebras $\mathcal{TC}_\varphi^*(\mathcal{G}, \Lambda)$ and $C_\varphi^*(\mathcal{G}, \Lambda)$ associated to a total 2-cocycle $\varphi \in C_{\text{Tot}}^2(\mathcal{G}, \Lambda; \mathbb{T})$. We prove that our cochain equivalence $\Psi^*: C_{\text{Tot}}^2(\mathcal{G}, \Lambda; \mathbb{T}) \rightarrow C_{\bowtie}^2(\mathcal{G}, \Lambda; \mathbb{T})$ induces isomorphisms $\mathcal{TC}_\varphi^*(\mathcal{G}, \Lambda) \cong \mathcal{TC}^*(\mathcal{G}, \Lambda, \Psi^*(\varphi))$ and $C_\varphi^*(\mathcal{G}, \Lambda) \cong C^*(\mathcal{G}, \Lambda; \Psi^*(\varphi))$.

The paper is organised as follows. In Section 2 we establish some background: on categories; on actions of one category on another; and on directed graphs and their path categories.

In Section 3 we discuss matched pairs of small categories. We show that each matched pair admits a Zappa–Szépp product, and discuss internal and external descriptions of this object and its relationship to strict factorisation systems. We show how the actions in a matched pair extend to actions on the categories of composable tuples in the categories involved. We give a number of concrete examples of matched pairs, including the key *model matched pairs* that serve as local models for composable tuples in arbitrary matched pairs.

In Section 4 we introduce the three homology theories for matched pairs. We first introduce categorical homology of a small category, described in terms of simplicial sets. We then introduce the *matched complex*—a double complex associated to a matched pair—in terms of a bisimplicial group, and show that the assignment of the matched complex to a matched pair is functorial. We then define the diagonal complex, the total complex, and the associated homology theories of a matched pair.

In Section 5 we prove our main homology theorem: categorical homology, total homology, and diagonal homology coincide. In Section 5.1, we describe the three chain maps that appear in our main theorem: the first is the Eilenberg–Zilber map for double complexes—we just give a formula for use in computations; the other two, $\Pi: C_{\bullet}^{\Delta} \rightarrow C_{\bullet}^{\bowtie}$ and $\Psi: C_{\bullet}^{\bowtie} \rightarrow C_{\bullet}^{\text{Tot}}$, are specific to our situation. In Section 5.2, we state the main theorem, Theorem 5.3, describe the Alexander–Whitney map, which induces the inverse of the Eilenberg–Zilber map, and outline the strategy of the proof. In Section 5.3, we show that our model matched pairs are acyclic in both diagonal and categorical homology, and describe functors from the Zappa–Szépp-product categories of model matched pairs into $\mathcal{C} \bowtie \mathcal{D}$ that realise all generators of each chain complex. In Section 5.4 we invoke the method of acyclic models to characterise chain equivalences between the diagonal and categorical complexes. In Subsection 5.5 we show that the concrete chain maps described in Section 5.1 are such chain equivalences and describe their inverses. Finally, in Section 5.6,

we describe a spectral sequence that computes the homology of a matched pair, and a Künneth theorem for matched pairs of monoids.

In Section 6, we compute the homology of a concrete class of examples: “graphs of odometers.” We consider a finite directed graph E together with a labelling $p: E^1 \rightarrow \{1, 2, \dots\}$ of its edges by strictly positive integers. We build the augmented graph, F that has a bundle $\{e\} \times \mathbb{Z}/p(e)\mathbb{Z}$ of $p(e)$ parallel edges for each edge $e \in E$. We consider a matched pair $(E^0 \times \mathbb{Z}, F^*)$ in which the copies of \mathbb{Z} behave, collectively, like odometers. In Section 6.1, we show that in a matched pair where the second factor is the path category of a graph, only the first two rows of the second page of the spectral sequence obtained above are nonzero. In Section 6.2, we show that for matched pairs where the first factor is a bundle of monoids, the homology groups each decompose as the direct sum of the corresponding homology groups (with appropriate coefficients) of the monoids. In Section 6.3, we prove that if the first factor is a bundle of copies of \mathbb{Z} , only the first two columns of the spectral sequence are nonzero, and the homology of each column is computable via a chain complex very similar to the bar resolution of \mathbb{Z} . In Section 6.4 we restrict to graphs of odometers, and write down an $E^0 \times E^1$ matrix over \mathbb{Z} whose kernel and cokernel, together with the homology of the graph E , compute the homology of the system (Theorem 6.15 and Corollary 6.16).

In Section 7, we consider twisted C^* -algebras associated to matched pairs. Section 7.1 deals with twists by categorical cocycles, and establishes some fundamental results about the associated C^* -algebras: we prove that the generators are all nonzero and give sufficient conditions under which the twisted C^* -algebra of \mathcal{G} embeds in each of $\mathcal{T}C^*(\mathcal{G}, \Lambda; c)$ and $C^*(\mathcal{G}, \Lambda; c)$ in Proposition 7.7; we prove our gauge-invariant uniqueness theorem, Corollary 7.10; and we show that the isomorphism classes of $\mathcal{T}C^*(\mathcal{G}, \Lambda; c)$ and $C^*(\mathcal{G}, \Lambda; c)$ only depend on the cohomology class of c . Section 7.2 describes twists by total cocycles, and shows that these correspond to twists by categorical 2-cocycles via the isomorphism of cohomology induced by our main theorem above (Theorem 7.15).

2. PRELIMINARIES

Throughout this article, \mathcal{C} and \mathcal{D} denote small categories. We identify each \mathcal{C} with its set of morphisms and write $\mathcal{C}^0 \subseteq \mathcal{C}$ for the set of identity morphisms (identified with objects). We write $r, s: \mathcal{C} \rightarrow \mathcal{C}^0$ for the maps assigning to $c \in \mathcal{C}$ (the identity morphisms at) its codomain and domain. For $n \geq 1$, we write \mathcal{C}^n for the set of composable n -tuples in \mathcal{C} and define $r, s: \mathcal{C}^n \rightarrow \mathcal{C}^0$ by $r(c_1, \dots, c_n) = r(c_1)$ and $s(c_1, \dots, c_n) = s(c_n)$. For $x, y \in \mathcal{C}^0$ we write $x\mathcal{C}^n := \{(c_1, \dots, c_n) \mid r(c_1) = x\}$, $\mathcal{C}^n y := \{(c_1, \dots, c_n) \mid s(c_n) = y\}$, and $x\mathcal{C}^n y := x\mathcal{C}^n \cap \mathcal{C}^n y$.

If $\mathcal{C}^0 = \mathcal{D}^0$ we define

$$\mathcal{C} * \mathcal{D} := \mathcal{C} \underset{s}{*} \underset{r}{*} \mathcal{D} = \{(c, d) \in \mathcal{C} \times \mathcal{D} \mid s(c) = r(d)\}.$$

If \mathcal{C} , \mathcal{C}' , and \mathcal{D} have the same objects and $f: \mathcal{C} \rightarrow \mathcal{C}'$ satisfies $s(f(c)) = f(s(c))$, then $f * 1_{\mathcal{D}}: \mathcal{C} * \mathcal{D} \rightarrow \mathcal{C}' * \mathcal{D}$ is the map $(f * 1_{\mathcal{D}})(c, d) := (f(c), d)$. Similarly, if $r(f(c)) = r(c)$, then $1_{\mathcal{D}} * f: \mathcal{D} * \mathcal{C} \rightarrow \mathcal{D} * \mathcal{C}'$ is the map $(1_{\mathcal{D}} * f)(d, c) := (d, f(c))$.

An *action* of a category \mathcal{C} on the left of a set X consists of maps $a: X \rightarrow \mathcal{C}^0$ and $\triangleright: \{(c, x) \mid s(c) = a(x)\} \rightarrow X$ such that $a(x) \triangleright x$ and $(c_1 c_2) \triangleright x = c_1 \triangleright (c_2 \triangleright x)$ for all $(c_1, c_2) \in \mathcal{C}^2$ and $x \in X$ with $s(c_2) = a(x)$. Right actions are defined similarly, and correspond to left actions of the opposite category \mathcal{C}^{op} . If $\mathcal{C}^0 = \mathcal{D}^0$ we only consider left actions for which $a = r: \mathcal{D} \rightarrow \mathcal{C}^0$, so \triangleright is a map $\triangleright: \mathcal{C} * \mathcal{D} \rightarrow \mathcal{D}$. Similarly, we only consider right actions of \mathcal{D} on \mathcal{C} for which $a = s: \mathcal{C} \rightarrow \mathcal{D}^0$.

A *groupoid* \mathcal{G} is a small category in which every morphism $g \in \mathcal{G}$ has an inverse $g^{-1} \in \mathcal{G}$ such that $gg^{-1} = r(g)$ and $g^{-1}g = s(g)$. The set of identity morphisms is called the *unit space* of \mathcal{G} . In this paper, \mathcal{G} always denotes a discrete groupoid.

A *directed graph* is a quadruple $E = (E^0, E^1, r, s)$, consisting of countable sets E^0 of *vertices* and E^1 of *edges*, and maps $r: E^1 \rightarrow E^0$ and $s: E^1 \rightarrow E^0$ called the *range* and *source* maps. For $n \geq 1$, we denote by $E^n := \{\mu = \mu_1 \cdots \mu_n \mid \mu_i \in E^1, s(\mu_i) = r(\mu_{i+1})\}$, the *paths of length n* in the graph E . If $\mu \in E^m$ and $\nu \in E^n$ with $s(\mu) = r(\nu)$, then $\mu\nu := \mu_1 \cdots \mu_m \nu_1 \cdots \nu_n \in E^{m+n}$ is the *concatenation* of μ and ν . The range and source maps extend to E^n : $r(\mu) = r(\mu_1)$ and $s(\mu) = s(\mu_n)$. We regard elements v of E^0 as paths of length 0 with $r(v) = s(v) = v$, and we extend concatenation by the formula $r(\mu)\mu = \mu = \mu s(\mu)$.

The *path category* of a directed graph E is the collection $E^* := \bigsqcup_{n=0}^{\infty} E^n$ of all finite paths in E . The objects of E^* are E^0 , and the range and source maps on the E^n extend to domain and codomain maps $r: E^* \rightarrow E^0$ and $s: E^* \rightarrow E^0$. Composition is concatenation. We use $|\mu|$ to denote the length of $\mu \in E^*$, so $|\mu| = n$ if and only if $\mu \in E^n$.

3. MATCHED PAIRS, ZAPPA-SZÉP PRODUCTS, AND FACTORISATION SYSTEMS

3.1. Matched pairs. In this subsection we introduce matched pairs of small categories and their associated Zappa–Szépproduct categories. We also examine how factorisation rules and strict factorisation systems are related to these constructions.

Definition 3.1. A *matched pair* is a quadruple $(\mathcal{C}, \mathcal{D}, \triangleright, \triangleleft)$ consisting of small categories \mathcal{C}, \mathcal{D} with $\mathcal{C}^0 = \mathcal{D}^0$, a left action $\triangleright: \mathcal{C} * \mathcal{D} \rightarrow \mathcal{D}$ of \mathcal{C} on \mathcal{D} , and a right action $\triangleleft: \mathcal{C} * \mathcal{D} \rightarrow \mathcal{C}$ of \mathcal{D} on \mathcal{C} such that for all $(c_1, c_2, d_1, d_2) \in \mathcal{C}^2 * \mathcal{D}^2$,

$$(MP1) \quad s(c_2 \triangleright d_1) = r(c_2 \triangleleft d_1),$$

$$(MP2) \quad c_2 \triangleright (d_1 d_2) = (c_2 \triangleright d_1)((c_2 \triangleleft d_1) \triangleright d_2), \text{ and}$$

$$(MP3) \quad (c_1 c_2) \triangleleft d = (c_1 \triangleleft (c_2 \triangleright d_1))(c_2 \triangleleft d_1).$$

We often just say that $(\mathcal{C}, \mathcal{D})$ is a matched pair, and suppress the actions $\triangleright, \triangleleft$.

The category MP of matched pairs has matched pairs as objects and morphisms $f = (f^L, f^R): (\mathcal{C}, \mathcal{D}) \rightarrow (\mathcal{C}', \mathcal{D}')$ consisting of pairs of functors $f^L: \mathcal{C} \rightarrow \mathcal{C}'$ and $f^R: \mathcal{D} \rightarrow \mathcal{D}'$ such that for all $(c, d) \in \mathcal{C} * \mathcal{D}$,

$$(i) \quad (f^L(c), f^R(d)) \in \mathcal{C}' * \mathcal{D}',$$

$$(ii) \quad f^L(c) \triangleright f^R(d) = f^R(c \triangleright d), \text{ and}$$

$$(iii) \quad f^L(c) \triangleleft f^R(d) = f^L(c \triangleleft d).$$

Remark 3.2. We are unsure of the provenance of the term *matched pair*. It is used for various related notions: matched pairs of groupoids in [AA05]; and matched pairs of Hopf algebras in [Sin72]. For matched pairs with $\mathcal{C}^0 \neq \mathcal{D}^0$, see [DL23, Definition 2.2].

Given a matched pair $(\mathcal{C}, \mathcal{D})$, we define $r: \mathcal{D} * \mathcal{C} \rightarrow \mathcal{C}^0$ and $s: \mathcal{D} * \mathcal{C} \rightarrow \mathcal{C}^0$ by $r(d, c) := r(d)$ and $s(d, c) := s(c)$.

Definition 3.3. Let \mathcal{C} and \mathcal{D} be small categories with $\mathcal{C}^0 = \mathcal{D}^0$. A *factorisation rule* on $(\mathcal{C}, \mathcal{D})$ is a map $\bowtie: \mathcal{C} * \mathcal{D} \rightarrow \mathcal{D} * \mathcal{C}$ such that

$$(FR1) \quad r(c \bowtie d) = r(c) \text{ and } s(c \bowtie d) = s(d) \text{ for all } (c, d) \in \mathcal{C} * \mathcal{D}, \text{ and}$$

(FR2) if $\mu_{\mathcal{C}}: \mathcal{C}^2 \rightarrow \mathcal{C}$ and $\mu_{\mathcal{D}}: \mathcal{D}^2 \rightarrow \mathcal{D}$ denote the composition maps, then the following diagrams commute:

$$\begin{array}{ccc} \mathcal{C}^2 * \mathcal{D} & \xrightarrow{1_{\mathcal{C}} * \bowtie} & \mathcal{C} * \mathcal{D} * \mathcal{C} & \xrightarrow{\bowtie * 1_{\mathcal{C}}} & \mathcal{D} * \mathcal{C}^2 & & \mathcal{C} * \mathcal{D}^2 & \xrightarrow{\bowtie * 1_{\mathcal{D}}} & \mathcal{D} * \mathcal{C} * \mathcal{D} & \xrightarrow{1_{\mathcal{D}} * \bowtie} & \mathcal{D}^2 * \mathcal{C} \\ \downarrow \mu_{\mathcal{C}} * 1_{\mathcal{D}} & & & & \downarrow 1_{\mathcal{D}} * \mu_{\mathcal{C}} & & \downarrow 1_{\mathcal{C}} * \mu_{\mathcal{D}} & & & & \downarrow \mu_{\mathcal{D}} * 1_{\mathcal{C}} \\ \mathcal{C} * \mathcal{D} & \xrightarrow{\quad \bowtie \quad} & \mathcal{D} * \mathcal{C} & & \mathcal{C} * \mathcal{D} & \xrightarrow{\quad \bowtie \quad} & \mathcal{D} * \mathcal{C} & & & & \mathcal{D} * \mathcal{C} \end{array}$$

The name *factorisation rule* becomes clear in the context of Zappa–Szép products (Definition 3.6). Matched pairs and factorisation rules are equivalent in the following sense.

Lemma 3.4. *Let \mathcal{C} and \mathcal{D} be small categories with the same object set. If $(\mathcal{C}, \mathcal{D}, \triangleright, \triangleleft)$ is a matched pair, then the formula*

$$c \bowtie d := (c \triangleright d, c \triangleleft d) \tag{3.1}$$

determines a factorisation rule $\bowtie: \mathcal{C} * \mathcal{D} \rightarrow \mathcal{D} * \mathcal{C}$. Conversely, if $\bowtie: \mathcal{C} * \mathcal{D} \rightarrow \mathcal{D} * \mathcal{C}$ is a factorisation rule and $p_{\mathcal{D}}: \mathcal{D} * \mathcal{C} \rightarrow \mathcal{D}$ and $p_{\mathcal{C}}: \mathcal{D} * \mathcal{C} \rightarrow \mathcal{C}$ are the coordinate projections, then $\triangleright: \mathcal{C} * \mathcal{D} \rightarrow \mathcal{D}$ and $\triangleleft: \mathcal{C} * \mathcal{D} \rightarrow \mathcal{C}$ given by

$$c \triangleright d := p_{\mathcal{D}}(c \bowtie d) \quad \text{and} \quad c \triangleleft d := p_{\mathcal{C}}(c \bowtie d) \tag{3.2}$$

make $(\mathcal{C}, \mathcal{D}, \triangleright, \triangleleft)$ a matched pair.

Proof. First suppose that $\bowtie: \mathcal{C} * \mathcal{D} \rightarrow \mathcal{D} * \mathcal{C}$ is a factorisation rule. Define $\triangleright, \triangleleft$ by (3.2). Since $c \bowtie d \in \mathcal{D} * \mathcal{C}$, (MP1) holds. The left-hand diagram of (FR2) implies that

$$((c_1 c_2) \triangleright d, (c_1 c_2) \triangleleft d) = (c_1 c_2) \bowtie d = (c_1 \triangleright (c_2 \triangleright d), (c_1 \triangleleft (c_2 \triangleright d))(c_2 \triangleleft d)),$$

so \triangleright is a left action and (MP3) holds. Symmetrically, \triangleleft is a right action and (MP2) holds.

Now suppose that $(\mathcal{C}, \mathcal{D}, \triangleright, \triangleleft)$ is a matched pair and define $\bowtie: \mathcal{C} * \mathcal{D} \rightarrow \mathcal{D} * \mathcal{C}$ by (3.1). Then $c \bowtie d \in \mathcal{D} * \mathcal{C}$ by (MP1). Since $\triangleright, \triangleleft$ are actions $r(c \triangleright d) = r(c)$ and $s(c \triangleleft d) = s(d)$, giving (FR1). For (FR2) we use (MP3) at the second equality to compute,

$$\begin{aligned} (c_1 c_2) \bowtie d &= (c_1 c_2 \triangleright d, c_1 c_2 \triangleleft d) = (c_1 \triangleright (c_2 \triangleright d), (c_1 \triangleleft (c_2 \triangleright d))(c_2 \triangleleft d)) \\ &= (1_{\mathcal{D}} * \mu_{\mathcal{C}}) \circ (\bowtie * 1_{\mathcal{C}}) \circ (1_{\mathcal{C}} * \bowtie)(c_1, c_2, d), \end{aligned}$$

and symmetrically $c \bowtie (d_1 d_2) = (\mu_{\mathcal{D}} * 1_{\mathcal{C}}) \circ (1_{\mathcal{D}} * \bowtie) \circ (\bowtie \circ 1_{\mathcal{D}})(c_1, d_2, d_2)$. \square

We use Lemma 3.4 without comment to move between matched pairs and factorisation rules. Importantly, (MP1)–(MP3) give the fibre product $\mathcal{D} * \mathcal{C}$ the structure of a category.

Lemma 3.5. *Suppose that $(\mathcal{C}, \mathcal{D})$ is a matched pair and let $\mu_{\mathcal{C}}$ and $\mu_{\mathcal{D}}$ denote the composition maps on \mathcal{C} and \mathcal{D} respectively. Define $\mu_{\bowtie}: (\mathcal{D} * \mathcal{C})^2 \rightarrow \mathcal{D} * \mathcal{C}$ by*

$$\mu_{\bowtie} := (\mu_{\mathcal{D}} * \mu_{\mathcal{C}}) \circ (1_{\mathcal{D}} * \bowtie * 1_{\mathcal{C}}).$$

Then for $(d_1, c_1), (d_2, c_2) \in \mathcal{D} * \mathcal{C}$ such that $s(c_1) = r(d_2)$,

$$\mu_{\bowtie}((d_1, c_1), (d_2, c_2)) = (d_1(c_1 \triangleright d_2), (c_1 \triangleleft d_2)c_2). \tag{3.3}$$

Moreover, $\mathcal{D} * \mathcal{C}$ is a small category with $(\mathcal{D} * \mathcal{C})^0 = \mathcal{C}^0 = \mathcal{D}^0$, $r(d, c) := r(d)$, $s(d, c) := s(c)$, and composition $(d_1, c_1)(d_2, c_2) := \mu_{\bowtie}((d_1, c_1), (d_2, c_2))$. The maps $\iota_{\mathcal{C}}: \mathcal{C} \rightarrow \mathcal{D} * \mathcal{C}$ and $\iota_{\mathcal{D}}: \mathcal{D} \rightarrow \mathcal{D} * \mathcal{C}$ defined by $\iota_{\mathcal{C}}(c) = (r(c), c)$ and $\iota_{\mathcal{D}}(d) = (d, s(d))$ are faithful functors.

Proof. Equation (3.3) follows from (3.1). For associativity, we calculate:

$$\begin{aligned}
& \mu_{\bowtie}((d_1, c_1), \mu_{\bowtie}((d_2, c_2), (d_3, c_3))) \\
&= \mu_{\bowtie}((d_1, c_1), (d_2(c_2 \triangleright d_3), (c_2 \triangleleft d_3)c_3)) \\
&= (d_1(c_1 \triangleright (d_2(c_2 \triangleright d_3))), c_1 \triangleleft (d_2(c_2 \triangleright d_3))(c_2 \triangleleft d_3)c_3) \\
&= (d_1(c_1 \triangleright d_2)((c_1 \triangleleft d_2)c_2 \triangleright d_3), ((c_1 \triangleleft d_2) \triangleleft (c_2 \triangleright d_3))(c_2 \triangleleft d_3)c_3) \\
&= (d_1(c_1 \triangleright d_2)((c_1 \triangleleft d_2)c_2 \triangleright d_3), (c_1 \triangleleft d_2(c_2 \triangleright d_3))(c_2 \triangleleft d_3)c_3) \\
&= \mu_{\bowtie}(d_1(c_1 \triangleright d_2), (c_1 \triangleleft d_2)c_2, d_3, c_3) \\
&= \mu_{\bowtie}(\mu_{\bowtie}((d_1, c_1), (d_2, c_2)), (d_3, c_3)).
\end{aligned}$$

For functoriality of $\iota_{\mathcal{C}}$ we calculate

$$\mu_{\bowtie}(\iota_{\mathcal{C}}(c_1), \iota_{\mathcal{C}}(c_2)) = (r(c_1)(c_1 \triangleright s(c_1)), (c_1 \triangleleft s(c_1))c_2) = (r(c_1), c_1 c_2) = \iota_{\mathcal{C}}(c_1 c_2).$$

Functoriality of $\iota_{\mathcal{D}}$ follows analogously. Faithfulness is clear. \square

Definition 3.6. We call the small category $\mathcal{D} * \mathcal{C}$ with the composition μ_{\bowtie} of Lemma 3.5 the *Zappa–Szép product* of \mathcal{C} and \mathcal{D} , and denote it $\mathcal{C} \bowtie \mathcal{D}$.

We identify \mathcal{C} and \mathcal{D} with the subcategories $\iota_{\mathcal{C}}(\mathcal{C})$ and $\iota_{\mathcal{D}}(\mathcal{D})$ of $\mathcal{C} \bowtie \mathcal{D}$. In particular, for $(d, c) \in \mathcal{D} * \mathcal{C}$ we write $dc := (d, c) \in \mathcal{C} \bowtie \mathcal{D}$. So for $c_i \in \mathcal{C}$ and $d_i \in \mathcal{D}$,

$$d_1 c_1 d_2 c_2 = d_1(c_1 \triangleright d_2)(c_1 \triangleleft d_2)c_2.$$

Example 3.7. If \mathcal{C} is a small category, then $(\mathcal{C}, \mathcal{C}^0)$ is a matched pair with actions $c \triangleright s(c) = r(c)$ and $c \triangleleft s(c) = c$. We have $\mathcal{C} \bowtie \mathcal{C}^0 \cong \mathcal{C} \cong \mathcal{C}^0 \bowtie \mathcal{C}$.

Example 3.8. Suppose that G and H are groups and suppose that $(G, H, \triangleright, \triangleleft)$ is a matched pair. Then $G \bowtie H$ is the Zappa–Szép product of G and H from [Zap42, Sze50]. If \triangleleft is the trivial right action of H on G , then for $h_i \in H$ and $g_i \in G$, we have

$$(h_1, g_1)(h_2, g_2) = (h_1(g_1 \triangleright h_2), g_1 g_2),$$

so $G \bowtie H$ is the semidirect product $G \rtimes H$.

Zappa–Szép products have the following universal property.

Proposition 3.9. *Suppose that $(\mathcal{C}, \mathcal{D})$ is a matched pair, let \mathcal{A} be a small category such that $\mathcal{A}^0 = \mathcal{C}^0 = \mathcal{D}^0$, and suppose that $j_{\mathcal{C}}: \mathcal{C} \rightarrow \mathcal{A}$ and $j_{\mathcal{D}}: \mathcal{D} \rightarrow \mathcal{A}$ are functors satisfying*

$$j_{\mathcal{C}}(c)j_{\mathcal{D}}(d) = j_{\mathcal{D}}(c \triangleright d)j_{\mathcal{C}}(c \triangleleft d) \tag{3.4}$$

for all $(c, d) \in \mathcal{C} * \mathcal{D}$. Then there exists a unique functor $j_{\mathcal{C}} \bowtie j_{\mathcal{D}}: \mathcal{C} \bowtie \mathcal{D} \rightarrow \mathcal{A}$ such that $(j_{\mathcal{C}} \bowtie j_{\mathcal{D}}) \circ \iota_{\mathcal{C}} = j_{\mathcal{C}}$ and $(j_{\mathcal{C}} \bowtie j_{\mathcal{D}}) \circ \iota_{\mathcal{D}} = j_{\mathcal{D}}$. If \mathcal{B} is a small category with $\mathcal{B}^0 = \mathcal{C}^0$ and $k_{\mathcal{C}}: \mathcal{C} \rightarrow \mathcal{B}$ and $k_{\mathcal{D}}: \mathcal{D} \rightarrow \mathcal{B}$ are functors satisfying (3.4) and with the same universal property, then $k_{\mathcal{C}} \bowtie k_{\mathcal{D}}$ is an isomorphism $\mathcal{C} \bowtie \mathcal{D} \rightarrow \mathcal{B}$.

Proof. Define $j_{\mathcal{C}} \bowtie j_{\mathcal{D}}: \mathcal{C} \bowtie \mathcal{D} \rightarrow \mathcal{A}$ by $(j_{\mathcal{C}} \bowtie j_{\mathcal{D}})(d, c) = j_{\mathcal{D}}(d)j_{\mathcal{C}}(c)$. Clearly, $(j_{\mathcal{C}} \bowtie j_{\mathcal{D}}) \circ \iota_{\mathcal{C}} = j_{\mathcal{C}}$ and $(j_{\mathcal{C}} \bowtie j_{\mathcal{D}}) \circ \iota_{\mathcal{D}} = j_{\mathcal{D}}$. For functoriality we compute,

$$\begin{aligned}
(j_{\mathcal{C}} \bowtie j_{\mathcal{D}})(d_1 c_1 d_2 c_2) &= j_{\mathcal{D}}(d_1)j_{\mathcal{D}}(c_1 \triangleright d_2)j_{\mathcal{C}}(c_1 \triangleleft d_2)j_{\mathcal{C}}(c_2) \\
&= j_{\mathcal{D}}(d_1)j_{\mathcal{C}}(c_1)j_{\mathcal{D}}(d_2)j_{\mathcal{C}}(c_2) = (j_{\mathcal{C}} \bowtie j_{\mathcal{D}})(d_1 c_1) (j_{\mathcal{C}} \bowtie j_{\mathcal{D}})(d_2 c_2).
\end{aligned}$$

If $f: \mathcal{C} \bowtie \mathcal{D} \rightarrow \mathcal{A}$ is a functor satisfying $f \circ \iota_{\mathcal{C}} = j_{\mathcal{C}}$ and $f \circ \iota_{\mathcal{D}} = j_{\mathcal{D}}$, then $f(dc) = f(\iota_{\mathcal{D}}(d)\iota_{\mathcal{C}}(c)) = j_{\mathcal{D}}(d)j_{\mathcal{C}}(c) = (j_{\mathcal{C}} \bowtie j_{\mathcal{D}})(dc)$. If $(\mathcal{B}, k_{\mathcal{C}}, k_{\mathcal{D}})$ has the same universal property, then that universal property applied to $\iota_{\mathcal{C}}$ and $\iota_{\mathcal{D}}$ yields a functor $\theta: \mathcal{B} \rightarrow \mathcal{C} \bowtie \mathcal{D}$ inverse to $k_{\mathcal{C}} \bowtie k_{\mathcal{D}}$. \square

Corollary 3.10. *The assignment $(\mathcal{C}, \mathcal{D}) \mapsto \mathcal{C} \bowtie \mathcal{D}$ is functorial: given a matched-pair morphism $(h^L, h^R): (\mathcal{C}, \mathcal{D}) \rightarrow (\mathcal{C}', \mathcal{D}')$, there is a functor $h: \mathcal{C} \bowtie \mathcal{D} \rightarrow \mathcal{C}' \bowtie \mathcal{D}'$ such that $h(dc) = h^R(d)h^L(c)$ for all $(d, c) \in \mathcal{D} * \mathcal{C}$. This functor satisfies $h \circ \iota_{\mathcal{C}} = \iota_{\mathcal{C}'} \circ h^L$ and $h \circ \iota_{\mathcal{D}} = \iota_{\mathcal{D}'} \circ h^R$. Conversely, if $h: \mathcal{C} \bowtie \mathcal{D} \rightarrow \mathcal{C}' \bowtie \mathcal{D}'$ is a functor such that $h(\mathcal{C}) \subseteq \mathcal{C}'$ and $h(\mathcal{D}) \subseteq \mathcal{D}'$, then $(h|_{\mathcal{C}}, h|_{\mathcal{D}}): (\mathcal{C}, \mathcal{D}) \rightarrow (\mathcal{C}', \mathcal{D}')$ is a matched pair morphism.*

Proof. To obtain h , apply Proposition 3.9 to $\iota_{\mathcal{C}} \circ h^L$ and $\iota_{\mathcal{D}} \circ h^R$. The second statement follows from a one-line calculation. \square

As with groups, we can take either an “external” or an “internal” view of Zappa–Szép products of categories. Recall that a *wide subcategory* of a category \mathcal{E} is a subcategory containing \mathcal{E}^0 .

Definition 3.11. A *strict factorisation system* for a category \mathcal{E} is a pair $[\mathcal{D}, \mathcal{C}]$ of wide subcategories of \mathcal{E} such that for every $e \in \mathcal{E}$ there are unique $d \in \mathcal{D}$ and $c \in \mathcal{C}$ satisfying $e = dc$.

Remark 3.12. Factorisation systems are related to distributive laws on monads: strict factorisation systems are equivalent to distributive laws in the category of spans [RW02, Theorem 3.8].

Proposition 3.13. *Let $(\mathcal{C}, \mathcal{D})$ be a matched pair. Then $[\mathcal{D}, \mathcal{C}]$ is a strict factorisation system for $\mathcal{C} \bowtie \mathcal{D}$. Conversely, let $[\mathcal{D}, \mathcal{C}]$ be a strict factorisation system for a small category \mathcal{E} . For $(c, d) \in \mathcal{C} * \mathcal{D}$, let $c \triangleright d \in \mathcal{D}$ and $c \triangleleft d \in \mathcal{C}$ be the unique elements such that $cd = (c \triangleright d)(c \triangleleft d)$. Then $(\mathcal{C}, \mathcal{D}, \triangleright, \triangleleft)$ is a matched pair and $(d, c) \mapsto dc$ is an isomorphism $\mathcal{C} \bowtie \mathcal{D} \cong \mathcal{E}$.*

Proof. Suppose that $(\mathcal{C}, \mathcal{D})$ is a matched pair. Since $\mathcal{C} \bowtie \mathcal{D} = \mathcal{D} * \mathcal{C}$ as sets, each $e \in \mathcal{C} \bowtie \mathcal{D}$ factors uniquely as $e = dc$.

Conversely, suppose that $[\mathcal{D}, \mathcal{C}]$ is a unique factorisation system for \mathcal{E} and fix $(c_1, c_2, d_1, d_2) \in \mathcal{C}^2 * \mathcal{D}^2$. Let $d', d'', d''' \in \mathcal{D}$ and $c', c'', c''' \in \mathcal{C}$ be the unique elements such that $c_1c_2d_1 = d'c'$, $c_2d_1 = d''c''$, and $c_1d'' = d'''c'''$. Then $d'''(c'''c'') = c_1d''c'' = c_1c_2d_1 = d'c'$, so uniqueness of factorisations gives $(c_1c_2) \triangleright d_1 = d' = d''' = c_1 \triangleright d'' = c_1 \triangleright (c_2 \triangleright d_1)$. So \triangleright is an action of \mathcal{C} on \mathcal{D} .

Now let $d', d'', d''' \in \mathcal{D}$ and $c', c'', c''' \in \mathcal{C}$ be the unique elements such that $c_2d_1d_2 = d'c'$, $c_2d_1 = d''c''$, and $c''d_2 = d'''c'''$. Then $d''d'''c''' = d'c'$, so uniqueness of factorisations gives $c_2 \triangleright (d_1d_2) = d' = d''d''' = (c_2 \triangleright d_1)(c'' \triangleright d_2) = (c_2 \triangleright d_1)((c_2 \triangleleft d_1) \triangleright d_2)$, verifying (MP2).

Symmetrically, \triangleleft defines a right action of \mathcal{D} on \mathcal{C} satisfying (MP3). Condition (MP1) follows from the composition laws in \mathcal{E} . \square

Remark 3.14. Proposition 3.13 says that the internal and external views of Zappa–Szép products are equivalent. Given a matched pair $(\mathcal{C}, \mathcal{D})$ we can equivalently: (a) build the concrete product $\mathcal{C} \bowtie \mathcal{D}$; or (b) say that \mathcal{E} is a Zappa–Szép product if it contains copies of \mathcal{D} and \mathcal{C} as wide subcategories such that $[\mathcal{D}, \mathcal{C}]$ is a strict factorisation system implementing the given actions.

For C^* -algebraic representations à la Spielberg [Spe20] it is important to know when a small category \mathcal{C} is *left cancellative* in the sense that if $c_1c_2 = c_1c_3$, then $c_2 = c_3$. The following lemma provides a sufficient condition under which Zappa–Szép products are left cancellative.

Lemma 3.15. *If $(\mathcal{C}, \mathcal{D})$ is matched pair in which \mathcal{C} and \mathcal{D} are both left cancellative and for each $c \in \mathcal{C}$ the map $c \triangleright \cdot: s(c)\mathcal{D} \rightarrow r(c)\mathcal{D}$ is injective, then $\mathcal{C} \bowtie \mathcal{D}$ is left cancellative.*

Proof. Suppose that $d_1c_1, d_2c_2 \in \mathcal{C} \bowtie \mathcal{D}$ satisfy $d_1c_1d_2c_2 = d_1c_1d_3c_3$. Then $d_1(c_1 \triangleright d_2) = d_1(c_1 \triangleright d_3)$ in \mathcal{D} and $(c_1 \triangleleft d_2)c_2 = (c_1 \triangleleft d_3)c_3$ in \mathcal{C} . Since \mathcal{D} is left cancellative, $c_1 \triangleright d_2 = c_1 \triangleright d_3$, and so injectivity of the left action gives $d_2 = d_3$. Consequently, $(c_1 \triangleleft d_2)c_2 = (c_1 \triangleleft d_2)c_3$. Left cancellation in \mathcal{C} implies that $c_2 = c_3$, so $\mathcal{C} \bowtie \mathcal{D}$ is left cancellative. \square

Example 3.16. If \mathcal{C} is a groupoid then it acts cancellatively on both itself and \mathcal{D} because it has inverses. So $\mathcal{C} \bowtie \mathcal{D}$ is left-cancellative whenever \mathcal{D} is.

3.2. Extending matched pairs to composable tuples. We define homology for matched pairs in terms of associated categories of composable tuple, so it is important to understand how \triangleright and \triangleleft extend to these categories.

Definition 3.17. The *free category* (or *path category*) of a small category \mathcal{C} is the category \mathcal{C}^* with morphisms $\bigcup_{k \geq 0} \mathcal{C}^k$, identity morphisms \mathcal{C}^0 , and composition (for non-identity morphisms) given by concatenation.

Remark 3.18. There is a subtlety here. The set \mathcal{C}^1 of 1-tuples in \mathcal{C}^* contains the 1-tuples $\{(v) \mid v \in \mathcal{C}^0\}$, but this disjoint from $\mathcal{C}^0 \subseteq \mathcal{C}^*$. This is reflected in the composition law: for $v \in \mathcal{C}^0$ and $(c_1, \dots, c_k) \in \mathcal{C}^k$ with $r(c_1) = v$, we have $v(c_1, \dots, c_k) = (c_1, \dots, c_k)$ while $(v)(c_1, \dots, c_k) = (v, c_1, \dots, c_k)$.

Lemma 3.19. Let $(\mathcal{C}, \mathcal{D})$ be a matched pair. Define $\bowtie_n: \mathcal{C} * \mathcal{D}^n \rightarrow \mathcal{D}^n * \mathcal{C}$ inductively by $\bowtie_1 := \bowtie$, and

$$\bowtie_n := (1_{\mathcal{D}^{n-1}} * \bowtie) \circ (1_{\mathcal{C}} * \bowtie_{n-1})$$

for $n \geq 2$. Define $\bowtie_*: \mathcal{C} * \mathcal{D}^* \rightarrow \mathcal{D}^* * \mathcal{C}$ by $\bowtie_*|_{\mathcal{C} * \mathcal{D}^n} := \bowtie_n$. Then

(i) for each $n \geq 1$ and $1 \leq p < n$,

$$\bowtie_n = (1_{\mathcal{D}^{n-p}} * \bowtie_p) \circ (\bowtie_{n-p} * 1_{\mathcal{D}^p}), \quad (3.5)$$

(ii) \bowtie_* is a factorisation rule, and

(iii) if $\mu_{\mathcal{D}}: \mathcal{D}^* \rightarrow \mathcal{D}$ is the map $\mu_{\mathcal{D}}(d_1, d_2, \dots, d_n) = d_1d_2 \cdots d_n$, then $(1_{\mathcal{C}}, \mu_{\mathcal{D}}): (\mathcal{C}, \mathcal{D}^*) \rightarrow (\mathcal{C}, \mathcal{D})$ is a matched-pair morphism.

Proof. (i) When $n = 1$, we have $p = 1$, so (3.5) is vacuous. For $n \geq 2$, equation (3.5) holds for $p = 1$ by definition of \bowtie_n . Fix $n_0 \geq 2$ and $1 \leq p_0 \leq n_0$, and suppose inductively, that (3.5) holds for all $n < n_0$ and $1 \leq p < p_0$. In the diagram

$$\begin{array}{ccccc} \mathcal{C} * \mathcal{D}^{n_0} & \xrightarrow{\bowtie_{n_0-p_0} * 1_{\mathcal{D}^{p_0}}} & \mathcal{D}^{n_0-p_0} * \mathcal{C} * \mathcal{D}^{p_0} & \xrightarrow{1_{\mathcal{D}^{n_0-p_0}} * \bowtie_{p_0}} & \mathcal{D}^{n_0} * \mathcal{C} \\ & \searrow \bowtie_{n_0-p_0+1} * 1_{\mathcal{D}^{p_0-1}} & \downarrow 1_{\mathcal{D}^{n_0-p_0}} * \bowtie * 1_{\mathcal{D}^{p_0-1}} & \nearrow 1_{\mathcal{D}^{n_0-p_0+1}} * \bowtie_{p_0-1} & \\ & & \mathcal{D}^{n_0-p_0+1} * \mathcal{C} * \mathcal{D}^{p_0-1} & & \end{array}$$

the left-hand triangle commutes by the inductive definition of $\bowtie_{n_0-p_0}$, and the right-hand triangle commutes by induction since $p_0 < n_0$. Since $p_0 - 1 < p_0$, the composition of the maps along the bottom of the triangle is \bowtie_{n_0} by induction. So (3.5) holds for all n and p .

(ii) A routine induction verifies (FR1). To see that the first diagram of (FR2) for \bowtie_* commutes, consider the following diagram.

$$\begin{array}{ccccc}
 \mathcal{C}^2 * \mathcal{D}^{n+1} & \xrightarrow{1_{\mathcal{C}} * \bowtie_{n+1}} & \mathcal{C} * \mathcal{D}^{n+1} * \mathcal{C} & \xrightarrow{\bowtie_{n+1} * 1_{\mathcal{C}}} & \mathcal{D}^{n+1} * \mathcal{C}^2 \\
 \downarrow \mu_{\mathcal{C}} * 1_{\mathcal{D}^{n+1}} & \searrow 1_{\mathcal{C}} * \bowtie_n * 1_{\mathcal{D}} & \nearrow 1_{\mathcal{C} * \mathcal{D}^n} * \bowtie & \searrow 1_{\mathcal{D}^n} * \bowtie & \downarrow 1_{\mathcal{D}^{n+1}} * \mu_{\mathcal{C}} \\
 & \mathcal{C} * \mathcal{D}^n * \mathcal{C} * \mathcal{D} & & \mathcal{D}^n * \mathcal{C} * \mathcal{D} * \mathcal{C} & \\
 & \searrow \bowtie_n * 1_{\mathcal{C} * \mathcal{D}} & & \nearrow 1_{\mathcal{D}^n} * \mathcal{C} * \bowtie & \\
 & \mathcal{D}^n * \mathcal{C}^2 * \mathcal{D} & & & \\
 \downarrow 1_{\mathcal{D}^n} * \mu_{\mathcal{C}} * 1_{\mathcal{D}} & & & & \downarrow 1_{\mathcal{D}^{n+1}} * \mu_{\mathcal{C}} \\
 \mathcal{C} * \mathcal{D}^{n+1} & \xrightarrow{\bowtie_n * 1_{\mathcal{D}}} & \mathcal{D}^n * \mathcal{C} * \mathcal{D} & \xrightarrow{1_{\mathcal{D}^n} * \bowtie} & \mathcal{D}^{n+1} * \mathcal{C}
 \end{array}$$

The central diamond clearly commutes, the top-left and top-right triangles commute by the definition of \bowtie_{n+1} , and the bottom-right pentagon commutes by (FR2) for \bowtie . The composition of the maps along the bottom row of the diagram is \bowtie_{n+1} by definition. So the whole diagram commutes if and only if the bottom-left pentagon commutes. An induction now shows that the first diagram of (FR2) commutes for \bowtie_* .

For $m, n \neq 0$, the multiplication map $\mu_{\mathcal{D}^*} : \mathcal{D}^m * \mathcal{D}^n \rightarrow \mathcal{D}^{m+n}$ is the obvious bijection. So the second diagram of (FR2) commutes by (i). Hence, \bowtie_* is a factorisation rule.

(iii) By Lemma 3.4 and Corollary 3.10, it suffices to show that $(\mu_{\mathcal{D}} * 1_{\mathcal{C}}) \circ \bowtie_n : \mathcal{C} * \mathcal{D}^n \rightarrow \mathcal{D} * \mathcal{C}$ and $\bowtie \circ (1_{\mathcal{C}} * \mu_{\mathcal{D}}) : \mathcal{C} * \mathcal{D}^n \rightarrow \mathcal{D} * \mathcal{C}$ are equal for all n . For $n = 1$ this is trivial, so suppose equality holds for $n - 1$, and consider the following diagram.

$$\begin{array}{ccccc}
 \mathcal{C} * \mathcal{D}^n & \xrightarrow{\bowtie_{n-1} * 1_{\mathcal{D}}} & \mathcal{D}^{n-1} * \mathcal{C} * \mathcal{D} & \xrightarrow{1_{\mathcal{D}^{n-1}} * \bowtie} & \mathcal{D}^n * \mathcal{C} \\
 1_{\mathcal{C}} * \mu_{\mathcal{D}} * 1_{\mathcal{D}} \downarrow & & \mu_{\mathcal{D}} * 1_{\mathcal{C} * \mathcal{D}} \downarrow & & \downarrow \mu_{\mathcal{D}} * 1_{\mathcal{D} * \mathcal{C}} \\
 \mathcal{C} * \mathcal{D}^2 & \xrightarrow{\bowtie * 1_{\mathcal{D}}} & \mathcal{D} * \mathcal{C} * \mathcal{D} & \xrightarrow{1_{\mathcal{D}} * \bowtie} & \mathcal{D}^2 * \mathcal{C} \\
 1_{\mathcal{C}} * \mu_{\mathcal{D}} \downarrow & & & & \downarrow \mu_{\mathcal{D}} * 1_{\mathcal{C}} \\
 \mathcal{C} * \mathcal{D} & \xrightarrow{\quad \quad \quad \bowtie \quad \quad \quad} & & & \mathcal{D} * \mathcal{C}
 \end{array}$$

The bottom pentagon commutes by (FR2) for \bowtie . The top-right square clearly commutes. The top left square commutes by the inductive hypothesis, and so the whole diagram commutes. The composition along the top row is equal to \bowtie_n , and the composition along the left and right columns are $1_{\mathcal{C}} * \mu_{\mathcal{D}}$ and $\mu_{\mathcal{D}} * 1_{\mathcal{C}}$. So $(\mu_{\mathcal{D}} * 1_{\mathcal{C}}) \circ \bowtie_n = \bowtie \circ (1_{\mathcal{C}} * \mu_{\mathcal{D}})$. \square

Lemmas 3.19 and 3.4 imply that $(\mathcal{C}, \mathcal{D}^*)$ is a matched pair. The left action of \mathcal{C} on \mathcal{D}^k is given explicitly by

$$\begin{aligned}
 c \triangleright (d_1, \dots, d_k) &:= (c \triangleright d_1, (c \triangleleft d_1) \triangleright (d_2, \dots, d_k)) \\
 &= (c_1 \triangleright d_1, (c_1 \triangleleft d_1) \triangleright d_2, (c_1 \triangleleft (d_1 d_2)) \triangleright d_3, \dots, (c_1 \triangleleft (d_1 \cdots d_{k-1})) \triangleright d_k).
 \end{aligned}$$

and the right action of \mathcal{D}^k on \mathcal{C} is given by $c \triangleleft (d_1, \dots, d_k) = c \triangleleft (d_1 \cdots d_k)$.

We can also define ${}_n \bowtie : \mathcal{C}^n * \mathcal{D} \rightarrow \mathcal{D} * \mathcal{C}^n$ inductively by ${}_1 \bowtie := \bowtie$, and

$$(c_1, \dots, c_n) {}_n \bowtie d = (\bowtie * 1_{\mathcal{C}^{n-1}})((c_2, \dots, c_n) {}_{n-1} \bowtie d),$$

and then ${}_* \bowtie : \mathcal{C}^* * \mathcal{D} \rightarrow \mathcal{D} * \mathcal{C}^*$ by ${}_* \bowtie|_{\mathcal{C}^n * \mathcal{D}} = {}_n \bowtie$. Lemma 3.19 applied to opposite categories implies that ${}_* \bowtie$ is a factorisation rule with properties analogous to those of \bowtie_* . The left action

of \mathcal{C}^k on \mathcal{D} is given by $(c_1, \dots, c_k) \triangleright d = (c_1 \cdots c_k) \triangleright d$ and the right action of \mathcal{D} on \mathcal{C}^k is given by

$$\begin{aligned} (c_1, \dots, c_k) \triangleleft d &= ((c_1, \dots, c_{k-1}) \triangleleft (c_k \triangleright d), c_k \triangleleft d) \\ &= (c_1 \triangleleft ((c_2 \cdots c_k) \triangleright d), \dots, c_{k-2} \triangleleft ((c_{k-1} c_k) \triangleright d), c_{k-1} \triangleleft (c_k \triangleright d), c_k \triangleleft d). \end{aligned}$$

Since $(\mathcal{C}^*, \mathcal{D})$ is itself a matched pair, $(\mathcal{C}^*, \mathcal{D}^*)$ can also be equipped with the structure of a matched pair via Lemma 3.19.

Proposition 3.20. *Let $(\mathcal{C}, \mathcal{D})$ be a matched pair. For $m, n \geq 1$ define $\bowtie_{m,n}: \mathcal{C}^m * \mathcal{D}^n \rightarrow \mathcal{D}^n * \mathcal{C}^m$ inductively by $\bowtie_{1,n} := \bowtie_n: \mathcal{C} * \mathcal{D}^n \rightarrow \mathcal{D}^n * \mathcal{C}$ and*

$$\bowtie_{m,n} := (\bowtie_n * 1_{\mathcal{C}^{m-1}}) \circ (1_{\mathcal{C}^*} \bowtie_{m-1,n}).$$

Define $\bowtie_{*,*}: \mathcal{C}^* * \mathcal{D}^* \rightarrow \mathcal{D}^* * \mathcal{C}^*$ by $\bowtie_{*,*}|_{\mathcal{C}^m * \mathcal{D}^n} = \bowtie_{m,n}$. Then

- (i) $\bowtie_{*,*}$ is a factorisation rule, and
- (ii) $(\mu_{\mathcal{C}}, \mu_{\mathcal{D}}): (\mathcal{C}^*, \mathcal{D}^*) \rightarrow (\mathcal{C}, \mathcal{D})$ is a matched-pair morphism.

Proof. That $(\mathcal{C}, \mathcal{D}^*)$ is a matched pair, together with an additional application of Lemma 3.19, gives (i). Two applications of Lemma 3.19(iii) give (ii). \square

We often write $\bowtie: \mathcal{C}^* * \mathcal{D}^* \rightarrow \mathcal{D}^* * \mathcal{C}^*$ for the map $\bowtie_{*,*}$ of Proposition 3.20, which implies that $\mathcal{C}^* \bowtie \mathcal{D}^*$ is a category with strict factorisation system $[\mathcal{D}^*, \mathcal{C}^*]$. We identify $\mathcal{C}^* \bowtie \mathcal{D}^*$ as a set with $\mathcal{D}^* * \mathcal{C}^*$. For each composable k -tuple $\gamma \in \mathcal{C}^* \bowtie \mathcal{D}^*$ there exist $p_i, q_i \geq 0$ such that $\gamma \in \mathcal{D}^{p_1} * \mathcal{C}^{q_1} * \cdots * \mathcal{D}^{p_k} * \mathcal{C}^{q_k}$; its product belongs to $\mathcal{D}^{p_1 + \cdots + p_k} * \mathcal{C}^{q_1 + \cdots + q_k}$.

The map $\bowtie_{m,n}$ can be computed in any order in the following sense.

Corollary 3.21. *Let $(\mathcal{C}, \mathcal{D})$ be a matched pair. For each $1 \leq p < m$ and $1 \leq q < n$, the diagram*

$$\begin{array}{ccc} \mathcal{C}^m * \mathcal{D}^n & \xrightarrow{\quad \bowtie_{m,n} \quad} & \mathcal{D}^n * \mathcal{C}^m \\ \downarrow 1_{\mathcal{C}^{m-p}} * \bowtie_{p,q} * 1_{\mathcal{D}^{n-q}} & & \uparrow 1_{\mathcal{D}^q} * \bowtie_{m-p,n-q} * 1_{\mathcal{C}^p} \\ \mathcal{C}^{m-p} * \mathcal{D}^q * \mathcal{C}^p * \mathcal{D}^{n-q} & \xrightarrow{\quad \bowtie_{m-p,q} * \bowtie_{p,n-q} \quad} & \mathcal{D}^q * \mathcal{C}^{m-p} * \mathcal{D}^{n-q} * \mathcal{C}^p \end{array} \quad (3.6)$$

commutes.

Proof. Elements of $\mathcal{C}^m * \mathcal{D}^n = \mathcal{C}^{m-p} * \mathcal{C}^p * \mathcal{D}^q * \mathcal{D}^{n-q}$ may be considered as composable 4-tuples in $\mathcal{C}^* \bowtie \mathcal{D}^*$. Since the 3-map composition around the bottom of (3.6) is an iterated product in $\mathcal{C}^* \bowtie \mathcal{D}^*$, uniqueness of factorisation implies that the diagram commutes. \square

Corollary 3.21 gives $\bowtie_{m,n} = (1_{\mathcal{D}^{n-1}} * \bowtie_{m,1}) \circ (\bowtie_{m,n-1} * 1_{\mathcal{D}})$. So we could also have applied Lemma 3.19 to $(\mathcal{C}, \mathcal{D}^*)$ to obtain the matched-pair structure on $(\mathcal{C}^*, \mathcal{D}^*)$ of Proposition 3.20.

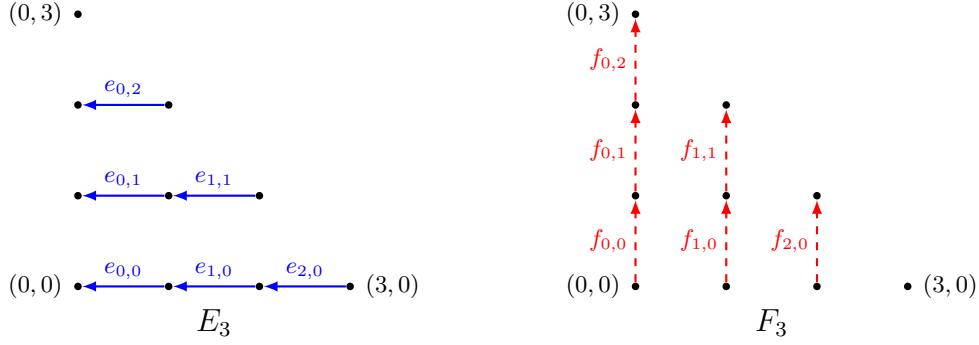
3.3. Model matched pairs. We introduce a class of model categories that will play a central role in our computation of homology (Theorem 5.3) below.

Let $X_n := \{(p, q) \in \mathbb{N} \times \mathbb{N} \mid 0 \leq p + q \leq n\}$. We denote elements of X_n using bold font. Given $\mathbf{a} \in X_n$ we write $\mathbf{a} = (a_L, a_R)$ to indicate the left and right coordinates of \mathbf{a} .

Definition 3.22. Let $\Gamma_n = \{(\mathbf{a}, \mathbf{b}) \in X_n \times X_n \mid a_L \leq b_L \text{ and } a_R \geq b_R\}$. Define $r, s: \Gamma_n \rightarrow X_n$ by $r(\mathbf{a}, \mathbf{b}) = \mathbf{a}$ and $s(\mathbf{a}, \mathbf{b}) = \mathbf{b}$. Identify X_n with $\{(\mathbf{a}, \mathbf{a}) \mid \mathbf{a} \in X_n\}$.

With composition defined by $(\mathbf{a}, \mathbf{b})(\mathbf{b}, \mathbf{c}) := (\mathbf{a}, \mathbf{c})$, the set Γ_n is a small category. It can also be realised as the Zappa–Szép product of the path categories of two graphs. Let E_n be the directed graph with $E_n^0 = X_n$ and $E_n^1 = \{e_{p,q}: (p, q) \in X_n \text{ and } p + q < n\}$, with $r(e_{p,q}) = (p, q)$ and $s(e_{p,q}) =$

$(p+1, q)$. Let F_n be the directed graph with $F_n^0 = X_n$ and $F_n^1 = \{f_{p,q} : (p, q) \in X_n \text{ and } p+q < n\}$, with $s(f_{p,q}) = (p, q)$ and $r(f_{p,q}) = (p, q+1)$. We draw E_n and F_n using coloured arrows (blue and solid for E , red and dashed for F).



Let $\mathcal{F}_n := F_n^*$ and $\mathcal{E}_n := E_n^*$ denote the path categories of F_n and E_n , respectively.

Lemma 3.23. *The subcategory $\{(p, q), (p', q) \mid p \leq p' \leq n, q \leq n\}$ of Γ_n is isomorphic to \mathcal{E}_n and the subcategory $\{(p, q), (p, q') \mid p \leq n, n \geq q \geq q'\}$ is isomorphic to \mathcal{F}_n . Moreover, $[\mathcal{F}_n, \mathcal{E}_n]$ is a strict factorisation system for Γ_n ; the pair $(\mathcal{E}_n, \mathcal{F}_n)$ is a matched pair, with*

$$e_{p,q} \triangleright f_{p+1,q-1} = f_{p,q-1} \quad \text{and} \quad e_{p,q} \triangleleft f_{p+1,q-1} = e_{p,q-1};$$

and $\Gamma_n \cong \mathcal{E}_n \bowtie \mathcal{F}_n$.

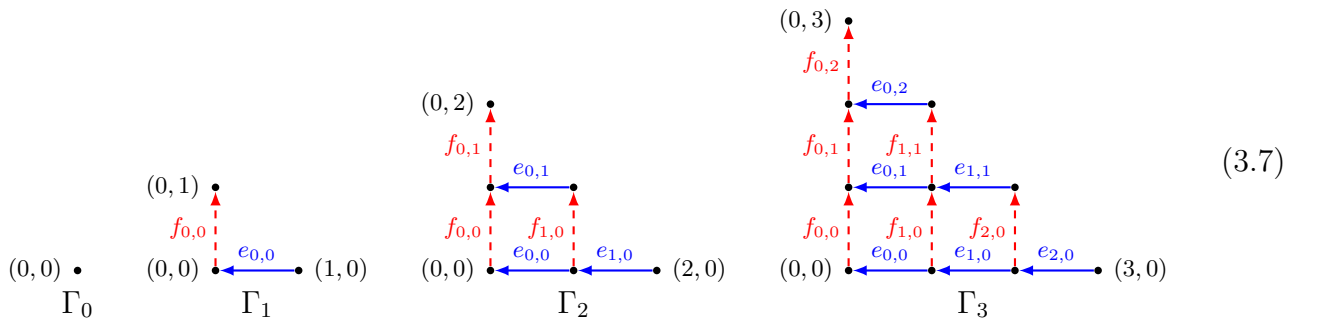
Proof. Since \mathcal{E}_n is freely generated by E_n , the map $e_{p,q} \mapsto ((p, q), (p+1, q))$ identifies \mathcal{E}_n with $\{(p, q), (p', q) \mid p \leq p'\}$. Similarly, $\mathcal{F}_n \cong \{(p, q), (p, q') \mid q \geq q'\}$ via $f_{p,q} \mapsto ((p, q+1), (p, q))$.

Both \mathcal{E}_n and \mathcal{F}_n are clearly wide subcategories. For each $(\mathbf{a}, \mathbf{b}) \in \Gamma_n$, $\alpha := ((a_L, a_R), (a_L, b_R))$ and $\beta := ((a_L, b_R), (b_L, b_R))$ are the unique elements of \mathcal{F}_n and \mathcal{E}_n , respectively such that $(\mathbf{a}, \mathbf{b}) = \alpha\beta$. So $[\mathcal{F}_n, \mathcal{E}_n]$ is a strict factorisation system for Γ_n .

The remaining statements follow from Proposition 3.13. \square

Definition 3.24. We refer to the matched pairs $(\mathcal{E}_n, \mathcal{F}_n)$ as *model matched pairs*.

Each Γ_n can be visualised as a commuting diagram incorporating both E_n and F_n .



For each n , we draw E_n and F_n on the same vertex set. Each picture in (3.7) is a commuting diagram in the corresponding Γ_n . A morphism $(\mathbf{a}, \mathbf{b}) \in \Gamma_n$ is equal to the composition of any of the paths in (3.7) from the vertex at \mathbf{b} to the one at \mathbf{a} .

The matched pairs $(\mathcal{E}_n, \mathcal{F}_n)$ are—in the following sense—*free* in the category MP.

Lemma 3.25. *Let $(\mathcal{C}, \mathcal{D})$ be a matched pair. For every $\gamma = (d_0c_0, \dots, d_{n-1}c_{n-1}) \in (\mathcal{C} \bowtie \mathcal{D})^n$, there is a unique matched pair morphism $h_\gamma: (\mathcal{E}_n, \mathcal{F}_n) \rightarrow (\mathcal{C}, \mathcal{D})$ such that for all $0 \leq k < n$,*

$$h_\gamma^L(e_{k,n-1-k}) = c_k \quad \text{and} \quad h_\gamma^R(f_{k,n-1-k}) = d_k. \quad (3.8)$$

Moreover, every matched pair morphism $(\mathcal{E}_n, \mathcal{F}_n) \rightarrow (\mathcal{C}, \mathcal{D})$ is of this form.

Proof. For each $0 \leq k < n$ let $d_{k,n-1-k} := d_k$ and $c_{k,n-1-k} := c_k$, and for each $0 \leq p+q < n-1$ define $d_{p,q} \in \mathcal{D}$ and $c_{p,q} \in \mathcal{C}$ inductively by $d_{p,q} := c_{p,q+1} \triangleright d_{p+1,q}$ and $c_{p,q} := c_{p,q+1} \triangleleft d_{p+1,q}$. Since \mathcal{E}_n is freely generated by edges, there is a unique functor $h_\gamma^L: \mathcal{E}_n \rightarrow \mathcal{C}$ satisfying $h_\gamma^L(e_{p,q}) = c_{p,q}$ for all $0 \leq p+q < n$. Similarly there is a unique functor $h_\gamma^R: \mathcal{F}_n \rightarrow \mathcal{D}$ satisfying $h_\gamma^R(f_{p,q}) = d_{p,q}$. Since $d_{p,q}c_{p,q} = c_{p,q+1}d_{p+1,q}$, it follows that $h_\gamma = (h_\gamma^L, h_\gamma^R)$ is a morphism of matched pairs.

For uniqueness, fix a matched pair morphism $h: (\mathcal{E}_n, \mathcal{F}_n) \rightarrow (\mathcal{C}, \mathcal{D})$. Then $h^L(e_{p,q}) = h^L(e_{p,q+1} \triangleleft f_{p+1,q}) = h^L(e_{p,q+1}) \triangleleft h^R(f_{p+1,q})$ and similarly $h^R(f_{p,q}) = h^L(e_{p,q+1}) \triangleright h^R(f_{p+1,q})$. Since $h^L: \mathcal{E}_n \rightarrow \mathcal{C}$ and $h^R: \mathcal{F}_n \rightarrow \mathcal{D}$ are functors, h^L and h^R are determined by the values $h^L(e_{k,n-1-k})$ and $h^R(f_{k,n-1-k})$ for $0 \leq k < n$. So h_γ is uniquely determined by (3.8). \square

Corollary 3.10 says that h_γ is the unique functor $\Gamma_n \rightarrow \mathcal{C} \bowtie \mathcal{D}$ such that $h_\gamma(f_{k,n-1-k}e_{k,n-1-k}) = d_kc_k$ for all $0 \leq k < n$.

3.4. Further examples.

3.4.1. *k-graphs.* Here we describe *k-graphs* [KP00] using matched pairs. The generalisations of higher-rank graphs of [LawV22] also fit into our framework, but we do not discuss them here.

Definition 3.26. A *k-graph* is a countable category Λ together with a functor $d: \Lambda \rightarrow \mathbb{N}^k$, called the *degree map*, which satisfies the following factorisation property: if $d(\lambda) = m+n$, then there exist unique elements $\mu, \nu \in \Lambda$ such that $\lambda = \mu\nu$, $d(\mu) = m$ and $d(\nu) = n$. For each $n \in \mathbb{N}^k$, we define $\Lambda^n := d^{-1}(n)$.

We show that every $(k_1 + k_2)$ -graph is a Zappa–Szépproduct of a k_1 -graph and a k_2 -graph.

Lemma 3.27. *Fix $k_1, k_2 \in \mathbb{N}$ and let Σ be a $(k_1 + k_2)$ -graph. Let $\Lambda := d^{-1}(\mathbb{N}^{k_1} \times \{0\})$ regarded as a k_1 -graph, and let $\Gamma := d^{-1}(\{0\} \times \mathbb{N}^{k_2})$ regarded as a k_2 -graph. There are unique actions \triangleright of Λ on Γ and \triangleleft of Γ on Λ such that $\lambda\gamma = (\lambda \triangleright \gamma)(\lambda \triangleleft \gamma)$ in Σ for all composable pairs $(\lambda, \gamma) \in \Lambda * \Gamma$. These make $(\Lambda, \Gamma, \triangleright, \triangleleft)$ a matched pair, and $(\gamma, \lambda) \mapsto \gamma\lambda$ is an isomorphism $\Lambda \bowtie \Gamma \rightarrow \Sigma$. We have $d(\lambda \triangleright \gamma) = d(\gamma)$ and $d(\lambda \triangleleft \gamma) = d(\lambda)$ for all λ, γ .*

Proof. Everything except the final statement follows the factorisation property and Proposition 3.13. The final statement follows from the factorisation property. \square

We now describe a converse to Lemma 3.27. An *edge* in a *k-graph* is a path e such that $d(e)$ is a standard generator of \mathbb{N}^k . We write $E(\Lambda)$ for the set of edges of Λ . Let $\Sigma, \Lambda, \Gamma, \triangleright$ and \triangleleft be as in Lemma 3.27, and write $d_\Lambda: \Lambda \rightarrow \mathbb{N}^{k_1}$ and $d_\Gamma: \Gamma \rightarrow \mathbb{N}^{k_2}$ for the degree functors. Then

- (K1) $s(\nu \triangleright \mu) = r(\nu \triangleleft \mu)$ for all $(\nu, \mu) \in E(\Lambda) * E(\Gamma)$;
- (K2) $\nu_1\nu_2 \triangleleft \mu = (\nu_1 \triangleleft (\nu_2 \triangleright \mu))(\nu_2 \triangleleft \mu)$ for all $(\nu_1, \nu_2, \mu) \in E(\Lambda) * E(\Lambda) * E(\Gamma)$,
- (K3) $\nu \triangleright \mu_1\mu_2 = (\nu \triangleright \mu_1)((\nu \triangleleft \mu_1) \triangleright \mu_2)$ for all $(\nu, \mu_1, \mu_2) \in E(\Lambda) * E(\Gamma) * E(\Gamma)$,
- (K4) $d_\Gamma(\nu \triangleright \mu) = d_\Gamma(\mu)$ and $d_\Lambda(\nu \triangleleft \mu) = d_\Lambda(\nu)$ for all $(\nu, \mu) \in E(\Lambda) * E(\Gamma)$, and
- (K5) for each $(\mu, \nu) \in E(\Gamma) * E(\Lambda)$ there exists a unique $\mu' \in E(\Gamma)$ and $\nu' \in E(\Lambda)$ such that $\mu = \nu' \triangleright \mu'$ and $\nu = \nu' \triangleleft \mu'$.

Lemma 3.28. *Let Λ be a k_1 -graph and let Γ be a k_2 -graph such that $\Lambda^0 = \Gamma^0$. Suppose that $\triangleright: \Lambda * \Gamma \rightarrow \Gamma$ and $\triangleleft: \Lambda * \Gamma \rightarrow \Gamma$ are actions satisfying (K1)–(K5). Then (Λ, Γ) is a matched pair, and the map $d: \Lambda \bowtie \Gamma \rightarrow \mathbb{N}^{k_1+k_2}$ given by $d(\gamma, \lambda) = (d_\Lambda(\lambda), d_\Gamma(\gamma))$ makes $\Lambda \bowtie \Gamma$ into a $(k_1 + k_2)$ -graph.*

Proof. Let E be directed graph with edges $E^1 = E(\Lambda) \sqcup E(\Gamma)$, vertices $E^0 = \Lambda^0 = \Gamma^0$ and range and source maps inherited from Λ and Γ . Define $c: E^1 \rightarrow \{1, \dots, k_1 + k_2\}$ by $c(\alpha) = i$ if $\alpha \in \Lambda^{e_i}$ and $c(\alpha) = k_1 + j$ if $\alpha \in \Gamma^{e_j}$, which we regard as a colouring of E^1 by $k_1 + k_2$ colours. Define a collection of squares in the sense of [HRSW13, LarV22] by $\alpha\beta \sim \beta'\alpha'$ if

- $\alpha\beta = \beta'\alpha'$ in one of Λ or Γ , or
- $\alpha \in E(\Lambda)$ and $\beta \in E(\Gamma)$ and $\alpha \triangleright \beta = \beta'$ and $\alpha \triangleleft \beta = \alpha'$, or
- $\alpha \in E(\Gamma)$ and $\beta \in E(\Lambda)$ and $\alpha' \triangleright \beta' = \beta$ and $\alpha' \triangleleft \beta' = \alpha$.

The factorisation properties and (K5) ensure that this is a complete collection of squares.

We claim that this is an associative collection of squares. For this we must check that if $\alpha, \beta, \gamma \in E^1$ are composable and of distinct colours, and if

$$\begin{array}{llll} \alpha\beta \sim \beta_1\alpha_1, & \alpha_1\gamma \sim \gamma_1\alpha_2, & \text{and} & \beta_1\gamma_1 \sim \gamma_2\beta_2; & \text{and} \\ \beta\gamma \sim \gamma^1\beta^1, & \alpha\gamma^1 \sim \gamma^2\alpha^1, & \text{and} & \alpha^1\beta^1 \sim \beta^2\alpha^2; & \end{array}$$

then $\alpha^2 = \alpha_2$, $\beta^2 = \beta_2$ and $\gamma^2 = \gamma_2$.

If α, β, γ all belong to either Λ or Γ , this follows from associativity of composition, so we just need to consider when this is not the case. We treat the case where $\alpha \in \Lambda$ and $\beta, \gamma \in \Gamma$; the calculations for the other cases are similarly straightforward. We have

$$\beta_1 = \alpha \triangleright \beta, \quad \alpha_1 = \alpha \triangleleft \beta, \quad \gamma_1 = \alpha_1 \triangleright \gamma, \quad \alpha_2 = \alpha_1 \triangleleft \gamma, \quad \text{and} \quad \beta_1\gamma_1 = \gamma_2\beta_2 \text{ in } \Gamma.$$

That is, $\gamma_2\beta_2 = (\alpha \triangleright \beta)((\alpha \triangleleft \beta) \triangleright \gamma) = \alpha \triangleright (\beta\gamma)$, and $\alpha_2 = (\alpha \triangleleft \beta) \triangleleft \gamma = \alpha \triangleleft (\beta\gamma)$. Similarly,

$$\beta\gamma = \gamma^1\beta^1, \quad \gamma^2 = \alpha \triangleright \gamma^1, \quad \alpha^1 = \alpha \triangleleft \gamma^1, \quad \beta^2 = \alpha^1 \triangleright \beta^1, \quad \text{and} \quad \alpha^2 = \alpha^1 \triangleleft \beta^1.$$

That is, $\gamma^2\beta^2 = \alpha \triangleright (\gamma^1\beta^1) = \alpha \triangleright (\beta\gamma)$, and $\alpha^2 = (\alpha \triangleleft \gamma^1) \triangleleft \beta^1 = \alpha \triangleleft (\gamma^1\beta^1) = \alpha \triangleleft (\beta\gamma)$. So $\gamma^2\beta^2 = \gamma_2\beta_2$ forcing $\gamma^2 = \gamma_2$ and $\beta^2 = \beta_2$ by uniqueness of factorisations in Γ , and $\alpha^2 = \alpha_2$.

By [HRSW13, Theorem 4.4] there is a unique $(k_1 + k_2)$ -graph Σ with skeleton E and the specified factorisation rules. Lemma 3.27, yields a k_1 -graph Λ' and a k_2 -graph Γ' such that $\Sigma \cong \Lambda' \bowtie \Gamma'$. By construction, Λ' has the same skeleton and factorisation rules as Λ so they are isomorphic by [HRSW13, Theorem 4.5] (see also [LarV22]), and likewise $\Gamma' \cong \Gamma$. These isomorphisms intertwine the actions of Λ' and Γ' on one another with those of Λ and Γ . \square

Taken together, Lemmas 3.27 and 3.28 prove the following.

Proposition 3.29. *Let Γ be a k_1 -graph and let Λ be a k_2 -graph with actions $\triangleright: \Lambda * \Gamma \rightarrow \Gamma$ and $\triangleleft: \Lambda * \Gamma \rightarrow \Gamma$ satisfying (K1)–(K5). The Zappa–Szépp product $\Lambda \bowtie \Gamma$ is a $(k_1 + k_2)$ -graph. Moreover, every $(k_1 + k_2)$ -graph Π is isomorphic to the Zappa–Szépp product of the k_1 -graph $\Lambda = d^{-1}(\mathbb{N}^{k_1+1} \times \{0\})$ and the k_2 -graph $\Gamma = d^{-1}(\{0\} \times \mathbb{N}^{k_2})$ with actions satisfying (K1)–(K5).*

3.4.2. Self-similar actions. We discuss self-similar actions of groupoids on k -graphs as in [ABRW19]. These include self-similar actions of groupoids and of groups on graphs as in [Nek05, EP17, LRRW14, LRRW18]. We show that each such self-similar action determines a matched pair in which the left action respects the degree map. Later we will study C^* -algebras associated to such matched pairs; the framework of matched pairs allows us to dispense with the faithfulness condition traditionally imposed in the study of self-similar actions.

Recall that an edge in a k -graph is a path f with $d(f) = e_i$ for some $i \leq k$.

Definition 3.30 ([LRRW18, Definition 3.3]). Let Λ be a k -graph and let \mathcal{G} be a groupoid with $\mathcal{G}^0 = \Lambda^0$. A *faithful self-similar action* of \mathcal{G} on Λ is a left action $\cdot : \mathcal{G} * \Lambda \rightarrow \Lambda$ of \mathcal{G} on Λ such that

- (SSA1) for each $n \in \mathbb{N}^k$ and $g \in \mathcal{G}$, we have $g \cdot (s(g)\Lambda^n) = r(g)\Lambda^n$, and
- (SSA2) $s(g_1) = s(g_2)$ and $g_1 \cdot \mu = g_2 \cdot \mu$ for all $\mu \in s(g_1)\Lambda$, then $g_1 = g_2$.
- (SSA3) for every $g \in \mathcal{G}$ and every edge $e \in s(g)\Lambda$ there exists $h \in s(e)\mathcal{G}$ such that $g \cdot (e\mu) = (g \cdot e)(h \cdot \mu)$ for all $\mu \in s(e)\Lambda$.

Remark 3.31. If \mathcal{G} acts self-similarly on Λ then (SSA2) implies that there is a unique h satisfying (SSA3). We denote this element by $g|_e$ and call it the *restriction* of g to μ .

As discussed immediately after Definition 3.3 in [ABRW19], the map $g \mapsto g|_e$ extends to a map $(g, \mu) \mapsto g|_\mu$ from $\mathcal{G} * \Lambda$ to Λ by the recursive formula $g|_{e\mu} = (g|_e)|_{(g|_e) \cdot \mu}$.

Proposition 3.32. *Let Λ be a k -graph and \mathcal{G} a groupoid with $\mathcal{G}^0 = \Lambda^0$. Suppose that $\cdot : \mathcal{G} * \Lambda \rightarrow \Lambda$ is a faithful self-similar action. Define $\triangleright : \mathcal{G} * \Lambda \rightarrow \Lambda$ by $g \triangleright \mu = g \cdot \mu$ and $\triangleleft : \mathcal{G} * \Lambda \rightarrow \mathcal{G}$ by $g \triangleleft \mu = g|_\mu$. Then $(\mathcal{G}, \Lambda, \triangleright, \triangleleft)$ is a matched pair such that*

- (i) if $g_1 \triangleright \mu = g_2 \triangleright \mu$ for all $\mu \in s(g_1)\Lambda$, then $g_1 = g_2$, and
- (ii) $d(g \triangleright \mu) = d(\mu)$ for all $(g, \mu) \in \mathcal{G} * \Lambda$.

Conversely, if $(\mathcal{G}, \Lambda, \triangleright, \triangleleft)$ is a matched pair satisfying (i) and (ii), then \triangleright defines a faithful self-similar action of \mathcal{G} on Λ with restriction map $g|_\mu := g \triangleleft \mu$.

Proof. First suppose that $\cdot : \mathcal{G} * \Lambda \rightarrow \Lambda$ is a faithful self-similar action. That $d(g \triangleright \mu) = d(\mu)$ for all $(g, \mu) \in \mathcal{G} * \Lambda$ follows from (SSA1), and [ABRW19, Lemma 3.4] implies that (\mathcal{G}, Λ) is a matched pair. Condition (i) follows from (SSA2).

Conversely, suppose that (\mathcal{G}, Λ) is a matched pair satisfying (i) and (ii). Then for each $g \in \mathcal{G}$ and each edge $e \in s(g)\Lambda$, the element $h := g \triangleright \mu$ satisfies (SSA3). The condition (MP2) gives (SSA3). That $d(g \triangleright \mu) = d(\mu)$ for all $(g, \mu) \in \mathcal{G} * \Lambda$ implies that $g \triangleright \cdot$ restricts to a map $g \triangleright \cdot : s(g)\Lambda^n \rightarrow r(g)\Lambda^n$. Invertibility of g implies that these maps are bijective, giving (SSA1). Condition (i) implies (SSA2). \square

Motivated by Proposition 3.32 we introduce a generalisation of the faithful self-similar actions of [ABRW19] (this is related to the definition in [LY21]).

Definition 3.33. A *self-similar action of a groupoid on a k -graph* is a matched pair (\mathcal{G}, Λ) in which \mathcal{G} is a groupoid, Λ is a k -graph, and $d(g \triangleright \mu) = d(\mu)$ for all $(g, \mu) \in \mathcal{G} * \Lambda$.

Example 3.34. Let $E = (E^0, E^1, r, s)$ be a directed graph as in Section 2. Then E^* is a 1-graph with degree map given by the length functor. Moreover, every 1-graph is of this form. The definition of a faithful self-similar action of \mathcal{G} on E^* as above reduces to the definition of a self-similar action of a groupoid on a graph in [LRRW18]. This in turn generalises the self-similar groups of automorphisms of trees discussed in, for example, [Nek05] (these correspond to the case where E has just one vertex). The definitions in [EP17] and [Yus23], which do not impose a faithfulness condition, are also instances Definition 3.33 with $k = 1$.

3.4.3. Graphs of groups and group actions on trees. An *undirected graph* $\Gamma = (\Gamma^0, \Gamma^1, r, s, \bar{\cdot})$ is a directed graph endowed with a map $\bar{\cdot} : \Gamma^1 \rightarrow \Gamma^1$ such that $\bar{\bar{e}} = e \neq \bar{e}$ and $r(\bar{e}) = s(e)$ for all e .

Definition 3.35. A *graph of groups* is a pair (Γ, G) consisting of: an undirected graph Γ ; assignments $v \mapsto G_v$ and $e \mapsto G_e$ of a group to each $v \in \Gamma^0$ and $e \in \Gamma^1$, such that $G_e = G_{\bar{e}}$ for all $e \in \Gamma^1$; and injective homomorphisms $\alpha_e: G_e \rightarrow G_{r(e)}$ for each $e \in \Gamma^1$.

The Bass–Serre Theorem [Bas93, Ser80] describes a duality between graphs of groups and edge-reversal-free actions of groups on trees.

Building on the observations of [MR21, Theorem 5.4], we show that every graph of groups (Γ, G) gives rise to a matched pair. For each $e \in \Gamma^1$, let Σ_e be a complete set of coset representatives for $G_{r(e)}/\alpha_e(G_e)$. We assume that $1_{G_{r(e)}} \in \Sigma_e$ so $1_{G_{r(e)}}$ is the representative of the coset $\alpha_e(G_e)$. There is a natural action of $G_{r(e)}$ on Σ_e : we define $g \cdot \mu$ to be the coset representative of $g\mu$ for all $g \in G_{r(e)}$ and $\mu \in \Sigma_e$.

Consider the groupoid $\mathcal{G} = \bigsqcup_{e \in \Gamma^1} G_e$, a bundle of groups over Γ^1 . Define a directed graph E by $E^0 := \Gamma^1$,

$$E^1 := \{e\mu f \mid \mu \in \Sigma_e, ef \in \Gamma^2, e = \bar{f} \implies \mu \neq 1_{G_{r(e)}}\},$$

$r(e\mu f) = e$, and $s(e\mu f) = f$. We identify each E^n with

$$\{e_0\mu_1e_1\mu_2e_2 \cdots \mu_n e_n \mid \mu_i \in \Sigma_{e_i}, e_0 \cdots e_n \in \Gamma^{n+1}, e_{i-1} = \bar{e}_i \implies \mu_i \neq 1_{r(e_i)}\}.$$

Consider the path category E^* of E . We show that (\mathcal{G}, E^*) can be made into matched pair (indeed, a self-similar groupoid action as in Example 3.34).

Fix $ef \in \Gamma^2$, $g \in G_e$ and $\mu \in \Sigma_f$. Then $g \blacktriangleright \mu := \alpha_{\bar{e}}(g) \cdot \mu \in \Sigma_f$ and $g \blacktriangleleft \mu := \alpha_f^{-1}((g \blacktriangleright \mu)^{-1} \alpha_{\bar{e}}(g) \mu) \in G_f$ are the unique elements such that $\alpha_{\bar{e}}(g) \mu = (g \blacktriangleright \mu) \alpha_f(g \blacktriangleleft \mu)$.

As in the proof of Proposition 3.13, for $g_1, g_2 \in G_e$, we have $(g_1 g_2) \blacktriangleright \mu = g_1 \cdot (g_2 \blacktriangleright \mu)$ and $(g_1 g_2) \blacktriangleleft \mu = (g_1 \blacktriangleleft (g_2 \blacktriangleright \mu))(g_2 \blacktriangleleft \mu)$. We define a left action $\triangleright: \mathcal{G} * E^* \rightarrow E^*$ inductively by

$$g \triangleright e_0 \mu_1 e_1 \mu_2 e_2 \cdots \mu_n e_n = (e_0 (g \blacktriangleright \mu_1) e_1) ((g \blacktriangleleft \mu_1) \triangleright e_1 \mu_2 e_2 \cdots \mu_n e_n),$$

and a right action $\blacktriangleleft: \mathcal{G} * E^* \rightarrow \mathcal{G}$ inductively by

$$g \blacktriangleleft e_0 \mu_1 e_1 \mu_2 e_2 \cdots \mu_n e_n = (g \blacktriangleleft \mu_1) \blacktriangleleft e_1 \mu_2 e_2 \cdots \mu_n e_n.$$

It is straightforward to verify that these actions turn (\mathcal{G}, E^*) into a matched pair. Moreover, $\triangleright, \blacktriangleleft$ satisfy (SSA1)–(SSA3) so \mathcal{G} acts self-similarly on E^* .

4. THREE HOMOLOGY THEORIES FOR MATCHED PAIRS

We describe three homology theories associated to a matched pair (the last two via a double complex). We show in Section 5 that they all coincide up to natural isomorphism.

4.1. The categorical complex and categorical homology.

Definition 4.1. Let \mathcal{C} be a small category. For each $k \geq 0$ let $C_k(\mathcal{C}) := \mathbb{Z}\mathcal{C}^k$ be the free abelian group generated by composable k -tuples. We write $[c_0, \dots, c_k] \in C_{k+1}(\mathcal{C})$ for the generator corresponding to $(c_0, \dots, c_k) \in \mathcal{C}^{k+1}$. For $k \geq 1$ define $d_k: C_{k+1}(\mathcal{C}) \rightarrow C_k(\mathcal{C})$ by

$$d_k[c_0, \dots, c_k] = [c_1, \dots, c_k] + \left(\sum_{i=1}^k (-1)^i [c_0, \dots, c_{i-1} c_i, \dots, c_k] \right) + (-1)^{k+1} [c_0, \dots, c_{k-1}]$$

and define $d_0: C_1(\mathcal{C}) \rightarrow C_0(\mathcal{C})$ by $d_0[c] = [s(c)] - [r(c)]$. Then $(C_\bullet(\mathcal{C}), d_\bullet)$ is a chain complex, and its homology, $H_\bullet(\mathcal{C})$, is called the *categorical homology of \mathcal{C}* .

For an abelian group A , let $C^k(\mathcal{C}; A) := \text{Hom}(C_k(\mathcal{C}), A)$. Define $d^k: C^k(\mathcal{C}; A) \rightarrow C^{k+1}(\mathcal{C}; A)$ by $d^k(f) = f \circ d_k$. Then $(C^\bullet(\mathcal{C}; A), d^\bullet)$ is a cochain complex, and its cohomology, $H^\bullet(\mathcal{C}; A)$, is called the *categorical cohomology of \mathcal{C} with coefficients in A* .

There are more-sophisticated definitions of categorical cohomology in terms of projective resolutions of \mathcal{C} -modules (cf. [GK18]). Our definition amounts to fixing a resolution, analogous to the *bar resolution* for group homology (cf. [Wei94, §6.5]), of a constant functor $\mathcal{C} \rightarrow A$ (see [GK18, Proposition 2.4])

Definition 4.2. The *categorical homology*, denoted $H_\bullet^\bowtie(\mathcal{C}, \mathcal{D})$, of a matched pair $(\mathcal{C}, \mathcal{D})$ is the categorical homology of the Zappa–Szép product category $\mathcal{C} \bowtie \mathcal{D}$. For each $k \geq 0$, $H_k^\bowtie: \text{MP} \rightarrow \text{Ab}$ is functor defined by the composition $(\mathcal{C}, \mathcal{D}) \mapsto \mathcal{C} \bowtie \mathcal{D} \mapsto H_k(\mathcal{C} \bowtie \mathcal{D})$.

For an abelian group A , the *categorical cohomology of $(\mathcal{C}, \mathcal{D})$ with coefficients in A* , denoted $H_\bowtie^\bullet(\mathcal{C}, \mathcal{D}; A)$, is the categorical cohomology of $\mathcal{C} \bowtie \mathcal{D}$ with coefficients in A .

We work with simplicial groups rather than chain complexes (see [Wei94, Ch.8]) to simplify calculations. The Dold–Kan Theorem [Wei94, Theorem 8.4.1] gives an equivalence of categories between simplicial abelian groups and chain complexes of abelian groups.

For each $k \geq 1$ and $0 \leq i \leq k+1$ we define the *face map* $\partial_k^i: C_{k+1}(\mathcal{C}) \rightarrow C_k(\mathcal{C})$ by

$$\partial_k^i[c_0, \dots, c_i, \dots, c_k] = \begin{cases} [c_1, \dots, c_k] & \text{if } i = 0 \\ [c_0, \dots, c_{i-1}c_i, \dots, c_k] & \text{if } 1 \leq i \leq k \\ [c_0, \dots, c_{k-1}] & \text{if } i = k+1. \end{cases} \quad (4.1)$$

We also define $\partial_0^0[c] = [s(c)]$ and $\partial_0^1[c] = [r(c)]$. In particular, $d_k = \sum_{i=0}^{k+1} (-1)^k \partial_k^i$.

To work with degeneracy maps, we use the following—slightly non-standard—notation.

Notation 4.3. If $(c_1, \dots, c_k) \in \mathcal{C}^k$ is a composable k -tuple, and $0 \leq i \leq k$, then we define

$$\begin{aligned} (c_1, \dots, c_{i-1}, _, c_{i+1}, \dots, c_k) &:= (c_1, \dots, c_{i-1}, s(c_{i-1}), c_{i+1}, \dots, c_k) \\ &= (c_1, \dots, c_{i-1}, r(c_{i+1}), c_{i+1}, \dots, c_k) \in \mathcal{C}^{k-1}. \end{aligned}$$

The identity morphism represented by any given instance of $_$ is determined by either of the neighbouring entries.

For each $k \geq 1$ and $0 \leq i \leq k$ we define the *degeneracy map* $\sigma_k^i: C_k(\mathcal{C}) \rightarrow C_{k+1}(\mathcal{C})$ by

$$\sigma_k^i[c_0, \dots, c_{k-1}] = \begin{cases} [_, c_0, \dots, c_{k-1}] & \text{if } i = 0, \\ [c_0, \dots, c_{i-1}, _, c_i, \dots, c_{k-1}] & \text{if } 0 < i < k, \\ [c_0, \dots, c_{k-1}, _] & \text{if } i = k, \end{cases}$$

with $\sigma_0^0[x] = [x]$ for $x \in \mathcal{C}^0$. These and the ∂_j^i satisfy the *simplicial identities*:

$$\begin{aligned} \partial_{k-1}^i \partial_k^j &= \partial_{k-1}^{j-1} \partial_k^i & \text{if } i < j \\ \sigma_{k+1}^i \sigma_k^j &= \sigma_{k+1}^{j+1} \sigma_k^i & \text{if } i \leq j, \text{ and} \\ \partial_k^i \sigma_k^j &= \begin{cases} \sigma_k^{j-1} \partial_k^i & \text{if } i < j \\ \text{id}_{C_k(\mathcal{C})} & \text{if } i = j \text{ or } i = j+1 \\ \sigma_k^j \partial_k^{i-1} & \text{if } i > j+1, \end{cases} \end{aligned}$$

so $(C_\bullet(\mathcal{C}), \partial, \sigma)$ is a simplicial abelian group.

If $(\mathcal{C}, \mathcal{D})$ is a matched pair, then Proposition 3.20 gives an action of \mathcal{C} on each \mathcal{D}^{k+1} . For $(c, d) \in \mathcal{C} * \mathcal{D}^{k+1}$, we write $c \triangleright [d]$ for the generator $[c \triangleright d]$ of $C_{k+1}(\mathcal{D})$. Similarly if $(c, d) \in \mathcal{C}^{k+1} * \mathcal{D}$ we write $[c] \triangleleft d$ for the generator $[c \triangleleft d]$.

Lemma 4.4. *Let $(\mathcal{C}, \mathcal{D})$ be a matched pair and take $k \in \mathbb{N}$. For $0 < i \leq k + 1$ and for $(c, d) \in \mathcal{C} * \mathcal{D}^{k+1}$, we have $\partial_k^i(c \triangleright [d]) = c \triangleright \partial_k^i[d]$ in $C_k(\mathcal{D})$. Similarly, for $0 \leq j < k + 1 \in \mathbb{N}$ and $(c, d) \in \mathcal{C}^{k+1} * \mathcal{D}$, we have $\partial_k^j([c] \triangleleft d) = \partial_k^j[c] \triangleleft d$ in $C_k(\mathcal{C})$.*

Proof. We prove the first statement; the second follows symmetrically. Since $i > 0$, we have

$$\begin{aligned} c \triangleright \partial_k^i[d] &= c \triangleright [d_0, \dots, d_i d_{i+1}, \dots, d_k] \\ &= [c \triangleright d_0, (c \triangleleft d_0) \triangleright d_1, \dots, (c \triangleleft d_0 \dots d_{i-1}) \triangleright d_i d_{i+1}, \dots, (c \triangleleft (d_0 \dots d_{k-1})) \triangleright d_k] \\ &= [c \triangleright d_0, (c \triangleleft d_0) \triangleright d_1, \dots, ((c \triangleleft d_0 \dots d_{i-1}) \triangleright d_i) ((c \triangleleft d_0 \dots d_i) \triangleright d_{i+1}), \\ &\quad \dots, (c \triangleleft (d_0 \dots d_{k-1})) \triangleright d_k] \\ &= \partial_k^i [c \triangleright d_0, (c \triangleleft d_0) \triangleright d_1, \dots, ((c \triangleleft d_0 \dots d_{i-1}) \triangleright d_i), ((c \triangleleft d_0 \dots d_i) \triangleright d_{i+1}), \\ &\quad \dots, (c \triangleleft (d_0 \dots d_{k-1})) \triangleright d_k] \\ &= \partial_k^i(c \triangleright [d]). \end{aligned} \quad \square$$

Remark 4.5. Lemma 4.4 is *only* valid for $i > 0$ and $j < k + 1$. The left action of \mathcal{C} on \mathcal{D}^{k+1} does not commute with ∂_k^0 , and the right action of \mathcal{D} on \mathcal{C}^{k+1} does not commute with ∂_k^{k+1} .

4.2. The matched complex. We associate a double complex to each matched pair $(\mathcal{C}, \mathcal{D})$. For $p, q \geq 0$, regard elements of $\mathcal{C}^p * \mathcal{D}^q$ as composable tuples in $(\mathcal{C} \bowtie \mathcal{D})^{p+q}$, whose first p terms belong to $\mathcal{C} \subseteq \mathcal{C} \bowtie \mathcal{D}$ and whose remaining q terms belong to $\mathcal{D} \subseteq \mathcal{C} \bowtie \mathcal{D}$.

Let $C_{p,q}(\mathcal{C}, \mathcal{D}) := \mathbb{Z}(\mathcal{C}^p * \mathcal{D}^q)$, the free abelian group generated by $\mathcal{C}^p * \mathcal{D}^q$. Let \triangleright be the action of \mathcal{C} on \mathcal{D}^q of Lemma 3.19. Define *horizontal face maps* $\partial_{p,q}^{h,i}: C_{p+1,q}(\mathcal{C}, \mathcal{D}) \rightarrow C_{p,q}(\mathcal{C}, \mathcal{D})$ as follows. For $q \geq 1$,

$$\partial_{p,q}^{h,i}[c_0, \dots, c_p, d_0, \dots, d_{q-1}] = \begin{cases} [c_1, \dots, c_p, d_0, \dots, d_{q-1}] & \text{if } i = 0 \\ [c_0, \dots, c_{i-1} c_i, \dots, c_p, d_0, \dots, d_{q-1}] & \text{if } 1 \leq i \leq p \\ [c_0, \dots, c_{p-1}, c_p \triangleright (d_0, \dots, d_{q-1})] & \text{if } i = p + 1, \end{cases}$$

while $\partial_{p,0}^{h,i} := \partial_p^i: C_{p+1}(\mathcal{C}) \rightarrow C_p(\mathcal{C})$ as in (4.1). For $0 \leq i \leq p$ we define the *horizontal degeneracy maps* $\sigma_{p,q}^{h,i}: C_{p,q}(\mathcal{C}, \mathcal{D}) \rightarrow C_{p+1,q}(\mathcal{C}, \mathcal{D})$ by

$$\sigma_{p,q}^{h,i}[c_0, \dots, c_{p-1}, d_0, \dots, d_{q-1}] = [c_0, \dots, c_i, _, c_{i+1}, \dots, c_{p-1}, d_0, \dots, d_{q-1}].$$

For each $q \geq 0$, the tuple $(C_{\bullet,q}(\mathcal{C}, \mathcal{D}), \partial_{\bullet,q}^h, \sigma_{\bullet,q}^h)$ is a simplicial abelian group.

Let \triangleleft be the action of \mathcal{D} on \mathcal{C}^* of Lemma 3.19. Define *vertical face maps* $\partial_{p,q+1}^{v,j}: C_{p,q+1}(\mathcal{C}, \mathcal{D}) \rightarrow C_{p,q}(\mathcal{C}, \mathcal{D})$ as follows. For $p > 0$,

$$\partial_{p,q+1}^{v,j}[c_0, \dots, c_{p-1}, d_0, \dots, d_q] = (-1)^p \begin{cases} [(c_0, \dots, c_{p-1}) \triangleleft d_0, d_1, \dots, d_q] & \text{if } j = 0 \\ [c_0, \dots, c_{p-1}, d_0, \dots, d_{j-1} d_j, \dots, d_q] & \text{if } 1 \leq j \leq q \\ [c_0, \dots, c_{p-1}, d_0, \dots, d_{q-1}] & \text{if } j = q + 1, \end{cases}$$

while $\partial_{0,q}^{v,i} := \partial_q^i: C_{q+1}(\mathcal{D}) \rightarrow C_q(\mathcal{D})$ as in (4.1). For $0 \leq j \leq q$ we define *vertical degeneracy maps* $\sigma_{p,q}^{v,j}: C_{p,q} \rightarrow C_{p,q+1}$ by

$$\sigma_{p,q}^{v,j}[c_0, \dots, c_{p-1}, d_0, \dots, d_{q-1}] = (-1)^p [c_0, \dots, c_{p-1}, d_0, \dots, d_j, _, d_{j+1}, d_{q-1}].$$

Then $(C_{p,\bullet}(\mathcal{C}, \mathcal{D}), \partial_{p,\bullet}^v, \sigma_{p,\bullet}^v)$ is also a simplicial abelian group.

For the next result, recall from [Wei94, §8.5] that a bisimplicial abelian group is a quintuple $(C_{\bullet,\bullet}, \partial^h, \partial^v, \sigma^h, \sigma^v)$ consisting of abelian groups $C_{p,q}$ and homomorphisms $\partial_{p,q}^{h,i}: C_{p+1,q} \rightarrow C_{p,q}$, $\sigma_{p,q}^{h,i}: C_{p,q} \rightarrow C_{p+1,q}$, $\partial_{p,q}^{v,j}: C_{p,q+1} \rightarrow C_{p,q}$, and $\sigma_{p,q}^{v,j}: C_{p,q} \rightarrow C_{p,q+1}$ such that each $(C_{p,\bullet}, \partial_{p,\bullet}^v, \sigma_{p,\bullet}^v)$ and each $(C_{\bullet,q}, \partial_{\bullet,q}^h, \sigma_{\bullet,q}^h)$ is a simplicial group, and

$$\partial_{p,q}^{v,i} \partial_{p,q+1}^{h,j} = -\partial_{p,q}^{h,j} \partial_{p+1,q}^{v,i} \quad \text{and} \quad \sigma_{p+1,q}^{v,i} \sigma_{p,q}^{h,j} = -\sigma_{p,q+1}^{h,j} \sigma_{p,q}^{v,i}.$$

Proposition 4.6. *The quintuple $(C_{\bullet,\bullet}, \partial^h, \partial^v, \sigma^h, \sigma^v)$ is a bisimplicial group. Define*

$$d_{p,q}^h = \sum_{i=0}^{p+1} (-1)^i \partial_{p,q}^{h,i}: C_{p+1,q} \rightarrow C_{p,q} \quad \text{and} \quad d_{p,q}^v = \sum_{i=0}^{q+1} (-1)^i \partial_{p,q}^{v,i}: C_{p,q+1} \rightarrow C_{p,q}.$$

Then

$$\begin{array}{ccccccc} & & \downarrow & & \downarrow & & \downarrow \\ C_{0,2} & \xleftarrow{d_{0,2}^h} & C_{1,2} & \xleftarrow{d_{1,2}^h} & C_{2,2} & \xleftarrow{\dots} & \\ \downarrow d_{0,1}^v & & \downarrow d_{1,1}^v & & \downarrow d_{2,1}^v & & \\ C_{0,1} & \xleftarrow{d_{0,1}^h} & C_{1,1} & \xleftarrow{d_{1,1}^h} & C_{2,1} & \xleftarrow{\dots} & \\ \downarrow d_{0,0}^v & & \downarrow d_{1,0}^v & & \downarrow d_{2,0}^v & & \\ C_{0,0} & \xleftarrow{d_{0,0}^h} & C_{1,0} & \xleftarrow{d_{1,0}^h} & C_{2,0} & \xleftarrow{\dots} & \end{array} \quad (4.2)$$

is a first-quadrant double chain complex satisfying $d^h d^v = -d^v d^h$.

Proof. Fix $p, q \geq 0$ and fix $i \leq p+1$ and $j \leq q+1$. We must show that $\partial_{p,q}^{v,i} \partial_{p,q+1}^{h,j} = -\partial_{p,q}^{h,j} \partial_{p+1,q}^{v,i}$. Fix $[c_0, \dots, c_p; d_0, \dots, d_q] \in C_{p+1,q+1}$. If $i \neq p+1$ or $j \neq 0$, then $\partial_{p,q}^{v,i}$ and $\partial_{p,q+1}^{h,j}$ concatenate or delete nonadjacent coordinates, as do $\partial_{p,q}^{h,j}$ and $\partial_{p+1,q}^{v,i}$, and so the factors of $(-1)^p$ and $(-1)^{p+1}$ in $\partial_{p,q}^{v,i}$ and $\partial_{p+1,q}^{v,i}$ give the desired anticommutation relation. If $i = p+1$ and $j = 0$, then

$$\begin{aligned} \partial_{p,q}^{v,0} \partial_{p,q+1}^{h,p+1}([c_0, \dots, c_p; d_0, \dots, d_q]) &= \partial_{p,q}^{v,0}([(c_0, \dots, c_p) \triangleleft d_0; d_1, \dots, d_q]) \\ &= \partial_{p,q}^{v,0}([(c_0, \dots, c_{p-1}) \triangleleft (c_p \triangleright d_0), c_p \triangleleft d_0; d_1, \dots, d_q]) \\ &= (-1)^p [(c_0, \dots, c_{p-1}) \triangleleft (c_p \triangleright d_0); (c_p \triangleleft d_0) \triangleright (d_1, \dots, d_q)], \end{aligned}$$

and

$$\begin{aligned} \partial_{p,q}^{h,p+1} \partial_{p+1,q}^{v,0}([c_0, \dots, c_p; d_0, \dots, d_q]) &= (-1)^{p+1} \partial_{p,q}^{v,0}([c_0, \dots, c_{p-1}; c_p \triangleright (d_0, \dots, d_q)]) \\ &= (-1)^{p+1} \partial_{p,q}^{v,0}([c_0, \dots, c_{p-1}; c_p \triangleright d_0, (c_p \triangleleft d_0) \triangleright (d_1, \dots, d_q)]) \\ &= (-1)^{p+1} [(c_0, \dots, c_{p-1}) \triangleleft (c_p \triangleright d_0); (c_p \triangleleft d_0) \triangleright (d_1, \dots, d_q)], \end{aligned}$$

which gives the desired relation. It follows that $d^v d^h = -d^h d^v$.

The anticommutation relation $\sigma_{p+1,q}^{v,i} \sigma_{p,q}^{h,j} = -\sigma_{p,q+1}^{h,j} \sigma_{p,q}^{v,i}$ also follows from direct computation. Routine calculation shows that (4.2) is a first-quadrant double chain complex. \square

Definition 4.7. We call the double chain complex $(C_{\bullet,\bullet}(\mathcal{C}, \mathcal{D}), d^h, d^v)$ the *matched complex* of the matched pair $(\mathcal{C}, \mathcal{D})$.

Lemma 4.8. *The assignment $C_{\bullet,\bullet}$ of a matched complex to each matched pair is a functor from the category MP of matched pairs to the category of double complexes of Abelian groups.*

Proof. Matched-pair morphisms intertwine the face and degeneracy maps $\partial_{p,q}^{\bullet,i}$ and $\sigma_{p,q}^{\bullet,i}$. \square

Notation 4.9. For the remainder of the paper we frequently omit the subscripts on face maps, degeneracy maps and boundary maps. For example, $\partial^{h,i}$ denotes any of the maps $\partial_{p,q}^{h,i}$; the values of p and q should be clear from context.

There are two chain complexes associated to each double complex: the *diagonal complex* and the *total complex* [Wei94, §8.5].

4.3. The diagonal complex and diagonal homology. Let $(C_{\bullet,\bullet}(\mathcal{C}, \mathcal{D}), d^h, d^v)$ be the matched complex of a matched pair $(\mathcal{C}, \mathcal{D})$. For each $k \geq 0$, let

$$C_k^\Delta(\mathcal{C}, \mathcal{D}) := C_{k,k}(\mathcal{C}, \mathcal{D})$$

and define $\partial_k^{\Delta,i}: C_{k+1}^\Delta(\mathcal{C}, \mathcal{D}) \rightarrow C_k^\Delta(\mathcal{C}, \mathcal{D})$ and $\sigma_k^{\Delta,i}: C_k^\Delta(\mathcal{C}, \mathcal{D}) \rightarrow C_{k+1}^\Delta(\mathcal{C}, \mathcal{D})$ by

$$\partial_k^{\Delta,i} := \partial_{k,k}^{h,i} \partial_{k+1,k}^{v,i} \quad \text{and} \quad \sigma_k^{\Delta,i} := \sigma_{k+1,k}^{v,i} \sigma_{k,k}^{h,i}.$$

Then $(C_{\bullet}^\Delta(\mathcal{C}, \mathcal{D}), \partial^\Delta, \sigma^\Delta)$ is a simplicial group [Wei94, §8.5]. Let $d_k^\Delta := \sum_{i=0}^{k+1} (-1)^i \partial_k^{\Delta,i}$.

Definition 4.10. The *diagonal complex* of $(\mathcal{C}, \mathcal{D})$ is the chain complex $(C_{\bullet}^\Delta(\mathcal{C}, \mathcal{D}), d_{\bullet}^\Delta)$. We denote the homology of this chain complex by $H_{\bullet}^\Delta(\mathcal{C}, \mathcal{D})$.

4.4. The total complex and total homology. Let $(C_{\bullet,\bullet}(\mathcal{C}, \mathcal{D}), d^h, d^v)$ be the matched complex of a matched pair $(\mathcal{C}, \mathcal{D})$. For each $k \geq 0$, let

$$C_k^{\text{Tot}}(\mathcal{C}, \mathcal{D}) := \bigoplus_{p+q=k} C_{p,q}(\mathcal{C}, \mathcal{D}).$$

Define $d_k^{\text{Tot}}: C_{k+1}^{\text{Tot}}(\mathcal{C}, \mathcal{D}) \rightarrow C_k^{\text{Tot}}(\mathcal{C}, \mathcal{D})$ by $d_k^{\text{Tot}} := d_k^v + d_k^h$.

Definition 4.11. The *total complex* of $(\mathcal{C}, \mathcal{D})$ is the chain complex $(C_{\bullet}^{\text{Tot}}(\mathcal{C}, \mathcal{D}), d_{\bullet}^{\text{Tot}})$. We denote the homology of this complex by $H_{\bullet}^{\text{Tot}}(\mathcal{C}, \mathcal{D})$.

5. EQUIVALENCE OF HOMOLOGY THEORIES

In this section we prove that the homology theories for matched pairs introduced in Subsections 4.1, 4.3, and 4.4 coincide. Specifically, we describe natural chain maps that induce isomorphisms between them and between the dual cohomology theories. We also give formulae for their inverses. The main result is Theorem 5.3. We start by defining the maps involved.

5.1. The natural chain maps. We begin by describing explicit formulae for natural chain maps $\nabla: C_{\bullet}^{\text{Tot}} \rightarrow C_{\bullet}^\Delta$, $\Pi: C_{\bullet}^\Delta \rightarrow C_{\bullet}^{\boxtimes}$, and $\Psi: C_{\bullet}^{\boxtimes} \rightarrow C_{\bullet}^{\text{Tot}}$.

5.1.1. *The map ∇ .* The map $\nabla: C_{\bullet}^{\text{Tot}} \rightarrow C_{\bullet}^{\Delta}$ is the *Eilenberg–Zilber map* [Wei94, § 8.5.4]. For $p, q \in \mathbb{N}$ a (p, q) -*shuffle* is a permutation β of $\{1, \dots, p+q\}$ such that

$$\beta(1) < \beta(2) < \dots < \beta(p) \quad \text{and} \quad \beta(p+1) < \beta(p+2) < \dots < \beta(p+q).$$

We write $\text{Sh}(p, q)$ for the collection of all (p, q) -shuffles, and $\text{sgn}(\beta)$ for the sign of a permutation β . The (p, q) -component $\nabla_{p,q}: C_{p,q} \rightarrow C_{p+q}^{\Delta}$ of the Eilenberg–Zilber map is

$$\nabla_{p,q} = \sum_{\beta \in \text{Sh}(p,q)} \text{sgn}(\beta) \sigma_{p+q-1, p+q}^{h, \beta(p+q)} \circ \dots \circ \sigma_{p, p+q}^{h, \beta(p+1)} \circ \sigma_{p, p+q-1}^{v, \beta(p)} \circ \dots \circ \sigma_{p, q}^{v, \beta(1)}. \quad (5.1)$$

5.1.2. *The map Π .* We describe $\Pi: C_{\bullet}^{\Delta} \rightarrow C_{\bullet}^{\bowtie}$. For $k \geq 1$ define $\bowtie^k: (\mathcal{C} * \mathcal{D})^k \rightarrow (\mathcal{D} * \mathcal{C})^k$ by

$$\bowtie^k(c_1, d_1, \dots, c_k, d_k) := (c_1 \bowtie d_1, \dots, c_k \bowtie d_k).$$

Set $\Pi_0 = \text{id}_{\mathcal{C}^0}$ and inductively define $\Pi_k: \mathcal{C}^k * \mathcal{D}^k \rightarrow (\mathcal{C} \bowtie \mathcal{D})^k = (\mathcal{D} * \mathcal{C})^k$ for $k \geq 1$ by

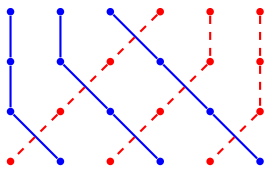
$$\Pi_k := \bowtie^k \circ (1_{\mathcal{C}} * \Pi_{k-1} * 1_{\mathcal{D}}). \quad (5.2)$$

These extend to homomorphisms $\Pi_k: C_k^{\Delta}(\mathcal{C}, \mathcal{D}) \rightarrow C_k^{\bowtie}(\mathcal{C}, \mathcal{D})$. For example,

$$\begin{aligned} \Pi_1[c_1, d_1] &= [c_1 \triangleright d_1, c_1 \triangleleft d_1] \\ \Pi_2[c_1, c_2, d_1, d_2] &= [c_1 c_2 \triangleright d_1, c_1 \triangleleft (c_2 \triangleright d_1), (c_2 \triangleleft d_1) \triangleright d_2, c_2 \triangleleft d_1 d_2] \\ \Pi_3[c_1, c_2, c_3, d_1, d_2, d_3] &= [c_1 c_2 c_3 \triangleright d_1, c_1 \triangleleft (c_2 c_3 \triangleright d_1), (c_2 c_3 \triangleleft d_1) \triangleright d_2, \\ &\quad c_2 \triangleleft (c_3 \triangleright d_1 d_2), (c_3 \triangleleft d_1 d_2) \triangleright d_3, c_3 \triangleleft d_1 d_2 d_3]. \end{aligned}$$

An induction on k , using that matched-pair morphisms respect left and right actions, shows that the Π_k extend to natural transformations $\Pi_k: C_k^{\Delta} \rightarrow C_k^{\bowtie}$.

Remark 5.1. The map Π_k can be described diagrammatically. We represent elements of \mathcal{C} by blue vertices, and elements of \mathcal{D} by red vertices; vertical lines are identity morphisms; and crossings are applications of \bowtie :



$$\begin{aligned} & [c_1, c_2, c_3, d_1, d_2, d_3] \\ & [c_1, c_2, c_3 \triangleright d_1, c_3 \triangleleft d_1, d_2, d_3] \\ & [c_1, c_2 c_3 \triangleright d_1, c_2 \triangleleft (c_3 \triangleright d_1), (c_3 \triangleleft d_1) \triangleright d_2, c_3 \triangleleft d_1 d_2, d_3] \\ & \Pi_3[c_1, c_2, c_3, d_1, d_2, d_3]. \end{aligned}$$

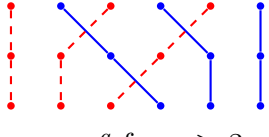
So starting with an element of $\mathcal{C}^k * \mathcal{D}^k$, we apply \bowtie to pairs of adjacent terms wherever possible until we obtain an element of $(\mathcal{D} * \mathcal{C})^k$.

5.1.3. *The map Ψ .* We now define $\Psi: C_{\bullet}^{\bowtie} \rightarrow C_{\bullet}^{\text{Tot}}$. For $q \geq 0$ define $\tau^q: (\mathcal{D} \bowtie \mathcal{C})^q \rightarrow \mathcal{D}^q * \mathcal{C}^q$ as follows: regard $(d_1 c_1, \dots, d_q c_q) \in (\mathcal{D} \bowtie \mathcal{C})^q$ as a composable q -tuple in $\mathcal{C}^* \bowtie \mathcal{D}^*$. By Proposition 3.20 there exist unique $d' \in \mathcal{D}^q$ and $c' \in \mathcal{C}^q$ such that $(d_1 c_1) \cdots (d_q c_q) = d' c' \in \mathcal{C}^p \bowtie \mathcal{D}^q$. For instance,

$$(d_1 c_1)(d_2 c_2)(d_3 c_3) = (d_1, c_1 \triangleright d_2, ((c_1 \triangleleft d_2) c_2) \triangleright d_3, c_1 \triangleleft (d_2 (c_2 \triangleright d_3)), c_2 \triangleleft d_3, c_3) \in \mathcal{D}^3 * \mathcal{C}^3.$$

We define $\tau^q(d_1 c_1, \dots, d_q c_q) := (d', c')$.

Remark 5.2. We can describe τ^q via a diagram using the same conventions as in Remark 5.1. For example τ^3 is represented by the diagram



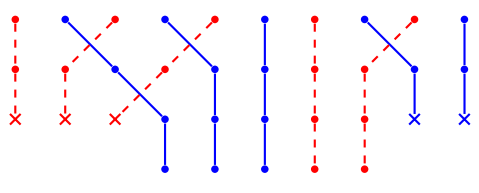
$$\begin{aligned}
 & (d_1, c_1, d_2, c_2, d_3, c_3) \\
 & (d_1, c_1 \triangleright d_2, c_1 \triangleleft d_2, c_2 \triangleright d_3, c_2 \triangleleft d_3, c_3) \\
 & (d_1, c_1 \triangleright d_2, ((c_1 \triangleleft d_2)c_2) \triangleright d_3, c_1 \triangleleft (d_2(c_2 \triangleright d_3)), c_2 \triangleleft d_3, c_3)
 \end{aligned}$$

The maps τ^q for $q \geq 3$ can be visualised similarly.

For $p, q \geq 0$ let $\rho_{p,q}: \mathcal{D}^p * \mathcal{C}^p * \mathcal{D}^q * \mathcal{C}^q \rightarrow \mathcal{C}^p * \mathcal{D}^q$ denote the projection onto the middle two factors. Define $\Psi_{p,q}: (\mathcal{C} \bowtie \mathcal{D})^{p+q} \rightarrow \mathcal{C}^p * \mathcal{D}^q$ by

$$\Psi_{p,q} = \rho_{p,q} \circ (\tau_p * \tau_q)$$

and extend it to a homomorphism $\Psi_{p,q}: C_{p+q}^{\bowtie}(\mathcal{C}, \mathcal{D}) \rightarrow C_{p,q}(\mathcal{C}, \mathcal{D})$. For example, we can represent $\Psi_{3,2}[d_1, c_1, \dots, d_5, c_5]$ diagrammatically by



$$\begin{aligned}
 & [d_1, c_1, \dots, d_5, c_5] \\
 & (\tau^3 * \tau^2)([d_1, c_1, \dots, d_5, c_5]) \\
 & \Psi_{3,2}([d_1, c_1, \dots, d_5, c_5])
 \end{aligned}$$

(crossed vertices like \times indicate omission of the corresponding entries).

We now define $\Psi_k: C_k^{\bowtie}(\mathcal{C}, \mathcal{D}) \rightarrow C_k^{\text{Tot}}(\mathcal{C}, \mathcal{D})$ by

$$\Psi_k = \sum_{p+q=k} \Psi_{p,q}. \quad (5.3)$$

Explicit formulae for low-degree terms are given by

$$\begin{aligned}
 \Psi_1[d_1 c_1] &= \Psi_{1,0}[d_1 c_1] + \Psi_{0,1}[d_1 c_1] = [c_1] + [d_1] \\
 \Psi_2[d_1 c_1, d_2 c_2] &= [c_1 \triangleleft d_2, c_2] + [c_1; d_2] + [d_1, c_1 \triangleright d_2] \\
 \Psi_3[d_1 c_1, d_2 c_2, d_3 c_3] &= [d_1, c_1 \triangleright d_2, ((c_1 \triangleleft d_2)c_2) \triangleright d_3] + [c_1; d_2, c_2 \triangleright d_3] \\
 &\quad + [c_1 \triangleleft d_2, c_2; d_3] + [c_1 \triangleleft (d_2(c_2 \triangleright d_3)), c_2 \triangleleft d_3, c_3].
 \end{aligned}$$

It is routine to verify that the Ψ_k extend to natural transformations $\Psi_k: C_k^{\bowtie} \rightarrow C_k^{\text{Tot}}$.

5.2. The statement of the main theorem. We state our main homology theorem and outline the proof. We write \mathbf{Ch}_\bullet for the category of abelian chain complexes and chain maps.

Theorem 5.3. *The formulae (5.1), (5.2), and (5.3) determine natural chain equivalences such that the diagram*

$$\begin{array}{ccc}
 C_\bullet^\Delta & \xleftarrow{\nabla} & C_\bullet^{\text{Tot}} \\
 & \searrow \Pi & \nearrow \Psi \\
 & C_\bullet^{\bowtie} &
 \end{array} \quad (5.4)$$

commutes up to natural chain homotopy. They induce natural isomorphisms

$$H_\bullet^{\bowtie} \cong H_\bullet^\Delta \cong H_\bullet^{\text{Tot}}$$

of functors from \mathbf{MP} to \mathbf{Ch}_\bullet . In particular, for any matched pair $(\mathcal{C}, \mathcal{D})$,

$$H_k^{\bowtie}(\mathcal{C}, \mathcal{D}) \cong H_k^\Delta(\mathcal{C}, \mathcal{D}) \cong H_k^{\text{Tot}}(\mathcal{C}, \mathcal{D}).$$

Before commencing the proof of this theorem, we record a corollary. Recall that for us, given a chain complex (C_\bullet, d_\bullet) and an abelian group A , the cohomology with coefficients in A is the cohomology of the dual cochain complex $(\text{Hom}(C_\bullet, A), d_\bullet^*)$.

Corollary 5.4. *For any fixed abelian group A , the duals of the natural chain maps ∇ , Π , and Ψ induce natural isomorphisms*

$$H_{\boxtimes}^\bullet(\cdot; A) \cong H_\Delta^\bullet(\cdot; A) \cong H_{\text{Tot}}^\bullet(\cdot; A)$$

of cohomology functors with coefficients in A .

Proof. Dualising all the maps in a chain homotopy diagram yields a cochain homotopy. \square

That ∇ induces a natural isomorphism is the content of a general form of the Eilenberg–Zilber Theorem. Following [Wei94, §8.5.4] the *Alexander–Whitney map* $\Delta: C_\bullet^\Delta \rightarrow C_\bullet^{\text{Tot}}$ is defined as follows: for p, q such that $p + q = n$, define $\Delta_{p,q}: C_n^\Delta \rightarrow C_{p,q}$ by

$$\Delta_{p,q} := \underbrace{\partial_{p+1}^h \cdots \partial_n^h}_{q \text{ terms}} \circ \underbrace{\partial_0^v \cdots \partial_0^v}_{p \text{ terms}}.$$

Then the map Δ is defined by

$$\Delta := \bigoplus_{p+q=n} \Delta_{p,q}. \quad (5.5)$$

Theorem 5.5 ([Wei94, Theorem 8.5.1]). *The map $\nabla: C_\bullet^{\text{Tot}} \rightarrow C_\bullet^\Delta$ of (5.1) induces a natural isomorphism $H_\bullet^\Delta \cong H_\bullet^{\text{Tot}}$, with inverse induced by the map $\Delta: C_\bullet^\Delta \rightarrow C_\bullet^{\text{Tot}}$ of (5.5).*

Given Theorem 5.5, to prove Theorem 5.3 it suffices to establish the natural isomorphism $H_\bullet^{\boxtimes} \cong H_\bullet^\Delta$. To do this we fill out a diagram

$$\begin{array}{ccc} C_\bullet^\Delta & \begin{array}{c} \xrightarrow{\Delta} \\ \xleftarrow{\nabla} \end{array} & C_\bullet^{\text{Tot}} \\ & \begin{array}{c} \xleftarrow{\Pi} \\ \xrightarrow{\Psi} \end{array} & \\ & C_\bullet^{\boxtimes} & \end{array} \quad (5.6)$$

of natural chain equivalences that commutes up to natural chain homotopy. We use the *method of acyclic models* (see [Rot88] for instance). The details occupy Subsections 5.3 and 5.4.

We show in Section 5.3 that the model matched pairs $(\mathcal{E}_{2k}, \mathcal{F}_{2k})$ satisfy $H_p^{\boxtimes}(\mathcal{E}_{2k}, \mathcal{F}_{2k}) = 0 = H_p^\Delta(\mathcal{E}_{2k}, \mathcal{F}_{2k})$ for all $p \geq 1$. So we can use these as the models in the method of acyclic models. We deduce that there exist natural chain equivalences between C_\bullet^{\boxtimes} and C_\bullet^Δ that induce natural isomorphisms on homology, and show how to recognise when given chain maps do the job.

In Subsection 5.5, we show that (5.2) and (5.3) are such chain maps. We also give explicit formulae for the remaining maps $\Pi := \nabla \circ \Psi$ and $\mathfrak{H} := \Pi \circ \nabla$.

5.3. Homological acyclicity of model matched pairs. The proof of Theorem 5.3 hinges on properties of the homology of the model matched pairs $(\mathcal{E}_n, \mathcal{F}_n)$. Recall that the object set of $\Gamma_n \cong \mathcal{E}_n * \mathcal{F}_n$ is $X_n = \{\mathbf{a} = (a_L, a_R) \in \mathbb{N} \times \mathbb{N} \mid 0 \leq a_L + a_R \leq n\}$. Each morphism of Γ_n is a pair $(\mathbf{a}, \mathbf{b}) \in X_n \times X_n$ such that $a_L \leq b_L$ and $a_R \geq b_R$.

The map $r \times s: \Gamma_n \rightarrow X_n \times X_n$ is injective. Hence,

$$\Gamma_n^k \ni (\gamma_1, \dots, \gamma_k) \mapsto (r(\gamma_1), s(\gamma_1), s(\gamma_2), \dots, s(\gamma_k)) \in X_n^{(k)}. \quad (5.7)$$

is a bijective correspondence between Γ_n^k and

$$X_n^{(k)} := \{(\mathbf{a}_0, \mathbf{a}_1, \dots, \mathbf{a}_k) \mid \mathbf{a}_i \in X_n, a_{i,L} \leq a_{i+1,L}, \text{ and } a_{i,R} \geq a_{i+1,R} \text{ for all } i\}.$$

Since $C_k^{\boxtimes}(\mathcal{E}_n, \mathcal{F}_n)$ is the free group $\mathbb{Z}\Gamma_n^k$, we have $C_k^{\boxtimes}(\mathcal{E}_n, \mathcal{F}_n) \cong \mathbb{Z}X_n^{(k)}$.

Let $\langle \mathbf{a}_0, \dots, \mathbf{a}_k \rangle$ denote the generator of $\mathbb{Z}X_n^{(k)}$ corresponding to $(\mathbf{a}_0, \dots, \mathbf{a}_k) \in X_n^{(k)}$. Using carats to denote elision of coordinates, the face and degeneracy maps on $C_k^{\boxtimes}(\mathcal{E}_n, \mathcal{F}_n)$ are

$$\partial_k^{\boxtimes, i} \langle \mathbf{a}_0, \dots, \mathbf{a}_k \rangle = \langle \mathbf{a}_0, \dots, \widehat{\mathbf{a}}_i, \dots, \mathbf{a}_k \rangle \quad \text{and} \quad \sigma_k^{\boxtimes, i} \langle \mathbf{a}_0, \dots, \mathbf{a}_k \rangle = \langle \mathbf{a}_0, \dots, \mathbf{a}_i, \mathbf{a}_i, \dots, \mathbf{a}_k \rangle.$$

A chain complex (C_\bullet, d_\bullet) is *acyclic* if $H_0(C_\bullet, d_\bullet) \cong \mathbb{Z}$ and $H_k(C_\bullet, d_\bullet) = 0$ for $k \geq 1$.

Recall that an *initial object* in a category \mathcal{C} is an object $v \in \mathcal{C}^0$ such that $w\mathcal{C}v$ has precisely one element for each $w \in \mathcal{C}^0$.

The following is well-known, but could not find an explicit reference.

Lemma 5.6. *Let \mathcal{C} be a small category with an initial object v . Let $\mathbb{1}$ be the category with a single morphism 1 . Let $\iota: \mathbb{1} \rightarrow \mathcal{C}$ be the functor such that $\iota(1) = v$. Let ρ be the unique functor from \mathcal{C} to $\mathbb{1}$. Then $\rho \circ \iota = \text{id}_{\mathbb{1}}$, and $(\iota \circ \rho)_\bullet: C_\bullet(\mathcal{C}) \rightarrow C_\bullet(\mathbb{1})$ is chain-homotopic to $\text{id}_{C_\bullet(\mathcal{C})}$. In particular, $(C_\bullet(\mathcal{C}), d_\bullet)$ is acyclic.*

Proof. Clearly, $\rho \circ \iota = \text{id}_{\mathbb{1}}$. For each $w \in \mathcal{C}^0$ let $\tau_w \in \mathcal{C}$ be the unique morphism from v to w . Fix $k \geq 0$. For $0 \leq i \leq k$ define $h^i: C_k(\mathcal{C}) \rightarrow C_{k+1}(\mathcal{C})$ by

$$h^i[c_0, \dots, c_{k-1}] = \begin{cases} [c_0, \dots, c_i, \tau_{s(c_i)}, v, \dots, v] & \text{if } i > 0 \\ [\tau_{r(c_0)}, v, \dots, v] & \text{if } i = 0. \end{cases}$$

To see that h is a simplicial homotopy we need to check that $\partial^0 h^0 = (\iota \circ \rho)_k$, that $\partial^{k+1} h^k = \text{id}_{C_k(\mathcal{C})}$, that $\partial^i h^j = h^{j-1} \partial^i$ for $i < j$, that $\partial^i h^j = h^j \partial^{i-1}$ for $i > j + 1$, that $\sigma^i h^j = h^{j+1} \sigma^i$ for $i \leq j$, and that $\sigma^i h^j = h^j \sigma^{i-1}$ for $i > j$. For the first two identities, we calculate

$$\begin{aligned} \partial^0 h^0[c_0, \dots, c_{k-1}] &= \partial^0[\tau_{r(c_0)}, v, \dots, v] = [v, \dots, v], \text{ and} \\ \partial^{k+1} h^k[c_0, \dots, c_{k-1}] &= \partial^{k+1}[c_0, \dots, c_{k-1}, \tau_{s(c_{k-1})}] = [c_0, \dots, c_{k-1}]. \end{aligned}$$

The remaining four conditions follow from similar calculations. For example, if $0 < i < j$, then

$$\begin{aligned} \partial^i h^j[c_0, \dots, c_{k-1}] &= \partial^i[c_0, \dots, c_j, \tau_{s(c_j)}, v, \dots, v] = [c_0, \dots, c_{i-1} c_i, \dots, c_j, \tau_{s(c_j)}, v, \dots, v] \\ &= h^{j-1}[c_0, \dots, c_{i-1} c_i, \dots, c_j, \dots, c_{k-1}] = h^{j-1} \partial^i[c_0, \dots, c_{k-1}]. \end{aligned}$$

Hence, the simplicial maps $(\iota \circ \rho)_\bullet$ and $\text{id}_{C_\bullet(\mathcal{C})}$ are simplicially homotopic. So $s_k := \sum_{i=0}^k (-1)^i h_k^i$ defines a chain homotopy s between $(\iota \circ \rho)_\bullet$ and $\text{id}_{C_\bullet(\mathcal{C})}$ [Wei94, Lemma 8.3.13].

The final statement follows from acyclicity of $(C_\bullet(\mathbb{1}), d_\bullet)$. □

Lemma 5.7. *For each $n \geq 0$ the chain complex $(C_\bullet^{\boxtimes}(\mathcal{E}_n, \mathcal{F}_n), d)$ is acyclic.*

Proof. The object $(n, 0)$ is an initial object in $\Gamma_n = \mathcal{E}_n \boxtimes \mathcal{F}_n$, so the result follows from Lemma 5.6. □

Chains in $C_{k,l}(\mathcal{E}_n, \mathcal{F}_n)$ also admit a tractable description. The formula

$$\begin{aligned} \mathcal{E}_n^k * \mathcal{F}_n^l \ni ((p_0, q_0), (p_1, q_0), \dots, (p_k, q_0), \dots, (p_k, q_{l-1}), (p_k, q_l)) \\ \mapsto (p_0, p_1, \dots, p_k; q_0, \dots, q_{l-1}, q_l) \in Y_n^{(k,l)} \end{aligned}$$

is a bijection between $\mathcal{E}_n^k * \mathcal{F}_n^l$ and

$$Y_n^{(k,l)} := \left\{ (p_0, \dots, p_k; q_0, \dots, q_l) \in \mathbb{N}^{2k} \mid p_i \leq p_{i+1}, q_i \geq q_{i+1}, \text{ and } p_k + q_0 \leq n \right\},$$

and induces an isomorphism $C_{k,l}(\mathcal{E}_n, \mathcal{F}_n) \cong \mathbb{Z}Y_n^{(k,l)}$.

We write $\langle p_0, \dots, p_k; q_0, \dots, q_l \rangle$ for the generator of $\mathbb{Z}Y_n^{(k,l)}$ that corresponds to the tuple $(p_0, \dots, p_k; q_0, \dots, q_l) \in Y_n^{(k,l)}$. The face maps in the double complex become

$$\begin{aligned} \partial^{h,i} \langle p_0, \dots, p_k; q_0, \dots, q_l \rangle &= \langle p_0, \dots, \widehat{p}_i, \dots, p_k; q_0, \dots, q_l \rangle \quad \text{and} \\ \partial^{v,i} \langle p_0, \dots, p_k; q_0, \dots, q_l \rangle &= \langle p_0, \dots, p_k; q_0, \dots, \widehat{q}_i, \dots, q_l \rangle. \end{aligned}$$

The degeneracy maps become

$$\begin{aligned} \sigma^{h,i} \langle p_0, \dots, p_k; q_0, \dots, q_l \rangle &= \langle p_0, \dots, p_{i-1}, p_i, p_i, p_{i+1}, \dots, p_k; q_0, \dots, q_l \rangle \quad \text{and} \\ \sigma^{v,i} \langle p_0, \dots, p_k; q_0, \dots, q_l \rangle &= \langle p_0, \dots, p_k; q_0, \dots, q_{i-1}, q_i, q_i, q_{i+1}, \dots, q_l \rangle. \end{aligned}$$

In particular, for the diagonal complex $C_\bullet^\Delta(\mathcal{E}_n, \mathcal{F}_n)$ the face and degeneracy maps are

$$\begin{aligned} \partial^{\Delta,i} \langle p_0, \dots, p_k; q_0, \dots, q_k \rangle &= \langle p_0, \dots, \widehat{p}_i, \dots, p_k; q_0, \dots, \widehat{q}_i, \dots, q_k \rangle \quad \text{and} \\ \sigma^{\Delta,i} \langle p_0, \dots, p_k; q_0, \dots, q_k \rangle &= \langle p_0, \dots, p_{i-1}, p_i, p_i, p_{i+1}, \dots, p_k; q_0, \dots, q_{i-1}, q_i, q_i, q_{i+1}, \dots, q_k \rangle. \end{aligned}$$

Lemma 5.8. *The diagonal complex $(C_\bullet^\Delta(\mathcal{E}_n, \mathcal{F}_n), d^\Delta)$ is acyclic.*

Proof. Consider the directed graph $G_n = 0 \xrightarrow{e_0} 1 \xrightarrow{e_1} \dots \xrightarrow{e_{n-1}} n$. Since n is an initial object for G_n^* , Lemma 5.6 implies that $(C_\bullet(G_n^*), d_\bullet)$ is acyclic. So it suffices to show that $(C_\bullet^\Delta(\mathcal{E}_n, \mathcal{F}_n), d^\Delta)$ is chain-homotopic to $(C_\bullet(G_n^*), d)$.

The group $C_k(G_n^*)$ is freely generated by k -tuples $\langle p_0, \dots, p_k \rangle$ where $0 \leq p_i \leq p_{i+1} \leq n$ for each $0 \leq i < k$. The functor $\iota: G_n^* \hookrightarrow \mathcal{E}_n$ given by $\iota(e_p) = e_{p,0}$ induces a chain map $\iota: C_\bullet(G_n^*) \rightarrow C_\bullet^\Delta(\mathcal{E}_n, \mathcal{F}_n)$ satisfying $\iota_k \langle p_0, \dots, p_k \rangle = \langle p_0, \dots, p_k; 0, \dots, 0 \rangle$. The functor $\rho: \mathcal{E}_n \rightarrow G_n^*$ defined by $\rho(e_{p,q}) = e_p$, induces a chain map $\rho: C_\bullet^\Delta(\mathcal{E}_n, \mathcal{F}_n) \rightarrow C_\bullet(G_n^*)$ satisfying $\rho_k \langle p_0, \dots, p_k; q_0, \dots, q_k \rangle = \langle p_0, \dots, p_k \rangle$. We have $\rho_k \circ \iota_k = \text{id}_{C_k(G_n^*)}$. For $0 \leq i \leq k$ define $h^i: C_k^\Delta(\mathcal{E}_n, \mathcal{F}_n) \rightarrow C_{k+1}^\Delta(\mathcal{E}_n, \mathcal{F}_n)$ by

$$h^i \langle p_0, \dots, p_k; q_0, \dots, q_k \rangle = \langle p_0, \dots, p_{i-1}, p_i, p_i, p_{i+1}, \dots, p_k; 0, \dots, 0, q_i, q_{i+1}, \dots, q_k \rangle.$$

Direct calculation shows that $\partial^0 h^0 = \text{id}_{C_k^\Delta(\mathcal{E}_n, \mathcal{F}_n)}$ and $\partial^{k+1} h^k = (\iota \circ \rho)_k$.

It is routine to check that $\partial^i h^j = h^{j-1} \partial^i$ for $i < j$ and $\partial^i h^j = h^j \partial^{i-1}$ for $i > j + 1$. Similarly, $\sigma^i h^j = h^{j+1} \sigma^i$ for $i \leq j$ and $\sigma^i h^j = h^j \sigma^{i-1}$ for $i > j$. It follows that the simplicial maps $(\iota \circ \rho)_\bullet$ and $\text{id}_{C_\bullet^\Delta(\mathcal{E}_n, \mathcal{F}_n)}$ are simplicially homotopic. \square

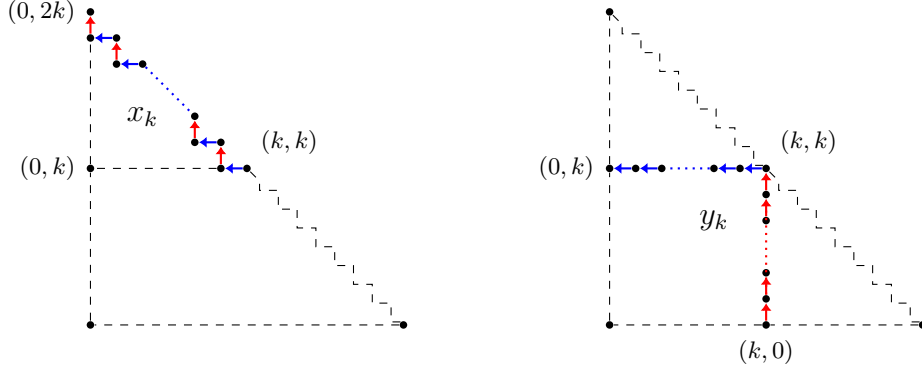
We identify some particularly useful chains in the categorical and diagonal homology of $(\mathcal{E}_{2k}, \mathcal{F}_{2k})$. For each $k \geq 0$ define $x_k \in C_k^{\boxtimes}(\mathcal{E}_{2k}, \mathcal{F}_{2k})$ and $y_k \in C_k^\Delta(\mathcal{E}_{2k}, \mathcal{F}_{2k})$ by

$$x_k := [f_{0,2k} e_{0,2k}, f_{1,2k-1} e_{1,2k-1}, \dots, f_{k-1,k+1} e_{k-1,k+1}] = \langle (0, 2k), (1, 2k-1), \dots, (k, k) \rangle \quad (5.8)$$

and

$$y_k := [e_{0,k}, e_{1,k}, \dots, e_{k-1,k}; f_{k,k-1}, \dots, f_{k,1}, f_{k,0}] = \langle 0, 1, \dots, k-1, k; k, k-1, \dots, 1, 0 \rangle. \quad (5.9)$$

Pictorially, x_k and y_k correspond to the following composable tuples in $\mathcal{E}_{2k} \bowtie \mathcal{F}_{2k}$:



By Corollary 3.10, a matched pair morphism $(\mathcal{E}_n, \mathcal{F}_n) \rightarrow (\mathcal{C}, \mathcal{D})$ corresponds to a functor $\Gamma_n \rightarrow \mathcal{C} \bowtie \mathcal{D}$ taking \mathcal{E}_n to \mathcal{C} and \mathcal{F}_n to \mathcal{D} . For each $\gamma = (d_0 c_0, \dots, d_{k-1} c_{k-1})$ in $(\mathcal{C} \bowtie \mathcal{D})^k$, Lemma 3.25 gives a morphism $h_\gamma^\bowtie: \Gamma_{2k} \rightarrow \mathcal{C} \bowtie \mathcal{D}$ such that

$$h_\gamma^\bowtie(f_{p, 2k-1-p} e_{p, 2k-1-p}) = \begin{cases} d_p c_p & \text{if } 0 \leq p < k \\ s(c_{k-1}) & \text{if } k \leq p < 2k. \end{cases}$$

For $\lambda = (c_0, \dots, c_{k-1}, d_0, \dots, d_{k-1}) \in \mathcal{C}^k * \mathcal{D}^k$, with $d'_i = r(c_i)$ and $c'_i = s(d_i)$, Lemma 3.25 applied to $(d'_0 c_0, \dots, d'_{k-1} c_{k-1}, d_0 c'_0, \dots, d_{k-1} c'_{k-1})$ yields a morphism $h_\lambda^\Delta: \Gamma_{2k} \rightarrow \mathcal{C} \bowtie \mathcal{D}$ such that

$$h_\lambda^\Delta(f_{p, 2k-1-p} e_{p, 2k-1-p}) = \begin{cases} c_p & \text{if } 0 \leq p < k \\ d_{p-k} & \text{if } k \leq p < 2k. \end{cases}$$

Lemma 5.9. *Let $(\mathcal{C}, \mathcal{D})$ be a matched pair. For $\gamma \in (\mathcal{C} \bowtie \mathcal{D})^k$ and $\lambda \in \mathcal{C}^k * \mathcal{D}^k$ we have $[\gamma] = C_k^\bowtie(h_\gamma^\bowtie)(x_k)$ and $[\lambda] = C_k^\Delta(h_\lambda^\Delta)(y_k)$. Moreover,*

$$\{C_k^\bowtie(h)(x_k) \mid h: (\mathcal{E}_{2k}, \mathcal{F}_{2k}) \rightarrow (\mathcal{C}, \mathcal{D})\} \quad \text{and} \quad \{C_k^\Delta(h)(y_k) \mid h: (\mathcal{E}_{2k}, \mathcal{F}_{2k}) \rightarrow (\mathcal{C}, \mathcal{D})\}$$

generate $C_k^\bowtie(\mathcal{C}, \mathcal{D})$ and $C_k^\Delta(\mathcal{C}, \mathcal{D})$ respectively.

Proof. That $[\gamma] = C_k^\bowtie(h_\gamma^\bowtie)(x_k)$ and $[\lambda] = C_k^\Delta(h_\lambda^\Delta)(y_k)$ follow immediately from the definitions of h_γ^\bowtie and h_λ^Δ . For the second statement, let $h: (\mathcal{E}_{2k}, \mathcal{F}_{2k}) \rightarrow (\mathcal{C}, \mathcal{D})$ be a matched pair morphism. Then $C_k^\bowtie(h)(x_k) = [h(f_{0, 2k} e_{0, 2k}), \dots, h(f_{k-1, k+1} e_{k-1, k+1})]$. So

$$\{C_k^\bowtie(h)(x_k) \mid h: (\mathcal{E}_{2k}, \mathcal{F}_{2k}) \rightarrow (\mathcal{C}, \mathcal{D})\} \supseteq \{C_k^\bowtie(h_\gamma^\bowtie)(x_k) \mid \gamma \in (\mathcal{C} \bowtie \mathcal{D})^k\} = \{[\gamma] \mid \gamma \in (\mathcal{C} \bowtie \mathcal{D})^k\},$$

which generates $C_k^\bowtie(\mathcal{C}, \mathcal{D})$. Similarly,

$$\{C_k^\Delta(h)(y_k) \mid h: (\mathcal{E}_{2k}, \mathcal{F}_{2k}) \rightarrow (\mathcal{C}, \mathcal{D})\} \supseteq \{[\lambda] \mid \lambda \in \mathcal{C}^k * \mathcal{D}^k\}$$

generates $C_k^\Delta(\mathcal{C}, \mathcal{D})$. \square

In the terminology of [Rot88, pp. 239–240], Lemma 5.9 says that the functors C_k^\bowtie and C_k^Δ from MP to \mathbf{Ch}_\bullet are *free* with bases $\{x_k\}$ and $\{y_k\}$, giving the following lemma.

Lemma 5.10 ([Rot88, Lemma 9.10]). *If $G: \mathbf{MP} \rightarrow \mathbf{Ab}$ is a functor and $g \in G(\mathcal{E}_{2k}, \mathcal{F}_{2k})$, then there is a unique natural transformation $\alpha: C_k^\bowtie \rightarrow G$ such that $\alpha_{(\mathcal{E}_{2k}, \mathcal{F}_{2k})}(x_k) = g$, and a unique natural transformation $\beta: C_k^\Delta \rightarrow G$ such that $\beta_{(\mathcal{E}_{2k}, \mathcal{F}_{2k})}(y_k) = g$.*

The proof of [Rot88, Lemma 9.10] describes the natural transformations of Lemma 5.10: for $[\gamma] \in C_k^\bowtie(\mathcal{C}, \mathcal{D})$, Lemma 5.9 gives $h_\gamma^\bowtie: (\mathcal{E}_{2k}, \mathcal{F}_{2k}) \rightarrow (\mathcal{C}, \mathcal{D})$ such that $h_\gamma^\bowtie(x_k) = [\gamma]$, and then $\alpha_{(\mathcal{C}, \mathcal{D})}([\gamma]) := G(h_\gamma^\bowtie)(g)$. Similarly, $\beta_{(\mathcal{C}, \mathcal{D})}([\lambda]) := G(h_\lambda^\Delta)(g)$ for $[\lambda] \in C_k^\Delta(\mathcal{C}, \mathcal{D})$.

5.4. Proof of the main theorem. To prove Theorem 5.3 we construct a chain equivalence between C_\bullet^\bowtie and C_\bullet^Δ inductively using [Rot88, Theorem 9.12].

Lemma 5.11. *The identity map $C_0^\Delta(\mathcal{C}, \mathcal{D}) := \mathbb{Z}X \xrightarrow{\text{id}} \mathbb{Z}X =: C_0^\bowtie(\mathcal{C}, \mathcal{D})$ induces a natural isomorphism $\tilde{\text{id}}: H_0^\bowtie \cong H_0^\Delta$.*

Proof. Fix a matched pair $(\mathcal{C}, \mathcal{D})$ with objects X . Identifying $C_0^\Delta(\mathcal{C}, \mathcal{D})$ with $C_0^\bowtie(\mathcal{C}, \mathcal{D})$ via the identity map on $\mathbb{Z}X$, it suffices to show that $\text{im}(d^\bowtie) = \text{im}(d^\Delta)$ in $\mathbb{Z}X$. If $[d, c] \in C_1^\bowtie(\mathcal{C}, \mathcal{D})$, then $d^\bowtie[d, c] = [s(c)] - [r(d)] = d^\Delta[c, s(c)] + d^\Delta[r(d), d] \in \text{im}(d^\Delta)$. If $[c, d] \in C_1^\Delta(\mathcal{C}, \mathcal{D})$, then $d^\Delta[c, d] = [s(d)] - [r(c)] = d^\bowtie[c \bowtie d] \in \text{im}(d^\bowtie)$. \square

Proposition 5.12. *There exist natural chain maps $\alpha: C_\bullet^\bowtie \rightarrow C_\bullet^\Delta$ and $\beta: C_\bullet^\Delta \rightarrow C_\bullet^\bowtie$ such that $\alpha \circ \beta$ is naturally chain-homotopic to $\text{id}_{C_\bullet^\Delta}$ and $\beta \circ \alpha$ is naturally chain-homotopic to $\text{id}_{C_\bullet^\bowtie}$ such that α and β lift the natural isomorphism $H_0^\bowtie \cong H_0^\Delta$, in the sense that the diagram*

$$\begin{array}{ccccccc}
 \cdots & \rightarrow & C_2^\bowtie & \xrightarrow{d_1^\bowtie} & C_1^\bowtie & \xrightarrow{d_0^\bowtie} & C_0^\bowtie & \longrightarrow & H_0^\bowtie & \longrightarrow & 0 \\
 & & \alpha_2 \downarrow & & \alpha_1 \downarrow & & \alpha_0 \downarrow & & \tilde{\text{id}} \downarrow & & \\
 \cdots & \rightarrow & C_2^\Delta & \xrightarrow{d_1^\Delta} & C_1^\Delta & \xrightarrow{d_0^\Delta} & C_0^\Delta & \longrightarrow & H_0^\Delta & \longrightarrow & 0 \\
 & & \beta_2 \downarrow & & \beta_1 \downarrow & & \beta_0 \downarrow & & \tilde{\text{id}} \downarrow & & \\
 \cdots & \rightarrow & C_2^\bowtie & \xrightarrow{d_1^\bowtie} & C_1^\bowtie & \xrightarrow{d_0^\bowtie} & C_0^\bowtie & \longrightarrow & H_0^\bowtie & \longrightarrow & 0
 \end{array} \tag{5.10}$$

of natural transformations commutes. If $\alpha': C_\bullet^\bowtie \rightarrow C_\bullet^\Delta$ and $\beta': C_\bullet^\Delta \rightarrow C_\bullet^\bowtie$ are chain maps that lift the natural isomorphism $H_0^\bowtie \cong H_0^\Delta$, then they are naturally chain-homotopic to α and β .

For $k \geq 0$, let x_k, y_k be as in (5.8) and (5.9). If for each $k \geq 0$, $\alpha_k: C_k^\Delta \rightarrow C_k^\bowtie$ is a natural transformation such that $d^\Delta \circ \alpha_k(x_k) = \alpha_{k-1} \circ d^\Delta(x_k)$, then $\alpha = (\alpha_k)$ is a natural chain equivalence from C_\bullet^Δ to C_\bullet^\bowtie . Similarly, if for each $k \geq 0$, $\beta_k: C_k^\bowtie \rightarrow C_k^\Delta$ is a natural transformations such that $d^\bowtie \circ \beta_k(y_k) = \beta_{k-1} \circ d^\bowtie(y_k)$, then $\beta = (\beta_k)$ is a natural chain equivalence from C_\bullet^Δ to C_\bullet^\bowtie .

The result is standard and follows from [Rot88, Theorem 9.12], but we include some details to describe the resulting isomorphisms in homology explicitly.

Proof. The morphisms α_0 and β_0 are induced by the identity maps on objects. We start by constructing α . Suppose that there exists maps α_n , for $n < k$ such that the right-most $n + 1$ squares of (5.10) commute. Consider the matched pair $(\mathcal{E}_{2k}, \mathcal{F}_{2k})$ and let $x_k \in C_k^\bowtie(\mathcal{E}_{2k}, \mathcal{F}_{2k})$ be as in (5.8). Commutativity of (5.10) implies that $d^\Delta \alpha_{k-1} d^\bowtie = (d^\Delta)^2 \alpha_{k-2} = 0$. Lemma 5.7 implies that $(C_\bullet^\bowtie(\mathcal{E}_{2k}, \mathcal{F}_{2k}), d^\bowtie)$ is acyclic, so there exists $\bar{x}_k \in C_k^\Delta(\mathcal{E}_{2k}, \mathcal{F}_{2k})$ such that $d^\Delta(\bar{x}_k) = \alpha_{k-1} d^\bowtie(x_k)$. So Lemma 5.10 yields a unique natural transformation $\alpha_k: C_k^\bowtie \rightarrow C_k^\Delta$ such that $\alpha_k^{(\mathcal{E}_{2k}, \mathcal{F}_{2k})}(x_k) = \bar{x}_k$ and $d^\Delta \alpha_k = \alpha_{k-1} d^\bowtie$.

A similar construction using y_k and Lemma 5.8 gives β_k . By [Rot88, Theorem 9.12] α_k and β_k induce natural chain equivalences α and β and these are, up to natural chain homotopy, the unique chain equivalences lifting the isomorphism of Lemma 5.11 \square

Proof of Theorem 5.3. The result follows from Proposition 5.12 and Theorem 5.5. \square

5.5. Explicit formulas for the natural isomorphisms between homology theories. Proposition 5.12 yields a natural isomorphism $C_{\bullet}^{\Delta} \cong C_{\bullet}^{\boxtimes}$, and its final statement says how to recognise chain maps α, β that induce such an isomorphism. We show that the map Π of (5.2) and $\Pi := \nabla \circ \Psi$ are such chain maps, and describe chain maps inducing the remaining arrows in (5.6). We first examine how Π_k behaves on the model matched pairs $(\mathcal{E}_m, \mathcal{F}_m)$.

Lemma 5.13. *Fix $m, k \geq 0$, and consider $\Pi_k : C_k^{\Delta}(\mathcal{E}_m, \mathcal{F}_m) \rightarrow C_k^{\boxtimes}(\mathcal{E}_m, \mathcal{F}_m)$. Then*

$$\Pi_k \langle p_0, \dots, p_k; q_0, \dots, q_k \rangle = \langle (p_0, q_0), \dots, (p_k, q_k) \rangle.$$

In particular, the element $y_k \in C_k^{\Delta}(\mathcal{E}_{2k}, \mathcal{F}_{2k})$ from (5.9), satisfies

$$\Pi_k(y_k) = [f_{0,k-1}e_{0,k-1}, f_{1,k-2}e_{1,k-2}, \dots, f_{k-2,1}e_{k-2,1}, f_{k-1,0}e_{k-1,0}]. \quad (5.11)$$

Proof. We begin by establishing (5.11). We proceed by induction on k . The case $k = 0$ is trivial. Suppose inductively that the analogue of (5.11) describes $\Pi_{k-1}(y_{k-1})$. Recall that

$$y_k = [e_{0,k}, e_{1,k}, \dots, e_{k-1,k}; f_{k,k-1}, \dots, f_{k,1}, f_{k,0}],$$

and let

$$\lambda_{k-1} := [e_{1,k}, \dots, e_{k-1,k}; f_{k,k-1}, \dots, f_{k,1}] \in C_k^{\Delta}(\mathcal{E}_{2(k-1)}, \mathcal{F}_{2(k-1)}).$$

By Lemma 5.9 there exists a morphism $h_{\lambda_{k-1}}^{\Delta} : (\mathcal{E}_{2(k-1)}, \mathcal{F}_{2(k-1)}) \rightarrow (\mathcal{E}_{2k}, \mathcal{F}_{2k})$ such that $\lambda_{k-1} = C_{k-1}^{\Delta}(h_{\lambda_{k-1}}^{\Delta})(y_{k-1})$. The inductive hypothesis gives

$$(1_{\mathcal{E}_{2k}} * \Pi_{k-1} * 1_{\mathcal{F}_{2k}})(y_k) = [e_{0,k}, f_{1,k-1}, e_{1,k-1}, f_{2,k-2}, \dots, e_{k-2,2}, f_{k-1,1}, e_{k-1,1}, f_{k,0}],$$

and applying \boxtimes^k yields (5.11).

For the second statement, let $\gamma = \langle p_0, \dots, p_k; q_0, \dots, q_k \rangle$. Lemma 5.9 gives a morphism $h_{\gamma}^{\Delta} : (\mathcal{E}_{2k}, \mathcal{F}_{2k}) \rightarrow (\mathcal{E}_m, \mathcal{F}_m)$ such that $C_k^{\Delta}(h_{\gamma}^{\Delta})(y_k) = \gamma$. Naturality of Π_k implies that $\Pi_k(\gamma) = C_k^{\boxtimes}(h_{\gamma}^{\Delta}) \circ \Pi_k(y_k)$. So the result follows from (5.11) and the definition of h_{γ}^{Δ} . \square

Proposition 5.14. *For $k \geq 0$, we have $\Pi_{k-1} \circ d^{\Delta}(y_k) = d^{\boxtimes} \circ \Pi_k(y_k)$. In particular, $\Pi : C_{\bullet}^{\Delta} \rightarrow C_{\bullet}^{\boxtimes}$ induces a natural isomorphism on homology.*

Proof. Fix $k \geq 0$. By Lemma 5.13, for each $0 \leq i \leq k$, we have

$$\begin{aligned} \Pi_{k-1} \circ \partial_i^{\Delta}(y_k) &= \Pi_{k-1} \langle 0, 1, \dots, \widehat{i}, \dots, k-1, k; k, k-1, \dots, \widehat{k-i}, \dots, 1, 0 \rangle \\ &= \langle (0, k), (1, k-1), \dots, (i, k-i), \dots, (k-1, 1), (k, 0) \rangle = \partial_i^{\boxtimes} \circ \Pi_k(y_k). \end{aligned}$$

A similar calculation gives $\Pi_{k+1} \circ \sigma_i^{\Delta}(y_k) = \sigma_i^{\boxtimes} \circ \Pi_k(y_k)$ and so $d^{\boxtimes} \circ \Pi_k(y_k) = \Pi_{k-1} \circ d^{\Delta}(y_k)$. The final statement follows from Proposition 5.12. \square

We next examine how the map Ψ of (5.3) behaves on the model matched pairs.

Lemma 5.15. *Fix $k, m \geq 0$ and $a = \langle \mathbf{a}_0, \dots, \mathbf{a}_k \rangle \in C_k^{\boxtimes}(\mathcal{E}_m, \mathcal{F}_m)$. We have*

$$\Psi_k \langle \mathbf{a}_0, \dots, \mathbf{a}_k \rangle = \sum_{i=0}^k \langle a_0^L, \dots, a_i^L; a_i^R, \dots, a_k^R \rangle. \quad (5.12)$$

In particular, $x_k \in C_k^{\boxtimes}(\mathcal{E}_{2k}, \mathcal{F}_{2k})$ as in (5.8), satisfies

$$\Psi_k(x_k) = \sum_{i=0}^k \langle 0, 1, \dots, i-1, i; 2k-i, 2k-i-1, \dots, k+1, k \rangle. \quad (5.13)$$

Proof. The formula (5.7) gives a bijection between composable q -tuples in $\Gamma_m = \mathcal{E}_m \bowtie \mathcal{F}_m$ and the set $X_m^{(q)}$. We claim that under this identification, if $(\mathbf{a}_0, \dots, \mathbf{a}_q) \in X_m^{(q)}$, then

$$\tau_q(\mathbf{a}_0, \dots, \mathbf{a}_q) = ((a_0^L, a_0^R), \dots, (a_0^L, a_{q-1}^R), (a_0^L, a_q^R), (a_1^L, a_q^R), \dots, (a_q^L, a_q^R)) \quad (5.14)$$

in Γ_m^q . The tuples $((a_0^L, a_0^R), (a_0^L, a_1^R), \dots, (a_0^L, a_q^R))$ and $((a_0^L, a_q^R), \dots, (a_{q-1}^L, a_q^R), (a_q^L, a_q^R))$ belong to $\mathcal{E}_m^q \subseteq \Gamma_m^q$ and $\mathcal{F}_m^q \subseteq \Gamma_m^q$. Since $r \times s: \Gamma_m \rightarrow X_m \times X_m$ is injective, and factorisation in $\mathcal{E}_m^* \bowtie \mathcal{F}_m^*$ is unique by Proposition 3.20, the formula (5.14) follows.

Since $\Psi_{p,q} = \rho_{p,q} \circ (\tau_p * \tau_q)$ by definition, and $\Psi_k = \sum_{i=0}^k \Psi_{i,k-i}$ the identity (5.12) holds; and (5.13) follows from (5.12) since $x_k = \langle (0, 2k), (1, 2k-1), \dots, (k, k) \rangle$. \square

Proposition 5.16. *For $k \geq 0$, we have $\Psi_{k-1} \circ d^{\boxtimes} = d^{\text{Tot}} \circ \Psi_k$. The chain maps $\Psi: C_{\bullet}^{\boxtimes} \rightarrow C_{\bullet}^{\text{Tot}}$ and $\Pi: C_{\bullet}^{\boxtimes} \rightarrow C_{\bullet}^{\Delta}$ are natural chain equivalences inducing isomorphisms in homology.*

Proof. Let $a = \langle \mathbf{a}_0, \dots, \mathbf{a}_k \rangle \in C_k^{\boxtimes}(\mathcal{E}_m, \mathcal{F}_m)$. For each $0 \leq i \leq k$,

$$\begin{aligned} \Psi_{k-1} \circ \partial_i^{\boxtimes}(a) &= \sum_{0 \leq p < i} \langle a_0^L, \dots, a_p^L; a_p^R, \dots, \widehat{a_i^R}, \dots, a_k^R \rangle + \sum_{i < p \leq k} \langle a_0^L, \dots, \widehat{a_i^L}, \dots, a_p^L; a_0^R, \dots, a_k^R \rangle \\ &= \sum_{0 \leq p < i} \partial_{i-p}^v \circ \Psi_{p,k-p}(a) + \sum_{i < p \leq k} \partial_i^h \circ \Psi_{p,k-p}(a). \end{aligned}$$

Consequently,

$$\Psi_{k-1} \circ d^{\boxtimes} = \sum_{0 \leq p < i \leq k} (-1)^i \partial_{i-p}^v \circ \Psi_{p,k-p} + \sum_{0 \leq i < p \leq k} (-1)^i \partial_i^h \circ \Psi_{p,k-p}.$$

Using at the third equality that, $\partial_0^v \circ \Psi_{0,k} = 0$ and $\partial_k^h \circ \Psi_{k,0} = 0$, we calculate:

$$\begin{aligned} d^{\text{Tot}} \circ \Psi_k &= \sum_{p=0}^k \left(\sum_{i=0}^{k-p} (-1)^{i+p} \partial_i^v \circ \Psi_{p,k-p} + \sum_{i=0}^p (-1)^i \partial_i^h \circ \Psi_{p,k-p} \right) \\ &= \sum_{0 \leq p < i \leq k} (-1)^i \partial_{i-p}^v \circ \Psi_{p,k-p} + \sum_{0 \leq i < p \leq k} (-1)^i \partial_i^h \circ \Psi_{p,k-p} \\ &\quad + \sum_{p=0}^k (-1)^p \partial_0^v \circ \Psi_{p,k-p} + (-1)^p \partial_p^h \circ \Psi_{p,k-p} \\ &= \Psi_{k-1} \circ d^{\boxtimes} + \sum_{t=0}^{k-1} (-1)^t (\partial_0^v \circ \Psi_{t,k-t} - \partial_{t+1}^h \circ \Psi_{t+1,k-t-1}). \end{aligned}$$

So $d^{\text{Tot}} \circ \Psi_k = \Psi_{k-1} \circ d^{\boxtimes}$ because

$$\partial_0^v \langle a_0^L, \dots, a_t^L; a_t^R, \dots, a_k^R \rangle = \langle a_0^L, \dots, a_t^L; a_{t+1}^R, \dots, a_k^R \rangle = \partial_{t+1}^h \langle a_0^L, \dots, a_{t+1}^L; a_{t+1}^R, \dots, a_k^R \rangle.$$

Since $\Pi = \nabla \circ \Psi$ and ∇ is a chain map, $d_{k-1}^{\Delta} \circ \Pi_k(x_k) = \Pi_{k-1} \circ d_k^{\boxtimes}(x_k)$. So Proposition 5.12 shows that Π is a natural chain equivalence inverse to Ψ . Theorem 5.5 implies that Ψ is also a natural chain equivalence. \square

To determine an explicit formula for Π_k we combine the formula (5.3) for Ψ_k with the formula (5.1) for the Eilenberg–Zilber map ∇_k . Explicit formulae for the first few Π_i are

$$\begin{aligned}\Pi_1[d_1c_1] &= [c_1; _] + [_ ; d_1] \\ \Pi_2[d_1c_1, d_2c_2] &= [c_1 \triangleleft d_2, c_2; _ , _] + [c_1, _ ; _ , d_2] - [_ , c_1; d_2, _] \\ &\quad + [_ , _ ; d_1, c_1 \triangleright d_2] \\ \Pi_3[d_1c_1, d_2c_2, d_3c_3] &= [d_1, c_1 \triangleright d_2, ((c_1 \triangleleft d_2)c_2) \triangleright d_3; _ , _ , _] + [_ , _ , c_1; d_2, c_2 \triangleright d_3, _] \\ &\quad - [_ , c_1, _ ; d_2, _ , c_2 \triangleright d_3] + [c_1, _ , _ ; _ , d_2, c_2 \triangleright d_3] \\ &\quad + [_ , c_1 \triangleleft d_2, c_2; d_3, _ , _] - [c_1 \triangleleft d_2, _ , c_2; _ , d_3, _] \\ &\quad + [c_1 \triangleleft d_2, c_2, _ ; _ , _ , d_3] + [_ , _ , _ ; c_1 \triangleleft (d_2(c_2 \triangleright d_3)), c_2 \triangleleft d_3, c_3].\end{aligned}$$

Remark 5.17. The formula (5.3) for Ψ_k was not initially obvious to us. We found formulae for Π_k for $k \leq 3$ using a computer-aided search predicated on formulae that involved factorisation in $\mathcal{C} \bowtie \mathcal{D}$ of the element $d_1c_1 \cdots d_kc_k$, interspersed with objects to obtain elements of $\mathcal{C}^k * \mathcal{D}^k$. We searched for, and found, integer coefficients that solved a \mathbb{Z} -linear equation ensuring a chain map that inverts Π_k on homology. With those in hand, we could guess, and then check, a general formula for Π_k , and then reverse-engineer a formula for Ψ_k .

We can also translate between categorical and total chains using the maps Ψ and $\mathfrak{H} = \Pi \circ \nabla$. For low-degree terms $\mathfrak{H}: \bigoplus_{p+q=k} \mathbb{Z}(\mathcal{C}^p * \mathcal{D}^q) \rightarrow C_k^{\text{cat}}(\mathcal{C}, \mathcal{D})$ is given explicitly by

$$\begin{aligned}\mathfrak{H}_1([c], [d]) &= [c] + [d] \\ \mathfrak{H}_2([c_1, c_2], [c_3; d_1], [d_2, d_3]) &= [c_1, c_2] + [c_3, d_1] - [c_3 \triangleright d_1, c_3 \triangleleft d_1] + [d_2, d_3] \\ \mathfrak{H}_3([c_1, c_2, c_3], [c_4, c_5; d_1], [c_6; d_2, d_3], [d_4, d_5, d_6]) &= [c_1, c_2, c_3] + [(c_4c_5) \triangleright d_1, c_4 \triangleleft (c_5 \triangleright d_1), c_5 \triangleright d_1] \\ &\quad - [c_4, c_5 \triangleright d_1, c_5 \triangleleft d_1] + [c_4, c_3, d_1] \\ &\quad + [c_6, d_2, d_3] - [c_6 \triangleright d_2, c_6 \triangleleft d_2, d_3] \\ &\quad + [c_6 \triangleleft d_2, (c_6 \triangleleft d_2) \triangleright d_3, c_6 \triangleleft (d_2d_3)] + [d_4, d_5, d_6].\end{aligned}$$

5.6. A spectral sequence and a Künneth Theorem.

5.6.1. *A spectral sequence.* There is a spectral sequence that computes the total homology of a double complex; this and Theorem 5.3 compute of the homology of $\mathcal{C} \bowtie \mathcal{D}$.

For fixed $p \in \mathbb{N}$, the sequence

$$\cdots \xrightarrow{d_{p,2}^v} C_{p,2} \xrightarrow{d_{p,1}^v} C_{p,1} \xrightarrow{d_{p,0}^v} C_{p,0} \quad (5.15)$$

(the p -th column of the double complex (4.2)) is a chain complex with homology groups

$$H_{p,q}^v(\mathcal{C}, \mathcal{D}) := H_q(C_{p,\bullet}, d_{p,\bullet}^v). \quad (5.16)$$

Since $d_{p,q}^v \circ d_{p,q+1}^h = -d_{p,q}^h \circ d_{p+1,q}^v$, the maps $d_{p,q}^h$ descend to homomorphisms $\tilde{d}_{p,q}^h: H_{p+1,q}^v(\mathcal{C}, \mathcal{D}) \rightarrow H_{p,q}^v(\mathcal{C}, \mathcal{D})$. For each $q \in \mathbb{N}$, the sequence

$$\cdots \xrightarrow{\tilde{d}_{2,q}^h} H_{2,q}^v(\mathcal{C}, \mathcal{D}) \xrightarrow{\tilde{d}_{1,q}^h} H_{1,q}^v(\mathcal{C}, \mathcal{D}) \xrightarrow{\tilde{d}_{0,q}^h} H_{0,q}^v(\mathcal{C}, \mathcal{D}) \quad (5.17)$$

is a chain complex. We define $H_p^h H_q^v(\mathcal{C}, \mathcal{D})$ to be the p -th homology group of this complex,

$$H_p^h H_q^v(\mathcal{C}, \mathcal{D}) := H_p(H_{\bullet,q}^v(\mathcal{C}, \mathcal{D}), \tilde{d}_{\bullet,q}^h).$$

We define $H_q^v H_p^h(\mathcal{C}, \mathcal{D})$ symmetrically by first considering rows of (4.2) and then columns:

$$H_q^v H_p^h(\mathcal{C}, \mathcal{D}) := H_q(H_{p,\bullet}^h(\mathcal{C}, \mathcal{D}), \tilde{d}_{p,\bullet}^v).$$

Corollary 5.18 (cf. [Wei94, §5.6]). *Let $C_{\bullet,\bullet}$ be the matched complex of a matched pair $(\mathcal{C}, \mathcal{D})$. Then there are homology spectral sequences $\{E_{p,q}^{hv,r}, d_{p,q}^{hv,r}\}$ and $\{E_{p,q}^{vh,r}, d_{p,q}^{vh,r}\}$ with first pages $E_{p,q}^{hv,1} = H_{p,q}^v(\mathcal{C}, \mathcal{D})$ and $E_{p,q}^{vh,1} = H_{p,q}^h(\mathcal{C}, \mathcal{D})$ with $d^{hv,1} = \tilde{d}^h$ and $d^{vh,1} = \tilde{d}^v$, and second pages*

$$E_{p,q}^{hv,2} = H_p^h H_q^v(\mathcal{C}, \mathcal{D}) \quad \text{and} \quad E_{p,q}^{vh,2} = H_q^v H_p^h(\mathcal{C}, \mathcal{D}),$$

that both converge to the categorical homology of $\mathcal{C} \bowtie \mathcal{D}$.

We will use these spectral sequences to compute the homology of examples in Section 6.

5.6.2. *The Künneth Theorem for products of monoids.* Let S and R be monoids. Define a matched pair (S, R) by $s \triangleright r = r$ and $s \triangleleft r = s$. The monoid $S \bowtie R$ is just $S \times R$. There is an isomorphism of double complexes $C_\bullet(S) \otimes_{\mathbb{Z}} C_\bullet(R) \cong C_{\bullet,\bullet}$ taking $[s_1, \dots, s_p] \otimes [r_1, \dots, r_q]$ to $[s_1, \dots, s_p, r_1, \dots, r_q]$.

Theorem 5.3 implies that $H_\bullet(S \times R) \cong H_{\bullet,\bullet}^{\text{Tot}}(S, R) = H^{\text{Tot}}(C_{\bullet,\bullet})$. So we recover the Künneth formula [Wei94, Theorem 3.6.3]: an unnaturally split exact sequence

$$0 \rightarrow \bigoplus_{p+q=n} H_p(S) \otimes H_q(R) \rightarrow H_n(S \times R) \rightarrow \bigoplus_{p+q=n-1} \text{Tor}_1^{\mathbb{Z}}(H_p(S), H_q(R)) \rightarrow 0.$$

6. EXAMPLES AND HOMOLOGY COMPUTATIONS

In this section (specifically in Section 6.4) we use our results to compute the homology of matched pairs that are like pullbacks of odometer actions over the path categories of directed graphs. The technical results we develop along the way apply to more-general systems such as Exel–Pardo self-similar systems and k -graphs.

We first consider, in Section 6.1, matched pairs $(\mathcal{C}, \mathcal{D})$ in which $\mathcal{D} = E^*$ for a directed graph E . We show that the vertical homology $H_{p,\bullet}^v(\mathcal{C}, E^*)$ of (5.16) vanishes above degree 1. This is unsurprising since directed graphs are 1-dimensional; but we could not find a general theorem that applies, so we prove that $H_{p,q}^v(\mathcal{C}, E^*) = 0$ for $q \geq 2$ by direct computation.

In Section 6.2 we consider matched pairs $(\mathcal{C}, \mathcal{D})$ where \mathcal{C} is a disjoint union $\bigsqcup_{u \in \mathcal{D}^0} \mathcal{C}_u$ of monoids. We describe an isomorphism between $H_{\bullet,q}^h(\mathcal{C}, \mathcal{D})$ and the direct sum $\bigoplus_{u \in \mathcal{D}^0} H_\bullet(\mathcal{C}_u; \mathbb{Z}u\mathcal{D}^q)$ of the homology of the monoids \mathcal{C}_u with coefficients in $\mathbb{Z}u\mathcal{D}^q$. The Universal Coefficient Theorem gives a short exact sequence that computes $H_{\bullet,q}^h(\mathcal{C}, \mathcal{D})$ as an extension of an appropriate Tor-group by $\bigoplus_{u \in \mathcal{D}^0} H_p(\mathcal{C}_u) \otimes_{\mathbb{Z}\mathcal{C}_u} \mathbb{Z}u\mathcal{D}^q$.

In Section 6.3 we restrict further to $\mathcal{C}_u \cong \mathbb{Z}$. We deduce that $H_{\bullet,q}^h(\mathbb{Z} \times \mathcal{D}^0, \mathcal{D})$ vanishes in degree 2 or more, and compute $H_{0,q}^h(\mathbb{Z} \times \mathcal{D}^0, \mathcal{D})$ and $H_{1,q}^h(\mathbb{Z} \times \mathcal{D}^0, \mathcal{D})$ in terms of the groups of invariants and coinvariants $\mathbb{Z}u\mathcal{D}^q$.

Finally, in Section 6.4 we compute $H_{\bullet,\bullet}^{\bowtie}(\mathbb{Z} \times F^0, F^*)$ for matched pairs consisting of a bundle of copies of \mathbb{Z} acting like odometers on the path category of a directed graph F . We show that in the second spectral sequence of Corollary 5.18, only $E_{0,0}^{vh,2}$, $E_{1,0}^{vh,2}$, $E_{0,1}^{vh,2}$, and $E_{1,1}^{vh,2}$ can be nonzero. So the sequence converges on its second page, yielding an explicit formula for $H_{\bullet,\bullet}^{\bowtie}(\mathbb{Z} \times F^0, F^*)$ in terms of a weighted incidence matrix in $M_{F^0, F^1}(\mathbb{Z})$ (Proposition 6.14).

6.1. Matched pairs involving path categories of directed graphs. Let E be a directed graph and suppose that (\mathcal{C}, E^*) is a matched pair. We say that (\mathcal{C}, E^*) is a *length-preserving* matched pair if $|c \triangleright \alpha| = |\alpha|$ for all $(c, \alpha) \in \mathcal{C} * E^*$.

We write $N_{p,q}$ for the subgroup of $C_{p,q}$ generated by the nondegenerate vertical chains: chains $[c; d_1, \dots, d_q]$ such that $d_i \notin \mathcal{D}^0$ for all $1 \leq i \leq q$. By [Wei94, Theorem 8.3.8], the group $H_{p,q}^v(\mathcal{C}, E^*)$ of (5.16) is isomorphic to the q -th homology group of $(N_{p,\bullet}, d_{p,\bullet}^v |_{N_{p,\bullet}})$.

For $\alpha \in E^*$ we write α^i for the i -th edge of α . So $\alpha = \alpha^1 \alpha^2 \cdots \alpha^{|\alpha|}$. For $0 \leq i \leq j \leq |\alpha|$, we define $\alpha^{[i,j]} \in E^{j-i}$ by $\alpha = \alpha' \alpha^{[i,j]} \alpha''$ for some $\alpha' \in E^i$ and $\alpha'' \in E^{|\alpha|-j}$. For example, $\alpha^{[i-1,i]} = \alpha^i$ for $1 \leq i \leq |\alpha|$, and $\alpha = \alpha^{[0,i-1]} \alpha^i \alpha^{[i,|\alpha|]}$ for each i .

Proposition 6.1. *Let \mathcal{C} be a small category and let E be a directed graph, and suppose that (\mathcal{C}, E^*) is a length-preserving matched pair. Taking the convention that the empty sum is zero, for $p \geq 0$ and $q \geq 1$, define $s_{p,q}^v: N_{p,q} \rightarrow N_{p,q+1}$ by*

$$s_{p,q}^v[c; \alpha] = - \sum_{i=1}^{|\alpha_1|-1} [c \triangleleft \alpha_1^{[0,i-1]}; \alpha_1^i, \alpha_1^{[i,|\alpha_1|]}, \alpha_2, \dots, \alpha_q]$$

for $c \in \mathcal{C}^p$ and $\alpha = (\alpha_1, \dots, \alpha_q) \in (E^*)^q$ with $s(c) = r(\alpha_1)$. Then for $q \geq 2$,

$$d_{p,q}^v s_{p,q}^v + s_{p,q-1}^v d_{p,q-1}^v = \text{id}_{N_{p,q}}. \quad (6.1)$$

In particular, for $q \geq 2$ we have $H_{p,q}^v(\mathcal{C}, E^*) = 0$, and there is a long exact sequence

$$\cdots \rightarrow H_{p+1}^\boxtimes(\mathcal{C}, E^*) \rightarrow E_{p+1,0}^{hv,2} \xrightarrow{d} E_{p-1,1}^{hv,2} \rightarrow H_p^\boxtimes(\mathcal{C}, E^*) \rightarrow E_{p,0}^{hv,2} \xrightarrow{d} E_{p-2,1}^{hv,2} \rightarrow H_{p-1}^\boxtimes(\mathcal{C}, E^*) \rightarrow \cdots$$

Proof. Fix $q \geq 2$. We write $d := \bigoplus_p d_{p,q}^v$, $s = \bigoplus_p s_{p,q}^v$, and $N_q := \bigoplus_p N_{p,q}$. The restrictions $\partial_j: N_q \rightarrow N_{q-1}$ of the face maps satisfy

$$\partial_j([c; \alpha_1, \dots, \alpha_q]) = \begin{cases} [c \triangleleft \alpha_1; \alpha_2, \dots, \alpha_q] & \text{if } j = 0 \\ [c; \alpha_1, \dots, \alpha_j \alpha_{j+1}, \dots, \alpha_q] & \text{if } 1 \leq j \leq q-1 \\ [c; \alpha_1, \dots, \alpha_{q-1}] & \text{if } j = q, \end{cases}$$

and $d_q = \sum_{j=0}^q (-1)^j \partial_j$.

We first claim that for $j \geq 3$ we have $\partial_j \circ s = s \circ \partial_{j-1}$. Indeed,

$$\begin{aligned} \partial_j(s([c; \alpha_1, \alpha_2, \dots, \alpha_q])) &= - \sum_{i=1}^{|\alpha_1|-1} \partial_j([c \triangleleft \alpha_1^{[0,i-1]}; \alpha_1^i, \alpha_1^{[i,|\alpha_1|]}, \alpha_2, \dots, \alpha_q]) \\ &= - \sum_{i=1}^{|\alpha_1|-1} [c \triangleleft \alpha_1^{[0,i-1]}; \alpha_1^i, \alpha_1^{[i,|\alpha_1|]}, \partial_{j-2}(\alpha_2, \dots, \alpha_q)] \\ &= s([c; \alpha_1, \partial_{j-2}(\alpha_2, \dots, \alpha_q)]) \\ &= s(\partial_{j-1}([c; \alpha_1, \alpha_2, \dots, \alpha_q])). \end{aligned}$$

Using this at the final equality, we obtain

$$\begin{aligned}
d \circ s + s \circ d &= \sum_{i=0}^{q+1} (-1)^i \partial_i \circ s + \sum_{j=0}^q (-1)^j s \circ \partial_j \\
&= \partial_0 \circ s - \partial_1 \circ s + \partial_2 \circ s + \left(\sum_{i=3}^{q+1} (-1)^i \partial_i \circ s \right) + s \circ \partial_0 - s \circ \partial_1 + \left(\sum_{j=2}^q (-1)^j s \circ \partial_j \right) \\
&= \partial_0 \circ s - \partial_1 \circ s + \partial_2 \circ s + s \circ \partial_0 - s \circ \partial_1 + \sum_{j=3}^{q+1} \left((-1)^j \partial_j \circ s - (-1)^j s \circ \partial_{j-1} \right) \\
&= \partial_0 \circ s - \partial_1 \circ s + \partial_2 \circ s + s \circ \partial_0 - s \circ \partial_1.
\end{aligned}$$

So it suffices to show that

$$\partial_0 \circ s - \partial_1 \circ s + \partial_2 \circ s + s \circ \partial_0 - s \circ \partial_1 = \text{id}_{N_q}. \quad (6.2)$$

For this, fix $x := [c; \alpha_1, \dots, \alpha_q] \in N_q$. Let $l := |\alpha_1|$. We claim that

$$(\partial_0 \circ s - \partial_1 \circ s)(x) = x - [c \triangleleft \alpha_1^{[0, l-1]}; \alpha_1^l, \alpha_2, \dots, \alpha_q]. \quad (6.3)$$

To see this, we compute:

$$\begin{aligned}
\partial_0 \circ s(x) &= \partial_0 \left(- \sum_{i=1}^{l-1} [c \triangleleft \alpha_1^{[0, i-1]}; \alpha_1^i, \alpha_1^{[i, l]}, \alpha_2, \dots, \alpha_q] \right) = - \sum_{i=1}^{l-1} [c \triangleleft \alpha_1^{[0, i]}; \alpha_1^{[i, l]}, \alpha_2, \dots, \alpha_q], \text{ and} \\
\partial_1 \circ s(x) &= \partial_1 \left(- \sum_{i=1}^{l-1} [c \triangleleft \alpha_1^{[0, i-1]}; \alpha_1^i, \alpha_1^{[i, l]}, \alpha_2, \dots, \alpha_q] \right) = - \sum_{i=1}^{l-1} [c \triangleleft \alpha_1^{[0, i-1]}; \alpha_1^{[i-1, l]}, \alpha_2, \dots, \alpha_q].
\end{aligned}$$

So

$$(\partial_0 \circ s - \partial_1 \circ s)(x) = \sum_{i=1}^{l-1} \left(- [c \triangleleft \alpha_1^{[0, i]}; \alpha_1^{[i, l]}, \alpha_2, \dots, \alpha_q] + [c \triangleleft \alpha_1^{[0, i-1]}; \alpha_1^{[i-1, l]}, \alpha_2, \dots, \alpha_q] \right)$$

telescopes to (6.3). Next, we claim that

$$(s \circ \partial_0 - s \circ \partial_1)(x) = \left(\sum_{i=1}^{l-1} [c \triangleleft \alpha_1^{[0, i-1]}; \alpha_1^i, \alpha_1^{[i, l]}, \alpha_2, \alpha_3, \dots, \alpha_q] \right) + [c \triangleleft \alpha_1^{[0, l-1]}; \alpha_1^l, \alpha_2, \alpha_3, \dots, \alpha_q]. \quad (6.4)$$

Let $m := |\alpha_2|$. Again, we compute:

$$s \circ \partial_0(x) = s([c \triangleleft \alpha_1; \alpha_2, \dots, \alpha_q]) = - \sum_{i=1}^{m-1} [c \triangleleft \alpha_2^{[0, i-1]}; \alpha_2^i, \alpha_2^{[i, m]}, \alpha_3, \dots, \alpha_q], \quad (6.5)$$

and

$$\begin{aligned}
s \circ \partial_1(x) &= s([c; \alpha_1 \alpha_2, \dots, \alpha_q]) \\
&= - \sum_{i=1}^{l+m-1} [c \triangleleft (\alpha_1 \alpha_2)^{[0, i-1]}; (\alpha_1 \alpha_2)^i, (\alpha_1 \alpha_2)^{[i, l+m]}, \alpha_3, \dots, \alpha_q] \\
&= - \left(\sum_{i=1}^{l-1} [c \triangleleft \alpha_1^{[0, i-1]}; \alpha_1^i, \alpha_1^{[i, l]} \alpha_2, \alpha_3, \dots, \alpha_q] \right) - [c \triangleleft \alpha_1^{[0, l-1]}; \alpha_1^l, \alpha_2, \dots, \alpha_q] \\
&\quad - \left(\sum_{i=1}^{m-1} [c \triangleleft (\alpha_1 \alpha_2)^{[0, i-1]}; \alpha_2^i, \alpha_2^{[i, m]}, \alpha_3, \dots, \alpha_q] \right).
\end{aligned}$$

Subtracting (6.5) from this equation yields (6.4). Now we add Equations (6.4) and (6.3), and the terms $[c \triangleleft \alpha_1^{[0,l-1]}; \alpha_1^l, \alpha_2, \alpha_3, \dots, \alpha_q]$ cancel, giving

$$(\partial_0 \circ s - \partial_1 \circ s + s \circ \partial_0 - s \circ \partial_1)(x) = x + \left(\sum_{i=1}^{l-1} [c \triangleleft \alpha_1^{[0,i-1]}; \alpha_1^i, \alpha_1^{[i,l]} \alpha_2, \alpha_3, \dots, \alpha_q] \right). \quad (6.6)$$

Finally, we compute

$$\partial_2 \circ s(x) = \partial_2 \left(- \sum_{i=1}^{l-1} [c \triangleleft \alpha_1^{[0,i-1]}; \alpha_1^i, \alpha_1^{[i,l]}, \alpha_2, \dots, \alpha_q] \right) = - \sum_{i=1}^{l-1} [c \triangleleft \alpha_1^{[0,i-1]}; \alpha_1^i, \alpha_1^{[i,l]} \alpha_2, \dots, \alpha_q].$$

Adding this to (6.6) gives $(\partial_0 \circ s - \partial_1 \circ s + \partial_2 \circ s + s \circ \partial_0 - s \circ \partial_1)(x) = x$, which gives (6.2).

Now fix $a \in \ker(d_{q-1}) \cap N_q$. Then (6.1) gives

$$a = d_q(s_q(a)) + s_{q-1}(d_{q-1}(a)) = d_q(s_q(a)) \in \text{ran}(d_q),$$

so $H_q(N_{p,\bullet}) = 0$. Theorem 8.3.8 of [Wei94] gives $H_{p,q}^v(\mathcal{C}, E^*) = H_q(N_{p,\bullet}) = 0$. The long exact sequence exists by definition of convergence of a spectral sequence [Wei94, p.124]. \square

Remark 6.2. The proof of the preceding lemma relies on treating the empty sum as zero, prompting a quick reality check of the edge-case where $[c; \alpha_1, \dots, \alpha_q] \in N_q$ with $\alpha_1 = e \in E^1$. Let $x = [c; e, \alpha_2, \dots, \alpha_q]$. Since $s_{p,q}^v(x) = 0$, (6.1) for x collapses to $s_{p,q-1}^v \circ d_{p,q-1}^v(x) = x$, so the cancellation that led to (6.2) appears to fall down.

But all is well: for $j \geq 3$ we have $\partial_j([c; e, \alpha_2, \dots, \alpha_q]) = [c; e, \kappa]$ for some $\kappa \in (E^*)^{q-1}$, and so $s_{p,q-1}^v(\partial_j(x)) = s_{p,q-1}^v([c; e, \kappa]) = 0$. So $s_{p,q-1}^v \circ d_{p,q-1}^v(x) = s_{p,q-1}^v([c \triangleleft e; \alpha_1, \dots, \alpha_q]) - s_{p,q-1}^v([c; e\alpha_1, \dots, \alpha_q])$ and all the terms in the resulting sums cancel except the first term $-(-[c; e, \alpha_1, \dots, \alpha_q]) = x$ of $s_{p,q-1}^v([c; e\alpha_1, \dots, \alpha_q])$.

Recall that $\{E_{p,q}^{vh,r}, d_{p,q}^{vh,r}\}$ is the spectral sequence of Corollary 5.18, which in our current setup satisfies $E_{p,q}^{vh,2} = H_p^v H_q^h(\mathcal{C}, E^*)$.

Lemma 6.3. *Let \mathcal{C} be a small category and let E be a directed graph, and suppose that (\mathcal{C}, E^*) is a length-preserving matched pair. For $p \geq 0$ and $q \geq 2$, we have $s_{p,q}^v \circ d_{p,q}^h = d_{p,q+1}^h \circ s_{p+1,q}^v$. We have $E_{p,q}^{vh,2} = 0$ for all $q \geq 2$, $H_0^{\boxtimes}(\mathcal{C}, E^*) \cong E_{0,0}^{vh,2}$, and for each $n \geq 1$ there is a short exact sequence*

$$0 \longrightarrow E_{0,n}^{vh,2} \longrightarrow H_n^{\boxtimes}(\mathcal{C}, E^*) \longrightarrow E_{1,n-1}^{vh,2} \longrightarrow 0.$$

Proof. Fix $(c; \alpha) \in \mathcal{C}^{p+1} * (E^*)^q$. We compute

$$d_{p,q+1}^h \circ s_{p+1,q}^v [c; \alpha] = \sum_{i=1}^{|\alpha_1|-1} \sum_{k=0}^{p+1} (-1)^k \partial_{p,q+1}^{h,k} [c \triangleleft \alpha_1^{[0,i-1]}; \alpha_1^i, \alpha_1^{[i,|\alpha_1|]}, \alpha_2, \dots, \alpha_q].$$

Analogously to Lemma 4.4, we observe that for $0 \leq k \leq p$,

$$\begin{aligned} \partial_{p,q+1}^{h,k} [c \triangleleft \alpha_1^{[0,i-1]}; \alpha_1^i, \alpha_1^{[i,|\alpha_1|]}, \alpha_2, \dots, \alpha_q] &= [\partial_{p,0}^{h,k} [c \triangleleft \alpha_1^{[0,i-1]}; \alpha_1^i, \alpha_1^{[i,|\alpha_1|]}, \alpha_2, \dots, \alpha_q] \\ &= [\partial_{p,0}^{h,k} [c \triangleleft \alpha_1^{[0,i-1]}; \alpha_1^i, \alpha_1^{[i,|\alpha_1|]}, \alpha_2, \dots, \alpha_q]. \end{aligned}$$

Write $(c_p \triangleright \alpha)_r$ for the r -th entry of $c \triangleright \alpha \in (E^*)^q$. For $k = p + 1$ we have

$$\begin{aligned} & \partial_{p,q+1}^{h,p+1} [c \triangleleft \alpha_1^{[1,i-1]}; \alpha_1^i, \alpha_1^{[i,|\alpha_1|]}, \alpha_2, \dots, \alpha_q] \\ &= [\partial_{p,0}^{h,p+1} [c \triangleleft \alpha_1^{[0,i-1]}; (c_p \triangleleft \alpha_1^{[0,i-1]}) \triangleright (\alpha_1^i, \alpha_1^{[i,|\alpha_1|]}, \alpha_2, \dots, \alpha_q)] \\ &= [\partial_{p,0}^{h,p+1} [c \triangleleft \alpha_1^{[1,i-1]}; (c_p \triangleright \alpha)_1^i, (c_p \triangleright \alpha)_1^{[i,|\alpha_1|]}, (c_p \triangleright \alpha)_2, \dots, (c_p \triangleright \alpha)_q]. \end{aligned}$$

Hence,

$$d_{p,q+1}^h \circ s_{p+1,q}^v [c; \alpha] = \sum_{k=0}^{p+1} (-1)^k s_{p,q}^v \circ \partial_{p,q}^{h,k} [c; \alpha] = s_{p,q}^v \circ d_{p,q}^h [c; \alpha].$$

Thus, the $s_{p,q}^v$ descend to sections of the differentials between the $E_{p,q}^{vh,1}$, and the resulting homology groups $E_{p,q}^{vh,2}$ vanish for $q \geq 2$. So the spectral sequence stabilises by the second page, and short exact sequences follow from the associated filtration. \square

6.2. Matched pairs involving bundles of monoids. In this section $(\mathcal{C}, \mathcal{D})$ is a matched pair in which $\mathcal{C} = \bigsqcup_{u \in \mathcal{D}^0} \mathcal{C}_u$ is a bundle of monoids over $\mathcal{C}^0 = \mathcal{D}^0$. For each $u \in \mathcal{D}^0$ and $q \geq 0$, the free \mathbb{Z} -module $\mathbb{Z}u\mathcal{D}^q$ generated by $u\mathcal{D}^q = \{d \in \mathcal{D}^q \mid r(d) = u\}$ is a left \mathcal{C}_u -module under the action $c \cdot \left(\sum_{d \in \mathcal{D}^q} n_d d \right) = \sum_{d \in \mathcal{D}^q} n_d (c \triangleright d)$.

For a monoid S , the categories of left (respectively right) S -modules and left (respectively right) $\mathbb{Z}[S]$ -modules are equivalent. Given a left S -module M we can compute the homology $H_\bullet(S; M)$ of S with coefficients in M as follows: fix a projective resolution $\dots \rightarrow P_1 \rightarrow P_0 \rightarrow \mathbb{Z} \rightarrow 0$ of the trivial S -module \mathbb{Z} . Then $H_\bullet(S; M)$ is the homology of the chain complex $\dots \rightarrow P_1 \otimes_{\mathbb{Z}[S]} M \rightarrow P_0 \otimes_{\mathbb{Z}[S]} M \rightarrow 0$: that is, $H_n(S; M) := \text{Tor}_n^{\mathbb{Z}[S]}(\mathbb{Z}; M)$.

By taking the *bar resolution* of the trivial (right) S -module \mathbb{Z} we arrive at a more familiar description. For $n \geq 1$ let B_n be the free S -module generated by $\{[s_1, \dots, s_n] \mid s_i \in S\}$. Let B_0 be the free S -module generated by the symbol $[\]$. Define $b_n: B_{n+1} \rightarrow B_n$ by

$$b_n[s_0, \dots, s_n] = [s_1, \dots, s_n] + \sum_{i=1}^n (-1)^i [s_0, \dots, s_{i-1} s_i, \dots, s_n] + (-1)^{n+1} [s_1, \dots, s_{n-1}] s_n \quad (6.7)$$

for $n \geq 1$ and $b_{-1}: B_0 \rightarrow \mathbb{Z}$ by $b_{-1}[\] = 1$; note that $b_0[s_0] = [\] - [s_0]$. The group of p -chains with values in M is

$$C_p(S; M) := B_p \otimes_{\mathbb{Z}[S]} M.$$

The boundary maps $d_p := b_p \otimes \text{id}_M: C_{p+1}(S; M) \rightarrow C_p(S; M)$ for $p \geq 1$ satisfy

$$\begin{aligned} d_p([s_0, \dots, s_p] \otimes m) &= [s_1, \dots, s_p] \otimes m + \sum_{i=1}^p (-1)^i [s_0, \dots, s_{i-1} s_i, \dots, s_p] \otimes m \\ &= (-1)^{p+1} [s_0, \dots, s_{p-1}] \otimes s_p \cdot m, \end{aligned}$$

and $d_0: C_1(S; M) \rightarrow C_0(S; M)$ satisfies $d_0([s_0] \otimes m) = [\] \otimes m - [s_0] \otimes m$. Then $H_n(S; M) \cong \ker(d_{n-1}) / \text{im}(d_n)$. Taking $M = \mathbb{Z}$, the trivial S -module, recovers Definition 4.1 if $\mathcal{C} = S$.

Given a matched pair $(\mathcal{C}, \mathcal{D})$ where \mathcal{C} is a bundle of monoids, we can compute the horizontal homology of the matched complex using the homology of the monoids \mathcal{C}_u .

Proposition 6.4. *Let \mathcal{D} be a small category and let $\mathcal{C} = \bigsqcup_{u \in \mathcal{D}^0} \mathcal{C}_u$ be a bundle of monoids over \mathcal{D}^0 such that $(\mathcal{C}, \mathcal{D})$ is a matched pair. Then $C_p(\mathcal{C}_u; \mathbb{Z}u\mathcal{D}^q) \ni [c_0, \dots, c_p] \otimes d \mapsto [c_0, \dots, c_{p-1}; c_p \triangleright d] \in$*

$C_{p,q}$ induces an isomorphism

$$H_{p,q}^h(\mathcal{C}, \mathcal{D}) \cong \bigoplus_{u \in \mathcal{D}^0} H_p(\mathcal{C}_u; \mathbb{Z}u\mathcal{D}^q),$$

and there is a short exact sequence

$$0 \rightarrow \bigoplus_{u \in \mathcal{D}^0} H_p(\mathcal{C}_u) \otimes_{\mathbb{Z}[\mathcal{C}_u]} \mathbb{Z}u\mathcal{D}^q \rightarrow H_{p,q}^h(\mathcal{C}, \mathcal{D}) \rightarrow \bigoplus_{u \in \mathcal{D}^0} \mathrm{Tor}_1^{\mathbb{Z}[\mathcal{C}_u]}(H_{p-1}(\mathcal{C}_u), \mathbb{Z}u\mathcal{D}^q) \rightarrow 0.$$

Proof. In $C_p(\mathcal{C}_u; \mathbb{Z}u\mathcal{D}^q) = B_p \otimes_{\mathbb{Z}[\mathcal{C}_u]} \mathbb{Z}u\mathcal{D}^q$ we have $[c_1, \dots, c_p] \otimes d = [c_1, \dots, c_{p-1}, s(c_{p-1})] \otimes c_p \cdot d$. Consequently, the map $\pi_{p,q}: \bigoplus_{u \in \mathcal{D}^0} C_p(\mathcal{C}_u; \mathbb{Z}u\mathcal{D}^q) \rightarrow C_{p,q}$ defined by

$$\pi_{p,q}([c_0, \dots, c_p] \otimes d) = [c_0, \dots, c_{p-1}; c_p \triangleright d]$$

for $[c_0, \dots, c_p] \otimes d \in C_p(\mathcal{C}_u; \mathbb{Z}u\mathcal{D}^q)$ and $u \in \mathcal{D}^0$, is an isomorphism. Moreover,

$$\begin{aligned} d_{p,q}^h \circ \pi_{p+1,q}([c_0, \dots, c_{p+1}] \otimes d) &= d_{p,q}^h[c_1, \dots, c_p, c_{p+1} \triangleright d] \\ &= [c_1, \dots, c_p; c_{p+1} \triangleright d] + \sum_{i=1}^p (-1)^i [c_0, \dots, c_{i-1}c_i, \dots, c_p; c_{p+1} \triangleright d] \\ &\quad + (-1)^{p+1} [c_0, \dots, c_{p-1}, c_p c_{p+1} \triangleright d] \\ &= \pi_{p,q} \circ d_{p,q}^u([c_0, \dots, c_{p+1}] \otimes d). \end{aligned}$$

So $H_{p,q}^h(\mathcal{C}, \mathcal{D}) \cong \bigoplus_{u \in \mathcal{D}^0} H_p(\mathcal{C}_u; \mathbb{Z}u\mathcal{D}^q)$.

Since \mathbb{Z} is a free \mathcal{C}_u -module the short exact sequence follows from [Wei94, Theorem 3.6.1]. \square

6.3. Matched pairs involving integer bundles. In this section we consider matched pairs $(\mathcal{C}, \mathcal{D})$ where $\mathcal{C} \cong \mathcal{D}^0 \times \mathbb{Z}$. If M is a \mathbb{Z} -module, then $M^{\mathbb{Z}} = \{m \in M \mid k \cdot m = m \text{ for all } k \in \mathbb{Z}\}$ is its submodule of *invariants* and $M_{\mathbb{Z}} = M / \langle k \cdot m - m \mid k \in \mathbb{Z}, m \in M \rangle$ is its module of *coinvariants*.

If X is a set and $\triangleright: \mathbb{Z} \times X \rightarrow X$ is a left action, we write $\mathbb{Z} \backslash X$ for the set of \mathbb{Z} -orbits in X . We denote the orbit of $x \in X$ by $\llbracket x \rrbracket \in \mathbb{Z} \backslash X$, and the set of periodic points of X by

$$\mathrm{Per}(X) := \{x \in X \mid \text{there exists } k \in \mathbb{Z} \setminus \{0\} \text{ such that } k \triangleright x = x\}.$$

Then $\mathbb{Z} \backslash \mathrm{Per}(X) \subseteq \mathbb{Z} \backslash X$ is the set of finite orbits in X .

Each left action $\triangleright: \mathbb{Z} \times X \rightarrow X$ induces a corresponding left action $\cdot: \mathbb{Z} \times \mathbb{Z}X \rightarrow \mathbb{Z}X$.

Lemma 6.5. *Let X be a set, and $\triangleright: \mathbb{Z} \times X \rightarrow X$ a left action. There are isomorphisms*

$$\phi_0: \mathbb{Z}(\mathbb{Z} \backslash X) \rightarrow (\mathbb{Z}X)_{\mathbb{Z}} \quad \text{and} \quad \phi_1: \mathbb{Z}(\mathbb{Z} \backslash \mathrm{Per}(X)) \rightarrow (\mathbb{Z}X)^{\mathbb{Z}}$$

such that $\phi_0(\llbracket d \rrbracket) = d + \langle k \cdot m - m \mid k \in \mathbb{Z}, m \in \mathbb{Z}X \rangle$ and $\phi_1(\llbracket d \rrbracket) = \sum_{d' \in \llbracket d \rrbracket} d'$.

Proof. For the first isomorphism, regard $\mathbb{Z}X$ and $\mathbb{Z}(\mathbb{Z} \backslash X)$ as the sets of finitely supported \mathbb{Z} -valued functions on X and $\mathbb{Z} \backslash X$ respectively. Define $\pi: \mathbb{Z}X \rightarrow \mathbb{Z}(\mathbb{Z} \backslash X)$ by $\pi(f)(\llbracket x \rrbracket) = \sum_{y \in \llbracket x \rrbracket} f(y)$. Then $\ker(\pi) = \langle k \cdot m - m \mid k \in \mathbb{Z}, m \in \mathbb{Z}X \rangle$, so π descends to an isomorphism $(\mathbb{Z}X)_{\mathbb{Z}} \cong \mathbb{Z}(\mathbb{Z} \backslash X)$ whose inverse is the desired map ϕ_0 .

For the second isomorphism, fix $m := \sum_{x \in X} a_x x \in \mathbb{Z}X$, where each $a_x \in \mathbb{Z}$. Then $\sum_{x \in X} a_x x = \sum_{\llbracket y \rrbracket \in \mathbb{Z} \backslash X} \sum_{x \in \llbracket y \rrbracket} a_x x$. For each $k \in \mathbb{Z}$,

$$k \cdot m - m = \sum_{\llbracket y \rrbracket \in \mathbb{Z} \backslash X} \left(\sum_{x \in \llbracket y \rrbracket} a_x (k \triangleright x) - a_x x \right).$$

Hence, $k \cdot m - m = 0$ for all $k \in \mathbb{Z}$ if and only if $a_x = a_y$ whenever $\llbracket x \rrbracket = \llbracket y \rrbracket$; that is, $a: x \mapsto a_x$ is constant on orbits. Since a is finitely supported, if $m \in (\mathbb{Z}X)^\mathbb{Z}$ then a is nonzero only on finite orbits. Hence, the formula for ϕ_1 determines an isomorphism. \square

Proposition 6.6. *Let $(\mathcal{C}, \mathcal{D})$ be a matched pair with $\mathcal{C} = \mathcal{D}^0 \times \mathbb{Z}$. There are isomorphisms $\alpha_0: \bigoplus_{u \in \mathcal{D}^0} \mathbb{Z}(\mathcal{C}_u \setminus u\mathcal{D}^q) \rightarrow H_{0,q}^h(\mathcal{C}, \mathcal{D})$ and $\alpha_1: \bigoplus_{u \in \mathcal{D}^0} \mathbb{Z}(\mathcal{C}_u \setminus \text{Per}(u\mathcal{D}^q)) \rightarrow H_{1,q}^h(\mathcal{C}, \mathcal{D})$ satisfying*

$$\alpha_0(\llbracket d \rrbracket) = [r(d); d] + \text{im}(d_{0,q}^h) \quad \text{and} \quad \alpha_1(\llbracket d \rrbracket) = \sum_{d' \in \llbracket d \rrbracket} [1; d'] + \text{im}(d_{1,q}^h). \quad (6.8)$$

Moreover,

$$H_{p,q}^h(\mathcal{C}, \mathcal{D}) \cong \begin{cases} \bigoplus_{u \in \mathcal{D}^0} (\mathbb{Z}u\mathcal{D}^q)_{\mathbb{Z}} & \text{if } p = 0 \\ \bigoplus_{u \in \mathcal{D}^0} (\mathbb{Z}u\mathcal{D}^q)^{\mathbb{Z}} & \text{if } p = 1 \\ 0 & \text{otherwise} \end{cases} \cong \begin{cases} \bigoplus_{u \in \mathcal{D}^0} \mathbb{Z}(\mathcal{C}_u \setminus u\mathcal{D}^q) & \text{if } p = 0 \\ \bigoplus_{u \in \mathcal{D}^0} \mathbb{Z}(\mathcal{C}_u \setminus \text{Per}(u\mathcal{D}^q)) & \text{if } p = 1 \\ 0 & \text{otherwise.} \end{cases} \quad (6.9)$$

Proof. We identify the group ring $\mathbb{Z}[\mathbb{Z}]$ with the ring of Laurent polynomials $\mathbb{Z}[t, t^{-1}]$. Let $\Sigma: \mathbb{Z}[t, t^{-1}] \rightarrow \mathbb{Z}$ be evaluation at 1, the homomorphism that sums coefficients.

As in [Wei94, Example 6.1.4],

$$\dots \longrightarrow 0 \longrightarrow \mathbb{Z}[t, t^{-1}] \xrightarrow{\times(t-1)} \mathbb{Z}[t, t^{-1}] \xrightarrow{\Sigma} \mathbb{Z} \longrightarrow 0, \quad (6.10)$$

is a projective resolution of \mathbb{Z} by $\mathbb{Z}[\mathbb{Z}]$ -modules. Since any projective resolution computes the homology of a group, it follows that $H_0(\mathbb{Z}; \mathbb{Z}u\mathcal{D}^q) \cong (\mathbb{Z}u\mathcal{D}^q)_{\mathbb{Z}}$ and $H_1(\mathbb{Z}; \mathbb{Z}u\mathcal{D}^q) \cong (\mathbb{Z}u\mathcal{D}^q)^{\mathbb{Z}}$, and that $H_n(\mathbb{Z}; \mathbb{Z}u\mathcal{D}^q) = 0$ for $n \geq 2$. The first isomorphism of (6.9) follows from Proposition 6.4; the second follows from Lemma 6.5.

Let $N_q := \langle k \cdot m - m \mid k \in \mathbb{Z}, m \in \mathbb{Z}u\mathcal{D}^q \rangle$. To establish (6.8) we describe the chain map connecting the bar resolution (6.7) with the resolution (6.10). Let δ_n be the generator of $\mathbb{Z}[\mathbb{Z}]$ corresponding to $n \in \mathbb{Z}$, and let $[n]$ be the basis element of the $\mathbb{Z}[\mathbb{Z}]$ -module B_1 (by definition, B_1 is the free $\mathbb{Z}[\mathbb{Z}]$ -module over \mathbb{Z} , but who wants to write $\mathbb{Z}[\mathbb{Z}][\mathbb{Z}]$?) Then the diagram

$$\begin{array}{ccccccccc} \dots & \longrightarrow & 0 & \longrightarrow & \mathbb{Z}[t, t^{-1}] & \xrightarrow{\times(t-1)} & \mathbb{Z}[t, t^{-1}] & \xrightarrow{\Sigma} & \mathbb{Z} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow t^n \mapsto [1] \cdot \delta_n & & \downarrow t^n \mapsto [1] \cdot \delta_n & & \downarrow \text{id} & & \\ \dots & \longrightarrow & B_2 & \xrightarrow{b_1} & B_1 & \xrightarrow{b_0} & B_0 & \xrightarrow{b_{-1}} & \mathbb{Z} & \longrightarrow & 0 \end{array} \quad (6.11)$$

commutes. The homology $H_\bullet(\mathbb{Z}; \mathbb{Z}u\mathcal{D}^q)$ as computed by each of these resolutions is obtained by tensoring by $\mathbb{Z}u\mathcal{D}^q$ on the right, replacing $\Sigma \otimes 1$ and $b_{-1} \otimes 1$ with 0, and taking homology.

Hence, for each $u \in \mathcal{D}^0$, the vertical map $t^n \mapsto [1] \cdot \delta_n$ in (6.11) induces an isomorphism $\text{coker}(\times(t-1) \otimes \text{id}_{\mathbb{Z}u\mathcal{D}^q}) \rightarrow H_0(\mathbb{Z}, \mathbb{Z}u\mathcal{D}^q)$ taking $t^0 \otimes d + \text{im}(\times(t-1) \otimes \text{id}_{\mathbb{Z}u\mathcal{D}^q})$ to $d + \text{im}(b_0)$. The isomorphism $(\mathbb{Z}u\mathcal{D}^q)_{\mathbb{Z}} \rightarrow \text{coker}(\times(t-1) \otimes \text{id}_{\mathbb{Z}u\mathcal{D}^q}) \rightarrow H_0(\mathbb{Z}, \mathbb{Z}u\mathcal{D}^q)$ of [Wei94, Example 6.1.4] carries $d + N_q$ to $t^0 \otimes d + \text{im}(\times(t-1) \otimes \text{id}_{\mathbb{Z}u\mathcal{D}^q})$. So composing these maps gives an isomorphism $\psi_{u,0}: (\mathbb{Z}u\mathcal{D}^q)_{\mathbb{Z}} \rightarrow H_0(\mathbb{Z}; \mathbb{Z}u\mathcal{D}^q)$ such that $\psi_{u,0}(d + N_q) = d + \text{im}(b_0 \otimes \text{id}_{\mathbb{Z}u\mathcal{D}^q}) = d + \text{im}(d_{0,q}^h)$.

Similarly, $t^n \mapsto [1] \cdot \delta_n$ restricts to an isomorphism $\ker(\times(t-1) \otimes \text{id}_{\mathbb{Z}u\mathcal{D}^q}) \rightarrow H_1(\mathbb{Z}, \mathbb{Z}u\mathcal{D}^q)$ taking $\sum_d k_d(t^0 \otimes d)$ to $\sum_d k_d[1; d] + \text{im}(d_{1,q}^h)$. The isomorphism $(\mathbb{Z}u\mathcal{D}^q)^{\mathbb{Z}} \rightarrow \ker(\times(t-1) \otimes \text{id}_{\mathbb{Z}u\mathcal{D}^q})$ of [Wei94, Example 6.1.4] carries $\sum k_d d$ to $\sum k_d(t^0 \otimes d)$. So composing these maps yields an isomorphism $\psi_{u,1}: (\mathbb{Z}u\mathcal{D}^q)^{\mathbb{Z}} \rightarrow H_1(\mathbb{Z}; \mathbb{Z}u\mathcal{D}^q)$ given by $\psi_{u,1}(\sum_d k_d d) = \sum_d k_d[1; d] + \text{im}(d_{1,q}^h)$.

Lemma 6.5 gives an isomorphism $\phi_{u,0}: \mathbb{Z}(\mathcal{C}_u \setminus u\mathcal{D}^q) \rightarrow (\mathbb{Z}u\mathcal{D}^q)_{\mathbb{Z}}$ such that $\phi_{u,0}(\llbracket d \rrbracket) = d + N_q$ and an isomorphism $\phi_{u,1}: \mathbb{Z}(\mathcal{C}_u \setminus \text{Per}(u\mathcal{D}^q)) \rightarrow (\mathbb{Z}u\mathcal{D}^q)^{\mathbb{Z}}$ such that $\phi_{u,1}(\llbracket d \rrbracket) = \sum_{d' \in \llbracket d \rrbracket} d'$. The maps $\alpha_i := \bigoplus_{u \in \mathcal{D}^0} \psi_{u,i} \circ \phi_{u,i}$ satisfy (6.8). \square

The next lemma helps to compute the terms $E_{p,q}^{vh}$ in Corollary 5.18.

Lemma 6.7. *Let $(\mathcal{C}, \mathcal{D})$ be a matched pair with $\mathcal{C} = \mathcal{D}^0 \times \mathbb{Z}$. For $d = (d_0, \dots, d_q) \in \text{Per}(\mathcal{D}^{q+1})$, let $O(d) := \min\{n \geq 1 \mid n \triangleright d = d\}$. For each $0 \leq i \leq q$, let $\partial_i: \mathcal{D}^{q+1} \rightarrow \mathcal{D}^q$ be the face map of (4.1). Regarding $O(d) \triangleleft d \in \{s(d)\} \times \mathbb{Z}$ as an integer, for $0 \leq i \leq q$ define*

$$\rho_i(d) := \begin{cases} (O(d) \triangleleft d_0)/O(\partial_0(d)) & \text{if } i = 0 \\ O(d)/O(\partial_i(d)) & \text{if } i \geq 1. \end{cases}$$

Then $\rho_i(d)$ is a nonnegative integer for each i .

Proof. First suppose that $i \geq 1$. It suffices to show that $O(d) \triangleright \partial_i(d) = \partial_i(d)$. By Lemma 4.4,

$$d = O(d) \triangleright d = (O(d) \triangleright d_0, (O(d) \triangleleft d_0) \triangleright d_1, \dots, (O(d) \triangleleft (d_0 \dots d_{q-1})) \triangleright d_q).$$

Hence, $(O(d) \triangleleft (d_0 \dots d_{j-1})) \triangleright d_j = d_j$ for all $1 \leq j \leq q$. In particular, for $i \leq q$, we have

$$(O(d) \triangleleft d_0 \dots d_{i-2}) \triangleright (d_{i-1}d_i) = (O(d) \triangleleft d_0 \dots d_{i-2}) \triangleright d_{i-1} \left((O(d) \triangleleft d_0 \dots d_{i-1}) \triangleright d_i \right) = d_{i-1}d_i.$$

Thus for $i \leq q$, we have

$$\begin{aligned} O(d) \triangleright \partial_i(d) &= (O(d) \triangleright d_0, \dots, (O(d) \triangleleft d_0 \dots d_{i-3}) \triangleright d_{i-2}, (O(d) \triangleleft d_0 \dots d_{i-2}) \triangleright (d_{i-1}d_i), \\ &\quad (O(d) \triangleleft d_0 \dots d_i) \triangleright d_{i+1}, \dots, (O(d) \triangleleft d_0 \dots d_{q-1}) \triangleright d_q) \\ &= (d_0, \dots, d_{i-2}, d_{i-1}d_i, d_{i+1}, \dots, d_q) = \partial_i(d). \end{aligned}$$

When $i = q + 1$, $O(d) \triangleright \partial_{q+1}(d) = \partial_{q+1}(O(d) \triangleright d) = \partial_{q+1}(d)$; and when $i = 0$,

$$\begin{aligned} (O(d) \triangleleft d_0) \triangleright \partial_0(d) &= (O(d) \triangleleft d_0) \triangleright (d_1, d_2, \dots, d_q) \\ &= \left((O(d) \triangleright d)_1, (O(d) \triangleright d)_2, \dots, (O(d) \triangleright d)_q \right) = (d_1, \dots, d_q) = \partial_0(d). \quad \square \end{aligned}$$

Proposition 6.8. *Let $(\mathcal{C}, \mathcal{D})$ be a matched pair with $\mathcal{C} = \mathcal{D}^0 \times \mathbb{Z}$. For $p \in \{0, 1\}$, let $d_{p,q}^1: E_{p,q+1}^{vh,1} \rightarrow E_{p,q}^{vh,1}$, $q \geq 0$, be the differentials in the first sheet of the spectral sequence $E_{p,q}^{vh}$ of Corollary 5.18. Let α_0, α_1 be as in Proposition 6.6. For $0 \leq i \leq q$, let $\partial_i: \mathcal{D}^{q+1} \rightarrow \mathcal{D}^q$ be the face map of (4.1), and let $\rho_i: \text{Per}(\mathcal{D}^q) \rightarrow \mathbb{Z}$ be as defined in Lemma 6.7. Define*

$$\begin{aligned} \Delta_{0,q} &: \bigoplus_{u \in \mathcal{D}^0} \mathbb{Z}(\mathcal{C}_u \setminus u\mathcal{D}^{q+1}) \rightarrow \bigoplus_{u \in \mathcal{D}^0} \mathbb{Z}(\mathcal{C}_u \setminus u\mathcal{D}^q) \quad \text{and} \\ \Delta_{1,q} &: \bigoplus_{u \in \mathcal{D}^0} \mathbb{Z}(\mathcal{C}_u \setminus \text{Per}(u\mathcal{D}^{q+1})) \rightarrow \bigoplus_{u \in \mathcal{D}^0} \mathbb{Z}(\mathcal{C}_u \setminus \text{Per}(u\mathcal{D}^q)) \end{aligned}$$

by

$$\Delta_{0,q}(\llbracket d \rrbracket) = \sum_{i=0}^q (-1)^i \llbracket \partial_i(d) \rrbracket, \quad \text{and} \quad \Delta_{1,q}(\llbracket d \rrbracket) = \sum_{i=0}^q (-1)^i \rho_i(d) \llbracket \partial_i(d) \rrbracket.$$

Then $\alpha_p \circ \Delta_{p,q} = d_{p,q}^1 \circ \alpha_p$ for $p = 0, 1$ and all $q \in \mathbb{N}$. In particular, $E_{0,\bullet}^{vh,2}$ is isomorphic to the homology of the chain complex $(\bigoplus_{u \in \mathcal{D}^0} \mathbb{Z}(\mathcal{C}_u \setminus u\mathcal{D}^\bullet), \Delta_{0,\bullet})$, and $E_{1,\bullet}^{vh,2}$ is isomorphic to the homology of the chain complex $(\bigoplus_{u \in \mathcal{D}^0} \mathbb{Z}(\mathcal{C}_u \setminus \text{Per}(u\mathcal{D}^\bullet)), \Delta_{1,\bullet})$.

Proof. To see that $\alpha_0 \circ \Delta_{0,q} = d_{0,q}^1 \circ \alpha_0$, we use that $d_{0,q}^v = d_{0,q}^1$ to compute:

$$d_{0,q}^1(\alpha_0(\llbracket d \rrbracket)) = d_{0,q}^v(\llbracket r(d); d \rrbracket) + \text{im}(d_{0,q+1}^h) = \alpha_0(\Delta_{0,q}(\llbracket d \rrbracket)).$$

To see that $\alpha_1 \circ \Delta_{1,q} = d_{1,q}^1 \circ \alpha_1$, we first claim that for $d \in \text{Per}(\mathcal{D}^q)$ and $n \geq 0$,

$$[n; d] + \text{im}(d_{1,q+1}^h) = \sum_{j=0}^{n-1} [1; j \triangleright d] + \text{im}(d_{1,q+1}^h) \in H_{1,q+1}^h(\mathcal{C}, \mathcal{D}). \quad (6.12)$$

We argue by induction. The case $n = 0$ is trivial: $[0; \llbracket d \rrbracket]$ is a sum of degenerate chains. Suppose inductively that (6.12) holds for n . We calculate, using (6.12) at the third equality:

$$\begin{aligned} 0 + \text{im}(d_{1,q+1}^h) &= d_{1,q+1}^h[1, n; d] + \text{im}(d_{1,q+1}^h) \\ &= [n; d] - [n+1; d] + [1; n \triangleright d] + \text{im}(d_{1,q+1}^h) \\ &= \sum_{j=0}^{n-1} [1; j \triangleright d] - [n+1; d] + [1; n \triangleright d] + \text{im}(d_{1,q+1}^h) \\ &= \sum_{j=0}^n [1; j \triangleright d] - [n+1; d] + \text{im}(d_{1,q+1}^h), \end{aligned}$$

and rearranging gives (6.12).

Since $\llbracket d \rrbracket = \{j \triangleright d : 0 \leq j \leq O(d) - 1\}$, Equation (6.12) gives

$$\alpha_1(\llbracket d \rrbracket) = \sum_{d' \in \llbracket d \rrbracket} [1; d'] + \text{im}(d_{1,q+1}^h) = \sum_{j=0}^{O(d)-1} [1; j \triangleright d] + \text{im}(d_{1,q+1}^h) = [O(d); d].$$

Using this at the first line, we calculate:

$$\begin{aligned} d_{1,q}^1(\alpha_1(\llbracket d \rrbracket)) &= d_{1,q}^1([O(d); d]) + \text{im}(d_{1,q}^h) \\ &= [O(d) \triangleleft d_0; \partial_0(d)] + \sum_{i=1}^{q+1} (-1)^i [O(d); \partial_i(d)] + \text{im}(d_{1,q}^h) \\ &= \sum_{i=0}^{q+1} (-1)^i [\rho_i(d) O(\partial_i(d)); \partial_i(d)] + \text{im}(d_{1,q}^h). \end{aligned} \quad (6.13)$$

For any $d' \in \mathcal{D}^q$ and any $n \geq 0$ we have

$$\begin{aligned} [nO(d'); d'] + \text{im}(d_{1,q}^h) &= \sum_{j=0}^{nO(d')-1} [1; j \triangleright d'] + \text{im}(d_{1,q}^h) \\ &= \sum_{k=0}^{n-1} \sum_{j=0}^{O(d')-1} [1; j \triangleright (kO(d') \triangleright d')] + \text{im}(d_{1,q}^h) \\ &= n \sum_{j=0}^{O(d')-1} [1; j \triangleright d'] + \text{im}(d_{1,q}^h) = n[O(d'); d'] + \text{im}(d_{1,q}^h). \end{aligned}$$

Applying this to each term of (6.13), we obtain

$$\begin{aligned} d_{1,q}^1(\alpha_1(\llbracket d \rrbracket)) &= \sum_{i=0}^{q+1} (-1)^i \rho_i(d) [O(\partial_i(d)); \partial_i(d)] + \text{im}(d_{1,q}^h) \\ &= \sum_{i=0}^{q+1} (-1)^i \rho_i(d) \alpha_1(\llbracket \partial_i(d) \rrbracket) = \alpha_1(\Delta_{1,q}(\llbracket d \rrbracket)). \end{aligned}$$

The remaining statements follow. □

6.4. Graphs of odometers. Here, we apply Proposition 6.8 and Theorem 5.3 to the following class of examples generalising the odometer action.

Set-up 6.9. Let E be a finite directed graph, and let $p: E^1 \rightarrow \mathbb{N} \setminus \{0\}$ be a function. Define $F = (F^0, F^1, r, s)$ by $F^0 = E^0$, $F^1 = \{(e, i) : e \in E^1, i \in \mathbb{Z}/p(e)\mathbb{Z}\}$, $r(e, i) = r(e)$, and $s(e, i) = s(e)$. We write $+_p$ for the group operation on $\mathbb{Z}/p\mathbb{Z}$. Let $\mathcal{G} := E^0 \times \mathbb{Z}$. We obtain a self-similar action of \mathcal{G} on the 1-graph F^* (in the sense of Definition 3.33) by the unique possible extension of the formulae

$$(r(e), 1) \triangleright (e, i) = (e, i +_{p(e)} 1) \quad \text{and} \quad (r(e), 1) \triangleleft (e, i) = \begin{cases} (s(e), 1) & \text{if } i = p(e) - 1 \\ (s(e), 0) & \text{otherwise.} \end{cases}$$

If $E^0 = \{v\}$, $E^1 = \{e\}$, and $p(e) = 2$, then (\mathcal{G}, F^*) is the binary odometer.

Extend p to a functor $p: E^* \rightarrow \mathbb{N}^\times$. Then $\Theta: F^* \rightarrow \{(\mu, i) \mid \mu \in E^*, i \in \mathbb{Z}/p(\mu)\mathbb{Z}\}$ given by

$$\Theta((e_1, m_1)(e_2, m_2) \cdots (e_k, m_k)) = (e_1 e_2 \cdots e_k, \sum_{j=1}^k m_j p(e_1 \cdots e_{j-1})).$$

is a bijection. Identifying F^* with $\{(\mu, i) \mid \mu \in E^*, i \in \mathbb{Z}/p(\mu)\mathbb{Z}\}$ via Θ , and writing $\lfloor \cdot \rfloor: \mathbb{R} \rightarrow \mathbb{Z}$ for the floor function $\lfloor x \rfloor = \max\{n \in \mathbb{Z} : n \leq x\}$, we have

$$(r(\mu), a) \triangleright (\mu, m) = (\mu, a +_{p(\mu)} m) \quad \text{and} \quad (r(\mu), a) \triangleleft (\mu, m) = (s(\mu), \lfloor (a + m)/p(\mu) \rfloor)$$

(in the second formula, m is regarded as an element of $\{0, \dots, p(\mu) - 1\}$ and the addition $a + m$ is computed in \mathbb{Z}). It is helpful to keep in mind the special case that

$$a \triangleright (\mu, 0) = (\mu, a \bmod p(\mu)), \quad \text{and} \quad a \triangleleft (\mu, 0) = \lfloor a/p(\mu) \rfloor. \quad (6.14)$$

Remark 6.10. The self-similar actions of Set-Up 6.9 are faithful self-similar actions as in Definition 3.30. They are also length-preserving matched pairs as in Subsection 6.1.

We use the symbols λ, μ, ν for paths in E and ξ, η, ζ for paths in F . So an element of F^* might be written as $\xi = (\mu, m)$. We write $\bar{p}: F^* \rightarrow \mathbb{N} \setminus \{0\}$ for the map $\bar{p}(\mu, m) = p(\mu)$.

Lemma 6.11. *In the situation of Set-up 6.9, we have $\text{Per}(uF^{*q}) = uF^{*q}$ for each $u \in F^0$ and $q \in E^0$. For each $u \in E^0$ the map $(\mu_0, \dots, \mu_{q-1}) \mapsto \mathcal{G}_u \triangleright ((\mu_0, 0), \dots, (\mu_{q-1}, 0))$ induces an isomorphism $\kappa_q: \mathbb{Z}uE^{*q} \rightarrow \mathbb{Z}(\mathcal{G}_u \setminus uF^{*q})$. The functions $O, \rho_i: (F^*)^{q+1} \rightarrow \mathbb{Z}$ satisfy*

$$O(\xi_0, \dots, \xi_q) = p(\xi_0 \xi_1 \cdots \xi_q) \quad \text{and} \quad \rho_i(\xi_0, \dots, \xi_q) = \begin{cases} 1 & \text{if } 0 \leq i < q \\ \bar{p}(\xi_q) & \text{if } i = q. \end{cases} \quad (6.15)$$

Proof. For $p_0, \dots, p_q > 0$, the odometer action Od of \mathbb{Z} on $\prod_{i=0}^q (\mathbb{Z}/p_i\mathbb{Z})$ is transitive, so the order of any point under Od is $\prod_i p_i$. For $u \in E^0$ and $\mu = (\mu_0, \dots, \mu_{q-1}) \in E^*u$, the action of \mathcal{G}_u on $\{((\mu_0, m_0), \dots, (\mu_q, m_q)) \mid m_i \in \mathbb{Z}/p(\mu_i)\mathbb{Z}\}$ is conjugate to this odometer with $p_i = p(\mu_i)$. So each $O(\xi_0, \dots, \xi_q) = \prod_{i=0}^q \bar{p}(\xi_i) = \bar{p}(\xi_0 \cdots \xi_q)$, we have $\text{Per}(uF^{*q}) = uF^{*q}$, and $(\mu_0, \dots, \mu_{q-1}) \mapsto \mathcal{G}_u \triangleright ((\mu_0, 0), \dots, (\mu_{q-1}, 0))$ is a bijection $uE^{*q} \rightarrow \mathcal{G}_u \setminus uF^{*q}$.

For the ρ_i , observe that if $i \geq 1$, then writing $\partial^i(\xi) = (\eta_0, \dots, \eta_{q-1})$, we have $\eta_0 \cdots \eta_{q-1} = \xi_0 \cdots \xi_q$ if $i \neq q$ and $\eta_0 \cdots \eta_{q-1} = \xi_0 \cdots \xi_{q-1}$ if $i = q$. Hence (6.15) implies that $\rho_i(\xi) = O(\xi)/O(\partial_i(\xi)) = 1$ if $i < q$, and $\rho_q(\xi) = O(\xi)/O(\partial_q(\xi)) = \bar{p}(\xi_q)$.

It remains to calculate $\rho_0(\xi)$. Since Od is transitive, $\text{id} \times \text{Od}$ is transitive on $\{(\mu_0, \dots, \mu_q)\} \times \prod_{i=0}^q \mathbb{Z}/p(\mu_i)\mathbb{Z}$. So it suffices to show that $\xi = ((\mu_0, 0), (\mu_1, 0), \dots, (\mu_q, 0))$ satisfies $\rho_0(\xi) = 1$. Applying (6.14), with $a = O(\xi) = p(\xi_0 \cdots \xi_q)$, gives

$$O(\xi) \triangleleft (\mu_0, 0) = \lfloor O(\xi)/p(\mu_0) \rfloor = \lfloor p(\mu_0 \cdots \mu_q)/p(\mu_0) \rfloor = p(\mu_1 \cdots \mu_q) = O(\partial_0(\xi)). \quad \square$$

Lemma 6.12. *With Set-up 6.9 let $\kappa_q: \mathbb{Z}uE^{*q} \rightarrow \mathbb{Z}(\mathcal{G}_u \setminus uF^{*q})$ be the isomorphism of Lemma 6.11. For each $0 \leq i \leq q$, let $\partial_i: E^{*(q+1)} \rightarrow E^{*q}$ be the face map of (4.1). Let $\Delta_{1,q}$ be as in Proposition 6.8, and define $\tilde{\Delta}_{1,q}: \bigoplus_{u \in E^0} \mathbb{Z}uE^{*(q+1)} \rightarrow \bigoplus_{u \in \mathcal{D}^0} \mathbb{Z}uE^{*q}$ by*

$$\tilde{\Delta}_{1,q}(\mu) = \left(\sum_{i=0}^{q-1} (-1)^i \partial_i(\mu) \right) + (-1)^q p(\mu_q) \partial_q(\mu). \quad (6.16)$$

Then $\kappa_q \circ \Delta_{1,q} = \tilde{\Delta}_{1,q} \circ \kappa_{q+1}$ for all q .

Proof. This follows directly from (6.16) and Lemma 6.11. \square

To compute homology for Set-up 6.9, we must compute $\tilde{\Delta}_{1,1} \left(\bigoplus_{u \in E^0} \mathbb{Z}uE^{*2} \right) \subseteq \bigoplus_{u \in \mathcal{D}^0} \mathbb{Z}uE^*$.

Lemma 6.13. *In the situation of Set-up 6.9, we have $\mathbb{Z}E^* = \mathbb{Z}E^1 + \text{im}(\tilde{\Delta}_{1,1})$, and $\text{im}(\tilde{\Delta}_{1,1}) \cap \mathbb{Z}E^1 = \{0\}$. In particular, $\mathbb{Z}E^* \cong \mathbb{Z}E^1 \oplus \text{im}(\tilde{\Delta}_{1,1})$.*

Proof. Since $\tilde{\Delta}_{1,0}(\mathbb{Z}E^0) = 0$ and $\tilde{\Delta}_{1,1}(\mathbb{Z}(E^0 * E^*)) = \tilde{\Delta}_{1,1}(\mathbb{Z}(E^* * E^0)) = \mathbb{Z}E^0$, it suffices to show that $\mathbb{Z}E^{\geq 1} = \mathbb{Z}E^1 + \text{im}(\tilde{\Delta}_{1,1}|_{\mathbb{Z}E^{\geq 1}})$ and $\text{im}(\tilde{\Delta}_{1,1}|_{\mathbb{Z}E^{\geq 1}}) \cap \mathbb{Z}E^1 = \{0\}$.

Recall that for $\mu \in E^{\geq 1}$ and $1 \leq i \leq |\mu|$, the elements $\mu^{[0, i-1]} \in E^{i-1}$, $\mu^i \in E^1$ and $\mu^{[i, |\mu|]} \in E^{|\mu|-i}$ are defined implicitly by $\mu = \mu^{[0, i-1]} \mu^i \mu^{[i, |\mu|]}$. Let $\tilde{\Delta} := \tilde{\Delta}_{1,1}$. To see that $\mathbb{Z}E^{\geq 1} = \mathbb{Z}E^1 + \text{im}(\tilde{\Delta}|_{\mathbb{Z}E^{\geq 1}})$, it suffices to show that for $\mu \in E^* \setminus E^0$,

$$\mu \in \left(\sum_{i=1}^{|\mu|} p(\mu^{[i, |\mu|]}) \mu^i \right) - \tilde{\Delta}(\mu^{[0, |\mu|-1]}, \mu^{|\mu|}) + \text{span}_{\mathbb{Z}} \{ \tilde{\Delta}(\alpha, \beta) : |\alpha\beta| < |\mu| \}. \quad (6.17)$$

in $\mathbb{Z}E^*$. We induct on $|\mu|$. If $|\mu| = 1$ then $\sum_{i=1}^{|\mu|} p(\mu^{[i, |\mu|]}) \mu^i = p(s(\mu))\mu = \mu$ and (6.17) is trivial. Now suppose that (6.17) holds for $|\mu| \leq n$ and fix $\mu \in E^{n+1}$. Write $\mu = \nu e$ with $e \in E^1$. Then $e = \mu^{[n, n+1]} = \mu^{n+1}$, and $\mu^{[i, n+1]} = \nu^{[i, n]} e$ and $\nu^i = \mu^i$ for $i \leq n$. We calculate (in $\mathbb{Z}E^*$):

$$\mu = -\tilde{\Delta}(\nu, e) + e + p(e)\nu = -\tilde{\Delta}(\nu, e) + p(\mu^{[n, n+1]})\mu^{n+1} + p(\mu^{n+1})\nu. \quad (6.18)$$

By the inductive hypothesis,

$$\begin{aligned} p(\mu^{n+1})\nu &\in p(\mu^{n+1}) \left(\left(\sum_{i=1}^n p(\nu^{[i, n]}) \nu^i \right) - \tilde{\Delta}(\nu^{[0, n-1]}, \nu^n) + \text{span}_{\mathbb{Z}} \{ \tilde{\Delta}(\alpha, \beta) : |\alpha\beta| < n \} \right) \\ &\subseteq \sum_{i=1}^n p(\mu^{n+1}) p(\nu^{[i, n]}) \nu^i + \text{span}_{\mathbb{Z}} \{ \tilde{\Delta}(\alpha, \beta) : |\alpha\beta| < n \}. \end{aligned}$$

Since p is multiplicative and each $\nu^{[i, n]} \mu^{n+1} = \mu^{[i, n+1]}$, we obtain

$$p(\mu^{n+1})\nu \in \sum_{i=1}^n p(\mu^{[i, n+1]}) \mu^i + \text{span}_{\mathbb{Z}} \{ \tilde{\Delta}(\alpha, \beta) : |\alpha\beta| < n+1 \}.$$

Substituting this into (6.18) completes the induction, proving the first statement.

For $\text{im}(\tilde{\Delta}|_{\mathbb{Z}E^{\geq 1}}) \cap \mathbb{Z}E^1 = \{0\}$, fix $(\mu, \nu), (\eta, \zeta) \in E^{\geq 1} * E^{\geq 1}$ with $\mu\nu = \eta\zeta$. We claim that

$$\tilde{\Delta}(\mu, \nu) - \tilde{\Delta}(\eta, \zeta) \in \text{span}_{\mathbb{Z}} \{ \tilde{\Delta}(\alpha, \beta) : |\alpha\beta| < |\mu\nu| \}. \quad (6.19)$$

We first show that if $\mu, \nu \in E^* \setminus E^0$ and $\mu\nu = \lambda$, then

$$\tilde{\Delta}(\mu, \nu) \in -\lambda + \sum_{i=1}^{|\lambda|} p(\lambda^{[i, |\lambda|]}) \lambda^i + \text{span}_{\mathbb{Z}}\{\tilde{\Delta}(\alpha, \beta) : |\alpha\beta| < |\lambda|\}. \quad (6.20)$$

For this, we calculate, applying (6.17) twice at the second step,

$$\begin{aligned} \tilde{\Delta}(\mu, \nu) &= \nu - \mu\nu + p(\nu)\mu \\ &\in -\mu\nu - \tilde{\Delta}(\nu^{[0, |\nu|-1]}, \nu^{|\nu|}) + \sum_{i=1}^{|\nu|} p(\nu^{[i, |\nu|]}) \nu^i - p(\nu) \left(\tilde{\Delta}(\mu^{[0, |\mu|-1]}, \mu^{|\mu|}) + \sum_{i=1}^{|\mu|} p(\mu^{[i, |\mu|]}) \mu^i \right) \\ &\quad + \text{span}_{\mathbb{Z}}\{\tilde{\Delta}(\alpha, \beta) : |\alpha\beta| < \max\{|\mu|, |\nu|\}\}. \end{aligned}$$

Since each $\mu^{[i, |\mu|]} \nu = (\mu\nu)^{[i, |\mu\nu|]} = \lambda^{[i, |\lambda|]}$ and since p is multiplicative, this gives

$$\begin{aligned} \tilde{\Delta}(\mu, \nu) &\in -\mu\nu + \sum_{i=1}^{|\lambda|} p(\lambda^{[i, |\lambda|]}) \lambda^i + -\tilde{\Delta}(\nu^{[0, |\nu|-1]}, \nu^{|\nu|}) - \tilde{\Delta}(p(\nu)\mu^{[0, |\mu|-1]}, \mu^{|\mu|}) \\ &\quad + \text{span}_{\mathbb{Z}}\{\tilde{\Delta}(\alpha, \beta) : |\alpha\beta| \leq \max\{|\mu|, |\nu|\}\}. \end{aligned}$$

Since $|\mu|, |\nu| < |\lambda|$, we obtain (6.20). Since the terms $-\lambda + \sum_{i=1}^{|\lambda|} p(\lambda^{[i, |\lambda|-1]}) \lambda^i$ in the right-hand side of (6.20) depend only on the product $\mu\nu$, we obtain (6.19).

Now, we suppose that $\text{im}(\tilde{\Delta}_{1,1}|_{\mathbb{Z}E^{\geq 1}}) \cap \mathbb{Z}E^1 \neq \{0\}$ and derive a contradiction. Let $l \in \mathbb{N}$ be minimal such that there exists $a \in \text{span}_{\mathbb{Z}}\{(\mu, \nu) : |\mu\nu| \leq l\}$ with $\tilde{\Delta}(a) \in \mathbb{Z}E^1 \setminus \{0\}$. Write $a = \sum a_{\mu, \nu}(\mu, \nu)$. For each (μ, ν) such that $|\mu\nu| = l$ and $0 \neq a_{\mu, \nu} \in \mathbb{Z}$, Equation (6.19) gives

$$\tilde{\Delta}\left(a_{\mu, \nu}\left((\mu^1, (\mu\nu)^{[1, l]}) - (\mu, \nu)\right)\right) \in \text{span}_{\mathbb{Z}}\{\tilde{\Delta}(\alpha, \beta) : |\alpha\beta| < l\}.$$

Hence,

$$a' := a + \sum_{|\mu\nu|=l} a_{\mu, \nu}\left((\mu^1, (\mu\nu)^{[1, l]}) - (\mu, \nu)\right)$$

satisfies $\tilde{\Delta}(a) - \tilde{\Delta}(a') \in \text{span}_{\mathbb{Z}}\{\tilde{\Delta}(\alpha, \beta) : |\alpha\beta| < l\}$. Fix $b \in \text{span}_{\mathbb{Z}}\{(\alpha, \beta) : |\alpha\beta| < l\}$ such that $\tilde{\Delta}(a) = \tilde{\Delta}(a') + \tilde{\Delta}(b) = \tilde{\Delta}(a' + b)$. Let $a'' := a' + b$; by construction, $a'' \in \text{span}_{\mathbb{Z}}\{(\mu, \nu) : |\mu\nu| \leq l\}$.

Write $a' = \sum a'_{\mu, \nu}(\mu, \nu)$. Then $a'_{\mu, \nu} = 0$ for all (μ, ν) such that $|\mu\nu| = l$ and $|\mu| > 1$. Write $b = \sum b_{\mu, \nu}(\mu, \nu)$. Then $b_{\mu, \nu} = 0$ for all μ, ν with $|\mu\nu| = l$. Hence $a''_{\mu, \nu} = 0$ whenever $|\mu\nu| = l$ and $|\mu| > 1$. We have $\tilde{\Delta}(a'') = \tilde{\Delta}(a) \in \mathbb{Z}E^1 \setminus \{0\}$, and since l is minimal there exist $\nu \in E^{l-1}$ and $e \in E^1 r(\nu)$ such that $a''_{e, \nu} \neq 0$. We have

$$\tilde{\Delta}(a'')_{e\nu} = \sum_{\alpha \in E^* r(e)} a''_{\alpha, e\nu} - \sum_{\eta\zeta=e\nu} a''_{\eta, \zeta} + \sum_{\tau \in s(e)E^*} p(\tau) a''_{e\nu, \tau}. \quad (6.21)$$

By construction of a'' , the only nonzero term in (6.21) is $-a''_{e, \nu}$ in the middle sum, so

$$\tilde{\Delta}(a'')_{e\nu} = -a''_{e, \nu} \neq 0,$$

which contradicts that $\tilde{\Delta}(a'') \in \mathbb{Z}E^1$. □

Define $M_s \in M_{E^0, E^1}(\mathbb{Z})$ by $M_s(v, e) = \delta_{v, s(e)}$, regarded as a group homomorphism from $\mathbb{Z}E^1$ to $\mathbb{Z}E^0$. Similarly, define $M_r \in M_{E^0, E^1}(\mathbb{Z})$ by $M_r(v, e) = \delta_{v, r(e)}$. Let $P \in M_{E^1}(\mathbb{Z})$ be the diagonal matrix $P(e, f) = \delta_{e, f}p(e)$. Finally, define $M: \mathbb{Z}E^1 \rightarrow \mathbb{Z}E^0$ by

$$M := M_r P - M_s. \quad (6.22)$$

In matrix form, $M \in M_{E^0, E^1}(\mathbb{Z})$ is given by

$$M(v, e) = p(e)\delta_{v, r(e)} - \delta_{v, s(e)} = \begin{cases} p(e) & \text{if } v = r(e) \text{ and } s(e) \neq r(e) \\ -1 & \text{if } v = s(e) \text{ and } s(e) \neq r(e) \\ p(e) - 1 & \text{if } v = r(e) = s(e) \\ 0 & \text{if } v \notin \{r(e), s(e)\}. \end{cases}$$

Proposition 6.14. *With Set-up 6.9, let $M: \mathbb{Z}E^1 \rightarrow \mathbb{Z}E^0$ be the homomorphism (6.22). The spectral sequence of Corollary 5.18 satisfies $E_{i,j}^{vh,2} = 0$ whenever $\max\{i, j\} \geq 2$,*

$$E_{0,1}^{vh,2} \cong H_1(E), \quad E_{1,1}^{vh,2} \cong \ker(M), \quad E_{0,0}^{vh,2} \cong H_0(E), \quad \text{and} \quad E_{1,0}^{vh,2} \cong \text{coker}(M).$$

Proof. Lemma 6.11 gives $\text{Per}((F^*)^q) = (F^*)^q$, so Proposition 6.8 implies that $E_{p,\bullet}^{vh,2}$ is isomorphic to the homology of the chain complex $(\bigoplus_{u \in E^0} \mathbb{Z}(\mathcal{G}_u \setminus uF^{*\bullet}), \Delta_{p,\bullet})$ for $p = 0, 1$. Lemma 6.12 gives isomorphisms $\kappa_q: E^{*q} \rightarrow \mathcal{G} \setminus F^{*q}$ intertwining the $\Delta_{0,\bullet}$ with the categorical-homology boundary maps for E , giving the descriptions of $E_{0,\bullet}^{vh,2}$.

Lemma 6.12 yields an isomorphism $\kappa_\bullet: (\bigoplus_{u \in E^0} \mathbb{Z}uE^{*\bullet}, \tilde{\Delta}_{1,\bullet}) \rightarrow (\bigoplus_{u \in E^0} \mathbb{Z}(\mathcal{G}_u \setminus uF^{*\bullet}), \Delta_{1,\bullet})$ of chain complexes induced by the κ_q . So $E_{1,\bullet}^{vh,2} \cong H_\bullet(\bigoplus_{u \in E^0} \mathbb{Z}uE^{*\bullet}, \tilde{\Delta}_{1,\bullet})$.

We have $\tilde{\Delta}_{1,0}(\mu) = p(\mu)r(\mu) - s(\mu)$, giving $E_{1,0}^{vh,2} = \mathbb{Z}E^0 / \text{span}_{\mathbb{Z}}\{p(\mu)r(\mu) - s(\mu) : \mu \in E^*\}$. Fix $\mu \in E^*$. The telescoping identity

$$p(\mu)r(\mu) - s(\mu) = \sum_{i=1}^{|\mu|} p(\mu^{i+1} \dots \mu^{|\mu|}) (p(\mu^i)r(\mu^i) - s(\mu^i)),$$

gives $\text{span}_{\mathbb{Z}}\{p(\mu)r(\mu) - s(\mu) : \mu \in E^*\} \subseteq \text{span}_{\mathbb{Z}}\{p(e)r(e) - s(e) : e \in E^1\}$. The reverse containment is trivial. Since each $p(e)r(e) - s(e) = Me$, we deduce that $E_{1,0}^{vh,2} = \text{coker}(M)$.

It remains to calculate $E_{1,1}^{vh,2} \cong \ker(\tilde{\Delta}_{1,0}) / \text{im}(\tilde{\Delta}_{1,1})$. Clearly, $(\ker(\tilde{\Delta}_{1,0}) \cap \mathbb{Z}E^1) + \text{im}(\tilde{\Delta}_{1,1}) \subseteq \ker(\tilde{\Delta}_{1,0})$. Conversely, if $a \in \ker(\tilde{\Delta}_{1,0})$, then Lemma 6.13 says that $a = a' + x$ for some $a' \in \mathbb{Z}E^1$ and $x \in \text{im}(\tilde{\Delta}_{1,1})$. Then $\tilde{\Delta}_{1,0}(a') = \tilde{\Delta}_{1,0}(a - x) = 0$, so $a' \in \ker(\tilde{\Delta}_{1,0}) \cap \mathbb{Z}E^1$. Hence,

$$\frac{\ker(\tilde{\Delta}_{1,0})}{\text{im}(\tilde{\Delta}_{1,1})} = \frac{(\ker(\tilde{\Delta}_{1,0}) \cap \mathbb{Z}E^1) + \text{im}(\tilde{\Delta}_{1,1})}{\text{im}(\tilde{\Delta}_{1,1})} \cong \frac{\ker(\tilde{\Delta}_{1,0}) \cap \mathbb{Z}E^1}{\text{im}(\tilde{\Delta}_{1,1}) \cap \mathbb{Z}E^1} = \ker(\tilde{\Delta}_{1,0}|_{\mathbb{Z}E^1}).$$

The restriction of $\tilde{\Delta}_{1,0}$ to $\mathbb{Z}E^1$ is M , so $E_{1,1}^{vh,2} \cong \ker(M)$. \square

We obtain a computation of the homology of matched pairs (\mathcal{G}, F^*) as in Set-up 6.9.

Theorem 6.15. *In the situation of Set-up 6.9, with $M \in M_{E^0, E^1}(\mathbb{Z})$ as in (6.22),*

$$H_0^\bowtie(\mathcal{G}, F^*) \cong H_0(E), \quad \text{and} \quad H_2^\bowtie(\mathcal{G}, F^*) \cong \ker(M),$$

and there is a short exact sequence

$$0 \rightarrow H_1(E) \rightarrow H_1^\bowtie(\mathcal{G}, F^*) \rightarrow \text{coker}(M) \rightarrow 0.$$

Proof. This follows immediately from Lemma 6.3 and Proposition 6.14. \square

Given a finite directed graph E , we define $\chi(E) := |E^0| - |E^1|$, the Euler characteristic of E .

Corollary 6.16. *In the situation of Set-up 6.9, suppose that $vE^*w \neq \emptyset$ for all $v, w \in E^0$, and that $E^1 \neq \emptyset$.*

(i) *If $p(e) = 1$ for all $e \in E^1$, then*

$$H_0^\times(\mathcal{G}, F^*) \cong \mathbb{Z}, \quad H_1^\times(\mathcal{G}, F^*) \cong \mathbb{Z}^{2-\chi(E)}, \quad \text{and} \quad H_2^\times(\mathcal{G}, F^*) \cong \mathbb{Z}^{1-\chi(E)}.$$

(ii) *If $p(e) > 1$ for some $e \in E^1$, then $\text{coker}(M)$ is a finite cyclic group,*

$$H_0^\times(\mathcal{G}, F^*) \cong \mathbb{Z} \quad \text{and} \quad H_2^\times(\mathcal{G}, F^*) \cong \mathbb{Z}^{-\chi(E)},$$

and there is a short exact sequence

$$0 \longrightarrow \mathbb{Z}^{1-\chi(E)} \longrightarrow H_1^\times(\mathcal{G}, F^*) \longrightarrow \text{coker}(M) \longrightarrow 0;$$

if $\gcd\{p(\mu) - p(\nu) : \mu, \nu \in E^, s(\mu) = s(\nu) \text{ and } r(\mu) = r(\nu)\} = 1$, then $\text{coker}(M) = 0$ and $H_1^\times(\mathcal{G}, F^*) \cong \mathbb{Z}^{1-\chi(E)}$.*

Proof. By [Mas91, p.194] (immediately after Theorem 3.4), $H_0(E)$ is the free abelian group generated by the connected components of E . Since $vE^*w \neq \emptyset$ for all $v, w \in E^0$, this is a singleton. So $H_0(E) = \mathbb{Z}$. This and [Mas91, Theorem 3.4] give $H_1(E) \cong \mathbb{Z}^{1-\chi(E)}$. We must compute $\ker(M)$ and $\text{coker}(M)$.

(i) Suppose that $p(e) = 1$ for all E . Lemma 6.11 gives $\rho_i \equiv 1$ for all i , and $\Delta_{1,q} = \Delta_{0,q}$ for all q . So $\ker(M) \cong E_{1,1}^{vh,2} \cong E_{0,1}^{vh,2} \cong H_1(E) \cong \mathbb{Z}^{1-\chi(E)}$ and $\text{coker}(M) \cong E_{1,0}^{vh,2} \cong E_{0,0}^{vh,2} \cong H_0(E) \cong \mathbb{Z}$. In particular, $\text{coker}(M)$ is free abelian, so the extension

$$0 \longrightarrow H_1(E) \longrightarrow H_1^\times(\mathcal{G}, F^*) \longrightarrow \text{coker}(M) \longrightarrow 0$$

of Theorem 6.15 splits, giving the desired formulae for $H_\bullet^\times(\mathcal{G}, F^*)$.

(ii) Now suppose that $p(e) > 1$ for some e . By assumption, there exists $\mu \in s(e)E^*r(e)$, and $p(e\mu) = p(e)p(\mu) > 1$. For $\nu \in E^*$, we have $p(\nu)r(\nu) - s(\nu) = \sum_{i=1}^{|\nu|} p(\nu^{i+1} \dots \nu^{|\nu|})(p(\nu^i)r(\nu^i) - s(\nu^i)) \in \text{im}(M)$. In particular, $(p(e\mu) - 1)r(e) = p(e\mu)r(e\mu) - s(e\mu) \in \text{im}(M)$, and so $r(e) + \text{im}(M)$ has finite order in $\text{coker}(M)$.

Fix $w \in E^0$. By assumption, there exists $\nu \in r(e)E^*w$ and so $w + \text{im}(M) = w + p(\nu)r(\nu) - s(\nu) + \text{im}(M) = p(\mu)r(e) + \text{im}(M)$. So $\text{coker}(M) = \mathbb{Z}r(e) + \text{im}(M)$ is a finite cyclic group. Hence, $\text{rank}(\text{im}(M)) = \text{rank}(\mathbb{Z}E^0) = |E^0|$, and Rank-Nullity for \mathbb{Z} -modules gives $\text{rank}(\ker(M)) = \text{rank}(\mathbb{Z}E^1) - \text{rank}(\mathbb{Z}E^0)$. Since $\ker(M)$ is a subgroup of a free abelian group, it is free abelian, so $\ker(M) \cong \mathbb{Z}^{-\chi(E)}$. The formulae for H_0 and H_2 and the exact sequence involving H_1 now follow from Theorem 6.15.

Finally, suppose that $\gcd\{p(\mu) - p(\nu) : \mu, \nu \in E^*, s(\mu) = s(\nu) \text{ and } r(\mu) = r(\nu)\} = 1$. As above, $a := r(e) + \text{im}(M)$ generates $\text{coker}(M)$. So it suffices to show that $O(a)$ divides $p(\mu) - p(\nu)$ whenever $s(\mu) = s(\nu)$ and $r(\mu) = r(\nu)$. Fix $v, w \in E^0$ and $\mu, \nu \in vE^*w$. We have $(p(\mu) - p(\nu))v = (p(\mu)r(\mu) - s(\mu)) - (p(\nu)r(\nu) - s(\nu)) \in \text{im}(M)$. Fix $\alpha \in vE^*r(e)$. Then $r(e) + \text{im}(M) = r(e) + (p(\alpha)r(\alpha) - s(\alpha)) + \text{im}(M) = p(\alpha)v + \text{im}(M)$. In particular, $(p(\mu) - p(\nu))r(e) + \text{im}(M) = p(\alpha)(p(\mu) - p(\nu))v + \text{im}(M) = 0 + \text{im}(M)$. So $O(v + \text{im}(M))$ divides $p(\mu) - p(\nu)$. \square

Remark 6.17. The situation when $p(e) = 1$ for all e in Corollary 6.16 boils down to $\mathcal{G} \bowtie F^* = \mathbb{Z} \times E^*$, so Corollary 6.16(i) is a nice reality check: it says that $H_p^\bowtie(\mathcal{G}, F^*) = \bigoplus_{i+j=p} H_i(\mathbb{Z}) \otimes H_j(E^*)$, in the spirit of the usual Künneth formula.

Example 6.18. Suppose that E is the directed graph with a single vertex v and a single edge e , so $\chi(E) = 0$. Fix $p(e) \in \mathbb{N} \setminus \{0\}$ and form the matched pair (\mathcal{G}, F^*) of Set-up 6.9. Then $H_0^\bowtie(\mathcal{G}, F^*) \cong H_0(E) \cong \mathbb{Z}$. The map $M: \mathbb{Z}E^1 \rightarrow \mathbb{Z}E^0$ is $\times(p(e) - 1)$, so we obtain an exact sequence

$$0 \rightarrow \mathbb{Z} \rightarrow H_1^\bowtie(\mathcal{G}, F^*) \rightarrow \mathbb{Z}/(p(e) - 1)\mathbb{Z} \rightarrow 0.$$

If $p(e) = 1$, then $H_1^\bowtie(\mathcal{G}, F^*) \cong \mathbb{Z}^2$ and $H_2^\bowtie(\mathcal{G}, F^*) \cong \mathbb{Z}$; if $p(e) = 2$ (the binary odometer), then $H_1^\bowtie(\mathcal{G}, F^*) \cong \mathbb{Z}$ and $H_2^\bowtie(\mathcal{G}, F^*) = 0$.

For $p(e) > 2$ the group cohomology $H^2(\mathbb{Z}/(p(e) - 1)\mathbb{Z}; \mathbb{Z}) \cong \mathbb{Z}/(p(e) - 1)\mathbb{Z}$, so the exact sequence in Corollary 6.16 (ii) does not completely determine $H_1^\bowtie(\mathcal{G}, F^*)$.

7. TWISTED C^* -ALGEBRAS OF SELF-SIMILAR GROUPOID ACTIONS ON k -GRAPHS

We give two constructions of a twisted C^* -algebra from a self-similar action of a groupoid on a k -graph as in Definition 3.33. This is a matched pair consisting of a groupoid and a k -graph in which the left action preserves the degree map. Recall from Proposition 3.32 that these generalise the faithful self-similar actions of groupoids on graphs and k -graphs of [LRRW18, ABRW19]. Our self-similar actions are the examples of [LawV22] in which the generalised higher-rank k -graphs are k -graphs.

Our first construction of such a C^* -algebra is twisted by a normalised 2-cocycle in $C_{\text{Tot}}^2(\mathcal{G}, \Lambda; \mathbb{T})$ and the second is twisted by a normalised 2-cocycle in $C_{\bowtie}^2(\mathcal{G}, \Lambda; \mathbb{T})$. We show that cohomologous cocycles yield isomorphic twisted C^* -algebras, and that our two constructions are compatible via the isomorphism of cohomology groups of Corollary 5.4. So all possible twisted C^* -algebras arise via total 2-cycles.

We first study C^* -algebras twisted by categorical cocycles, and establish some elementary structure theory, including a gauge-invariant uniqueness theorem.

Recall from [KP00] that a k -graph Λ is *row-finite* and has *no sources* if $0 < |v\Lambda^n| < \infty$ for all $v \in \Lambda^0$ and $n \in \mathbb{N}^k$. Following [RS05] (see also [RSY04, Remark 2.3]), if Λ is a k -graph and $\mu, \nu \in \Lambda$ we define

$$\text{MCE}(\mu, \nu) := \mu\Lambda \cap \nu\Lambda \cap \Lambda^{d(\mu) \vee d(\nu)}.$$

Elements of $\text{MCE}(\mu, \nu)$ are called *minimal common extensions* of μ and ν .

We adopt the usual conventions from the theory of C^* -algebras that homomorphisms are $*$ -homomorphisms, and that ideals are closed 2-sided ideals.

7.1. Twists by categorical cocycles. Given a normalised categorical 2-cocycle $c: \mathcal{C}^2 \rightarrow \mathbb{T}$ and a subcategory $\mathcal{C}' \subseteq \mathcal{C}$, we write c for the restriction of c to $(\mathcal{C}')^2 \subseteq \mathcal{C}^2$. Given a self similar action of a groupoid \mathcal{G} on a k -graph Λ , we regard \mathcal{G} and Λ as subsets of $\mathcal{G} \bowtie \Lambda$; so $g\lambda = (g \triangleleft \lambda)(g \triangleright \lambda)$ for $(g, \lambda) \in \mathcal{G} * \Lambda$.

Definition 7.1 (cf. [Yus23, ABRW19]). Let (\mathcal{G}, Λ) be a self-similar action of a groupoid on a row-finite k -graph with no sources. Let $c: (\mathcal{G} \bowtie \Lambda)^2 \rightarrow \mathbb{T}$ be a normalised categorical 2-cocycle. A *Toeplitz–Cuntz–Krieger $(\mathcal{G}, \Lambda; c)$ -family* in a C^* -algebra A is a function $t: \mathcal{G} \bowtie \Lambda \rightarrow A$, such that

$$(\text{TCK1}) \quad t_\zeta t_\eta = \delta_{s(\zeta), r(\eta)} c(\zeta, \eta) t_{\zeta\eta} \text{ for all } (\zeta, \eta) \in (\mathcal{G} \bowtie \Lambda)^{*2},$$

(TCK2) $t_{s(\zeta)} = t_\zeta^* t_\zeta$ for all $\zeta \in \mathcal{G} \rtimes \Lambda$,

(TCK3) for all $\mu, \nu \in \Lambda$ we have $t_\mu t_\mu^* t_\nu t_\nu^* = \sum_{\lambda \in \text{MCE}(\mu, \nu)} t_\lambda t_\lambda^*$.

We call t a *Cuntz–Krieger* $(\mathcal{G}, \Lambda; c)$ -family if, in addition

(CK) $t_v = \sum_{\lambda \in v\Lambda^n} t_\lambda t_\lambda^*$ for all $v \in \Lambda^0$ and $n \in \mathbb{N}^k$.

We write $C^*(t) := C^*(\{t_\zeta \mid \zeta \in \mathcal{G} \rtimes \Lambda\}) \subseteq A$.

Remark 7.2. Relation (TCK2) for $\zeta = v \in \Lambda^0$ is $t_v^* t_v = t_v$, so t_v is a projection. Now (TCK2) for any ζ implies that t_ζ is a partial isometry.

Relation (TCK1) implies that the t_v , for $v \in \Lambda^0$, are mutually orthogonal [KP00, Remarks 1.6(vi)]. So (TCK1)–(TCK3) say that $\{t_\lambda : \lambda \in \Lambda\}$ is a Toeplitz–Cuntz–Krieger (Λ, c) -family as in [SWW14], and so induces a homomorphism $t_\Lambda^t : \mathcal{TC}^*(\Lambda, c) \rightarrow A$, which descends to a homomorphism $t_\Lambda^t : C^*(\Lambda, c) \rightarrow A$ if t is a Cuntz–Krieger $(\mathcal{G}, \Lambda; c)$ -family.

If $\mu, \nu \in \Lambda$ satisfy $d(\mu) = d(\nu)$, then $\text{MCE}(\mu, \nu) = \{\mu\}$ if $\mu = \nu$ and \emptyset otherwise (see [SWW14, Lemma 3.2]). So (TCK3) gives $t_\mu t_\mu^* t_\nu t_\nu^* = \delta_{\mu, \nu} t_\nu t_\nu^*$. Since c is normalised, (TCK1) implies that $t_{r(\mu)} t_\mu = t_\mu$, so $t_\mu t_\mu^* \leq t_{r(\mu)}$ for all $\mu \in \Lambda$. Hence, as in [SWW14, Remark 3.4], every Toeplitz–Cuntz–Krieger $(\mathcal{G}, \Lambda; c)$ -family satisfies

(TCK4) $t_v \geq \sum_{\lambda \in v\Lambda^n} t_\lambda t_\lambda^*$ for all $v \in \Lambda^0$ and $n \in \mathbb{N}^k$.

Relation (TCK1) for $g \in \mathcal{G}$ gives $t_g t_{g^{-1}} = c(g, g^{-1}) t_{r(g)}$, so $\overline{t_g c(g, g^{-1})} t_{g^{-1}} = t_{r(g)}$. Uniqueness of quasi-inverses in an inverse semigroup then forces $c(g, g^{-1}) t_{g^{-1}} = t_g^*$, so $g \mapsto t_g$ is a twisted unitary representation of \mathcal{G} à la [Ren80], and so induces a homomorphism $t_{\mathcal{G}}^t : C^*(\mathcal{G}, c) \rightarrow A$.

The following standard arguments [Spe20, SWW14] show that every $(\mathcal{G}, \Lambda; c)$ admits a Toeplitz–Cuntz–Krieger-family of nonzero partial isometries.

Example 7.3. By Example 3.16 and left cancellativity of k -graphs, $\mathcal{G} \rtimes \Lambda$ is left cancellative. Hence, for each $\zeta \in \mathcal{G} \rtimes \Lambda$ there is a partial isometry, $L_\zeta \in \mathcal{B}(\ell^2(\mathcal{G} \rtimes \Lambda))$ such that $L_\zeta e_\eta = \delta_{s(\zeta), r(\eta)} c(\zeta, \eta) e_{\zeta\eta}$ for all $\eta \in \mathcal{G} \rtimes \Lambda$. Routine calculations show that this determines a Toeplitz–Cuntz–Krieger $(\mathcal{G}, \Lambda; c)$ -family $L : \mathcal{G} \rtimes \Lambda \rightarrow \mathcal{B}(\ell^2(\mathcal{G} \rtimes \Lambda))$.

We claim that $\{L_\mu L_g L_\nu^* : \mu, \nu \in \Lambda, g \in \mathcal{G}_{s(\mu)}^{s(\nu)}\}$ is linearly independent. To see this fix a linear combination $a = \sum_{\mu, g, \nu} a_{\mu, g, \nu} L_\mu L_g L_\nu^*$ with at least one nonzero coefficient. Fix (μ, g, ν) such that $a_{\mu, g, \nu} \neq 0$ and $a_{\mu', g', \nu'} = 0$ whenever $d(\nu') < d(\nu)$. Then $L_{\nu'}^* e_\nu = 0$ whenever $a_{\mu', g', \nu'} \neq 0$ and $\nu' \neq \nu$. Hence, $\|a\| \geq |(a e_\nu \mid e_{\mu g})| = |a_{\mu, g, \nu}| \neq 0$.

Proposition 7.4. *Let (\mathcal{G}, Λ) be a self-similar action of a groupoid on a row-finite k -graph with no sources. Let $c : (\mathcal{G} \rtimes \Lambda)^2 \rightarrow \mathbb{T}$ be a normalised categorical 2-cocycle. There is a C^* -algebra $\mathcal{TC}^*(\mathcal{G}, \Lambda; c)$ generated by a Toeplitz–Cuntz–Krieger $(\mathcal{G}, \Lambda; c)$ -family t that is universal for Toeplitz–Cuntz–Krieger $(\mathcal{G}, \Lambda; c)$ -families: if T is a Toeplitz–Cuntz–Krieger $(\mathcal{G}, \Lambda; c)$ -family, then there is a unique homomorphism $\pi^T : \mathcal{TC}^*(\mathcal{G}, \Lambda; c) \rightarrow C^*(T)$ such that $T = \pi^T \circ t$.*

Consider the ideal I of $\mathcal{TC}^(\mathcal{G}, \Lambda; c)$ generated by $\{t_v - \sum_{\lambda \in v\Lambda^n} t_\lambda t_\lambda^* \mid v \in \Lambda^0\}$. Then $s : \zeta \mapsto t_\zeta + I$ is a Cuntz–Krieger $(\mathcal{G}, \Lambda; c)$ -family in $C^*(\mathcal{G}, \Lambda; c) := \mathcal{TC}^*(\mathcal{G}, \Lambda; c)/I$, and is universal for Cuntz–Krieger $(\mathcal{G}, \Lambda; c)$ -families: if S is a Cuntz–Krieger $(\mathcal{G}, \Lambda; c)$ -family, then there is a unique homomorphism $\pi^S : C^*(\mathcal{G}, \Lambda; c) \rightarrow C^*(S)$ such that $S = \pi^S \circ s$.*

To prove Proposition 7.4, we follow the standard construction of [Bla85, Rae05, Lor10]. We first need the following technical lemma.

Lemma 7.5. *Let (\mathcal{G}, Λ) be a self-similar action of a groupoid on a row-finite k -graph with no sources. Let $c: (\mathcal{G} \rtimes \Lambda)^2 \rightarrow \mathbb{T}$ be a normalised categorical 2-cocycle. Fix (μ, g, ν) and (η, h, ζ) in $\Lambda * \mathcal{G} * \Lambda$, and for each (α, β) such that $\nu\alpha = \eta\beta \in \text{MCE}(\nu, \eta)$, define*

$$\begin{aligned} \omega(\alpha, \beta) &:= \overline{c(\nu, \alpha)}c(\eta, \beta)c(g, \alpha)c(h^{-1}, h)\overline{c(h^{-1}, \beta)}\overline{c(g \triangleleft \alpha, g \triangleright \alpha)} \\ &\quad \times c(h^{-1} \triangleright \beta, h^{-1} \triangleleft \beta)c(\mu, g \triangleright \alpha)\overline{c(\zeta, h^{-1} \triangleright \beta)} \\ &\quad \times \overline{c(h^{-1} \triangleleft \beta, (h^{-1} \triangleleft \beta)^{-1})}c(g \triangleleft \alpha, (h^{-1} \triangleleft \beta)^{-1}). \end{aligned} \tag{7.1}$$

Then for any Toeplitz–Cuntz–Krieger $(\mathcal{G}, \Lambda; c)$ -family T , we have

$$T_\mu T_g T_\nu^* T_\eta T_h T_\zeta^* = \sum_{\nu\alpha = \eta\beta \in \text{MCE}(\nu, \eta)} \omega_{\alpha, \beta} T_{\mu(g \triangleright \alpha)} T_{(g \triangleleft \alpha)(h^{-1} \triangleleft \beta)^{-1}} T_{\zeta(h^{-1} \triangleright \beta)}^*.$$

Proof. Relation (TCK3) implies that each

$$T_\nu^* T_\eta = T_\nu^* (T_\nu T_\nu^* T_\eta T_\eta^*) T_\eta = \sum_{\nu\alpha = \eta\beta \in \text{MCE}(\nu, \eta)} T_\nu^* T_{\nu\alpha} T_{\eta\beta}^* T_\eta.$$

For fixed (α, β) in the above sum, relations (TCK1) and then (TCK2) give

$$T_\nu^* T_{\nu\alpha} T_{\eta\beta}^* T_\eta = \overline{c(\nu, \alpha)}c(\eta, \beta) T_\nu^* T_\nu T_\alpha T_\beta^* T_\eta^* T_\eta = \overline{c(\nu, \alpha)}c(\eta, \beta) T_\alpha T_\beta^*.$$

Fix $\nu\alpha = \eta\beta \in \text{MCE}(\nu, \eta)$. Remark 7.2 gives $T_\beta^* T_h T_\zeta^* = c(h^{-1}, h) T_\beta^* T_{h^{-1}}^* T_\zeta^*$. We have

$$T_g T_\alpha = c(g, \alpha) T_{g\alpha} = c(g, \alpha) T_{(g \triangleright \alpha)(g \triangleleft \alpha)} = c(g, \alpha) \overline{c(g \triangleright \alpha, g \triangleleft \alpha)} T_{g \triangleright \alpha} T_{g \triangleleft \alpha},$$

and similarly,

$$T_\beta^* T_{h^{-1}}^* = \overline{c(h^{-1}, \beta)}c(h^{-1} \triangleright \beta, h^{-1} \triangleleft \beta) T_{h^{-1} \triangleleft \beta}^* T_{h^{-1} \triangleright \beta}^*.$$

We have

$$T_\mu T_{g \triangleright \alpha} = c(\mu, g \triangleright \alpha) T_{\mu(g \triangleright \alpha)} \quad \text{and} \quad T_{h^{-1} \triangleright \beta}^* T_\zeta^* = \overline{c(\zeta, h^{-1} \triangleright \beta)} T_{\zeta(h^{-1} \triangleright \beta)}^*.$$

Finally,

$$\begin{aligned} T_{g \triangleright \alpha} T_{h^{-1} \triangleleft \beta}^* &= \overline{c(h^{-1} \triangleleft \beta, (h^{-1} \triangleleft \beta)^{-1})} T_{g \triangleright \alpha} T_{(h^{-1} \triangleleft \beta)^{-1}} \\ &= \overline{c(h^{-1} \triangleleft \beta, (h^{-1} \triangleleft \beta)^{-1})} c(g \triangleright \alpha, (h^{-1} \triangleleft \beta)^{-1}) T_{(g \triangleright \alpha)(h^{-1} \triangleleft \beta)^{-1}}. \end{aligned}$$

Putting all of these identities together gives

$$\begin{aligned} T_\mu T_g T_\nu^* T_\eta T_h T_\zeta^* &= \sum_{\nu\alpha = \eta\beta \in \text{MCE}(\nu, \eta)} \overline{c(\nu, \alpha)}c(\eta, \beta) T_\mu T_g T_\alpha T_\beta^* T_h T_\zeta^* \\ &= \sum_{\nu\alpha = \eta\beta \in \text{MCE}(\nu, \eta)} \omega_{\alpha, \beta} T_{\mu(g \triangleright \alpha)} T_{(g \triangleleft \alpha)(h^{-1} \triangleleft \beta)^{-1}} T_{\zeta(h^{-1} \triangleright \beta)}^*. \end{aligned} \quad \square$$

Corollary 7.6 (cf. [LRRW18, Proposition 4.5]). *Let (\mathcal{G}, Λ) be a self-similar action of a groupoid on a row-finite k -graph with no sources, and let $c: (\mathcal{G} \rtimes \Lambda)^2 \rightarrow \mathbb{T}$ be a normalised categorical 2-cocycle. If T is a Toeplitz–Cuntz–Krieger $(\mathcal{G}, \Lambda; c)$ -family, then $C^*(T) = \overline{\text{span}}\{T_\mu T_g T_\nu^* : (\mu, g, \nu) \in \Lambda * \mathcal{G} * \Lambda\}$.*

Proof. The set $X := \overline{\text{span}}\{T_\mu T_g T_\nu^* : (\mu, g, \nu)\}$ is a closed subspace of $C^*(T)$. It is closed under adjoints since $T_\mu T_g T_\nu^* = c(g, g^{-1}) T_\nu^* T_{g^{-1}} T_\mu^*$, and Lemma 7.5 shows that it is closed under multiplication, so X is a C^* -subalgebra of $C^*(T)$. Fix $\zeta \in \mathcal{G} \rtimes \Lambda$. Proposition 3.13 gives a unique factorisation $\zeta = \mu g$ with $\mu \in \Lambda$ and $g \in \mathcal{G}$. So $T_\zeta = c(\mu, g) T_\mu T_g T_{s(g)}^* \in X$. So X contains the generators of $C^*(T)$ giving $X = C^*(T)$. \square

Proof of Proposition 7.4. Consider the vector space $V := C_c(\Lambda * \mathcal{G} * \Lambda)$ of finitely supported complex-valued functions on $\Lambda * \mathcal{G} * \Lambda$, which has basis the indicator functions $\theta_{\mu,g,\nu}$.

These $\theta_{\mu,g,\nu}$ are linearly independent, so there is a conjugate-linear map $*$: $V \rightarrow V$ such that

$$\theta_{\mu,g,\nu}^* = \overline{c(g, g^{-1})} \theta_{\nu,g,\mu}, \quad (7.2)$$

and there is a bilinear map \cdot : $V \times V \rightarrow V$ such that, for the scalars $\omega_{\alpha,\beta}$ defined in (7.1),

$$\theta_{\mu,g,\nu} \theta_{\eta,h,\zeta} = \sum_{\nu\alpha=\eta\beta \in \text{MCE}(\nu,\eta)} \omega_{\alpha,\beta} \theta_{\mu(g \triangleright \alpha), (g \triangleleft \alpha)(h^{-1} \triangleleft \beta)^{-1}, \zeta(h^{-1} \triangleright \beta)}. \quad (7.3)$$

In the Toeplitz–Cuntz–Krieger $(\mathcal{G}, \Lambda; c)$ -family L of Example 7.3, the $L_\mu L_g L_\mu^*$ are linearly independent, so there is a linear injection $\varphi_L: V \rightarrow \mathcal{B}(\ell^2(\mathcal{G} \rtimes \Lambda))$ satisfying $\varphi_L(\theta_{\mu,g,\nu}) = L_\mu L_g L_\nu^*$. Lemma 7.5 and bilinearity of multiplication in $\mathcal{B}(\ell^2(\mathcal{G} \rtimes \Lambda))$ shows that φ_L intertwines (7.3) with multiplication. Remark 7.2 shows that it carries (7.2) to the adjoint in $\mathcal{B}(\ell^2(\mathcal{G} \rtimes \Lambda))$.

Since $\mathcal{B}(\ell^2(\mathcal{G} \rtimes \Lambda))$ is a $*$ -algebra, we deduce that the operations we have defined on V satisfy the $*$ -algebra axioms, so V is a $*$ -algebra. The $\theta_{\mu,g,\nu}$ are linearly independent, so for any Toeplitz–Cuntz–Krieger $(\mathcal{G}, \Lambda; c)$ -family T there is a linear map $\varphi_T: V \rightarrow \overline{\text{span}}\{T_\mu T_g T_\nu^* : (\mu, g, \nu) \in \Lambda * \mathcal{G} * \Lambda\}$ such that $\varphi_T(\theta_{\mu,g,\nu}) = T_\mu T_g T_\nu^*$. Lemma 7.5 shows that φ_T is a homomorphism. The T_ζ are partial isometries, so for $a = \sum_{(\mu,g,\nu)} a_{\mu,g,\nu} \theta_{\mu,g,\nu} \in V$, we have $\|\varphi_T(a)\| \leq \sum_{(\mu,g,\nu)} |a_{\mu,g,\nu}|$. The map $\rho: V \rightarrow [0, \infty)$, given by

$$\rho(a) := \sup\{\|\varphi_T(a)\| : T \text{ is a Toeplitz–Cuntz–Krieger } (\mathcal{G}, \Lambda; c)\text{-family}\}$$

is a pre- C^* -seminorm. Quotienting by $N := \ker \rho$ and completing gives a C^* -algebra $\mathcal{TC}^*(\mathcal{G}, \Lambda; c)$.

The map $t: \mu g \mapsto \overline{c(\mu, g)} \theta_{\mu,g,s(g)} + N$ is a Toeplitz–Cuntz–Krieger $(\mathcal{G}, \Lambda; c)$ -family in $\mathcal{TC}^*(\mathcal{G}, \Lambda; c)$ because N contains the obstructions to relations (TCK1)–(TCK3). This t is universal: for any family T and any $a \in V$, we have $\|\varphi_T(a)\| \leq \|\varphi_t(a)\|$, so φ_T factors through a norm-decreasing homomorphism from $\varphi_t(V)$ to $\varphi_T(V)$, which extends to a homomorphism $\pi^T: \mathcal{TC}^*(\Lambda, \mathcal{G}; c) \rightarrow C^*(T)$ of C^* -algebras by continuity.

By definition of I , the map s is a Cuntz–Krieger $(\mathcal{G}, \Lambda; c)$ -family. Given any Cuntz–Krieger $(\mathcal{G}, \Lambda; c)$ -family S , the kernel of the homomorphism $\pi^S: \mathcal{TC}^*(\Lambda, \mathcal{G}; c) \rightarrow C^*(S)$ contains I , so π^S descends to a homomorphism $C^*(\Lambda, \mathcal{G}; c) \rightarrow C^*(S)$. \square

Since Λ and \mathcal{G} are subcategories of $\mathcal{G} \rtimes \Lambda$, if $c: (\Lambda * \mathcal{G})^2 \rightarrow \mathbb{T}$ is a 2-cocycle then $c|_{\Lambda^2}$ and $c|_{\mathcal{G}^2}$ are 2-cocycles on Λ and \mathcal{G} , which we continue to denote by c .

Our next steps are to show that if \mathcal{G} is amenable, then $\iota_{\mathcal{G}}^t: C^*(\mathcal{G}, c) \rightarrow \mathcal{TC}^*(\mathcal{G}, \Lambda; c)$ from Remark 7.2 is always injective, and follow Yusnitha’s analysis [Yus23] to see that her joint-faithfulness condition implies that $\iota_{\mathcal{G}}^s: C^*(\mathcal{G}, c) \rightarrow C^*(\mathcal{G}, \Lambda; c)$ is faithful. We also show that $\iota_\Lambda^t: \mathcal{TC}^*(\Lambda, c) \rightarrow \mathcal{TC}^*(\mathcal{G}, \Lambda; c)$ and $\iota_\Lambda^s: C^*(\Lambda, c) \rightarrow C^*(\mathcal{G}, \Lambda; c)$ are always injective.

If (\mathcal{G}, Λ) is a self-similar action of a groupoid on a row-finite k -graph with no sources, then the degree map $d_\Lambda: \Lambda \rightarrow \mathbb{N}^k$ determines a function $d_{\mathcal{G} \rtimes \Lambda}: \mathcal{G} \rtimes \Lambda \rightarrow \mathbb{N}^k$ by $d_{\mathcal{G} \rtimes \Lambda}(\mu, g) = d_\Lambda(\mu)$ for all $(\mu, g) \in \Lambda * \mathcal{G} = \mathcal{G} \rtimes \Lambda$. We will just write d for both d_Λ and $d_{\mathcal{G} \rtimes \Lambda}$ unless the subscript is needed for clarity. Since $d(g \triangleright \mu) = d(\mu)$ for all $(g, \mu) \in \mathcal{G} * \Lambda$, for each $((\mu, g), (\nu, h)) \in (\mathcal{G} \rtimes \Lambda)^2$,

$$d((\mu, g)(\nu, h)) = d(\mu(g \triangleright \nu), (g \triangleleft \nu)h) = d(\mu(g \triangleright \nu)) = d(\mu\nu) = d(\mu, g) + d(\nu, h).$$

So $d: \mathcal{G} \rtimes \Lambda \rightarrow \mathbb{N}^k$ is a functor.

For each $z \in \mathbb{T}^k$, the function $\gamma_z(t): \mathcal{G} \rtimes \Lambda \rightarrow \mathcal{TC}^*(\mathcal{G}, \Lambda; c)$ defined by $\gamma_z(t)(\zeta) = z^{d(\zeta)} t_\zeta$ is a Toeplitz–Cuntz–Krieger $(\mathcal{G}, \Lambda; c)$ -family. By the universal property, γ_z extends to an endomorphism

of $\mathcal{TC}^*(\mathcal{G}, \Lambda; c)$. Since $\gamma_z \circ \gamma_w(s_\zeta) = \gamma_{zw}(s_\zeta)$ for all $\zeta \in \mathcal{G} \rtimes \Lambda$, this γ is an action by automorphisms. An $\varepsilon/3$ -argument shows that it is strongly continuous. We call $\gamma: \mathbb{T}^k \rightarrow \text{Aut}(\mathcal{TC}^*(\mathcal{G}, \Lambda; c))$ the *gauge action*, and write $\mathcal{TC}^*(\mathcal{G}, \Lambda; c)^\gamma$ for the fixed-point algebra $\{a \in \mathcal{TC}^*(\mathcal{G}, \Lambda; c) \mid \gamma_z(a) = a \text{ for all } z \in \mathbb{T}\}$.

The same argument yields a strongly continuous action, also denoted γ and called the gauge action, of \mathbb{T} on $C^*(\mathcal{G}, \Lambda; c)$ such that $\gamma_z(s_\zeta) = z^{d(\zeta)}s_\zeta$ for all ζ , and we likewise write $C^*(\mathcal{G}, \Lambda; c)^\gamma$ for the resulting fixed-point algebra.

Proposition 7.7 (cf. [Yus23, Proposition 3.6]). *Let (\mathcal{G}, Λ) be a self-similar action of a groupoid on a row-finite k -graph with no sources, and let $c: (\mathcal{G} \rtimes \Lambda)^2 \rightarrow \mathbb{T}$ be a normalised categorical 2-cocycle. The generators t_ζ of $\mathcal{TC}^*(\mathcal{G}, \Lambda; c)$ and s_ζ of $C^*(\mathcal{G}, \Lambda; c)$ are all nonzero. The homomorphisms $\iota_\Lambda^t: \mathcal{TC}^*(\Lambda, c) \rightarrow \mathcal{TC}^*(\mathcal{G}, \Lambda; c)$ and $\iota_\Lambda^s: C^*(\Lambda, c) \rightarrow C^*(\mathcal{G}, \Lambda; c)$ are injective. If \mathcal{G} is amenable, then $\iota_{\mathcal{G}}^t: C^*(\mathcal{G}, c) \rightarrow \mathcal{TC}^*(\mathcal{G}, \Lambda; c)$ is injective. If, in addition, for every $v \in \Lambda^0$ and every $n \in \mathbb{N}^k$, there exists $\lambda \in v\Lambda^n$ such that $g \mapsto (g \triangleleft \lambda, g \triangleright \lambda)$ is injective on \mathcal{G}_v^v , then $\iota_{\mathcal{G}}^s: C^*(\mathcal{G}, c) \rightarrow C^*(\mathcal{G}, \Lambda; c)$ is injective.*

Proof. Let $L: \mathcal{G} \rtimes \Lambda \rightarrow \mathcal{B}(\ell^2(\mathcal{G} \rtimes \Lambda))$ be the Toeplitz–Cuntz–Krieger $(\mathcal{G}, \Lambda; c)$ -family of Example 7.3. Since the L_v are nonzero, the universal property of $\mathcal{TC}^*(\mathcal{G}, \Lambda; c)$ implies that the t_v are nonzero. As each $\|t_\zeta\|^2 = \|t_\zeta^*t_\zeta\| = \|t_{s(\zeta)}\|$, the t_ζ are all nonzero. For each $v \in \Lambda^0$ and $n \in \mathbb{N}^k$, we have $(L_v - \sum_{\mu \in v\Lambda^n} L_\mu L_\mu^*)e_v = e_v \neq 0$, so [SWW14, Theorem 3.15] implies that $\Pi^L \circ \iota_\Lambda^t$ is injective, and hence ι_Λ^t itself is injective.

To see that the s_ζ are all nonzero, we follow the argument of [Yus23]. For each $v \in \Lambda^0$ and $n \in \mathbb{N}^k$, the projection $\Delta_{n,v} := L_v - \sum_{\lambda \in v\Lambda^n} L_\lambda L_\lambda^*$ vanishes on $\overline{\text{span}}\{e_{\lambda g} : d(\lambda) \geq n\}$. A direct calculation using that $d(g \triangleright \lambda) = d(\lambda)$ for all λ , shows that $L_\lambda L_g L_\mu^* \Delta_{n,v} L_\nu L_h L_\eta^* e_{\zeta g} = 0$ whenever $d(\zeta) > d(\eta)$ and $d(\zeta) - d(\eta) + d(\nu) \geq n$. In particular, for $a \in \text{span}\{L_\lambda L_g L_\mu^* \Delta_{n,v} L_\nu L_h L_\eta^* : \lambda, \mu, \nu, \eta \in \Lambda, g, h \in \mathcal{G}, n \in \mathbb{N}^k, v \in \Lambda^0\}$, regarding \mathbb{N}^k as a directed set, $\lim_{n \in \mathbb{N}^k} \|a|_{\overline{\text{span}}\{e_{\zeta g} : \zeta \in \Lambda^n, g \in \mathcal{G}\}}\| = 0$. An approximation argument gives

$$\lim_{n \in \mathbb{N}^k} \left\| \pi^L(a)|_{\overline{\text{span}}\{e_{\zeta g} : \zeta \in \Lambda^n, g \in \mathcal{G}\}} \right\| = 0 \quad \text{for all } a \in I. \quad (7.4)$$

Fix $v \in \Lambda^0$, $n \in \mathbb{N}^k$, and $\zeta \in v\Lambda^n$. Then $\|L_v e_\zeta\| = \|e_\zeta\| = 1$, and so

$$\lim_{n \in \mathbb{N}^k} \left\| \pi^L(t_v)|_{\overline{\text{span}}\{e_{\zeta g} : \zeta \in \Lambda^n, g \in \mathcal{G}\}} \right\| = 1.$$

Hence, $t_v \notin I$, so $s_v = t_v + I \neq 0$. Now by (TCK2), each $\|s_\zeta\|^2 = \|s_\zeta s_\zeta^*\| = \|s_{s(\zeta)}\| > 0$. The homomorphism ι_Λ^s intertwines the gauge actions of \mathbb{T}^k on $C^*(\Lambda, c)$ and $C^*(\mathcal{G}, \Lambda; c)$, so the gauge-invariant uniqueness theorem [KPS15, Corollary 7.7] implies that ι_Λ^s is injective.

The subspace $\ell^2(\mathcal{G}) \subseteq \ell^2(\mathcal{G} \rtimes \Lambda)$ is invariant for $\pi_{\mathcal{G}}^L: C^*(\mathcal{G}, c) \rightarrow \mathcal{B}(\ell^2(\mathcal{G} \rtimes \Lambda))$, and the reduction of π^L to $\ell^2(\mathcal{G})$ is the left regular representation λ of $C^*(\mathcal{G}, c)$. Since \mathcal{G} is amenable, λ is faithful, so $\pi_{\mathcal{G}}^L = \pi^L \circ \iota_{\mathcal{G}}^t$ is injective, and hence $\iota_{\mathcal{G}}^t$ is injective.

Finally, fix $v \in \Lambda^0$, $n \in \mathbb{N}^k$, and $\lambda \in \Lambda$ such that $g \mapsto (g \triangleright \lambda, g \triangleleft \lambda)$ is injective on $\mathcal{G}_{r(\lambda)}^{r(\lambda)}$. Again following Yusnitha [Yus23, Proposition 3.6], the space $\mathcal{H}_{n,\lambda} := \overline{\text{span}}\{e_{g\lambda} : g \in \mathcal{G}_v^v\}$ is invariant for $\{L_g : g \in \mathcal{G}_v^v\}$, and $e_{g\lambda} \mapsto \overline{c(g, \lambda)}e_g$ induces an isomorphism $U: \mathcal{H}_{n,\lambda} \rightarrow \ell^2(\mathcal{G}_v^v)$ that intertwines the reduction of $\pi^L|_{C^*(\mathcal{G}_v^v, c)}$ with the regular representation. Since \mathcal{G} is amenable, so is \mathcal{G}_v^v and so the reduction of $\pi^L|_{C^*(\mathcal{G}_v^v, c)}$ to $\mathcal{H}_{n,\lambda}$ is faithful. So, for $a \in C^*(\mathcal{G}_v^v, c)$, we have $\left\| \pi^L(a)|_{\overline{\text{span}}\{e_{\zeta g} : \zeta \in \Lambda^n, g \in \mathcal{G}\}} \right\| = \|a\|$ for all n . So by (7.4), $a \notin I$, so $\iota_{\mathcal{G}}^s$ is injective on each $C^*(\mathcal{G}_v^v, c)$.

Fix a subset $V \subseteq \mathcal{G}^0$ that intersects each \mathcal{G} -orbit exactly once. Then $P_V = \sum_{v \in V} \delta_v \in \mathcal{M}C^*(\mathcal{G}, c)$ is a full projection, and $P_V C^*(\mathcal{G}, c) P_V \cong \bigoplus_{v \in V} C^*(\mathcal{G}_v, c)$. Since $\iota_s^{\mathcal{G}}$ is injective on this full corner, it is injective on all of $C^*(\mathcal{G}, c)$. \square

Remark 7.8. Let (\mathcal{G}, Λ) be a self-similar action of an amenable groupoid on a row-finite k -graph with no sources, and let $c : (\mathcal{G} \rtimes \Lambda)^2 \rightarrow \mathbb{T}$ be a normalised categorical 2-cocycle. Proposition 7.7 shows that $\mathcal{T}C^*(\mathcal{G}, \Lambda; c)$ is generated copies of $\mathcal{T}C^*(\Lambda; c)$ and $C^*(\mathcal{G}, c)$. It would be interesting to determine when $\mathcal{T}C^*(\mathcal{G}, \Lambda; c)$ is a C^* -blend of these two subalgebras in the sense of [Exe13]; or when $C^*(\mathcal{G}, \Lambda; c)$ is a blend of $C^*(\Lambda, c)$ and $C^*(\mathcal{G}, c)$ (the corresponding result for Zappa–Szépproducts of Fell bundles over groupoids appears in [DL23, Theorem 5.4]). For example, it seems likely that a contracting condition like that of [LRRW18, Section 9] or [Nek05, Section 2.11] implies that each spanning element $s_\mu s_g s_\nu^*$ of $C^*(\mathcal{G}, \Lambda; c)$ belongs to $\overline{\text{span}}\{s_\alpha s_\beta^* s_g \mid \alpha, \beta \in \Lambda, g \in \mathcal{G}\}$. But we do not pursue this question here.

We now prove a gauge-invariant uniqueness theorem for $C^*(\mathcal{G}, \Lambda; c)$. This by-now ubiquitous tool in the study of C^* -algebras of graphs and related objects goes back to [aHR97]. Our argument in the context of twisted C^* -algebras of self-similar actions on k -graphs generalises those of [LRRW18, ABRW19, Yus23] for untwisted actions on graphs and k -graphs; our analysis of the core is heavily based on Yusnitha’s [Yus23].

Proposition 7.9 (cf. [Yus23, Lemma 4.10]). *Let (\mathcal{G}, Λ) be a self-similar groupoid action on a row-finite k -graph with no sources. Let $c : (\mathcal{G} \rtimes \Lambda)^2 \rightarrow \mathbb{T}$ be a normalised categorical 2-cocycle. Then*

$$C^*(\mathcal{G}, \Lambda; c)^\gamma = \overline{\text{span}}\{s_\mu s_g s_\nu^* : \mu, \nu \in \Lambda, d(\mu) = d(\nu), g \in \mathcal{G}_{s(\nu)}^{s(\mu)}\}.$$

Let S be a Cuntz–Krieger $(\Lambda, \mathcal{G}; c)$ -family, and suppose that $\pi^S : C^*(\mathcal{G}, \Lambda; c) \rightarrow C^*(S)$ is injective on $\iota_{\mathcal{G}}^S(C^*(\mathcal{G}, c))$. Then π^S is injective on $C^*(\mathcal{G}, \Lambda; c)^\gamma$.

Proof. Let $\Phi : C^*(\mathcal{G}, \Lambda; c) \rightarrow C^*(\mathcal{G}, \Lambda; c)^\gamma$ be the faithful conditional expectation satisfying $\Phi(a) = \int_{\mathbb{T}^k} \gamma_z(a) dz$ [Rae05, Proposition 3.2]. Then $\Phi(s_\mu s_g s_\nu^*) = \delta_{d(\mu), d(\nu)} s_\mu s_g s_\nu^*$, and so

$$\begin{aligned} C^*(\mathcal{G}, \Lambda; c)^\gamma &= \Phi(C^*(\mathcal{G}, \Lambda; c)) = \Phi\left(\overline{\text{span}}\{s_\mu s_g s_\nu^* : \mu, \nu \in \Lambda, g \in \mathcal{G}_{s(\nu)}^{s(\mu)}\}\right) \\ &= \overline{\text{span}}\{s_\mu s_g s_\nu^* : \mu, \nu \in \Lambda, d(\mu) = d(\nu), g \in \mathcal{G}_{s(\nu)}^{s(\mu)}\}. \end{aligned}$$

For fixed $n \in \mathbb{N}^k$,

$$F_n := \overline{\text{span}}\{s_\mu s_g s_\nu^* : \mu, \nu \in \Lambda^n, g \in \mathcal{G}_{s(\nu)}^{s(\mu)}\}$$

is a C^* -subalgebra of $C^*(\mathcal{G}, \Lambda; c)^\gamma$. If $m, n \in \mathbb{N}^k$, $\mu, \nu \in \Lambda^m$, and $g \in \mathcal{G}_{s(\nu)}^{s(\mu)}$, then

$$\begin{aligned} s_\mu s_g s_\nu^* &= s_\mu s_g \sum_{\tau \in s(g)\Lambda^n} s_\tau s_\tau^* s_\nu^* = \sum_{\tau \in s(g)\Lambda^n} c(g, \tau) \overline{c(g \triangleright \tau, g \triangleleft \tau)} s_\mu s_{g \triangleright \tau} s_{g \triangleleft \tau}^* s_\nu^* \\ &= \sum_{\tau \in s(g)\Lambda^n} c(\mu, g \triangleright \tau) \overline{c(\nu, \tau)} c(g, \tau) \overline{c(g \triangleright \tau, g \triangleleft \tau)} s_{\mu g \triangleright \tau} s_{g \triangleleft \tau}^* s_\nu^* \in F_{m+n}. \end{aligned}$$

So $F_m \subseteq F_{m+n}$ and $C^*(\mathcal{G}, \Lambda; c)^\gamma = \overline{\bigcup_n F_n}$. So to see that π^S is injective on $C^*(\mathcal{G}, \Lambda; c)^\gamma$, it suffices to show that it is injective, and hence isometric, on each F_n .

Fix $n \in \mathbb{N}^k$. Consider the equivalence relation \sim on Λ^n such that $\lambda \sim \mu$ if and only if $\mathcal{G}_{s(\mu)}^{s(\lambda)} \neq \emptyset$. Let $K \subseteq \Lambda^n$ be a set of representatives of Λ^n / \sim . For $\lambda \in \Lambda^n$, there exists $\mu \in K$ with $\mu \sim \lambda$, say $g \in \mathcal{G}_{s(\mu)}^{s(\lambda)}$. So $s_\lambda s_\lambda^* = s_\lambda u_g s_\mu^* (s_\mu s_\mu^*) s_\mu u_g^* s_\lambda$. Since $\sum_{\lambda \in \Lambda^n} s_\lambda s_\lambda^*$ is an approximate identity for F_n it

follows that $P_K := \sum_{\lambda \in K} s_\lambda s_\lambda^*$ is a full projection in $\mathcal{M}F_n$. So it suffices to show that π^S is injective on $P_K F_n P_K$.

For distinct $\lambda, \mu \in K$ we have $s_\lambda s_\lambda^* F_k s_\mu s_\mu^* = \{0\}$ by definition of \sim , and so $P_k F_k P_k \cong \bigoplus_{\lambda \in K} s_\lambda s_\lambda^* F_k s_\lambda s_\lambda^*$. So it suffices to show that π^S is injective on each $s_\lambda s_\lambda^* F_k s_\lambda s_\lambda^*$.

Fix $\lambda \in K$, and let $v := s(\lambda)$. Since the $s_\mu s_\mu^*$, for $\mu \in \Lambda^n$, are mutually orthogonal, $s_\lambda s_\lambda^* F_k s_\lambda s_\lambda^* = \overline{\text{span}}\{s_\lambda s_g s_\lambda^* : g \in \mathcal{G}_v^v\}$. Conjugation by s_λ is an isomorphism of this subalgebra onto $\overline{\text{span}}\{s_g : g \in \mathcal{G}_v^v\}$. Similarly, $\overline{\text{span}}\{S_\lambda S_g S_\lambda^* : g \in \mathcal{G}_v^v\} \cong \overline{\text{span}}\{S_g : g \in \mathcal{G}_v^v\}$ via conjugation by S_λ . Since $\pi^S(s_\lambda) = S_\lambda$, and π^S is injective on $\overline{\text{span}}\{s_g : g \in \mathcal{G}_v^v\}$, the result follows. \square

We obtain a version of an Huef and Raeburn's gauge-invariant uniqueness theorem [aHR97].

Corollary 7.10 (The Gauge-Invariant Uniqueness Theorem). *Let (\mathcal{G}, Λ) be a self-similar action of a groupoid on a row-finite k graph with no sources, and let $c : (\mathcal{G} \rtimes \Lambda)^2 \rightarrow \mathbb{T}$ be a normalised categorical 2-cocycle. Let S be a Cuntz–Krieger $(\mathcal{G}, \Lambda; c)$ family in a C^* -algebra A . If there is a strongly-continuous action $\beta : \mathbb{T}^k \rightarrow \text{Aut}(A)$ such that $\beta_z(S_\zeta) = z^{d(\zeta)} s_\zeta$ for all $\zeta \in \mathcal{G} \rtimes \Lambda$, and if $\pi^S : C^*(\mathcal{G}, \Lambda; c) \rightarrow C^*(S)$ is injective on $\mathcal{I}_{\mathcal{G}}^s(C^*(\mathcal{G}, c))$, then π^S is injective.*

Proof. The assumptions combined with Proposition 7.9 show that π^S is injective on $C^*(\mathcal{G}, \Lambda; c)^\gamma$. Define $\Gamma : A \rightarrow A$ by $\Gamma(a) = \int_{\mathbb{T}^k} \beta_z(a) dz$. Since $\beta_z \circ \pi^S = \pi^S \circ \gamma_z$ for all z , we have $\pi^S \circ \Phi = \Gamma \circ \pi^S$, and then [SWW14, Lemma 3.14] shows that π^S is injective. \square

We now show that the isomorphism class of the twisted C^* -algebra of a self-similar action of a groupoid on a k -graph depends only on the cohomology class of the twisting 2-cocycle. The argument is standard; see, for example, [KPS15, Proposition 5.6].

Proposition 7.11. *Let (\mathcal{G}, Λ) be a self-similar action on a row-finite k -graph with no sources, and let $c_1, c_2 : (\mathcal{G} \rtimes \Lambda)^2 \rightarrow \mathbb{T}$ be normalised categorical 2-cocycles. Suppose that $b : \mathcal{G} \rtimes \Lambda \rightarrow \mathbb{T}$ is a categorical 1-cochain such that $d_{\rtimes}^1(b)c_1 = c_2$. For $i = 1, 2$ let t^i be the universal Toeplitz–Cuntz–Krieger family in $\mathcal{T}C^*(\mathcal{G}, \Lambda; c_i)$. Then there is an isomorphism $\theta_b : \mathcal{T}C^*(\mathcal{G}, \Lambda; c_2) \rightarrow \mathcal{T}C^*(\mathcal{G}, \Lambda; c_1)$ such that $\theta_b(t_\zeta^2) = b(\zeta)t_\zeta^1$ for all $\zeta \in \mathcal{G} \rtimes \Lambda$. This isomorphism descends to an isomorphism $\tilde{\theta}_b : C^*(\mathcal{G}, \Lambda; c_2) \rightarrow C^*(\mathcal{G}, \Lambda; c_1)$.*

Proof. Define $bt : \mathcal{G} \rtimes \Lambda \rightarrow \mathcal{T}C^*(\mathcal{G}, \Lambda; c_1)$ by $(bt)_\zeta := b(\zeta)t_\zeta^1$. For $(\zeta, \eta) \in (\mathcal{G} \rtimes \Lambda)^2$,

$$(bt)_\zeta (bt)_\eta = b(\zeta)t_\zeta^1 b(\eta)t_\eta^1 = b(\zeta\eta)d^1(b)(\zeta, \eta)c_1(\zeta, \eta)t_{\zeta\eta}^1 = c_2(\zeta, \eta)(bt)_{\zeta\eta}.$$

So bt satisfies (TCK1). It also satisfies (TCK2) and (TCK3), and satisfies (CK) if and only if t does, because the factors of $b(\zeta)$ and $\overline{b(\zeta)}$ in these relations cancel.

The universal property of $\mathcal{T}C^*(\mathcal{G}, \Lambda; c_2)$ gives a homomorphism $\theta_b : \mathcal{T}C^*(\mathcal{G}, \Lambda; c_2) \rightarrow \mathcal{T}C^*(\mathcal{G}, \Lambda; c_1)$ such that $\theta_b(t_\zeta^2) = b(\zeta)t_\zeta^1$ for all $\zeta \in \mathcal{G} \rtimes \Lambda$, which descends to a homomorphism $\tilde{\theta}_b : C^*(\mathcal{G}, \Lambda; c_2) \rightarrow C^*(\mathcal{G}, \Lambda; c_1)$. Since $d^1(\tilde{b})c_2 = c_1$, there is a corresponding homomorphism $\theta_{\tilde{b}} : \mathcal{T}C^*(\mathcal{G}, \Lambda; c_1) \rightarrow \mathcal{T}C^*(\mathcal{G}, \Lambda; c_2)$ such that $\theta_{\tilde{b}}(t_\zeta^1) = \overline{b(\zeta)}t_\zeta^2$, which also descends to Cuntz–Krieger algebras. Since $\theta_b \circ \theta_{\tilde{b}}$ and $\theta_{\tilde{b}} \circ \theta_b$ fix the generators t_ζ^i , they are the identity homomorphisms, and this descends to Cuntz–Krieger algebras as well. \square

7.2. Twists by total 2-cocycles. We describe the twisted C^* -algebra of a self-similar action on k -graph with respect to a total 2-cocycle, and show that we obtain the same class of C^* -algebras as for categorical cohomology.

Definition 7.12. Let (\mathcal{G}, Λ) be a self-similar action of a groupoid on a row-finite k -graph with no sources. A function $\varphi: \mathcal{G}^2 \sqcup (\mathcal{G} * \Lambda) \sqcup \Lambda^2 \rightarrow \mathbb{T}$ is a *normalised total \mathbb{T} -valued 2-cocycle* on (\mathcal{G}, Λ) , if $\varphi_{2,0} := \varphi|_{\mathcal{G}^2}$, $\varphi_{1,1} := \varphi|_{\mathcal{G} * \Lambda}$ and $\varphi_{0,2} := \varphi|_{\Lambda^2}$ satisfy

- (i) $\varphi_{2,0}: \mathcal{G}^2 \rightarrow \mathbb{T}$ is a normalised \mathbb{T} -valued 2-cocycle in the sense of [Ren80];
- (ii) $\varphi_{0,2}: \Lambda^2 \rightarrow \mathbb{T}$ is a normalised \mathbb{T} -valued categorical 2-cocycle in the sense of [KPS15]; and
- (iii) $\varphi_{1,1}(h, \lambda) = 1$ whenever $h \in \mathcal{G}^0$, or $\lambda \in \Lambda^0$, and for $(g, h, \lambda, \mu) \in \mathcal{G} * \mathcal{G} * \Lambda * \Lambda$,

$$\begin{aligned} \varphi_{1,1}(h \triangleleft \lambda, \mu) \overline{\varphi_{1,1}(h, \lambda \mu)} \varphi_{1,1}(h, \lambda) \varphi_{0,2}(\lambda, \mu) \overline{\varphi_{0,2}(h \triangleright (\lambda, \mu))} &= 1 \quad \text{and} \\ \varphi_{2,0}((g, h) \triangleleft \lambda) \overline{\varphi_{2,0}(g, h)} \overline{\varphi_{1,1}(h, \lambda)} \varphi_{1,1}(gh, \lambda) \overline{\varphi_{1,1}(g, h \triangleright \lambda)} &= 1. \end{aligned}$$

Remark 7.13. In defining a normalised total \mathbb{T} -valued 2-cocycle we have just written out explicitly what it means for φ to be a cocycle in $C_{\text{Tot}}^2(\mathcal{G}, \Lambda; \mathbb{T})$. This can be verified by computing what it means for a cochain to be in the kernel $d_{\text{Tot}}^2: C_{\text{Tot}}^2(\mathcal{G}, \Lambda; \mathbb{T}) \rightarrow C_{\text{Tot}}^3(\mathcal{G}, \Lambda; \mathbb{T})$ which satisfies $d_{\text{Tot}}^2 \circ \varphi = \varphi \circ d_2^{\text{Tot}}$, where d_2^{Tot} is from Section 4.4.

Definition 7.14. Let (\mathcal{G}, Λ) be a self-similar action of a groupoid on a row-finite k -graph with no sources. Fix a normalised cocycle $\varphi \in C_{\text{Tot}}^2(\mathcal{G}, \Lambda; \mathbb{T})$ and let A be a C^* -algebra. A pair of functions $\mathfrak{t}: \Lambda \rightarrow A$ and $\mathfrak{w}: \mathcal{G} \rightarrow A$ is a *Toeplitz φ -pair* if:

- (T1) \mathfrak{t} is a Toeplitz–Cuntz–Krieger $(\Lambda, \varphi_{0,2})$ -family in the sense of [SWW14],
- (T2) \mathfrak{w} is a unitary representation of $(\mathcal{G}, \varphi_{2,0})$, and
- (T3) $\mathfrak{w}_g \mathfrak{t}_\lambda = \varphi_{1,1}(g, \lambda) \mathfrak{t}_{g \triangleright \lambda} \mathfrak{w}_{g \triangleleft \lambda}$ for all $(g, \lambda) \in \mathcal{G} * \Lambda$.

We call $(\mathfrak{w}, \mathfrak{t})$ a *Cuntz–Krieger φ -pair* if \mathfrak{t} is a Cuntz–Krieger $(\Lambda, \varphi_{0,2})$ -family.

With $\Psi_\bullet: C_\bullet^{\text{Tot}}(\mathcal{G}, \Lambda) \rightarrow C_\bullet^{\text{Tot}}(\mathcal{G}, \Lambda)$ as in Subsection 5.1.3, define $\Psi^\bullet: C_{\text{Tot}}^\bullet(\mathcal{G}, \Lambda; \mathbb{T}) \rightarrow C_{\text{Tot}}^\bullet(\mathcal{G}, \Lambda; \mathbb{T})$ by $\Psi^k = \varphi \circ \Psi_k$. Then Ψ^\bullet induces an isomorphism on cohomology. On 2-cochains,

$$\Psi^2(\varphi)(\lambda g, \mu h) = \varphi_{2,0}(g \triangleleft \mu, h) \varphi_{1,1}(g, \mu) \varphi_{0,2}(\lambda, g \triangleright \mu)$$

for all $(\lambda, g, \mu, h) \in \Lambda * \mathcal{G} * \Lambda * \mathcal{G}$.

We show that $\mathcal{TC}^*(\mathcal{G}, \Lambda; \Psi^2(\varphi))$ is universal for Toeplitz φ -pairs, and $C^*(\mathcal{G}, \Lambda; \Psi^2(\varphi))$ is universal for Cuntz–Krieger φ -pairs.

Theorem 7.15. *Let (\mathcal{G}, Λ) be a self-similar action of a groupoid on a row-finite k -graph with no sources. Fix a normalised cocycle $\varphi \in C_{\text{Tot}}^2(\mathcal{G}, \Lambda; \mathbb{T})$ and let $c := \Psi^2(\varphi) \in C_{\text{Tot}}^2(\mathcal{G}, \Lambda; \mathbb{T})$.*

- (i) *Let $t: \mathcal{G} \rtimes \Lambda \rightarrow \mathcal{TC}^*(\mathcal{G}, \Lambda; c)$ be the universal Toeplitz–Cuntz–Krieger $(\mathcal{G}, \Lambda; c)$ -family. There is a Toeplitz φ -pair $\mathfrak{t}, \mathfrak{w}$ in $\mathcal{TC}^*(\mathcal{G}, \Lambda; c)$ given by $\mathfrak{t} = t|_\Lambda$ and $\mathfrak{w} = t|_{\mathcal{G}}$. Moreover $\mathcal{TC}^*(\mathcal{G}, \Lambda; c)$ is generated by the ranges of \mathfrak{t} and \mathfrak{w} , and is universal in the sense that given any Toeplitz φ -pair $\mathfrak{t}', \mathfrak{w}'$ in a C^* -algebra A , there is a homomorphism $\rho: \mathcal{TC}^*(\mathcal{G}, \Lambda; c) \rightarrow A$ such that $\rho \circ \mathfrak{t} = \mathfrak{t}'$ and $\rho \circ \mathfrak{w} = \mathfrak{w}'$.*
- (ii) *Let $s: \mathcal{G} \rtimes \Lambda \rightarrow C^*(\mathcal{G}, \Lambda; c)$ be the universal Cuntz–Krieger $(\mathcal{G}, \Lambda; c)$ -family. Then there is a Cuntz–Krieger φ -pair $\mathfrak{s}, \mathfrak{u}$ in $C^*(\mathcal{G}, \Lambda; c)$ given by $\mathfrak{s} = s|_\Lambda$ and $\mathfrak{u} = s|_{\mathcal{G}}$. Moreover $C^*(\mathcal{G}, \Lambda; c)$ is generated by the ranges of \mathfrak{s} and \mathfrak{u} , and is universal in the sense that given any Cuntz–Krieger φ -pair $\mathfrak{s}', \mathfrak{u}'$ in a C^* -algebra A , there is a homomorphism $\rho: C^*(\mathcal{G}, \Lambda; c) \rightarrow A$ such that $\rho \circ \mathfrak{s} = \mathfrak{s}'$ and $\rho \circ \mathfrak{u} = \mathfrak{u}'$.*

The theorem follows from the following correspondence between φ pairs and $(\mathcal{G}, \Lambda; c)$ -families.

Lemma 7.16. *Let (\mathcal{G}, Λ) be a self-similar action of a groupoid on a row-finite k -graph with no sources. Fix a normalised cocycle $\varphi \in C_{\text{Tot}}^2(\mathcal{G}, \Lambda; \mathbb{T})$ and let $c := \Psi^2(\varphi) \in C_{\bowtie}^2(\mathcal{G}, \Lambda; \mathbb{T})$. If \mathbf{t}, \mathbf{w} is a Toeplitz φ -pair in a C^* -algebra A , then*

$$t_{\lambda g} := \mathbf{t}_\lambda \mathbf{w}_g$$

defines a Toeplitz–Cuntz–Krieger $(\mathcal{G}, \Lambda; c)$ -family in A . If t is a Toeplitz–Cuntz–Krieger $(\mathcal{G}, \Lambda; c)$ -family in A , then $\mathbf{t}_\lambda := t_\lambda$ for $\lambda \in \Lambda$ and $\mathbf{w}_g := t_g$ for $g \in \mathcal{G}$ defines a Toeplitz φ -pair. Moreover, \mathbf{t}, \mathbf{w} is a Cuntz–Krieger φ -pair if and only if t is a Cuntz–Krieger $(\mathcal{G}, \Lambda; c)$ -family.

Proof. For $g \in \mathcal{G}$ and $\lambda \in s(g)\Lambda$,

$$c(g, \lambda) = c(r(g)g, \lambda s(\lambda)) = \varphi_{2,0}(g \triangleleft \lambda, s(\lambda)) \varphi_{1,1}(g, \lambda) \varphi_{0,2}(r(g), g \triangleright \lambda) = \varphi_{1,1}(g, \lambda).$$

Also, for $h \in \mathcal{G}$ and $\mu \in \Lambda r(h)$, since $v \triangleleft v = v \triangleright v = v$ for $v \in \Lambda^0 = \mathcal{G}^0$, we have

$$c(\mu, h) = c(\mu s(\mu), r(h)h) = \varphi_{2,0}(r(h), h) \varphi_{1,1}(s(\mu), r(h)) \varphi_{0,2}(\mu, s(\mu)) = 1.$$

Hence, if $g \in \mathcal{G}$ and $\lambda \in s(g)\Lambda$, we have

$$c(g, \lambda) \overline{c(g \triangleright \lambda, g \triangleleft \lambda)} = c(g, \lambda) 1 = \varphi_{1,1}(g, \lambda). \quad (7.5)$$

Suppose that \mathbf{t}, \mathbf{w} is a Toeplitz φ -pair. If $\lambda g, \mu h$ are composable in $\mathcal{G} \bowtie \Lambda$, then

$$\begin{aligned} \mathbf{t}_\lambda \mathbf{w}_g \mathbf{t}_\mu \mathbf{w}_h &= \varphi_{1,1}(g, \mu) \mathbf{t}_\lambda \mathbf{t}_{g \triangleright \mu} \mathbf{w}_{g \triangleleft \mu} \mathbf{w}_h \\ &= \varphi_{2,0}(\lambda, g \triangleright \mu) \varphi_{1,1}(g, \mu) \varphi_{0,2}(g \triangleleft \mu, h) \mathbf{t}_{\lambda(g \triangleright \mu)} \mathbf{w}_{(g \triangleleft \mu)h} \\ &= c(\lambda g, \mu h) \mathbf{t}_{\lambda(g \triangleright \mu)} \mathbf{w}_{(g \triangleleft \mu)h}, \end{aligned}$$

so $t: \lambda g \mapsto \mathbf{t}_\lambda \mathbf{w}_g$ satisfies (TCK1). For (TCK2), we calculate

$$t_{\lambda g}^* t_{\lambda g} = \mathbf{w}_g^* \mathbf{t}_\lambda^* \mathbf{t}_\lambda \mathbf{w}_g = \mathbf{w}_g^* \mathbf{t}_{s(\lambda)} \mathbf{w}_g = \mathbf{w}_g^* \mathbf{w}_g = \mathbf{w}_{s(g)} = s_{s(\lambda g)}.$$

Relation (TCK3) follows from (T1) by definition of a Toeplitz–Cuntz–Krieger $(\Lambda, \varphi_{0,2})$ -family.

Now suppose that t is a Toeplitz–Cuntz–Krieger $(\mathcal{G}, \Lambda; c)$ -family, and define $\mathbf{t} = t|_{\mathcal{G}}$ and $\mathbf{w} = t|_{\Lambda}$. We have $\mathbf{t}_v = \mathbf{w}_v$ for $v \in \Lambda^0$ because $\mathcal{G}^0 = \Lambda^0$.

Remark 7.2 implies that \mathbf{t} and \mathbf{w} satisfy (T1) and (T2). For (T3), we calculate

$$\mathbf{w}_g \mathbf{t}_\lambda = t_g t_\lambda \stackrel{(7.5)}{=} c(g, \lambda) \overline{c(g \triangleright \lambda, g \triangleleft \lambda)} t_{g \triangleright \lambda} t_{g \triangleleft \lambda} = \varphi_{1,1}(g, \lambda) \mathbf{t}_{g \triangleright \lambda} \mathbf{w}_{g \triangleleft \lambda}.$$

So \mathbf{t}, \mathbf{w} is a Toeplitz φ -pair. For the final assertion, observe that

$$\sum_{\lambda \in v\Lambda^n} t_\lambda t_\lambda^* = \sum_{\lambda \in v\Lambda^n} \mathbf{t}_\lambda \mathbf{w}_{s(\lambda)} \mathbf{w}_{s(\lambda)}^* \mathbf{t}_\lambda^* = \sum_{\lambda \in v\Lambda^n} \mathbf{t}_\lambda \mathbf{t}_\lambda^*. \quad \square$$

Proof of Theorem 7.15. (i) The second statement of Lemma 7.16 shows that \mathbf{t}, \mathbf{w} is a Toeplitz φ -pair. It generates $\mathcal{TC}^*(\mathcal{G}, \Lambda; c)$ because each $t_{\lambda g} = c(\lambda, g) \mathbf{t}_\lambda \mathbf{w}_g = c(\lambda, g) \mathbf{t}_\lambda \mathbf{w}_g$; and given a Toeplitz φ -pair \mathbf{t}', \mathbf{w}' , the first statement of Lemma 7.16 shows that $t'_{\lambda g} := \mathbf{t}'_\lambda \mathbf{w}'_g$ defines a Toeplitz–Cuntz–Krieger $(\mathcal{G}, \Lambda; c)$ -family. So the universal property of $\mathcal{TC}^*(\mathcal{G}, \Lambda; c)$ gives a homomorphism ρ such that $\rho(t_{\lambda g}) = t'_{\lambda g}$. In particular, $\rho(\mathbf{t}_\lambda) = \mathbf{t}'_\lambda$, and $\rho(\mathbf{w}_g) = \mathbf{w}'_g$.

(ii) Apply (i) together with the final statement of Lemma 7.16. □

REFERENCES

- [ABRW19] Z. Afsar, N. Brownlowe, J. Ramagge and M.F. Whittaker, *C*-algebras of self-similar actions of groupoids on higher-rank graphs and their equilibrium states*, preprint 2019 (arXiv:1910.02472 [math.OA]).
- [AA05] M. Aguiar and N. Andruskiewitsch, *Representations of matched pairs of groupoids and applications to weak Hopf algebras*, Algebraic structures and their representations, Contemp. Math. **376**, American Mathematical Society, Providence, RI, (2005), 127–173.
- [Bas93] H. Bass, *Covering theory for graphs of groups*, J. Pure Appl. Algebra **89** (1993), 3–47.
- [BKQS18] E. Bédos E., S. Kaliszewski, J. Quigg and J. Spielberg, *On finitely aligned left cancellative small categories, Zappa–Szép products and Exel–Pardo algebras*, Theory Appl. Categ. **33** (2018), 1346–1406.
- [Bla85] B. Blackadar, *Shape theory for C*-algebras*, Math. Scand. **56** (1985), 249–275.
- [Bri05] M.G. Brin, *On the Zappa–Szép product*, Comm. in Algebra **33** (2005), 393–424.
- [BPRRW17] N. Brownlowe, D. Pask, J. Ramagge, D. Robertson, and M.F. Whittaker, *Zappa–Szép product groupoids and C*-blends*, Semigroup Forum **94** (2017), 500–519.
- [Dea21] V. Deaconu, *On groupoids and C*-algebras from self-similar actions*, New York J. Math. **27** (2021), 923–942.
- [DL23] A. Duwenig and B. Li, *Imprimitivity theorems and self-similar actions on Fell bundles*, preprint 2023 (arXiv:2208.01124 [math.OA]).
- [Exe13] R. Exel, *Blends and alloys*, C. R. Math. Acad. Sci. Soc. R. Can. **35** (2013), 77–113.
- [EP17] R. Exel and E. Pardo, *Self-similar graphs, a unified treatment of Katsura and Nekrashevych algebras*, Adv. Math. **306** (2017), 1046–1129.
- [GK18] E. Gillaspy and A. Kumjian, *Cohomology for small categories: k-graphs and groupoids*, Banach J. Math. Anal. **12** (2018), 572–599.
- [Gri80] R.I. Grigorchuk, *On Burnside’s problem on periodic groups*, Func. Anal. Appl. **14** (1980), 53–54.
- [Gri84] R.I. Grigorchuk, *Degrees of growth of finitely generated groups and the theory of invariant means* (Russian), Izv. Akad. Nauk SSSR Ser. Mat. **48** (1984), 939–985.
- [HRSW13] R. Hazlewood, I. Raeburn, A. Sims and S.B.G. Webster, *Remarks on some fundamental results about higher-rank graphs and their C*-algebras*, Proc. Edinb. Math. Soc. (2) **56** (2013), 575–597.
- [aHR97] A. an Huef and I. Raeburn, *The ideal structure of Cuntz–Krieger algebras*, Ergodic Theory Dynam. Systems **17** (1997), 611–624.
- [KP00] A. Kumjian and D. Pask, *Higher rank graph C*-algebras*, New York J. Math. **6** (2000), 1–20.
- [KPS15] A. Kumjian, D. Pask and A. Sims, *On twisted higher-rank graph C*-algebras*, Trans. Amer. Math. Soc. **367** (2015), 5177–5216.
- [LRRW14] M. Laca, I. Raeburn, J. Ramagge and M.F. Whittaker, *Equilibrium states on the Cuntz–Pimsner algebras of self-similar actions*, J. Funct. Anal. **266** (2014), 6619–6661.
- [LRRW18] M. Laca, I. Raeburn, J. Ramagge and M.F. Whittaker, *Equilibrium states on operator algebras associated to self-similar actions of groupoids on graphs*, Adv. Math. **331** (2018), 268–325.
- [LarV22] N. Larsen and A. Vdovina, *Higher dimensional digraphs from cube complexes and their spectral theory*, Groups Geom. Dyn., to appear (arXiv:2111.09120v2 [math.OA]).
- [Law08] M.V. Lawson, *A correspondence between a class of monoids and self-similar group actions I*, Semigroup Forum **76** (2008), 489–517.
- [LawV22] M.V. Lawson and A. Vdovina, *A generalisation of higher-rank graphs*, Bull. Austral. Math. Soc. **105** (2022), 257–266.
- [LY21] H. Li and D. Yang, *Self-similar k-graph C*-algebras*, Int. Math. Res. Not. **2021** (2021), 11270–11305.
- [Lor10] T.A. Loring, *C*-algebra relations*, Math. Scand. **107** (2010), 43–72.
- [Mas91] W.S. Massey, “A basic course in algebraic topology,” Grad. Texts in Math., vol 127, Springer-Verlag, New York, 1991, xvi+428pp.
- [MR21] A. Munday and A. Rennie, *A Cuntz–Pimsner model for the C*-algebra of a graph of groups*, J. Math. Anal. Appl. **496** (2021), 124838.

- [Nek05] V. Nekrashevych, “Self-Similar Groups,” Math. Surveys Monogr., vol. 117, Amer. Math. Soc., Providence, 2005.
- [Nek18] V. Nekrashevych, *Palindromic subshifts and simple periodic groups of intermediate growth*, Ann. Math. **187** (2018), 667–719.
- [OPT80] D. Olesen, G.K. Pedersen and M. Takesaki, *Ergodic actions of compact abelian groups*, J. Operator Th. **3** (1980), 237–269.
- [PO22] E. Pardo and E. Ortega, *Zappa-Szép products for partial actions of groupoids on left cancellative small categories*, J. Noncomm. Geom. to appear (arXiv:2110.05872v1 [math.OA]).
- [Rae05] I. Raeburn, “Graph algebras,” Published for the Conference Board of the Mathematical Sciences, Washington, DC, 2005, vi+113.
- [RS05] I. Raeburn and A. Sims, *Product systems of graphs and the C^* -algebras of higher-rank graphs*, J. Operator Th. **53** (2005), 399–429.
- [RSY04] I. Raeburn, A. Sims and T. Yeend, *The C^* -algebras of finitely aligned higher-rank graphs*, J. Funct. Anal. **213** (2004), 206–240.
- [Ren80] J. Renault, “A groupoid approach to C^* -algebras,” Lecture Notes in Mathematics, vol. 793, Springer, Berlin, 1980, ii+160pp.
- [RW02] R. Rosebrugh and R.J. Wood, *Distributive laws and factorization*, J. Pure Appl. Algebra **175** (2002), 327–353.
- [Rot88] J.J. Rotman, “An introduction to algebraic topology,” Graduate Texts in Mathematics **119**, Springer-Verlag, New York, 1988, ISBN: 0-387-96678-1, DOI: 10.1007/978-1-4612-4576-6.
- [Ser80] J.-P. Serre, *Trees*. Springer-Verlag, Berlin-New York, 1980, Translated from the French by John Stillwell.
- [SWW14] A. Sims, B. Whitehead and M.F. Whittaker, *Twisted C^* -algebras associated to finitely aligned higher-rank graphs*, Documenta Math. **19** (2014), 831–866.
- [Sin72] W.M. Singer, *Extension theory for connected Hopf algebras*, J. Algebra **21** (1972), 1–16.
- [Spe20] J. Spielberg, *Groupoids and C^* -algebras for left cancellative small categories*, Indiana Univ. Math. J. **69** (2020), 1579–1626.
- [Sze50] J. Szép, *On factorisable, not simple groups*, Acta Univ. Szeged. Sect. Sci. Math. **13** (1950), 239–241.
- [Wei94] C.A. Weibel, “An introduction to homological algebra,” Cambridge Studies in Advanced Mathematics **38**, Cambridge University Press, Cambridge, 1994, ISBN: 0-521-43500-5, DOI: 10.1017/CBO9781139644136.
- [Yus23] I. Yusnitha, *C^* -algebras of self-similar action of groupoids on row-finite directed graphs*, Bull. Aust. Math. Soc. **108** (2023), 150–161.
- [Zap42] G. Zappa, *Sulla costruzione dei gruppi prodotto di due dati sottogruppi permutabili tra loro*, In Atti Secondo Congresso Un. Mat. Ital., Bologna (1942), 119–125.

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