ULTRAGRAPH C^*-ALGEBRAS VIA TOPOLOGICAL QUIVERS

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Abstract. Given an ultragraph in the sense of Tomforde, we construct a topological quiver in the sense of Muhly and Tomforde in such a way that the universal C^*-algebras associated to the two objects coincide. We apply results of Muhly and Tomforde for topological quiver algebras and of Katsura for topological graph C^*-algebras to study the K-theory and gauge-invariant ideal structure of ultragraph C^*-algebras.

1. Introduction

Our objective in this paper is to show how the theory of *ultragraph C^*-algebras*, first proposed by Tomforde in [13, 14], can be formulated in the context of topological graphs [6] and topological quivers [11] in a fashion that reveals the K-theory and ideal theory (for gauge-invariant ideals) of these algebras. The class of graph C^*-algebras has attracted enormous attention in recent years. The graph C^*-algebra associated to a directed graph E is generated by projections p_v associated to the vertices v of E and partial isometries s_e associated to the edges e of E. Graph C^*-algebras, which, in turn, are a generalization of the Cuntz-Krieger algebras of [2], were first studied using groupoid methods [9, 8]. An artifact of the initial groupoid approach is that the original theory was restricted to graphs which are *row-finite* and have *no sinks* in the sense that each vertex emits at least one and at most finitely many edges[1].

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[1]A refinement of the analysis in [9], due to Paterson [12], extends the groupoid approach to cover non-row-finite graphs. One can find a groupoid approach that

The connection between Cuntz-Krieger algebras and graph C^*-algebras is that each directed graph can be described in terms of its edge matrix, which is a $\{0,1\}$-matrix indexed by the edges of the graph; a 1 in the (e,f) entry indicates that the range of e is equal to the source of f. The terminology *row-finite* refers to the fact that in any row of the edge matrix of a row-finite graph, there are at most finitely many nonzero entries.

The two points of view, graph and matrix, led to two versions of Cuntz-Krieger theory for non-row-finite objects. In [5], C^*-algebras were associated to arbitrary graphs in such a way that the construction agrees with the original definition in the row-finite case. In [3] C^*-algebras — now called Exel-Laca algebras — were associated to arbitrary $\{0,1\}$ matrices, once again in such a way that the definitions coincide for row-finite matrices. The fundamental difference between the two classes of algebras is that a graph C^*-algebra is generated by a collection containing a partial isometry for each edge and a projection for each vertex, while an Exel-Laca algebra is generated by a collection containing a partial isometry for each row in the matrix (and in the non-row-finite case there are rows in the matrix corresponding to an infinite collection of edges with the same source vertex). Thus, although these two constructions agree in the row-finite case, there are C^*-algebras of non-row-finite graphs that are not isomorphic to any Exel-Laca algebra, and there are Exel-Laca algebras of non-row-finite matrices that are not isomorphic to the C^*-algebra of any graph [14].

In order to bring graph C^*-algebras of non-row-finite graphs and Exel-Laca algebras together under one theory, Tomforde introduced the notion of an ultragraph and described how to associate a C^*-algebra to such an object [13, 14]. His analysis not only brought the two classes of C^*-algebras under one rubric, but it also showed that there are ultragraph C^*-algebras that belong to neither of these classes. Ultragraphs are basically directed graphs in which the range of each edge is non-empty *set* of vertices rather than a single vertex — thus in an ultragraph each edge points from a single vertex to a set of vertices, and directed graphs are the special case where the range of each edge is a singleton set. Many of the fundamental results for graph C^*-algebras, such as the well-known Cuntz-Krieger Uniqueness Theorem and the Gauge-Invariant Uniqueness Theorem, can be proven in the setting

handles sinks and extends the whole theory to higher rank graphs in [4]. A groupoid approach to ultragraph C^*-algebras may be found in [10].

of ultragraphs [13]. However, other results, such as K-theory computations and ideal structure are less obviously amenable to traditional graph C^*-algebra techniques.

Recently, Katsura [6] and Muhly and Tomforde [11] studied the notions of topological graphs and topological quivers, respectively. These structures consist of second countable locally compact Hausdorff spaces E^0 and E^1 of vertices and edges respectively with range and source maps $r, s : E^1 \to E^0$ which satisfy appropriate topological hypotheses. The main point of difference between the two (apart from a difference in edge-direction conventions) is that in a topological graph the source map is assumed to be a local homeomorphism so that $s^{-1}(v)$ is discrete, whereas in a topological quiver the range map (remember the edge-reversal!) is only assumed to be continuous and open and a system $\lambda = \{\lambda_v\}_{v \in E^0}$ of Radon measures λ_v on $r^{-1}(v)$ satisfying some natural conditions (see [11, Definition 3.1]) is supplied as part of the data. It is worth pointing out that given E^0, E^1, r and s, with r open, such a system of Radon measures will always exist. A topological graph can be regarded as a topological quiver by reversing the edges and taking each λ_v to be counting measure; the topological graph C^*-algebra and the topological quiver C^*-algebra then coincide. One can regard an ordinary directed graph as either a topological graph or a topological quiver by endowing the edge and vertex sets with the discrete topology, and then the topological graph C^*-algebra and topological quiver algebra coincide with the original graph C^*-algebra.

In this article we show how to build a topological quiver $\mathcal{Q}(\mathcal{G})$ from an ultragraph $\mathcal{G}$ in such a way that the ultragraph C^*-algebra $C^*(\mathcal{G})$ and the topological quiver algebra $C^*(\mathcal{Q}(\mathcal{G}))$ coincide. We then use results of [6] and [11] to compute the K-theory of $C^*(\mathcal{G})$, to produce a listing of its gauge-invariant ideals, and to provide a version of Condition (K) under which all ideals of $C^*(\mathcal{G})$ are gauge-invariant.

It should be stressed that the range map in $\mathcal{Q}(\mathcal{G})$ is always a local homeomorphism, so $\mathcal{Q}(\mathcal{G})$ can equally be regarded as a topological graph; indeed our analysis in some instances requires results regarding topological graphs from [6] that have not yet been generalized to topological quivers. We use the notation and conventions of topological quivers because the edge-direction convention for quivers in [11] is compatible with the edge-direction convention for ultragraphs [13, 14].

The paper is organized as follows: In Section 2 we describe the commutative C^*-algebra $\mathfrak{A}_{\mathcal{G}} \subset C^*(\mathcal{G})$ generated by the projections $\{p_A : A \in \mathcal{G}^0\}$. In Section 3 we provide two alternative formulations of the defining relations among the generators of an ultragraph C^*-algebra which will prove more natural in our later analysis. In Section 4 we describe the spectrum of $\mathfrak{A}_\mathcal{G}$. We use this description in Section 5 to define the quiver $\mathcal{Q}(\mathcal{G})$, show that its C^*-algebra is isomorphic to $C^*(\mathcal{G})$, and compute its K-theory in terms of the structure of $\mathcal{G}$ using results from [6]. In Section 6 we use the results of [11] to produce a listing of the gauge-invariant ideals of $C^*(\mathcal{G})$ in terms of the structure of $\mathcal{G}$, and in Section 7 we use a theorem of [7] to provide a condition on $\mathcal{G}$ under which all ideals of $C^*(\mathcal{G})$ are gauge-invariant.

2. The commutative C^*-algebra $\mathfrak{A}_\mathcal{G}$ and its representations

For a set X, let $\mathcal{P}(X)$ denote the collection of all subsets of X.

Definition 2.1. An *ultragraph* $\mathcal{G} = (G^0, \mathcal{G}^1, r, s)$ consists of a countable set of vertices G^0, a countable set of edges $\mathcal{G}^1$, and functions $s\colon \mathcal{G}^1 \to G^0$ and $r\colon \mathcal{G}^1 \to \mathcal{P}(G^0) \setminus \{\emptyset\}$.

Definition 2.2. For an ultragraph $\mathcal{G} = (G^0, \mathcal{G}^1, r, s)$, we denote by $\mathfrak{A}_\mathcal{G}$ the C^*-subalgebra of $\ell^\infty(G^0)$ generated by the point masses $\{\delta_v : v \in G^0\}$ and the characteristic functions $\{\chi_{r(e)} : e \in \mathcal{G}^1\}$.

Let us fix an ultragraph $\mathcal{G} = (G^0, \mathcal{G}^1, r, s)$, and consider the representations of $\mathfrak{A}_\mathcal{G}$.

Definition 2.3. For a set X, a subcollection $\mathcal{C}$ of $\mathcal{P}(X)$ is called a *lattice* if

(i) $\emptyset \in \mathcal{C}$
(ii) $A \cap B \in \mathcal{C}$ and $A \cup B \in \mathcal{C}$ for all $A, B \in \mathcal{C}$.

An *algebra* is a lattice $\mathcal{C}$ that also satisfies the additional condition

(iii) $A \setminus B \in \mathcal{C}$ for all $A, B \in \mathcal{C}$.

Definition 2.4. For an ultragraph $\mathcal{G} = (G^0, \mathcal{G}^1, r, s)$, we let $\mathcal{G}^0$ denote the smallest algebra in $\mathcal{P}(G^0)$ containing the singleton sets and the sets $\{r(e) : e \in \mathcal{G}^1\}$.

Remark 2.5. In [13], $\mathcal{G}^0$ was defined to be the smallest lattice — not algebra — containing the singleton sets and the sets $\{r(e) : e \in \mathcal{G}^1\}$. The change to the above definition causes no problem when defining Cuntz-Krieger $\mathcal{G}$-families (see the final paragraph of Section 3). Furthermore, this new definition is convenient for us in a variety of situations: It relates $\mathcal{G}^0$ to the C^*-algebra $\mathfrak{A}_\mathcal{G}$ described in Proposition 2.6, it allows us too see immediately that the set $r(\lambda, \mu)$ of Definition 2.8 is in $\mathcal{G}^0$, and — most importantly — it aids in our description of the gauge-invariant ideals in Definition 6.1 and Lemma 6.2. For additional justification for the change in definition, we refer the reader to [10, Section 2].

Proposition 2.6. *We have* $\mathcal{G}^0 = \{A \subset G^0 : \chi_A \in \mathfrak{A}_\mathcal{G}\}$ *and*

$$\mathfrak{A}_\mathcal{G} = \overline{\operatorname{span}}\{\chi_A : A \in \mathcal{G}^0\}. \tag{2.1}$$

Proof. We begin by proving (2.1). Since $\{A \subset G^0 : \chi_A \in \mathfrak{A}_\mathcal{G}\}$ is an algebra containing $\{v\}$ and $r(e)$, we have $\mathcal{G}^0 \subset \{A \subset G^0 : \chi_A \in \mathfrak{A}_\mathcal{G}\}$. Hence we get

$$\mathfrak{A}_\mathcal{G} \supset \overline{\operatorname{span}}\{\chi_A : A \in \mathcal{G}^0\}.$$

Since $\mathcal{G}^0$ is closed under intersections, the set $\overline{\operatorname{span}}\{\chi_A : A \in \mathcal{G}^0\}$ is closed under multiplication, and hence is a C^*-algebra containing $\{\delta_v\}$ and $\{\chi_{r(e)}\}$. Hence $\mathfrak{A}_\mathcal{G} \subset \overline{\operatorname{span}}\{\chi_A : A \in \mathcal{G}^0\}$, establishing (2.1).

We must now show that $\mathcal{G}^0 = \{A \subset G^0 : \chi_A \in \mathfrak{A}_\mathcal{G}\}$. We have already seen $\mathcal{G}^0 \subset \{A \subset G^0 : \chi_A \in \mathfrak{A}_\mathcal{G}\}$. Let $A \subset G^0$ with $\chi_A \in \mathfrak{A}_\mathcal{G}$. By (2.1), we have $\|\chi_A - \sum_{k=1}^m z_k \chi_{A_k}\| < 1/2$ for some $A_1, \dots, A_m \in \mathcal{G}^0$ and $z_1, \dots, z_m \in \mathbb{C}$; moreover, since $\mathcal{G}^0$ is an algebra, we may assume that $j \neq k$ implies $A_j \cap A_k = \emptyset$. But then $x \in A$ if and only if $x \in A_k$ for some (unique) k with $|1 - z_k| < 1/2$. That is, $A = \bigcup_{|1-z_k|<1/2} A_k \in \mathcal{G}^0$. □

Definition 2.7. A *representation* of a lattice $\mathcal{C}$ on a C^*-algebra $\mathfrak{B}$ is a collection of projections $\{p_A\}_{A\in\mathcal{C}}$ of $\mathfrak{B}$ satisfying $p_\emptyset = 0$, $p_A p_B = p_{A\cap B}$, and $p_{A\cup B} = p_A + p_B - p_{A\cap B}$ for all $A, B \in \mathcal{C}$.

When $\mathcal{C}$ is an algebra, the last condition of a representation can be replaced by the equivalent condition that $p_{A\cup B} = p_A + p_B$ for all $A, B \in \mathcal{C}$ with $A \cap B = \emptyset$.

Note that we define representations of lattices here, rather than just of algebras, so that our definition of $C^*(\mathcal{G})$ agrees with the original definition given in [13] (see the final paragraph of Section 3).

Definition 2.8. For a fixed ultragraph $\mathcal{G} = (G^0, \mathcal{G}^1, r, s)$ we define

$$X = \{(\lambda, \mu) : \lambda, \mu \text{ are finite subsets of } \mathcal{G}^1 \text{ with } \lambda \cap \mu = \emptyset \text{ and } \lambda \neq \emptyset\}.$$

For $(\lambda, \mu) \in X$, we define $r(\lambda, \mu) \subset G^0$ by

$$r(\lambda, \mu) := \bigcap_{e\in\lambda} r(e) \setminus \bigcup_{f\in\mu} r(f).$$

Definition 2.9. Let $\mathcal{G} = (G^0, \mathcal{G}^1, r, s)$ be an ultragraph. A collection of projections $\{p_v\}_{v\in G^0}$ and $\{q_e\}_{e\in\mathcal{G}^1}$ is said to satisfy *Condition (EL)* if the following hold:

(1) the elements of $\{p_v\}_{v\in G^0}$ are pairwise orthogonal,
(2) the elements of $\{q_e\}_{e\in\mathcal{G}^1}$ pairwise commute,
(3) $p_v q_e = p_v$ if $v \in r(e)$, and $p_v q_e = 0$ if $v \notin r(e)$,

(4) $\prod_{e\in\lambda} q_e \prod_{f\in\mu}(1 - q_f) = \sum_{v\in r(\lambda,\mu)} p_v$ for all $(\lambda, \mu) \in X$ with $|r(\lambda, \mu)| < \infty$.

From a representation of $\mathcal{G}^0$, we get a collection satisfying Condition (EL). We prove a slightly stronger statement.

Lemma 2.10. *Let $\mathcal{C}$ be a lattice in $\mathcal{P}(G^0)$ which contains the singleton sets and the sets $\{r(e) : e \in \mathcal{G}^1\}$, and let $\{p_A\}_{A\in\mathcal{C}}$ be a representation of $\mathcal{C}$. Then the collection $\{p_{\{v\}}\}_{v\in G^0}$ and $\{p_{r(e)}\}_{e\in\mathcal{G}^1}$ satisfies Condition (EL).*

Proof. From the condition $p_\emptyset = 0$ and $p_A p_B = p_{A\cap B}$, it is easy to show that the collection satisfies the conditions (1), (2) and (3) in Definition 2.9. To see condition (4) let $(\lambda, \mu) \in X$ with $|r(\lambda, \mu)| < \infty$. Define $A, B \subset G^0$ by $A = \bigcap_{e\in\lambda} r(e)$ and $B = \bigcup_{f\in\mu} r(f)$. Then we have $A, B \in \mathcal{C}$, and from the definition of a representation, we obtain

$$p_A = \prod_{e\in\lambda} p_{r(e)} \qquad \text{and} \qquad 1 - p_B = \prod_{f\in\mu}(1 - p_{r(f)}).$$

Since $r(\lambda, \mu)$ is a finite set, $r(\lambda, \mu) \in \mathcal{C}$ and $p_{r(\lambda,\mu)} = \sum_{v\in r(\lambda,\mu)} p_{\{v\}}$. Also, because $r(\lambda, \mu) = A \setminus B$, we obtain $r(\lambda, \mu) \cup B = A \cup B$ and $r(\lambda, \mu) \cap B = \emptyset$. Hence $p_{A\cup B} = p_{r(\lambda,\mu)} + p_B$. Since $p_{A\cup B} = p_A + p_B - p_{A\cap B}$, we get $p_{r(\lambda,\mu)} = p_A - p_{A\cap B}$. Hence we have,

$$\sum_{v\in r(\lambda,\mu)} p_{\{v\}} = p_A - p_{A\cap B} = p_A(1 - p_B) = \prod_{e\in\lambda} p_{r(e)} \prod_{f\in\mu}(1 - p_{r(f)}).$$

□

We will prove that from a collection satisfying Condition (EL), we can construct a $*$-homomorphism from $\mathfrak{A}_\mathcal{G}$ onto the C^*-subalgebra generated by that collection. To this end, we fix a listing $\mathcal{G}^1 = \{e_i\}_{i=1}^\infty$, and for each positive integer n define a C^*-subalgebra $\mathfrak{A}_\mathcal{G}^{(n)}$ of $\mathfrak{A}_\mathcal{G}$ to be the C^*-algebra generated by $\{\delta_v : v \in G^0\}$ and $\{\chi_{r(e_i)} : i = 1, 2, \ldots, n\}$. Note that the union of the increasing family $\{\mathfrak{A}_\mathcal{G}^{(n)}\}_{n=1}^\infty$ is dense in $\mathfrak{A}_\mathcal{G}$.

Definition 2.11. Let n be a positive integer. Let $0^n := (0, 0, \ldots, 0) \in \{0, 1\}^n$. For $\omega = (\omega_1, \omega_2, \ldots, \omega_n) \in \{0, 1\}^n \setminus \{0^n\}$, we set

$$r(\omega) := \bigcap_{\omega_i=1} r(e_i) \setminus \bigcup_{\omega_j=0} r(e_j).$$

Lemma 2.12. *Let n be a positive integer. For each $\omega \in \{0, 1\}^n$, we define $\lambda_\omega, \mu_\omega \subset \mathcal{G}^1$ by $\lambda_\omega = \{e_i : \omega_i = 1\}$ and $\mu_\omega = \{e_i : \omega_i = 0\}$. Then the map*

$$\omega \mapsto (\lambda_\omega, \mu_\omega)$$

is a bijection between $\{0,1\}^n \setminus \{0^n\}$ *and* $\big\{(\lambda,\mu) \in X : \lambda \cup \mu = \{e_1,\dots,e_n\}\big\}$*, and we have* $r(\omega) = r(\lambda_\omega, \mu_\omega)$*.*

Proof. The map $\omega \mapsto (\lambda_\omega, \mu_\omega)$ is a bijection because $(\lambda,\mu) \mapsto \chi_\lambda$ provides an inverse, and $r(\omega) = r(\lambda_\omega, \mu_\omega)$ by definition. □

Definition 2.13. We define $\Delta_n := \big\{\omega \in \{0,1\}^n \setminus \{0^n\} : |r(\omega)| = \infty\big\}$.

Lemma 2.14. *For each* $i = 1, 2, \dots, n$*, the set* $r(e_i)$ *is a disjoint union of the infinite sets* $\{r(\omega)\}_{\omega \in \Delta_n, \omega_i = 1}$ *and the finite set* $\bigcup_{\omega \notin \Delta_n, \omega_i = 1} r(\omega)$*.*

Proof. First note that $r(\omega) \cap r(\omega') = \emptyset$ for distinct $\omega, \omega' \in \{0,1\}^n \setminus \{0^n\}$. For $v \in r(e_i)$, define $\omega^v \in \{0,1\}^n$ by $\omega^v_j = \chi_{r(e_j)}(v)$ for $1 \le j \le n$. Since $v \in r(e_i)$, $\omega^v \neq 0^n$, and $v \in r(\omega^v)$ by definition. Hence

$$r(e_i) = \bigcup_{v \in r(e_i)} r(\omega^v) = \bigcup_{\omega_i = 1} r(\omega)$$

Since $r(\omega)$ is a finite set for $\omega \in \{0,1\}^n \setminus \{0^n\}$ with $\omega \notin \Delta_n$, the result follows. □

Proposition 2.15. *The* C^**-algebra* $\mathfrak{A}_{\mathcal{G}}^{(n)}$ *is generated by* $\{\delta_v : v \in G^0\} \cup \{\chi_{r(\omega)} : \omega \in \Delta_n\}$*.*

Proof. For each $\omega \in \Delta_n$, we have

$$\chi_{r(\omega)} = \prod_{\omega_i = 1} \chi_{r(e_i)} \prod_{\omega_j = 0} \big(1 - \chi_{r(e_j)}\big) \in \mathfrak{A}_{\mathcal{G}}^{(n)},$$

giving inclusion in one direction. It follows from Lemma 2.14 that the generators of $\mathfrak{A}_{\mathcal{G}}^{(n)}$ all belong to the C^*-algebra generated by $\{\delta_v : v \in G^0\} \cup \{\chi_{r(\omega)} : \omega \in \Delta_n\}$, establishing the reverse inclusion. □

For each $\omega \in \Delta_n$, the C^*-subalgebra of $\mathfrak{A}_{\mathcal{G}}^{(n)}$ generated by $\{\delta_v : v \in r(\omega)\}$ and $\chi_{r(\omega)}$ is isomorphic to the unitization of $c_0(r(\omega))$. Since the C^*-algebra $\mathfrak{A}_{\mathcal{G}}^{(n)}$ is a direct sum of such C^*-subalgebras indexed by the set Δ_n and the C^*-subalgebra $c_0\big(G^0 \setminus \bigcup_{\omega \in \Delta_n} r(\omega)\big)$ (recall that the $r(\omega)$'s are pairwise disjoint), we have the following:

Lemma 2.16. *For two families* $\{p_v\}_{v \in G^0}$ *and* $\{q_\omega\}_{\omega \in \Delta_n}$ *of mutually orthogonal projections in a* C^**-algebra* $\mathfrak{B}$ *satisfying*

$$p_v q_\omega = \begin{cases} p_v & \text{if } v \in r(\omega), \\ 0 & \text{if } v \notin r(\omega), \end{cases}$$

there exists a $*$*-homomorphism* $\pi_n \colon \mathfrak{A}_{\mathcal{G}}^{(n)} \to \mathfrak{B}$ *with* $\pi_n(\delta_v) = p_v$ *for* $v \in G^0$ *and* $\pi_n(\chi_{r(\omega)}) = q_\omega$ *for* $\omega \in \Delta_n$*.*

Proposition 2.17. *Let $\mathcal{G} = (G^0, \mathcal{G}^1, r, s)$ be an ultragraph, and $\mathfrak{B}$ be a C^*-algebra. Then there exist natural bijections among the following sets:*

(i) *the set of $*$-homomorphisms from $\mathfrak{A}_{\mathcal{G}}$ to $\mathfrak{B}$,*
(ii) *the set of representations of $\mathcal{G}^0$ on $\mathfrak{B}$,*
(iii) *the set of collections of projections in $\mathfrak{B}$ satisfying Condition (EL).*

Specifically, if $\pi\colon \mathfrak{A}_{\mathcal{G}} \to \mathfrak{B}$ is a $*$-homomorphism, then $p_A := \pi(\chi_A)$ for $A \in \mathcal{G}^0$ gives a representation of $\mathcal{G}^0$; if $\{p_A\}_{A\in\mathcal{G}^0}$ is a representation of $\mathcal{G}^0$ on $\mathfrak{B}$, then $\{p_{\{v\}}\}_{v\in G^0} \cup \{p_{r(e)}\}_{e\in\mathcal{G}^1}$ satisfies Condition (EL); and if a collection of projections $\{p_v\}_{v\in G^0} \cup \{q_e\}_{e\in\mathcal{G}^1}$ satisfies Condition (EL), then there exists a unique $*$-homomorphism $\pi\colon \mathfrak{A}_{\mathcal{G}} \to \mathfrak{B}$ such that $\pi(\delta_v) = p_v$ and $\pi(\chi_{r(e)}) = q_e$ for all $v \in G^0$ and $e \in \mathcal{G}^1$.

Proof. Clearly we have the map from (i) to (ii), and by Lemma 2.10 we have the map from (ii) to (iii). Suppose that $\{p_v\}_{v\in G^0}$ and $\{q_e\}_{e\in\mathcal{G}^1}$ is a collection of projections satisfying Condition (EL). Fix a positive integer n. For each $\omega \in \{0,1\}^n \setminus \{0^n\}$, we define $q_\omega = \prod_{\omega_i=1} q_{e_i} \prod_{\omega_j=0}(1 - q_{e_j}) \in \mathfrak{B}$. Then $\{q_\omega : \omega \in \{0,1\}^n \setminus \{0^n\}\}$ is mutually orthogonal. By Definition 2.9(3), we have

$$p_v q_\omega = \begin{cases} p_v & \text{if } v \in r(\omega), \\ 0 & \text{if } v \notin r(\omega). \end{cases}$$

Hence by Lemma 2.16, there exists a $*$-homomorphism $\pi_n\colon \mathfrak{A}_{\mathcal{G}}^{(n)} \to \mathfrak{B}$ such that $\pi_n(\delta_v) = p_v$ for $v \in G^0$ and $\pi_n(\chi_{r(\omega)}) = q_\omega$ for $\omega \in \Delta_n$. For $\omega \in \{0,1\}^n \setminus \{0^n\}$ with $|r(\omega)| < \infty$, we have $\pi_n(\chi_{r(\omega)}) = \sum_{v\in r(\omega)} p_v = q_\omega$ by Definition 2.9(4). Hence we obtain

$$\pi_n(\chi_{r(e_i)}) = \pi_n\Bigg(\sum_{\substack{\omega\in\{0,1\}^n \\ \omega_i=1}} \chi_{r(\omega)} \Bigg) = \sum_{\substack{\omega\in\{0,1\}^n \\ \omega_i=1}} q_\omega = q_{e_i},$$

for $i = 1, \ldots, n$. Thus for each n, the $*$-homomorphism $\pi_n\colon \mathfrak{A}_{\mathcal{G}}^{(n)} \to \mathfrak{B}$ satisfies $\pi_n(\delta_v) = p_v$ for $v \in G^0$ and $\pi_n(\chi_{r(e_i)}) = q_{e_i}$ for $i = 1, \ldots, n$. Since there is at most one $*$-homomorphism of $\mathfrak{A}_{\mathcal{G}}^{(n)} \to \mathfrak{B}$ with this property, the restriction of the $*$-homomorphism $\pi_{n+1} : \mathfrak{A}_{\mathcal{G}}^{(n+1)} \to \mathfrak{B}$ to $\mathfrak{A}_{\mathcal{G}}^{(n)}$ coincides with π_n. Hence there is a $*$-homomorphism $\pi\colon \mathfrak{A}_{\mathcal{G}} \to \mathfrak{B}$ such that $\pi(\delta_v) = p_v$ for $v \in G^0$ and $\pi(\chi_{r(e)}) = q_e$ for $e \in \mathcal{G}^1$. The $*$-homomorphism π is unique because $\mathfrak{A}_{\mathcal{G}}$ is generated by $\{\delta_v : v \in G^0\} \cup \{\chi_{r(e)} : e \in \mathcal{G}^1\}$. □

Corollary 2.18. *Let* $\mathcal{G}^1 = \{e_i\}_{i=1}^{\infty}$ *be some listing of the edges of* $\mathcal{G}$*. To check that a family of projections* $\{p_v\}_{v\in G^0} \cup \{q_e\}_{e\in\mathcal{G}^1}$ *satisfies Condition (EL), it suffices to verify that Definition 2.9(1)–(3) hold and that (4) holds for* $(\lambda,\mu) \in X$ *with* $|r(\lambda,\mu)| < \infty$ *and* $\lambda \cup \mu = \{e_1,\ldots,e_n\}$ *for some* n*.*

We conclude this section by computing the K-groups of the C^*-algebra $\mathfrak{A}_{\mathcal{G}}$.

Definition 2.19. For an ultragraph $\mathcal{G} = (G^0, \mathcal{G}^1, r, s)$, we denote by $Z_{\mathcal{G}}$ the (algebraic) subalgebra of $\ell^\infty(G^0,\mathbb{Z})$ generated by $\{\delta_v : v \in G^0\} \cup \{\chi_{r(e)} : e \in \mathcal{G}^1\}$.

An argument similar to the proof of Proposition 2.6 shows that

$$Z_{\mathcal{G}} = \bigg\{ \sum_{k=1}^{n} z_k \chi_{A_k} : n \in \mathbb{N}, z_k \in \mathbb{Z}, A_k \in \mathcal{G}^0 \bigg\}.$$

Proposition 2.20. *We have* $K_0(\mathfrak{A}_{\mathcal{G}}) \cong Z_{\mathcal{G}}$ *and* $K_1(\mathfrak{A}_{\mathcal{G}}) = 0$*.*

Proof. For each $n \in \mathbb{N} \setminus \{0\}$, let $Z_{\mathcal{G}}^{(n)}$ be the subalgebra of $\ell^\infty(G^0,\mathbb{Z})$ generated by $\{\delta_v : v \in G^0\} \cup \{\chi_{r(e_i)} : i = 1,2,\ldots,n\}$. By an argument similar to the paragraph following Proposition 2.15, we see that $Z_{\mathcal{G}}^{(n)}$ is a direct sum of the unitizations (as algebras) of $c_0(r(\omega),\mathbb{Z})$'s and $c_0\big(G^0 \setminus \bigcup_{\omega\in\Delta_n} r(\omega),\mathbb{Z}\big)$. Hence the description of $\mathfrak{A}_{\mathcal{G}}^{(n)}$ in the paragraph following Proposition 2.15 shows that there exists an isomorphism $K_0(\mathfrak{A}_{\mathcal{G}}^{(n)}) \to Z_{\mathcal{G}}^{(n)}$ which sends $[\delta_v], [\chi_{r(\omega)}] \in K_0(\mathfrak{A}_{\mathcal{G}}^{(n)})$ to $\delta_v, \chi_{r(\omega)} \in Z_{\mathcal{G}}^{(n)}$. By taking inductive limits, we get an isomorphism $K_0(\mathfrak{A}_{\mathcal{G}}) \to Z_{\mathcal{G}}$ which sends $[\chi_A] \in K_0(\mathfrak{A}_{\mathcal{G}})$ to $\chi_A \in Z_{\mathcal{G}}$ for $A \in \mathcal{G}^0$. That $K_1(\mathfrak{A}_{\mathcal{G}}) = 0$ follows from the fact that $K_1(\mathfrak{A}_{\mathcal{G}}^{(n)}) = 0$ for each n, and by taking direct limits. □

Remark 2.21. It is not difficult to see that the isomorphism in Proposition 2.20 preserves the natural order and scaling.

3. C^*-algebras of ultragraphs

Definition 3.1. Let $\mathcal{G} = (G^0, \mathcal{G}^1, r, s)$ be an ultragraph. A vertex $v \in G^0$ is said to be *regular* if $0 < |s^{-1}(v)| < \infty$. The set of all regular vertices is denoted by $G^0_{\mathrm{rg}} \subset G^0$.

Definition 3.2. For an ultragraph $\mathcal{G} = (G^0, \mathcal{G}^1, r, s)$, a *Cuntz-Krieger* $\mathcal{G}$*-family* is a representation $\{p_A\}_{A\in\mathcal{G}^0}$ of $\mathcal{G}^0$ in a C^*-algebra $\mathfrak{B}$ and a collection of partial isometries $\{s_e\}_{e\in\mathcal{G}^1}$ in $\mathfrak{B}$ with mutually orthogonal ranges that satisfy

(1) $s_e^* s_e = p_{r(e)}$ for all $e \in \mathcal{G}^1$,
(2) $s_e s_e^* \leq p_{s(e)}$ for all $e \in \mathcal{G}^1$,
(3) $p_v = \sum_{s(e)=v} s_e s_e^*$ for all $v \in G^0_{\mathrm{rg}}$,

where we write p_v in place of $p_{\{v\}}$ for $v \in G^0$.

The C^*-algebra $C^*(\mathcal{G})$ is the C^*-algebra generated by a universal Cuntz-Krieger $\mathcal{G}$-family $\{p_A, s_e\}$.

We will show that this definition of $C^*(\mathcal{G})$ and the following natural generalization of the definition of Exel-Laca algebras in [3] are both equivalent to the original definition of $C^*(\mathcal{G})$ in [13, Definition 2.7].

Definition 3.3. For an ultragraph $\mathcal{G} = (G^0, \mathcal{G}^1, r, s)$, an *Exel-Laca $\mathcal{G}$-family* is a collection of projections $\{p_v\}_{v \in G^0}$ and partial isometries $\{s_e\}_{e \in \mathcal{G}^1}$ with mutually orthogonal ranges for which

(1) the collection $\{p_v\}_{v \in G^0} \cup \{s_e^* s_e\}_{e \in \mathcal{G}^1}$ satisfies Condition (EL),
(2) $s_e s_e^* \leq p_{s(e)}$ for all $e \in \mathcal{G}^1$,
(3) $p_v = \sum_{s(e)=v} s_e s_e^*$ for $v \in G^0_{\mathrm{rg}}$.

Proposition 3.4. *For each Cuntz-Krieger $\mathcal{G}$-family $\{p_A, s_e\}$, the collection $\{p_v, s_e\}$ is an Exel-Laca $\mathcal{G}$-family. Conversely, for each Exel-Laca $\mathcal{G}$-family $\{p_v, s_e\}$, there exists a unique representation $\{p_A\}$ of $\mathcal{G}^0$ on the C^*-algebra generated by $\{p_v, s_e\}$ such that $p_{\{v\}} = p_v$ for $v \in G^0$ and $\{p_A, s_e\}$ is a Cuntz-Krieger $\mathcal{G}$-family.*

Proof. This follows from Proposition 2.17. □

Corollary 3.5. *Let $\{p_v, s_e\}$ be the Exel-Laca $\mathcal{G}$-family in $C^*(\mathcal{G})$. For an Exel-Laca $\mathcal{G}$-family $\{P_v, S_e\}$ on a C^*-algebra $\mathfrak{B}$, there exists a $*$-homomorphism $\phi\colon C^*(\mathcal{G}) \to \mathfrak{B}$ such that $\phi(p_v) = P_v$ and $\phi(s_e) = S_e$. The $*$-homomorphism ϕ is injective if $P_v \neq 0$ for all $v \in G^0$ and there exists a strongly continuous action β of $\mathbb{T}$ on $\mathfrak{B}$ such that $\beta_z(P_v) = P_v$ and $\beta_z(S_e) = zS_e$ for $v \in G^0$, $e \in \mathcal{G}^1$, and $z \in \mathbb{T}$.*

Proof. The first part follows from Proposition 3.4, and the latter follows from [13, Theorem 6.8] because $\phi(p_A) \neq 0$ for all non-empty A if $\phi(p_v) = P_v \neq 0$ for all $v \in G^0$. □

It is easy to see that Proposition 3.4 is still true if we replace $\mathcal{G}^0$ by any lattice contained in $\mathcal{G}^0$ and containing $\{v\}$ and $r(e)$ for all $v \in G^0$ and $e \in \mathcal{G}^1$ (see Lemma 2.10). Hence the restriction gives a natural bijection from Cuntz-Krieger $\mathcal{G}$-families in the sense of Definition 3.2 to the Cuntz-Krieger $\mathcal{G}$-families of [13, Definition 2.7]. Thus the C^*-algebra $C^*(\mathcal{G})$ is naturally isomorphic to the C^*-algebra defined in [13, Theorem 2.11].

4. The spectrum of the commutative C^*-algebra $\mathfrak{A}_{\mathcal{G}}$

Let $\mathcal{G} = (G^0, \mathcal{G}^1, r, s)$ be an ultragraph. In this section, we describe the spectrum of the commutative C^*-algebra $\mathfrak{A}_{\mathcal{G}}$ concretely. Fix a listing $\mathcal{G}^1 = \{e_i\}_{i=1}^{\infty}$ of the edges of $\mathcal{G}$. As described in the paragraph following the proof of Lemma 2.10, the C^*-algebra $\mathfrak{A}_{\mathcal{G}}$ is equal to the inductive limit of the increasing family $\{\mathfrak{A}_{\mathcal{G}}^{(n)}\}_{n=1}^{\infty}$, where $\mathfrak{A}_{\mathcal{G}}^{(n)}$ is the C^*-subalgebra of $\mathfrak{A}_{\mathcal{G}}$ generated by $\{\delta_v : v \in G^0\} \cup \{\chi_{r(e_i)} : i = 1, 2, \ldots, n\}$. In order to compute the spectrum of $\mathfrak{A}_{\mathcal{G}}$, we first compute the spectrum of $\mathfrak{A}_{\mathcal{G}}^{(n)}$ for a positive integer n.

Definition 4.1. For $n \in \mathbb{N} \setminus \{0\}$, we define a topological space $\Omega_{\mathcal{G}}^{(n)}$ such that $\Omega_{\mathcal{G}}^{(n)} = G^0 \sqcup \Delta_n$ as a set and $A \sqcup Y$ is open in $\Omega_{\mathcal{G}}^{(n)}$ for $A \subset G^0$ and $Y \subset \Delta_n$ if and only if $|r(\omega) \setminus A| < \infty$ for all $\omega \in Y$.

For each $v \in G^0$, $\{v\}$ is open in $\Omega_{\mathcal{G}}^{(n)}$, and a fundamental system of neighborhoods of $\omega \in \Delta_n \subset \Omega_{\mathcal{G}}^{(n)}$ is

$$\{A \sqcup \{\omega\} : A \subset G^0, |r(\omega) \setminus A| < \infty\}.$$

Hence G^0 is a discrete dense subset of $\Omega_{\mathcal{G}}^{(n)}$. Note that $\Omega_{\mathcal{G}}^{(n)}$ is a disjoint union of the finitely many compact open subsets $r(\omega) \sqcup \{\omega\}$ for $\omega \in \Delta_n$ and the discrete set $G^0 \setminus \bigcup_{\omega \in \Delta_n} r(\omega)$. This fact and the paragraph following Proposition 2.15 show the following:

Lemma 4.2. *There exists an isomorphism $\pi^{(n)} : \mathfrak{A}_{\mathcal{G}}^{(n)} \to C_0(\Omega_{\mathcal{G}}^{(n)})$ such that $\pi^{(n)}(\delta_v) = \delta_v$ and $\pi^{(n)}(\chi_{r(\omega)}) = \chi_{r(\omega) \sqcup \{\omega\}}$ for $v \in G^0$ and $\omega \in \Delta_n$.*

Lemma 4.3. *For $i = 1, 2, \ldots, n$, the closure $\overline{r(e_i)}$ of $r(e_i) \subset \Omega_{\mathcal{G}}^{(n)}$ is the compact open set $r(e_i) \sqcup \{\omega \in \Delta_n : \omega_i = 1\}$, and we have $\pi^{(n)}(\chi_{r(e_i)}) = \chi_{\overline{r(e_i)}}$.*

Proof. This follows from Lemma 2.14. □

Let $\widetilde{\Delta}_n := \Delta_n \cup \{0^n\}$. We can define a topology on $\widetilde{\Omega}_{\mathcal{G}}^{(n)} := G^0 \sqcup \widetilde{\Delta}_n$ similarly as in Definition 4.1 so that $\widetilde{\Omega}_{\mathcal{G}}^{(n)}$ is the one-point compactification of $\Omega_{\mathcal{G}}^{(n)}$. The restriction map $\{0,1\}^{n+1} \to \{0,1\}^n$ induces a map $\widetilde{\Delta}_{n+1} \to \widetilde{\Delta}_n$, and hence a map $\widetilde{\Omega}_{\mathcal{G}}^{(n+1)} \to \widetilde{\Omega}_{\mathcal{G}}^{(n)}$. It is routine to check that this map is a continuous surjection, and the induced $*$-homomorphism $C_0(\Omega_{\mathcal{G}}^{(n)}) \to C_0(\Omega_{\mathcal{G}}^{(n+1)})$ coincides with the inclusion $\mathfrak{A}_{\mathcal{G}}^{(n)} \hookrightarrow \mathfrak{A}_{\mathcal{G}}^{(n+1)}$ via the isomorphisms in Lemma 4.2.

For each element

$$\omega = (\omega_1, \omega_2, \ldots, \omega_i, \ldots) \in \{0,1\}^{\infty},$$

we define $\omega|_n \in \{0,1\}^n$ by $\omega|_n = (\omega_1, \omega_2, \ldots, \omega_n)$. The space $\{0,1\}^\infty$ is a compact space with the product topology, and it is homeomorphic to $\varprojlim \{0,1\}^n$.

Definition 4.4. We define

$$\widetilde{\Delta}_\infty := \bigl\{\omega \in \{0,1\}^\infty : \omega|_n \in \widetilde{\Delta}_n \text{ for all } n\bigr\},$$

and $\Delta_\infty := \widetilde{\Delta}_\infty \setminus \{0^\infty\}$ where $0^\infty := (0,0,\ldots) \in \{0,1\}^\infty$.

Since $\widetilde{\Delta}_\infty$ is a closed subset of $\{0,1\}^\infty$, the space Δ_∞ is locally compact, and its one-point compactification is homeomorphic to $\widetilde{\Delta}_\infty$. By definition, $\widetilde{\Delta}_\infty \cong \varprojlim \widetilde{\Delta}_n$.

Definition 4.5. We define a topological space $\Omega_{\mathcal{G}}$ as follows: $\Omega_{\mathcal{G}} = G^0 \sqcup \Delta_\infty$ as a set and $A \sqcup Y$ is open in $\Omega_{\mathcal{G}}$ for $A \subset G^0$ and $Y \subset \Delta_\infty$ if and only if for each $\omega \in Y$ there exists an integer n satisfying

(i) if $\omega' \in \Delta_\infty$ and $\omega'|_n = \omega|_n$, then $\omega' \in Y$; and
(ii) $|r(\omega|_n) \setminus A| < \infty$.

Equivalently, $A \sqcup Y \subset \Omega_{\mathcal{G}}$ is closed if and only if $Y \subset \Delta_\infty$ is closed in the product topology on $\{0,1\}^\infty$, and for each $\omega \in \Delta_\infty$ such that $|r(\omega|_n) \cap A| = \infty$ for all n, we have $\omega \in Y$.

We can define a topology on $\widetilde{\Omega}_{\mathcal{G}} := G^0 \sqcup \widetilde{\Delta}_\infty$ similarly as in the definition above, so that $\widetilde{\Omega}_{\mathcal{G}}$ is the one-point compactification of $\Omega_{\mathcal{G}}$ and $\widetilde{\Omega}_{\mathcal{G}} \cong \varprojlim \widetilde{\Omega}_{\mathcal{G}}^{(n)}$.

Lemma 4.6. *In the space $\Omega_{\mathcal{G}}$, the closure $\overline{r(e_i)}$ of $r(e_i) \subset \Omega_{\mathcal{G}}$ is the compact open set $r(e_i) \sqcup \{\omega \in \Delta_\infty : \omega_i = 1\}$.*

Proof. This follows from the homeomorphism $\widetilde{\Omega}_{\mathcal{G}} \cong \varprojlim \widetilde{\Omega}_{\mathcal{G}}^{(n)}$ combined with Lemma 4.3. □

Proposition 4.7. *There exists an isomorphism $\pi\colon \mathfrak{A}_{\mathcal{G}} \to C_0(\Omega_{\mathcal{G}})$ such that $\pi(\delta_v) = \delta_v$ and $\pi(\chi_{r(e)}) = \chi_{\overline{r(e)}}$ for $v \in G^0$ and $e \in \mathcal{G}^1$.*

Proof. Taking the inductive limit of the isomorphisms $\pi^{(n)}$ in Lemma 4.2 produces an isomorphism

$$\pi\colon \mathfrak{A}_{\mathcal{G}} \to \varinjlim C_0(\Omega_{\mathcal{G}}^{(n)}) \cong C_0(\Omega_{\mathcal{G}}).$$

This isomorphism satisfies the desired condition by Lemma 4.6. □

By the isomorphism π in the proposition above, we can identify the spectrum of $\mathfrak{A}_{\mathcal{G}}$ with the space $\Omega_{\mathcal{G}}$.

5. Topological quivers and K-groups

In this section we will construct a topological quiver $\mathcal{Q}(\mathcal{G})$ from $\mathcal{G}$, and show that the C^*-algebra $C^*(\mathcal{G})$ is isomorphic to the C^*-algebra $C^*(\mathcal{Q}(\mathcal{G}))$ of [11]. Fix an ultragraph $\mathcal{G} = (G^0, \mathcal{G}^1, r, s)$, and define

$$\mathcal{Q}(\mathcal{G}) := (E(\mathcal{G})^0, E(\mathcal{G})^1, r_{\mathcal{Q}}, s_{\mathcal{Q}}, \lambda_{\mathcal{Q}})$$

as follows. Let $E(\mathcal{G})^0 := \Omega_{\mathcal{G}}$ and

$$E(\mathcal{G})^1 := \{(e, x) \in \mathcal{G}^1 \times \Omega_{\mathcal{G}} : x \in \overline{r(e)}\},$$

where $\mathcal{G}^1$ is considered as a discrete set, and $\overline{r(e)} \subset \Omega_{\mathcal{G}}$ are compact open sets (see Lemma 4.6).

We define a local homeomorphism $r_{\mathcal{Q}} \colon E(\mathcal{G})^1 \to E(\mathcal{G})^0$ by $r_{\mathcal{Q}}(e, x) := x$, and a continuous map $s_{\mathcal{Q}} \colon E(\mathcal{G})^1 \to E(\mathcal{G})^0$ by $s_{\mathcal{Q}}(e, x) := s(e) \in G^0 \subset E(\mathcal{G})^0$. Since $r_{\mathcal{Q}}$ is a local homeomorphism, we have that $r_{\mathcal{Q}}^{-1}(x)$ is discrete and countable for each $x \in E(\mathcal{G})^0$. For each $x \in E(\mathcal{G})^0$ we define the measure λ_x on $r_{\mathcal{Q}}^{-1}(x)$ to be counting measure, and set $\lambda_{\mathcal{Q}} = \{\lambda_x : x \in E(\mathcal{G})^0\}$.

Reversing the roles of the range and source maps, we can also regard $\mathcal{Q}(\mathcal{G})$ as a topological graph $E(\mathcal{G})$ in the sense of [6], and its C^*-algebra $\mathcal{O}(E(\mathcal{G}))$ is naturally isomorphic to $C^*(\mathcal{Q}(\mathcal{G}))$ (see [11, Example 3.19]). Since some of the results about $C^*(\mathcal{Q}(\mathcal{G}))$ which we wish to apply have only been proved in the setting of [6] to date, we will frequently reference these results; the reversal of edge-direction involved in regarding $\mathcal{Q}(\mathcal{G})$ as a topological graph is implicit in these statements. We have opted to use the notation and conventions of [11] throughout, and to reference the results of [11] where possible only because the edge-direction conventions there agree with those for ultragraphs [13].

We let $E(\mathcal{G})^0_{\mathrm{rg}}$ denote the largest open subset of $E(\mathcal{G})^0$ such that the restriction of $s_{\mathcal{Q}}$ to $s_{\mathcal{Q}}^{-1}(E(\mathcal{G})^0_{\mathrm{rg}})$ is surjective and proper.

Lemma 5.1. *We have* $E(\mathcal{G})^0_{\mathrm{rg}} = G^0_{\mathrm{rg}}$.

Proof. Since the image of $s_{\mathcal{Q}}$ is contained in $G^0 \subset E(\mathcal{G})^0$, we have $E(\mathcal{G})^0_{\mathrm{rg}} \subset G^0$. For each $v \in G^0$, we see that $v \in E(\mathcal{G})^0_{\mathrm{rg}}$ if and only if $s_{\mathcal{Q}}^{-1}(v)$ is non-empty and compact because $\{v\}$ is open in $E(\mathcal{G})^0$. Since $\{e\} \times \overline{r(e)} \subset E(\mathcal{G})^1$ is compact for all $e \in \mathcal{G}^1$, it follows that $s_{\mathcal{Q}}^{-1}(v) = \{(e, x) \in E(\mathcal{G})^1 : s(e) = v\}$ is non-empty and compact if and only if $\{e \in \mathcal{G}^1 : s(e) = v\}$ is non-empty and finite; that is, if and only if $v \in G^0_{\mathrm{rg}}$. □

For the statement of the following theorem, we identify $C_0(E(\mathcal{G})^0)$ with $\mathfrak{A}_{\mathcal{G}}$ via the isomorphism in Proposition 4.7, and denote by $\chi_e \in$

$C_c(E(\mathcal{G})^1)$ the characteristic function of the compact open subset $\{e\}\times\overline{r(e)} \subset E(\mathcal{G})^1$ for each $e \in \mathcal{G}^1$. We denote by $(\psi_{\mathcal{Q}(\mathcal{G})}, \pi_{\mathcal{Q}(\mathcal{G})})$ the universal generating $\mathcal{Q}(\mathcal{G})$-pair, and by $\{p_A^{\mathcal{G}}, s_e^{\mathcal{G}} : A \in \mathcal{G}^0, e \in \mathcal{G}^1\}$ the universal generating Cuntz-Krieger $\mathcal{G}$-family.

Theorem 5.2. *There is an isomorphism from $C^*(\mathcal{G})$ to $C^*(\mathcal{Q}(\mathcal{G}))$ which is canonical in the sense that it takes $p_A^{\mathcal{G}}$ to $\pi_{\mathcal{Q}(\mathcal{G})}(\chi_A)$ and $s_e^{\mathcal{G}}$ to $\psi_{\mathcal{Q}(\mathcal{G})}(\chi_e)$ for all for all $A \in \mathcal{G}^0$ and $e \in \mathcal{G}^1$. Moreover, this isomorphism is equivariant for the gauge actions on $C^*(\mathcal{G})$ and $C^*(\mathcal{Q}(\mathcal{G}))$.*

Proof. It is easy to check using Lemma 5.1 and Proposition 3.4 that:

1. for each Cuntz-Krieger $\mathcal{G}$-family $\{p_A, s_e : A \in \mathcal{G}^0, e \in \mathcal{G}^1\}$ there is a unique $\mathcal{Q}(\mathcal{G})$-pair $(\pi_{q,t}, \psi_{p,s})$ satisfying $\pi_{p,s}(\chi_A) = p_A$ for each $A \in \mathcal{G}^0$ and $\psi_{p,s}(\chi_e) = s_e$ for each $e \in \mathcal{G}^1$; and
2. for each $\mathcal{Q}(\mathcal{G})$-pair (π, ψ), the formulae $p_A^{\pi,\psi} := \pi(\chi_A)$ and $s_e^{\pi,\psi} := \psi(\chi_e)$ determine a Cuntz-Krieger $\mathcal{G}$-family $\{p_A^{\pi,\psi}, s_e^{\pi,\psi} : A \in \mathcal{G}^0, e \in \mathcal{G}^1\}$.

The result then follows from the universal properties of the two C^*-algebras $C^*(\mathcal{G})$ and $C^*(\mathcal{Q}(\mathcal{G}))$. □

Remark 5.3. To prove Theorem 5.2, one could alternatively use the gauge-invariant uniqueness theorems of ultragraphs [13, Theorem 6.8] or of topological graphs [6, Theorem 4.5], or of topological quivers [11, Theorem 6.16].

Theorem 5.4. *Let $\mathcal{G} = (G^0, \mathcal{G}^1, r, s)$ be an ultragraph. We define $\partial\colon \mathbb{Z}^{G^0_{\mathrm{rg}}} \to Z_{\mathcal{G}}$ by $\partial(\delta_v) = \delta_v - \sum_{e\in s^{-1}(v)} \chi_{r(e)}$ for $v \in G^0_{\mathrm{rg}}$. Then we have $K_0(C^*(\mathcal{G})) \cong \operatorname{coker}(\partial)$ and $K_1(C^*(\mathcal{G})) \cong \ker(\partial)$.*

Proof. Since $C_0(E(\mathcal{G})^0_{\mathrm{rg}}) \cong c_0(G^0_{\mathrm{rg}})$ and $C_0(E(\mathcal{G})^0) \cong \mathfrak{A}_{\mathcal{G}}$, we have $K_0(C_0(E(\mathcal{G})^0_{\mathrm{rg}})) \cong \mathbb{Z}^{G^0_{\mathrm{rg}}}$, $K_0(C_0(E(\mathcal{G})^0)) \cong Z_{\mathcal{G}}$ and $K_1(C_0(E(\mathcal{G})^0_{\mathrm{rg}})) = K_1(C_0(E(\mathcal{G})^0)) = 0$ by Proposition 2.20. It is straightforward to see that the map $[\pi_r]\colon K_0(C_0(E(\mathcal{G})^0_{\mathrm{rg}})) \to K_0(C_0(E(\mathcal{G})^0))$ in [6, Corollary 6.10] satisfies $[\pi_r](\delta_v) = \sum_{e\in s^{-1}(v)} \chi_{r(e)}$ for $v \in G^0_{\mathrm{rg}}$. Hence the conclusion follows from [6, Corollary 6.10]. □

6. Gauge-invariant ideals

In this section we characterize the gauge-invariant ideals of $C^*(\mathcal{G})$ for an ultragraph $\mathcal{G}$ in terms of combinatorial data associated to $\mathcal{G}$.

For the details of the following, see [11]. Let $\mathcal{Q} = (E^0, E^1, r, s, \lambda)$ be a topological quiver. We say that a subset $U \subset E^0$ is *hereditary* if, whenever $e \in E^1$ satisfies $s(e) \in U$, we have $r(e) \in U$. We say that

U is *saturated* if, whenever $v \in E^0_{\mathrm{rg}}$ satisfies $r(s^{-1}(v)) \subset U$, we have $v \in U$.

Suppose that $U \subset E^0$ is open and hereditary. Then

$$\mathcal{Q}_U := (E^0_U, E^1_U, r|_{E^1_U}, s|_{E^1_U}, \lambda|_{E^0_U})$$

is a topological quiver, where $E^0_U = E^0 \setminus U$ and $E^1_U = E^1 \setminus r^{-1}(U)$.

We say that a pair (U, V) of subsets of E^0 is *admissible* if

(1) U is a saturated hereditary open subset of E^0; and
(2) V is an open subset of E^0_U with $E^0_{\mathrm{rg}} \setminus U \subset V \subset (E^0_U)_{\mathrm{rg}}$.

It follows from [11, Theorem 8.22] that the gauge-invariant ideals of $C^*(\mathcal{Q})$ are in bijective correspondence with the admissible pairs (U, V) of $\mathcal{Q}$.

Let $\mathcal{G} = (G^0, \mathcal{G}^1, r, s)$ be an ultragraph. We define admissible pairs of $\mathcal{G}$ in a similar way as above, and show that these are in bijective correspondence with the gauge-invariant ideals of $C^*(\mathcal{G})$.

Definition 6.1. A subcollection $\mathcal{H} \subset \mathcal{G}^0$ is said to be an *ideal* if it satisfies

(1) $A, B \in \mathcal{H}$ implies $A \cup B \in \mathcal{H}$,
(2) $A \in \mathcal{H}$, $B \in \mathcal{G}^0$ and $B \subset A$ imply $B \in \mathcal{H}$.

Let $\pi\colon \mathfrak{A}_{\mathcal{G}} \to C_0(\Omega_{\mathcal{G}})$ be the isomorphism in Proposition 4.7. For an ideal $\mathcal{H}$ of $\mathcal{G}^0$, the set $\overline{\mathrm{span}}\{\chi_A : A \in \mathcal{H}\}$ is an ideal of the C^*-algebra $\mathfrak{A}_{\mathcal{G}}$. Hence there exists an open subset $U_{\mathcal{H}}$ of $\Omega_{\mathcal{G}}$ with

$$C_0(U_{\mathcal{H}}) = \pi\big(\overline{\mathrm{span}}\{\chi_A : A \in \mathcal{H}\}\big).$$

Lemma 6.2. *The correspondence $\mathcal{H} \mapsto U_{\mathcal{H}}$ is a bijection from the set of all ideals of $\mathcal{G}^0$ to the set of all open subsets of $\Omega_{\mathcal{G}}$.*

Proof. Since $\mathfrak{A}_{\mathcal{G}}$ is an AF-algebra, every ideal of $\mathfrak{A}_{\mathcal{G}}$ is generated by its projections. From this fact, we see that $\mathcal{H} \mapsto \overline{\mathrm{span}}\{\chi_A : A \in \mathcal{H}\}$ is a bijection from the set of all ideals of $\mathcal{G}^0$ to the set of all ideals of $\mathfrak{A}_{\mathcal{G}}$. Hence the conclusion follows from the well-known fact that $U \mapsto C_0(U)$ is a bijection from the set of all open subsets of $\Omega_{\mathcal{G}}$ to the set of all ideals of $C_0(\Omega_{\mathcal{G}})$. □

Remark 6.3. The existence of this bijection is one of the advantages of changing the definition of $\mathcal{G}^0$ from that given in [13].

Lemma 6.4. *Let $\mathcal{H}$ be an ideal of $\mathcal{G}^0$, and let $U_{\mathcal{H}} \subset \Omega_{\mathcal{G}}$ be the corresponding open set. Then for $v \in G^0$, $\{v\} \in \mathcal{H}$ if and only if $\overline{v} \in U_{\mathcal{H}}$, and for $e \in \mathcal{G}^1$, $r(e) \in \mathcal{H}$ if and only if $\overline{r(e)} \subset U_{\mathcal{H}}$.*

Proof. This follows from Proposition 4.7. □

Definition 6.5. We say that an ideal $\mathcal{H} \subset \mathcal{G}^0$ is *hereditary* if, whenever $e \in \mathcal{G}^1$ satisfies $\{s(e)\} \in \mathcal{H}$, we have $r(e) \in \mathcal{H}$, and that it is *saturated* if, whenever $v \in G^0_{\mathrm{rg}}$ satisfies $r(e) \in \mathcal{H}$ for all $e \in s^{-1}(v)$, we have $\{v\} \in \mathcal{H}$.

Proposition 6.6. *An ideal $\mathcal{H}$ of $\mathcal{G}^0$ is hereditary (resp. saturated) if and only if the corresponding open subset $U_{\mathcal{H}} \subset \Omega_{\mathcal{G}} = E(\mathcal{G})^0$ is hereditary (resp. saturated) in the topological quiver $\mathcal{Q}(\mathcal{G})$.*

Proof. An open subset $U \subset \Omega_{\mathcal{G}} = E(\mathcal{G})^0$ is hereditary if and only if, whenever $(e, x) \in E(\mathcal{G})^1$ satisfies $s_{\mathcal{Q}}(e, x) = s(e) \in U$, we have $r_{\mathcal{Q}}(e, x) = x \in U$. This is equivalent to the statement that, whenever $e \in \mathcal{G}^1$ satisfies $s(e) \in U$, we have $\overline{r(e)} \subset U$. Thus Lemma 6.4 shows that an ideal $\mathcal{H}$ is hereditary if and only if $U_{\mathcal{H}}$ is hereditary.

An open subset $U \subset \Omega_{\mathcal{G}} = E(\mathcal{G})^0$ is saturated if and only if, whenever $v \in E(\mathcal{G})^0_{\mathrm{rg}} = G^0_{\mathrm{rg}}$ satisfies $r_{\mathcal{Q}}(s_{\mathcal{Q}}^{-1}(v)) \subset U$, we have $v \in U$. For $v \in G^0_{\mathrm{rg}}$, we have $r_{\mathcal{Q}}(s_{\mathcal{Q}}^{-1}(v)) = \bigcup_{e \in s^{-1}(v)} \overline{r(e)}$. Hence U is saturated if and only if, whenever $v \in G^0_{\mathrm{rg}}$ satisfies $\overline{r(e)} \subset U$ for all $e \in s^{-1}(v)$, we have $v \in U$. Thus Lemma 6.4 again shows that an ideal $\mathcal{H}$ is saturated if and only if $U_{\mathcal{H}}$ is saturated. ☐

Definition 6.7. Let $\mathcal{H}$ be a hereditary ideal of $\mathcal{G}^0$. For $v \in G^0$, we define $s^{-1}_{\mathcal{G}/\mathcal{H}}(v) \subset \mathcal{G}^1$ by

$$s^{-1}_{\mathcal{G}/\mathcal{H}}(v) := \{e \in \mathcal{G}^1 : s(e) = v \text{ and } r(e) \notin \mathcal{H}\}.$$

We define $(G^0_{\mathcal{H}})_{\mathrm{rg}} \subset G^0$ by

$$(G^0_{\mathcal{H}})_{\mathrm{rg}} := \{v \in G^0 : s^{-1}_{\mathcal{G}/\mathcal{H}}(v) \text{ is non-empty and finite}\}.$$

Since $\mathcal{H}$ is hereditary, if $\{v\} \in \mathcal{H}$ then we have $s^{-1}_{\mathcal{G}/\mathcal{H}}(v) = \emptyset$ and hence $v \notin (G^0_{\mathcal{H}})_{\mathrm{rg}}$.

Lemma 6.8. *A hereditary ideal $\mathcal{H}$ of $\mathcal{G}^0$ is saturated if and only if, whenever $v \in G^0_{\mathrm{rg}}$ satisfies $\{v\} \notin \mathcal{H}$, we have $v \in (G^0_{\mathcal{H}})_{\mathrm{rg}}$.*

Proof. An element $v \in G^0_{\mathrm{rg}}$ is in $(G^0_{\mathcal{H}})_{\mathrm{rg}}$ if and only if $s^{-1}_{\mathcal{G}/\mathcal{H}}(v) \subset s^{-1}(v)$ is non-empty, which occurs if and only if there exists $e \in s^{-1}(v)$ with $r(e) \notin \mathcal{H}$. ☐

Let $\mathcal{H}$ be a hereditary ideal of $\mathcal{G}^0$, and $U_{\mathcal{H}} \subset \Omega_{\mathcal{G}} = E(\mathcal{G})^0$ be the corresponding open subset which is hereditary by Proposition 6.6. As in the beginning of this section, we obtain a topological quiver $\mathcal{Q}(\mathcal{G})_{U_{\mathcal{H}}}$.

Lemma 6.9. *We have $(E(\mathcal{G})^0_{U_{\mathcal{H}}})_{\mathrm{rg}} = (G^0_{\mathcal{H}})_{\mathrm{rg}}$.*

Proof. Since the image of $s_{\mathcal{Q}}|_{E(\mathcal{G})^1_{U_{\mathcal{H}}}}$ is contained in $G^0 \setminus U_{\mathcal{H}}$, we have $(E(\mathcal{G})^0_{U_{\mathcal{H}}})_{\mathrm{rg}} \subset G^0 \setminus U_{\mathcal{H}}$. For $v \in G^0$, $v \in U_{\mathcal{H}}$ implies $\{v\} \in H$ by Lemma 6.4, and this implies $v \notin (G^0_{\mathcal{H}})_{\mathrm{rg}}$ as remarked after Definition 6.7. Hence we have $(G^0_{\mathcal{H}})_{\mathrm{rg}} \subset G^0 \setminus U_{\mathcal{H}}$. An element $v \in G^0 \setminus U_{\mathcal{H}}$ is in $(E(\mathcal{G})^0_{U_{\mathcal{H}}})_{\mathrm{rg}}$ if and only if $s_{\mathcal{Q}}^{-1}(v) \cap E(\mathcal{G})^1_{U_{\mathcal{H}}}$ is non-empty and compact because $\{v\}$ is open in $E(\mathcal{G})^0_{U_{\mathcal{H}}}$. Since

$$s_{\mathcal{Q}}^{-1}(v) \cap E(\mathcal{G})^1_{U_{\mathcal{H}}} = \{(e,x) \in E(\mathcal{G})^1 : s(e) = v \text{ and } x \notin U_{\mathcal{H}}\},$$

$s_{\mathcal{Q}}^{-1}(v) \cap E(\mathcal{G})^1_{U_{\mathcal{H}}}$ is non-empty and compact if and only if

$$\{e \in \mathcal{G}^1 : s(e) = v \text{ and } \overline{r(e)} \not\subset U_{\mathcal{H}}\}$$

is non-empty and finite. This set is equal to $s_{\mathcal{G}/\mathcal{H}}^{-1}(v)$ by Lemma 6.4. Therefore an element $v \in G^0 \setminus U_{\mathcal{H}}$ is in $(E(\mathcal{G})^0_{U_{\mathcal{H}}})_{\mathrm{rg}}$ if and only if $v \in (G^0_{\mathcal{H}})_{\mathrm{rg}}$. Thus $(E(\mathcal{G})^0_{U_{\mathcal{H}}})_{\mathrm{rg}} = (G^0_{\mathcal{H}})_{\mathrm{rg}}$ as required. □

By Lemma 6.9, the subset $(E(\mathcal{G})^0_{U_{\mathcal{H}}})_{\mathrm{rg}} \subset E(\mathcal{G})^0_{U_{\mathcal{H}}}$ is discrete.

Definition 6.10. Let $\mathcal{G} = \{G^0, \mathcal{G}^1, r, s\}$ be an ultragraph. We say that a pair $(\mathcal{H}, V)$ consisting of an ideal $\mathcal{H}$ of $\mathcal{G}^0$ and a subset V of G^0 is *admissible* if $\mathcal{H}$ is hereditary and saturated and $V \subset (G^0_{\mathcal{H}})_{\mathrm{rg}} \setminus G^0_{\mathrm{rg}}$.

Definition 6.11. For an admissible pair $(\mathcal{H}, V)$ of an ultragraph $\mathcal{G}$, we define an ideal $I_{(\mathcal{H},V)}$ of $\mathcal{C}^*(\mathcal{G})$ to be the ideal generated by the projections

$$\{p_A : A \in \mathcal{H}\} \cup \Big\{p_v - \sum_{e \in s_{\mathcal{G}/\mathcal{H}}^{-1}(v)} s_e s_e^* : v \in V\Big\}.$$

For an ideal I of $\mathcal{C}^*(\mathcal{G})$, we define $\mathcal{H}_I := \{A \in \mathcal{G}^0 : p_A \in I\}$ and

$$V_I := \Big\{v \in (G^0_{\mathcal{H}_I})_{\mathrm{rg}} \setminus G^0_{\mathrm{rg}} : p_v - \sum_{e \in s_{\mathcal{G}/\mathcal{H}_I}^{-1}(v)} s_e s_e^* \in I\Big\}.$$

Theorem 6.12. *Let $\mathcal{G}$ be an ultragraph. Then the correspondence $I \mapsto (\mathcal{H}_I, V_I)$ is a bijection from the set of all gauge-invariant ideals of $C^*(\mathcal{G})$ to the set of all admissible pairs of $\mathcal{G}$, whose inverse is given by $(\mathcal{H}, V) \mapsto I_{(\mathcal{H},V)}$.*

Proof. By Theorem 5.2, the gauge-invariant ideals of $C^*(\mathcal{G})$ are in bijective correspondence with the gauge-invariant ideals of $C^*(\mathcal{Q}(\mathcal{G}))$. We know that the latter are indexed by admissible pairs (U, V) of $\mathcal{Q}(\mathcal{G})$ by [11, Theorem 8.22]. Proposition 6.6 and Lemma 6.9 show that $(\mathcal{H}, V) \mapsto (U_{\mathcal{H}}, V \cup (G^0_{\mathrm{rg}} \setminus U_{\mathcal{H}}))$ is a bijection from the set of all admissible pairs of $\mathcal{G}$ to the one of $\mathcal{Q}(\mathcal{G})$. Thus we get bijective correspondences between the set of all gauge-invariant ideals of $C^*(\mathcal{G})$ and the set of all admissible pairs of $\mathcal{G}$. By keeping track of the arguments in [11, Section 8], we see that the bijective correspondences are given by $I \mapsto (\mathcal{H}_I, V_I)$ and $(\mathcal{H}, V) \mapsto I_{(\mathcal{H},V)}$. □

Remark 6.13. The theorem above naturally generalizes [1, Theorem 3.6].

7. Condition (K)

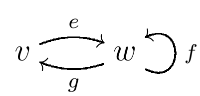
In this section we define a version of Condition (K) for ultragraphs, and show that this condition characterizes ultragraphs $\mathcal{G}$ such that every ideal of $C^*(\mathcal{G})$ is gauge-invariant.

Let $\mathcal{G}$ be an ultragraph. For $v \in G^0$, a *first-return path based at* v in $\mathcal{G}$ is a path $\alpha = e_1 e_2 \cdots e_n$ such that $s(\alpha) = v$, $v \in r(\alpha)$, and $s(e_i) \neq v$ for $i = 2, 3, \ldots, n$. When α is a first-return path based at v, we say that v *hosts the first-return path* α.

Note that there is a subtlety here: a first-return path based at v may pass through other vertices $w \neq v$ more than once (that is, we may have $s(e_i) = s(e_j)$ for some $1 < i, j \leq n$ with $i \neq j$), but no edge other than e_1 may have source v.

Definition 7.1. Let $\mathcal{G}$ be an ultragraph. We say that $\mathcal{G}$ satisfies Condition (K) if every $v \in \mathcal{G}^0$ which hosts a first-return path hosts at least two distinct first-return paths.

Example 7.2. The graph



satisfies Condition (K) because v hosts infinitely many first-return paths $eg, efg, effg, \ldots$, and w hosts two first-return paths f and ge. Note that all first-return paths based at v except eg pass through the vertex w more than once.

Proposition 7.3. *Let $\mathcal{G} = (G^0, \mathcal{G}^1, r, s)$ be an ultragraph. Then every ideal of $C^*(\mathcal{G})$ is gauge-invariant if and only if $\mathcal{G}$ satisfies Condition (K).*

Proof. In the same way as above, we can define first-return paths in the topological graph $E(\mathcal{G})$. It is straightforward to see that for each $v \in G^0$, first-return paths $\alpha = e_1 e_2 \cdots e_n$ based at v in $\mathcal{G}$ correspond bijectively to first-return paths

$$l = (e_1, s(e_2))(e_2, s(e_3)) \cdots (e_n, s(e_1))$$

based at $v \in G^0 \subset E(\mathcal{G})^0$ in $E(\mathcal{G})$.

Recall (see [7, Definition 7.1] and the subsequent paragraph for details) that $\mathrm{Per}(E(\mathcal{G}))$ denotes the collection of vertices $v \in E(\mathcal{G})^0$ such that v hosts exactly one first-return path in $E(\mathcal{G})$, and v is isolated in

$$\{s_{\mathcal{Q}}(l) : l \text{ is a path in } E(\mathcal{G}) \text{ with } r_{\mathcal{Q}}(l) = v\}$$

(recall that the directions of paths are reversed when passing from the quiver $\mathcal{Q}(\mathcal{G})$ to the topological graph $E(\mathcal{G})$). Wee see that [7, Theorem 7.6] implies that every ideal of $\mathcal{O}(E(\mathcal{G}))$ is gauge invariant if and only if $\mathrm{Per}(E(\mathcal{G}))$ is empty. Since the isomorphism of $C^*(\mathcal{G})$ with $\mathcal{O}(E(\mathcal{G}))$ is gauge equivariant, it therefore suffices to show that $\mathrm{Per}(E(\mathcal{G}))$ is empty if and only if $\mathcal{G}$ satisfies Condition (K).

The image of $s_{\mathcal{Q}}$ is contained in the discrete set $G^0 \subset E(\mathcal{G})^0$. Thus $v \in E(\mathcal{G})^0$ belongs to $\mathrm{Per}(E(\mathcal{G}))$ if and only if $v \in G^0 \subset E(\mathcal{G})^0$ and v hosts exactly one first-return path in $E(\mathcal{G})$. By the first paragraph of this proof, $v \in G^0 \subset E(\mathcal{G})^0$ hosts exactly one first-return path in $E(\mathcal{G})$ if and only if v hosts exactly one first-return path in $\mathcal{G}$. Hence $\mathrm{Per}(E(\mathcal{G}))$ is empty if and only if $\mathcal{G}$ satisfies Condition (K). □

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