A MACKEY NORMAL SUBGROUP ANALYSIS FOR GROUPOIDS

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ABSTRACT. Given a normal subgroup bundle $\mathcal{A}$ of the isotropy bundle of a groupoid Σ, we obtain a twisted action of the quotient groupoid $\Sigma/\mathcal{A}$ on the bundle of group C^*-algbras determined by $\mathcal{A}$ whose twisted crossed product recovers the groupoid C^*-algebra $C^*(\Sigma)$. Restricting to the case where $\mathcal{A}$ is abelian, we describe $C^*(\Sigma)$ as the C^*-algebra associated to a **T**-groupoid over the action groupoid obtained from the canonical action of $\Sigma/\mathcal{A}$ on the Pontryagin dual space of $\mathcal{A}$. We give some illustrative examples of this result.

INTRODUCTION

The objective of this paper is to develop tools for analyzing C^*-algebras of very general groupoids via a version of the Mackey machine, and also to construct Cartan subalgebras in C^*-algebras of large classes of groupoids that are not necessarily étale. Specifically, one of the fundamental components of the modern Mackey Subgroup Analysis for group representations comes from Green's [Gre78, Proposition 1]: if H is a normal subgroup of a locally compact group G, then $C^*(G)$ is a twisted crossed product $C^*(G, C^*(H), \kappa)$ of $C^*(H)$ by G. The idea is then to describe the representation theory and the structure of $C^*(G)$ in terms of representations of H and the action of G on the spectrum of $C^*(H)$ induced by conjugation. Variants of this approach are sometimes called "Mackey's Little Group Method".

Our main theorem generalizes Green's theorem to the situation of a locally compact Hausdorff groupoid Σ and a normal subgroup bundle $\mathcal{A}$ of the isotropy groupoid of Σ. Our result says that the full C^*-algebra of Σ coincides with a twisted crossed product of $C^*(\mathcal{A})$ by Σ. If $\mathcal{A}$ is amenable, this descends to an isomorphism of reduced C^*-algebras. We pay particular attention to the situation when $\mathcal{A}$ is abelian, and hence has a Pontryagin dual space $\hat{\mathcal{A}}$. The quotient $\mathcal{G} = \Sigma/\mathcal{A}$ is a topological groupoid, and we use our main theorem to prove that $C^*(\Sigma)$ is isomorphic to the restricted C^*-algebra $C^*(\hat{\mathcal{A}} \rtimes \mathcal{G}; \widehat{\Sigma})$ of a **T**-groupoid $\widehat{\Sigma}$ over an action groupoid $\hat{\mathcal{A}} \rtimes \mathcal{G}$. As the theory of **T**-groupoids—also called twists—is well studied, this provides powerful tools for studying $C^*(\Sigma)$. This was the approach for the main result in [MRW96], and in fact, our isomorphism result is a significant generalization of, and is motivated by, [MRW96, Proposition 4.5]. As an illustrative example, we discuss

the special case of a locally compact Hausdorff group Σ with a closed normal abelian subgroup $\mathcal{A}$; even in this special case, our results have something new to say.

Accordingly, in Section 2, we consider a second countable locally compact groupoid and subgroup bundle $\mathcal{A}$ of the isotropy groupoid of Σ whose unit space coincides with that of Σ. Then $\mathcal{A}$ acts on the left and right of Σ and we say that $\mathcal{A}$ is normal if the orbits $\sigma\mathcal{A}$ and $\mathcal{A}\sigma$ coincide for all $\sigma \in \Sigma$. If $\mathcal{A}$ is normal, then the quotient $\mathcal{G} = \Sigma/\mathcal{A}$ is a groupoid and we can view Σ as an extension of $\mathcal{A}$ by $\mathcal{G}$. Assuming that both Σ and $\mathcal{A}$ have Haar systems, Σ acts by automorphisms on $C^*(\mathcal{A})$ and we can show that there is a Green–Renault twisting map κ for this action. This allows us to form a twisted crossed product of which Green's twisted crossed products are a special case. When $\mathcal{A}$ is an abelian group bundle these twisted crossed products are the same as those in [Ren91, Ren87]. Our first main result extends Green's result by proving that $C^*(\Sigma)$ is isomorphic to the twisted crossed product. A similar result is obtained for the reduced norm when $\mathcal{A}$ is amenable.

In Section 3 we restrict to the case that $\mathcal{A}$ is an abelian group bundle. Then the Gelfand dual, $\hat{\mathcal{A}}$, of $C^*(\mathcal{A})$ is a locally compact space right $\mathcal{G}$-space and we can form the action groupoid $\hat{\mathcal{A}} \rtimes \mathcal{G}$. We can build a natural $\mathbf{T}$-groupoid, $\widehat{\Sigma}$, over $\hat{\mathcal{A}} \rtimes \mathcal{G}$. Our second main result is that $C^*(\Sigma)$ is isomorphic to the restricted groupoid C^*-algebra $C^*(\hat{\mathcal{A}} \rtimes \mathcal{G}, \widehat{\Sigma})$ of this $\mathbf{T}$-groupoid with a similar assertion for the reduced norms. It follows from [BL17] that $C^*(\Sigma)$ belongs to the UCT class if and only if it is nuclear.

In Section 4, we pause to see that the construction of the $\mathbf{T}$-groupoid in Section 3 is natural in that it is an example of a useful "pushout" construction for groupoids that has intrinsic universal properties. In addition to being an interesting construct in its own right, this brings our current results into line with our earlier work in [IKSW19].

In Section 5, we give a number of applications and examples of our main results. The first is to groups. We consider the situation of a closed normal abelian subgroup H of a locally compact group G; our results appear to be new even in this special case. The quotient G/H acts on H by conjugation, and hence also on the Pontryagin dual $\widehat{H}$ viewed as a topological space. Our construction yields a $\mathbf{T}$-groupoid $\widehat{\Sigma}$ over $\widehat{H} \rtimes (G/H)$. Our main theorem identifies the group C^*-algebra $C^*(G)$ with the C^*-algebra $C^*(\widehat{H} \rtimes (G/H); \widehat{\Sigma})$ of this $\mathbf{T}$-groupoid, and similarly at the level of reduced C^*-algebras. This generalises Zeller-Meier's results [ZM68] for clopen normal abelian subgroups. As specific examples of these results we demonstrate how to recover two standard descriptions of the integer Heisenberg group: Anderson–Paschke's description as the section algebra of a field of rotation algebras; and the description as a crossed product of $C(\mathbf{T}^2)$ by $\mathbf{Z}$. Our second class of examples is that of extensions of effective étale groupoids $\mathcal{G}$ by bundles $\mathcal{A}$ of abelian groups. In this situation, the semidirect product $\hat{\mathcal{A}} \rtimes \mathcal{G}$ is also étale, and we deduce that $C_0(\hat{\mathcal{A}})$ embeds as a Cartan subalgebra of $C^*_r(\Sigma)$. We conclude Section 5 with a brief discussion of how this applies to C^*-algebras of row-finite k-graphs with no sources, and relates to [BNR$^+$16].

1. Preliminaries

Our results require that we work with Green–Renault twisted crossed products of C^*-algebras by locally compact Hausdorff groupoids as well as the theory of $\mathbf{T}$-groupoids and their restricted groupoid C^*-algebras. In order to take a unified approach, we use the theory

of Fell bundles over groupoids and their associated C^*-algebras. For convenience, we give a brief overview of the required background with selected references.

1.1. **Upper semicontinuous Banach bundles.** An excellent source for continuous Banach bundles $p : \mathscr{E} \to X$ can be found in §§13–14 in [FD88a, Chap. II]. When the axiom that $a \mapsto \|a\|$ be continuous is relaxed to require only that this map be upper-semicontinuous, then $\mathscr{E}$ is called an upper semicontinuous Banach bundle. In the past, such bundles were also called "loose" or H-bundles and studied in [DG83] and [Hof77]. For more details see [MW08a, Appendix A]. A more detailed exposition in the the C^*-bundle case can be found in [Wil07, Appendix C]. If $p : \mathscr{E} \to X$ is an upper semicontinuous Banach bundle, we will write $\Gamma_0(X; \mathscr{E})$ for the continuous sections of $\mathscr{E}$ which vanish at infinity. This $\Gamma_0(X; \mathscr{E})$ is a Banach space with respect to the supremum norm. A primary motivation for working with upper-semicontinuous bundles is that a $C_0(X)$-algebra A is always $C_0(X)$-isomorphic to $\Gamma_0(X, \mathscr{A})$ for an appropriate upper-semicontinuous C^*-bundle $\mathscr{A}$ [Wil07, Theorem C.26].

1.2. **Fell bundles and their C^*-algebras.** Fell bundles over groupoids are a natural generalization of Fell's Banach $*$-algebraic bundles from [FD88b, Chapter VIII]. They were introduced in [Yam87]. For more details, see [Kum98, MW08a].

Roughly speaking a Fell bundle $\mathscr{B}$ over a locally compact Hausdorff groupoid $\mathcal{G}$ is an upper-semicontinuous Banach bundle $p : \mathscr{B} \to \mathcal{G}$ endowed with a continuous involution $b \mapsto b^*$ and a continuous multiplication $(a, b) \mapsto ab$ from $\mathscr{B}^{(2)} = \{ (b, b') : (p(b), p(b') \in \mathcal{G}^{(2)} \}$ to $\mathscr{B}$ such that—with respect to the operations, actions, and inner products induced by the involution and multiplication—the fibres $B(x) = p^{-1}(x)$ over units $x \in \mathcal{G}^{(0)}$ are C^*-algebras and such that each fibre $B(\gamma)$ is a $B(r(\gamma))$–$B(s(\gamma))$-imprimitivity bimodule. We write $\Gamma_c(\mathcal{G}; \mathscr{B})$ for the $*$-algebra of continuous compactly supported sections of $\mathscr{B}$ under convolution and involution.

Each Fell bundle has a both a full and a reduced C^*-algebra. The full C^*-algebra $C^*(\mathcal{G}, \mathscr{B})$ is universal for representations of the bundle that are continuous in the inductive-limit topology. The reduced C^*-algebra is obtained from a representation of $C_c(\mathcal{G}; \mathscr{B})$ on the Hilbert module $\mathcal{H}(\mathscr{B})$ defined as follows (see [Kum98, MT16, SW13, Hol17]).

The restriction of $\mathscr{B}$ to a bundle over the unit space is an upper-semicontinuous C^*-bundle, and we write $\Gamma_0(\mathcal{G}^{(0)}, \mathscr{B})$ for the C^*-algebra of C_0-sections of this bundle. Define $\langle \cdot, \cdot \rangle_{C_0(\mathcal{G}^{(0)}, \mathscr{B})}$ on $C_c(\mathcal{G}; \mathscr{B})$ by $\langle f, g \rangle_{C_0(\mathcal{G}^{(0)}, \mathscr{B})} = (f^* * g)|_{\mathcal{G}^{(0)}}$. This is a pre-inner product, and the completion

$$\mathcal{H}(\mathscr{B}) := \overline{C_c(\mathcal{G}; \mathscr{B})}^{\|\cdot\|_{C_0(\mathcal{G}^{(0)}, \mathscr{B})}}$$

is a Hilbert module. Left multiplication in $C_c(\mathcal{G}; \mathscr{B})$ extends to a left action $\phi : C_c(\mathcal{G}; \mathscr{B}) \to \mathcal{L}(\mathcal{H}(\mathscr{B}))$ by adjointable operators, and the reduced norm on $C_c(\mathcal{G}; \mathscr{B})$ is given by $\|f\|_r = \|\phi(f)\|_{\mathcal{L}(\mathcal{H}(\mathscr{B}))}$.

1.3. **Twisted groupoid crossed products.** The notion of a twisted groupoid crossed product is the natural analogue of the ordinary twisted crossed product construction (for group actions), to the setting of actions of groupoids on upper semicontinuous C^*-bundles. These objects were studied intensively in [Ren91, Ren87] (in the continuous bundle case). A very good summary of the upper semicontinuous case can be found in [Goe09, §3.2]. A shorter treatment can be found in [MW08b, §4]. Renault incorporated a Green-style twist

in his constructions in [Ren91, Ren87] providing a natural generalization of Green's twisted crossed products. However, to make our approach as self-contained as possible, we view Green–Renault twists using Fell bundles as described in [MW08a, Example 2.5]. Groupoid crossed products then arise as a special case—see [MW08a, Example 2.1]. Since the treatment in [MW08a, §2] is a bit terse, we have included additional details in Appendices B and C for completeness.

For our isomorphism result, we will need the special case of the twisted crossed product associated to a twist $\mathcal{G}^{(0)} \times \mathbf{T} \longrightarrow \Sigma \longrightarrow \mathcal{G}$. The associated full and reduced C^*-algebras coincide with the full and reduced C^*-algebras of the Fell line-bundle over $\mathcal{G}$ determined by Σ (see [MW08a, Example 2.3]).[1] Hence the reduced norm on $\Gamma_c(\mathcal{G}; \Sigma)$ is realized by the representation of $\Gamma_c(\mathcal{G}; \Sigma)$ on the Hilbert module $\mathcal{H}(\mathcal{G}; \Sigma)$ given by taking the completion of $\Gamma_c(\mathcal{G}; \Sigma)$ with respect to $\langle f, g \rangle := (f^* * g)|_{\mathcal{G}^{(0)}}$. Restricting to the special case where the twist Σ is the trivial twist $\mathcal{G} \times \mathbf{T}$, we see that the reduced norm on $C_c(\mathcal{G})$ is realized by the representation of $C_c(\mathcal{G})$ on the Hilbert module $\mathcal{H}(\mathcal{G})$ obtained as the completion of $C_c(\mathcal{G})$ with respect to $\langle f, g \rangle = (f^* * g)|_{\mathcal{G}^{(0)}}$.

2. Extensions of Group Bundles

Let Σ be a second-countable locally compact Hausdorff groupoid with a Haar system $\lambda = \{\lambda^u\}_{u \in \Sigma^{(0)}}$. Let $\mathcal{A}$ be a closed subgroupoid of the the isotropy group bundle $\Sigma' := \{\sigma \in \Sigma : s(\sigma) = r(\sigma)\}$ such that $\mathcal{A}^{(0)} = \Sigma^{(0)}$. We summarize this by saying that $\mathcal{A}$ *is a wide subgroup bundle of* Σ. We let $p_{\mathcal{A}} = r|_{\mathcal{A}} = s|_{\mathcal{A}}$ be the projection from $\mathcal{A}$ onto $\Sigma^{(0)}$ and write $A(u)$ for the fibre of $\mathcal{A}$ over u.[2] Note that $\mathcal{A}$ acts freely and properly on the left as well as on the right of Σ. We assume that $\mathcal{A}$ has a Haar system $\beta = \{\beta^u\}_{u \in \Sigma^{(0)}}$.

In this section we will construct a groupoid dynamical system $(\mathscr{E}, \Sigma, \vartheta)$ (see Proposition 2.8) from Σ and $\mathcal{A}$. By realising ϑ as an action by conjugation for a map $\kappa : \mathcal{A} \to \bigcup_u UM(C^*(\mathcal{A}(u)))$, we obtain a Green–Renault twist and an associated algebra $C_c(\mathcal{G}, \Sigma, \mathscr{E}, \kappa)$ of equivariant sections $\Sigma \to \mathscr{E}$. We construct a *-algebra homomorphism from $C_c(\Sigma)$ to $C_c(\mathcal{G}, \Sigma, \mathscr{E}, \kappa)$ (see Lemma 2.10), and then our main result, Theorem 3.4, shows that this homomorphism extends to isomorphisms $C^*(\Sigma) \cong C^*(\mathcal{G}, \Sigma, \mathscr{E}, \kappa)$ and, if $\mathcal{A}$ is amenable, $C_r^*(\Sigma) \cong C_r^*(\mathcal{G}, \Sigma, \mathscr{E}, \kappa)$. Although Σ has a Haar system by assumption, $\mathcal{A}$ has a Haar system only if the fibres of $\mathcal{A}$ are well-behaved as described in the following result taken from [Ren91, §1] (see also [Wil19, Theorem 6.12]).

Lemma 2.1. *Let $\mathcal{A}$ be a closed subgroup bundle of Σ. Then the following are equivalent.*
  (a) *There is a Haar system $\beta = \{\beta^u\}_{u \in \Sigma^{(0)}}$ for $\mathcal{A}$.*
  (b) *The projection map $p_{\mathcal{A}} : \mathcal{A} \to \Sigma^{(0)}$ is open.*
  (c) *The map $u \mapsto A(u)$ is continuous from $\Sigma^{(0)}$ into the space of closed subgroups of Σ in the Fell topology.*

If $p_{\mathcal{A}}$ is open then the orbit maps for $\mathcal{A}$-actions are open [Wil19, Proposition 2.12], so both orbit spaces $\mathcal{A}\backslash\Sigma$ and $\Sigma/\mathcal{A}$ are locally compact Hausdorff in our situation. We say

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[1]The complex conjugate appearing in the transformation equation (2.1) in [MW08a, Example 2.3] is a typo and should not appear there.

[2]We use the roman font for the fibre groups of a group bundle $\mathcal{A}$.

that $\mathcal{A}$ is *normal* if the orbits $\mathcal{A}\sigma$ and $\sigma\mathcal{A}$ coincide for all $\sigma \in \Sigma$. Note that $\mathcal{A}$ is normal if and only if Σ acts on $\mathcal{A}$ by conjugation.

While the isotropy bundle $\mathcal{A} = \Sigma' := \bigcup_{u \in \Sigma^{(0)}} \Sigma_u^u$ is always a normal subgroup bundle, Lemma 2.1 implies that Σ' has a Haar system only when the isotropy map $u \mapsto \Sigma(u) := \Sigma_u^u$ is continuous. Nevertheless, there are interesting examples where $\mathcal{A} \subsetneq \Sigma'$ has a Haar system even when Σ' does not. A number of such instances are given in Section 5.2.

Lemma 2.2. *Suppose that $\mathcal{A}$ is wide normal subgroup bundle of Σ with a Haar system. Consider the orbit space $\mathcal{G} = \mathcal{A}\backslash\Sigma = \Sigma/\mathcal{A}$. Define $\mathcal{G}^{(2)} = \{(\sigma_1\mathcal{A}, \sigma_2\mathcal{A}) : s(\sigma_1) = r(\sigma_2)\}$, and define a multiplication map $\mathcal{G}^{(2)} \to \mathcal{G}$ by $(\sigma_1\mathcal{A})(\sigma_2\mathcal{A}) = \sigma_1\sigma_2\mathcal{A}$. Then $\mathcal{G}$ is a locally compact groupoid, and $\mathcal{G}^{(0)} \cong \Sigma^{(0)}$ via $u\mathcal{A} \mapsto u$. Equivalently, $\mathcal{A}$ is a normal subgroup bundle of Σ if and only if we have a spectrum-fixing groupoid extension*

(2.1)

where ι is the inclusion map and p is the orbit map. To ease notation, we often write $\dot{\sigma}$ in place of $p(\sigma) = \sigma\mathcal{A}$.

In the situation of Lemma 2.2, we will use its second assertion to identify $\mathcal{G}^{(0)}$ with $\Sigma^{(0)}$ without further comment.

Remark 2.3 (Extensions). The extension in (2.1) is restricted to the case that ι is the *inclusion map*. We will look at more general types of extensions in Section 4.

Lemma 2.4. *There is a continuous homomorphism $\delta : \Sigma \to \mathbf{R}^+$, called the* modular map *for the extension* (2.1), *such that*

$$\int_{\mathcal{A}} f(\sigma a \sigma^{-1}) \, d\beta^{s(\sigma)}(a) = \delta(\sigma) \int_{\mathcal{A}} f(a) \, d\beta^{r(\sigma)}(a).$$

Note that if $a \in A(u)$, then $\delta(a) = \Delta_{A(u)}(a)$.

Remark 2.5. Note that $\delta(\sigma)$ is the inverse of the similar constant $\omega(\sigma)$ used in [MRW96], and δ has the advantage that it restricts to the modular function on each $A(u)$.

Recall that $b \in C_{c,p}(\Sigma)$ if b is continuous and $\operatorname{supp} b \cap p^{-1}(K)$ is compact for all $K \subset \mathcal{G}$ compact.

Lemma 2.6. *There exists $b \in C_{c,p}^+(\Sigma)$, called a* Bruhat section, *such that*

(2.2)
$$\int_{\mathcal{A}} b(\sigma a) \, d\beta^{s(\sigma)}(a) = 1 \quad \text{for all } \sigma \in \Sigma.$$

There is a surjection $Q : C_c(\Sigma) \to C_c(\mathcal{G})$ given by

$$Q(f)(\dot{\sigma}) = \int_{\mathcal{A}} f(\sigma a) \, d\beta^{s(\sigma)}(a).$$

Proof. By left invariance of the measures $\beta^{s(\sigma)} = \beta^{s(\dot{\sigma})}$, if $f \in C_c(\Sigma)$ and $\dot{\sigma} = \dot{\tau} \in \mathcal{G}$ then $\int_{\mathcal{A}} f(\sigma a) d\beta^{s(\sigma)}(a) = \int_{\mathcal{A}} f(\tau a) d\beta^{s(\tau)}(a)$. Hence for $f \in C_c(\Sigma)$ there is a well-defined map $\gamma \mapsto m^{\gamma}(f)$ from $\mathcal{G}$ to $\mathbf{C}$ such that

$$m^{\dot{\sigma}}(f) = \int_{\mathcal{A}} f(\sigma a) \, d\beta^{s(\sigma)}(a) \quad \text{for all } \sigma \in \Sigma.$$

The collection $m = \{m^{\gamma}\}_{\gamma \in \mathcal{G}}$ is a p-system of measures by [Wil19, Lemma 2.21(b)]. The existence of b follows from the paracompactness of $\mathcal{G}$ as in [Wil19, Proposition 3.18]. $\qquad\square$

Lemma 2.7 ([MRW96, Lemma 4.2]). *There is a Haar system $\alpha = \{\alpha_u\}_{u \in \Sigma^{(0)}}$ on $\mathcal{G}$ such that for all $f \in C_c(\Sigma)$ and $u \in \Sigma^{(0)}$ we have*

$$(2.3) \qquad \int_{\Sigma} f(\sigma) \, d\lambda^u(\sigma) = \int_{\mathcal{G}} \int_{\mathcal{A}} f(\sigma a) \, d\beta^{s(\sigma)}(a) \, d\alpha^u(\dot{\sigma}).$$

Proof. To show that a Radon measure α^u exists satisfying (2.3), it will suffice to see that whenever

$$(2.4) \qquad \int_{\mathcal{A}} f(\sigma a) \, d\beta^{s(\sigma)}(a) = 0 \quad \text{for all } \sigma \in r^{-1}(u)$$

it follows that

$$\int_{\Sigma} f(\sigma) \, d\lambda^u(\sigma) = 0.$$

(Then we can define a linear functional on $C_c(\mathcal{G})$ by $\alpha^u(Q(f)) = \lambda^u(f)$.) But if (2.4) holds, then for any $h \in C_c(\Sigma)$,

$$(2.5) \qquad \int_{\mathcal{A}} f * h(a) \, d\beta^u(a) = \int_{\mathcal{A}} \int_{\Sigma} f(a\sigma) h(\sigma^{-1}) \, d\lambda^u(\sigma) \, d\beta^u(a)$$
$$= \int_{\Sigma} \left(\delta(\sigma)^{-1} \int_{\mathcal{A}} f(\sigma a) \, d\beta^{s(\sigma)}(a) \right) h(\sigma^{-1}) \, d\lambda^u(\sigma) = 0.$$

Taking h to be a multiple of b in (2.2) by an appropriate function in $C_c(\Sigma)$, we can assume that

$$\int_{\mathcal{A}} h(\sigma^{-1} a) \, d\beta^u(a) = 1$$

for all $\sigma \in \text{supp } f$. Then the left-hand side of (2.5) is exactly

$$\int_{\Sigma} f(\sigma) \, d\lambda^u(\sigma).$$

If $F \in C_c(\mathcal{G})$, then

$$\int_{\mathcal{G}} F(\gamma) \, d\alpha^u(\gamma) = \int_{\Sigma} F(\dot{\sigma}) b(\sigma) \, d\lambda^u(\sigma).$$

Hence $u \mapsto \alpha^u(F)$ is continuous. For left-invariance, suppose that $\eta = \dot{\tau}$. Then

$$\int_{\mathcal{G}} F(\eta\gamma) \, d\alpha^{s(\eta)}(\gamma) = \int_{\Sigma} F(p(\tau\sigma))b(\sigma) \, d\lambda^{s(\tau)}(\sigma)$$

$$= \int_{\Sigma} F(p(\sigma))b(\tau^{-1}\sigma) \, d\lambda^{r(\tau)}(\sigma)$$

$$= \int_{\mathcal{G}} F(\gamma) \, d\alpha^{r(\eta)}(\gamma). \qquad \square$$

To see that Σ acts continuously on $C^*(\mathcal{A})$ by automorphisms as in [MW08b, Definition 4.1], we need to exhibit $C^*(\mathcal{A})$ as the section algebra of an upper-semicontinuous C^*-bundle as in [Wil07, Appendix C.2]. Note that $C^*(\mathcal{A}) = C^*(\mathcal{A}, \beta)$ is a $C_0(\Sigma^{(0)})$-algebra with fibres $C^*(\mathcal{A})(u)$ identified with the group C^*-algebras $C^*(A(u))$.[3]

Using [Wil07, Theorem C.26], we can equip

$$\mathscr{E} := \coprod_{u \in \Sigma^{(0)}} C^*(A(u))$$

with a topology such that it becomes an upper-semicontinuous C^*-bundle over $\Sigma^{(0)}$ such that each $f \in C_c(\mathcal{A})$ defines a continuous section in $\check{f} \in \Gamma_c(\Sigma^{(0)}; \mathscr{E})$ given by $\check{f}(u)(a) = f(a)$. Then $C^*(\mathcal{A})$ can be identified with $\Gamma_0(\Sigma^{(0)}; \mathscr{E})$.

For each $\sigma \in \Sigma$, we get an isomorphism

$$\vartheta_\sigma : C^*\big(A(s(\sigma))\big) \to C^*\big(A(r(\sigma))\big)$$

given on $h \in C_c(A(s(\sigma)))$ by

$$\vartheta_\sigma(h)(a) = \delta(\sigma)h(\sigma^{-1}a\sigma).$$

Proposition 2.8 (cf., [Goe12, Proposition 2.6]). *With the notation just established, writing $\vartheta = \{\vartheta_\sigma\}$, the triple $(\mathscr{E}, \Sigma, \vartheta)$ is a groupoid dynamical system.*

Proof. We need to verify the axioms in [MW08b, Definition 4.1]. We have already established that each ϑ_σ is an isomorphism, and it is easy to check that $\vartheta_{\sigma\tau} = \vartheta_\sigma \circ \vartheta_\tau$. It only remains to check the continuity of $(\sigma, s) \mapsto \vartheta_\sigma(s)$ from $\Sigma * \mathscr{E}$ to $\mathscr{E}$. We establish this using [MW08b, Lemma 4.3] (see also [KMRW98, Lemma 2.13]).

We must consider the pullbacks[4]

$$r^*\big(C^*(\mathcal{A})\big) = \Gamma_0(\Sigma; r^*\mathscr{E}) \quad \text{and} \quad s^*\big(C^*(\mathcal{A})\big) = \Gamma_0(\Sigma; s^*\mathscr{E}).$$

But we can clearly identify sections in $\Gamma(\Sigma; r^*\mathscr{E})$ with continuous functions $f : \Sigma \to \mathscr{E}$ such that $p_{\mathscr{E}}(f(\sigma)) = r(\sigma)$, and similarly for sections of $s^*\mathscr{E}$. Hence we can define $\vartheta : \Gamma_0(\Sigma; r^*\mathscr{E}) \to \Gamma_0(\Sigma; s^*\mathscr{E})$ by $\vartheta(f)(\sigma) = \vartheta_\sigma\big(f(\sigma)\big)$. Then ϑ is an isomorphism, and the continuity of the action follows from [MW08b, Lemma 4.3]. $\square$

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[3]Here we treat $A(u)$ as a groupoid with a single unit. This means that the modular function $\Delta_{A(u)}$ does not appear in the formula for the involution on $C_c(A(u))$.

[4]As in [RW85, Proposition 1.3], we can identify $r^*\big(C^*(\mathcal{A})\big)$ and $s^*\big(C^*(\mathcal{A})\big)$ with the usual C^*-pullbacks $C_0(\Sigma) \otimes_{C_0(\Sigma^{(0)})} C^*(\mathcal{A})$ and $C_0(\Sigma) \otimes_{C_0(\Sigma^{(0)})} C^*(\mathcal{A})$, respectively. But this is not required here.

If $t \in A(u)$, then we can define $\kappa(t) : C_c(A(u)) \to C_c(A(u))$ by

$$\kappa(t)h(a) = \delta(t)^{\frac{1}{2}}h(t^{-1}a) = \Delta_{A(u)}(t)^{\frac{1}{2}}h(t^{-1}a).$$

Since $(\kappa(t)h)^* * (\kappa(t)k) = h^* * k$, it follows that $\kappa(t)$ extends to a unitary in the multiplier algebra $M\big(C^*(A(u))\big)$. It is straightforward to check that $\kappa(tt') = \kappa(t)\kappa(t')$.

Lemma 2.9. *The action of $\mathcal{A}$ on $\mathscr{E}$ is continuous. That is, the map $(t,s) \mapsto \kappa(t)s$ is continuous from $\mathcal{A} * \mathscr{E} \to \mathscr{E}$.*

Proof. Suppose that we have a net $(t_i, s_i) \to (t_0, s_0)$ in $\mathcal{A} * \mathscr{E}$. We need to see that $\kappa(t_i)s_i \to \kappa(t_0)s_0$ in $\mathscr{E}$. Let $u_i = p_{\mathcal{A}}(t_i) = p_{\mathscr{E}}(s_i)$. Using [Wil07, Proposition C.20], it will suffice to see that given $\epsilon > 0$, there are $s_i' \to s_0'$ in $\mathscr{E}$ such that $p_{\mathscr{E}}(s_i') = u_i$, and such that we eventually have $\|s_i' - \kappa(t_i)s_i\| < \epsilon$.

Let $f \in C_c(\mathcal{A})$ be such that $\|\check{f}(u_0) - s_0\| < \epsilon$. Then we eventually have $\|\check{f}(u_i) - s_i\| < \epsilon$. Since $\mathcal{A}$ acts by unitaries, we eventually have $\|\kappa(t_i)\check{f}(u_i) - \kappa(t_i)s_i\| < \epsilon$.

Therefore it will suffice to see that $s_i' := \kappa(t_i)\check{f}(u_i) \to s_0' := \kappa(t_0)\check{f}(u_0)$ in $\mathscr{E}$. For this, we form the pullback bundle $p_{\mathcal{A}}^*\mathscr{E} = \{(t,s) \in \mathcal{A} \times \mathscr{E} : p_{\mathcal{A}}(t) = p_{\mathscr{E}}(s)\}$. If $F \in C_c(\mathcal{A} * \mathcal{A})$, then we get a section of $p_{\mathcal{A}}^*\mathscr{E}$ given by $\check{F}(t) = (t, F(t, \cdot)) \in C_c(A(p_{\mathcal{A}}(t))$. If $F(t,a) = f(t)g(a)$ for $f, g \in C_c(\mathcal{A})$, then we clearly have $\check{F} \in \Gamma(\mathcal{A}; p_{\mathcal{A}}^*\mathscr{E})$. Since finite sums of such functions are dense in the inductive-limit topology on $C_c(\mathcal{A} * \mathcal{A})$, we have $\check{F} \in \Gamma(\mathcal{A}; p_{\mathcal{A}}^*\mathscr{E})$ for all $F \in C_c(\mathcal{A} * \mathcal{A})$.

We can assume that all the t_i are in a compact neighborhood V of t_0. Then if $\phi \in C_c^+(\mathcal{A})$ is such that $\phi \equiv 1$ on V, then $F(t,a) = \phi(t)\kappa(t)f(a)$ defines an element of $C_c(\mathcal{A} * \mathcal{A})$ and $\check{F}(t_i) = (t_i, \kappa(t_i)\check{f}(u_i))$. Since projection on the second factor is continuous from $p_{\mathcal{A}}^*\mathscr{E}$ to $\mathscr{E}$, the result follows. $\square$

Since $h\kappa(t)^*(a) = \delta(t)^{\frac{1}{2}}h(at)$, it is routine to check that $\vartheta_t(s) = \kappa(t)s\kappa(t)^*$ for all $(t,s) \in \mathcal{A} * \mathscr{E}$, and that $\kappa(\sigma t\sigma^{-1}) = \overline{\vartheta}_\sigma\big(\kappa(t)\big)$ for all $(\sigma, t) \in \Sigma * \mathcal{A}$. It follows from Lemma 2.9 that κ is a twisting map for $(\mathscr{E}, \Sigma, \vartheta)$ as in Appendix C. Then as in Appendix C, we can form the Fell bundle $\mathscr{B}(\mathscr{E}, \vartheta, \kappa) = \mathscr{E}^\Sigma$ associated to the twist. Recall that $\mathscr{B}(\mathscr{E}, \vartheta, \kappa)$ is the quotient of $r^*\mathscr{E} = \{(\sigma, a) \in \Sigma \times \mathcal{A} : r(\sigma) = p_{\mathscr{E}}(a)\}$ by the $\mathcal{A}$-action $t \cdot (\sigma, a) = (t\sigma, a\kappa(t)^*)$.

We can identify $\Gamma_c(\mathcal{G}; \mathscr{B}(\mathscr{E}, \vartheta, \kappa))$ with the collection $C_c(\mathcal{G}, \Sigma, \mathscr{E}, \kappa)$ of continuous functions $f : \Sigma \to \mathscr{E}$ such that $p_{\mathscr{E}}(f(\sigma)) = r(\sigma)$ and

$$(2.6) \qquad f(a\sigma) = f(\sigma)\kappa(a)^*$$

whose support has compact image in $\mathcal{G}$. The $*$-algebra structure on $C_c(\mathcal{G}, \Sigma, \mathscr{E}, \kappa)$ is given by

$$(2.7) \qquad f * g(\sigma) = \int_{\mathcal{G}} f(\tau)\vartheta_\tau(g(\tau^{-1}\sigma)) \, d\alpha^{r(\sigma)}(\dot{\tau}) \quad \text{and} \quad f^*(\sigma) = \vartheta_\sigma(f(\sigma^{-1})^*).$$

We denote the associated Fell bundle C^*-algebra by $C^*(\mathcal{G}, \Sigma, \mathscr{E}, \kappa)$.

If $f \in C_c(\Sigma)$ and $\sigma \in \Sigma$, we let $j(f)(\sigma)$ be the element of $C_c(A(r(\sigma)))$ given by $a \mapsto \delta(\sigma)^{\frac{1}{2}}f(a\sigma)$. In particular, $j(f) \in \Gamma_c(\Sigma; r^*\mathscr{E})$ and a quick computation verifies that it satisfies (2.6) and is an element of $C_c(\mathcal{G}, \Sigma, \mathscr{E}, \kappa)$.

Lemma 2.10. *The map $f \mapsto j(f)$ is a $*$-homomorphism of $C_c(\Sigma)$ into $C_c(\mathcal{G}, \Sigma, \mathscr{E}, \kappa)$.*

Proof. Using (2.7) we have

$$j(f * g)(\sigma)(a') = \delta(\sigma)^{\frac{1}{2}} f * g(a'\sigma)$$

$$= \delta(\sigma)^{\frac{1}{2}} \int_{\Sigma} f(\tau) g(\tau^{-1} a' \sigma) \, d\lambda^{r(\sigma)}(\tau)$$

$$= \delta(\sigma)^{\frac{1}{2}} \int_{\mathcal{G}} \int_{\mathcal{A}} f(\tau a) g(a^{-1} \tau^{-1} a' \sigma) \, \beta^{s(\tau)}(a) \, d\alpha^{r(\sigma)}(\dot\tau)$$

$$= \delta(\sigma)^{\frac{1}{2}} \int_{\mathcal{G}} \delta(\tau) \int_{\mathcal{A}} f(a\tau) g(\tau^{-1} a^{-1} a' \sigma) \, d\beta^{r(\tau)}(a) \, d\alpha^{r(\sigma)}(\dot\tau)$$

$$= \int_{\mathcal{G}} \int_{\mathcal{A}} \delta(\tau)^{\frac{1}{2}} f(a\tau) \delta(\tau) \delta(\tau^{-1}\sigma)^{\frac{1}{2}} g(\tau^{-1} a^{-1} a' \sigma) \, d\beta^{r(\tau)}(a) \, d\alpha^{r(\sigma)}(\dot\tau)$$

$$= \int_{\mathcal{G}} \int_{\mathcal{A}} j(f)(\tau)(a) \vartheta_\tau \big(j(g)(\tau^{-1}\sigma) \big)(a^{-1} a') \, \beta^{r(\tau)}(a) \, d\alpha^{r(\sigma)}(\dot\tau)$$

$$= \int_{\mathcal{G}} j(f)(\tau) * \vartheta_\tau \big(j(g)(\tau^{-1}\sigma) \big)(a') \, d\alpha^{r(\sigma)}(\dot\tau)$$

which, arguing as in [Wil07, Lemma 1.108], is

$$= \left(\int_{\mathcal{G}} j(f)(\tau) * \vartheta_\tau \big(j(g)(\tau^{-1}\sigma) \big) \, d\alpha^{r(\sigma)}(\dot\tau) \right)(a').$$

Thus $j(f * g)(\sigma) = j(f) * j(g)(\sigma)$ as required.

Similarly,

$$j(f^*)(\sigma)(a) = \delta(\sigma)^{\frac{1}{2}} f^*(a\sigma) = \delta(\sigma)^{\frac{1}{2}} \overline{f(\sigma^{-1} a^{-1} \sigma \sigma^{-1})}$$

$$= \overline{\vartheta_\sigma \big(j(f)(\sigma^{-1}) \big)(a^{-1})} = \big(\vartheta_\sigma \big(j(f)(\sigma^{-1}) \big) \big)^*(a).$$

Thus $j(f^*) = j(f)^*$. □

Remark 2.11. Let $\{g_i\}$ be a net in $C_c(\mathcal{G}, \Sigma, \mathscr{E}, \kappa)$. We say that $f_i \to f$ in the inductive-limit topology if $f_i \to f$ uniformly and the supports of the f_i are all contained in some B such that $p(B)$ is compact in $\mathcal{G}$. This implies that $\check{f}_i \to \check{f}$ in the inductive-limit topology on $\Gamma_c(\mathcal{G}, \mathscr{B}(\mathscr{E}, \vartheta, \kappa))$. Now suppose that $f_i \to f$ in the inductive-limit topology on Σ so that there is a compact set $K \subset \Sigma$ such that $\operatorname{supp} f_i \subset K$ for all i and $f_i \to f$ uniformly. We claim that $j(f_i) \to j(f)$ in the inductive-limit topology on $C_c(\mathcal{G}, \Sigma, \mathscr{E}, \kappa)$. Certainly we have the supports of the $j(f_i)$ all contained in the image of K. Moreover if $\sigma \in K$, then $\operatorname{supp} j(f_i)(\sigma) \subset K^{-1} K \cap \mathcal{A}$. Since $u \mapsto \beta^u(K^{-1} K \cap \mathcal{A})$ is bounded and δ is bounded on K, it follows that $j(f_i) \to j(f)$ uniformly on K. Since $\|j(f_i)(\sigma) - j(f)(\sigma)\|$ depends only on $\dot\sigma$, the claim follows.

Theorem 2.12. *The $*$-homomorphism $j : C_c(\Sigma) \to C_c(\mathcal{G}, \Sigma, \mathscr{E}, \kappa)$ defined just before Lemma 2.10 is isometric for the (universal) C^*-norm and therefore extends to an isomorphism $j : C^*(\Sigma) \to C^*(\mathcal{G}, \Sigma, \mathscr{E}, \kappa)$. If $\mathcal{A}$ is amenable, then j is also isometric for reduced norms and extends to an isomorphism $j_r : C_r^*(\Sigma) \to C_r^*(\mathcal{G}, \Sigma, \mathscr{E}, \kappa)$.*

Remark 2.13. If $\mathcal{A}$ is not amenable, the situation is complicated. One might expect to replace $C^*(\mathcal{A})$ with $C_r^*(\mathcal{A})$ in the construction of $\mathscr{E}$ to obtain a bundle $\mathscr{E}_r$ and an isomorphism $C_r^*(\Sigma) \cong C_r^*(\mathcal{G}, \Sigma, \mathscr{E}_r, \kappa_r)$. However, as shown in [Wil15, Arm19], while $C_r^*(\mathcal{A})$ is a

$C_0(\mathcal{G}^{(0)})$-algebra, its fibre over u need not be isomorphic to $C_r^*(A(u))$. We have not pursued this subtlety further.

To prove Theorem 2.12 we will need the following technical result from [Ren91, Corollary 1.8]. We have included the details for completeness.

Lemma 2.14. *If μ is a quasi-invariant measure on $\Sigma^{(0)}$ with respect to Σ, then μ is also quasi-invariant with respect to $\mathcal{G}$. If $\underline{\Delta}$ is a modular function on $\mathcal{G}$ for μ, then $\Delta(\sigma) = \delta(\sigma)\underline{\Delta}(\dot\sigma)$ is a modular function on Σ for μ. In particular, we an assume both $\underline{\Delta}$ and Δ are homomorphisms into $\mathbf{R}^+$.*

Proof. Let $b \in C_{c,p}^+(\Sigma)$ be a Bruhat section as in Lemma 2.6. Suppose $f \in C_c(\mathcal{G})$. Then

$$
\begin{aligned}
\nu_{\mathcal{G}}(f) &= \int_{\Sigma^{(0)}} \int_{\mathcal{G}} f(\lambda)\, d\alpha^u(\lambda)\, d\mu(u) \\
&= \int_{\Sigma^{(0)}} \int_{\mathcal{G}} \int_{\mathcal{A}} f(\dot\sigma)b(\sigma a)\, d\beta^{s(\sigma)}(a)\, d\alpha^u(\dot\sigma)\, d\mu(u) \\
&= \int_{\Sigma^{(0)}} \int_{\Sigma} f(\dot\sigma)b(\sigma)\, d\lambda^u(\sigma)\, d\mu(u).
\end{aligned}
$$

So for any modular function Δ for μ on Σ, we obtain

$$
\nu_{\mathcal{G}}(f) = \int_{\Sigma^{(0)}} \int_{\Sigma} f(\dot\sigma^{-1})b(\sigma^{-1})\Delta(\sigma^{-1})\, d\lambda^u(\sigma)\, d\mu(u)
$$

$$
(2.8) \qquad = \int_{\Sigma^{(0)}} \int_{\mathcal{G}} f(\dot\sigma^{-1}) \int_{\mathcal{A}} b(a^{-1}\sigma^{-1})\Delta(a^{-1}\sigma^{-1})\, d\beta^{s(\sigma)}(a)\, d\alpha^u(\dot\sigma)\, d\mu(u).
$$

Define $B : \mathcal{G} \to \mathbf{C}$ by

$$
B(\dot\sigma) = \int_{\mathcal{A}} b(a^{-1}\sigma^{-1})\Delta(a^{-1}\sigma^{-1})\, d\beta^{s(\sigma)}(a).
$$

Then (2.8) gives

$$
\nu_{\mathcal{G}}(f) = \int_{\Sigma^{(0)}} \int_{\mathcal{G}} f(\gamma^{-1})B(\gamma)\, d\sigma^u(\gamma)\, d\mu(u) = \nu_{\mathcal{G}}^{-1}(fB^*).
$$

Since δ agrees with $\Delta_{\mathcal{A}(u)}$, we have

$$
\begin{aligned}
B(\dot\sigma) &= \delta(\sigma)^{-1} \int_{\mathcal{A}} b(\sigma^{-1}a^{-1})\Delta(\sigma^{-1}a^{-1})\, d\beta^{r(\sigma)}(a) \\
&= \int_{\mathcal{A}} b(\sigma^{-1}a)\Delta(\sigma^{-1}a)\delta(\sigma^{-1}a^{-1})\, d\beta^{r(\sigma)}(a).
\end{aligned}
$$

Since Δ and δ never vanish, it follows from (2.2) that B never vanishes. Hence $\nu_{\mathcal{G}}$ and $\nu_{\mathcal{G}}^{-1}$ are equivalent. Thus, by definition, μ is $\mathcal{G}$-quasi-invariant.

Let $\underline{\Delta}$ be a modular function for $\mathcal{G}$ (with respect to μ). Then

$$\int_{\Sigma^{(0)}} \int_{\Sigma} f(\sigma^{-1}) \underline{\Delta}(\dot{\sigma}^{-1}) \delta(\sigma)^{-1} \, d\lambda^u(\sigma) \, d\mu(u)$$

$$= \int_{\Sigma^{(0)}} \int_{\mathcal{G}} \underline{\Delta}(\dot{\sigma}^{-1}) \delta(\sigma)^{-1} \int_{\mathcal{A}} f(a^{-1}\sigma^{-1}) \delta(a)^{-1} \, d\beta^{s(\sigma)}(a) \, d\alpha^u(\dot{\sigma}) \, d\mu(u)$$

$$= \int_{\Sigma^{(0)}} \int_{\mathcal{G}} Q(f)(\dot{\sigma}^{-1}) \underline{\Delta}(\dot{\sigma}^{-1}) \, d\alpha^u(\dot{\sigma}) \, d\mu(u)$$

$$= \int_{\Sigma^{(0)}} \int_{\mathcal{G}} Q(f)(\dot{\sigma}) \, d\alpha^u(\dot{\sigma}) \, d\mu(u)$$

$$= \int_{\Sigma^{(0)}} \int_{\Sigma} f(\sigma) \, d\lambda^u(\sigma) \, d\mu(u).$$

Hence $\underline{\Delta}(\dot{\sigma})\delta(\sigma)$ is a modular function for Σ. Work of Ramsay—see [Wil19, Proposition 7.6]—implies we can take $\underline{\Delta}$ to be a homomorphism. Then we can let $\Delta(\sigma) = \underline{\Delta}(\dot{\sigma})\delta(\sigma)$, and we're done. $\qquad\square$

Proof of Theorem 2.12. Suppose that $f_i \to f$ in the inductive-limit topology on $C_c(\Sigma)$. Then as in Remark 2.11, $j(f_i) \to j(f)$ in the inductive-limit topology on $C_c(\mathcal{G}, \Sigma, \mathscr{E}, \kappa)$. Thus if $\underline{L}$ is a nondegenerate representation of $\Gamma_c(\mathcal{G}; \mathscr{B}(\mathscr{E}, \vartheta, \kappa))$, then $\underline{L} \circ j$ is continuous in the inductive-limit topology on $C_c(\Sigma)$ and therefore bounded with respect to the C^*-norm:

$$\|\underline{L}(j(f))\| \leq \|f\| \quad \text{for all } f \in C_c(\Sigma).$$

Since $\underline{L}$ is arbitrary, j extends to a homomorphism $j : C^*(\Sigma) \to C^*(\mathcal{G}, \Sigma, \mathscr{E}, \kappa)$.

Proving that j is isometric for universal norms requires considerably more work. The idea is straightforward. Given a nondengenerate representation L of $C_c(\Sigma)$, it will suffice to produce a representation $\underline{L}$ of $\Gamma_c(\mathcal{G}; \mathscr{B}(\mathscr{E}, \vartheta, \kappa))$ such that $L(f) = \underline{L}(j(f))$. Applying this to a faithul L will show that j is isometric.

We can assume that L is the integrated form of a unitary representation $(\mu, \Sigma^{(0)} * \mathscr{H}, \hat{L})$ where μ is a quasi-invariant measure on $\Sigma^{(0)}$, $\Sigma^{(0)} * \mathscr{H}$ is a Borel Hilbert bundle over $\Sigma^{(0)}$, and $\hat{L}$ is a groupoid homomorphism of Σ into $\mathrm{Iso}(\Sigma^{(0)} * \mathscr{H})$ of the form $\hat{L}(\sigma) = (r(\sigma), L_\sigma, s(\sigma))$. By Lemma 2.14, μ is quasi-invariant with respect to $\mathcal{G}$ and we may assume that Δ is given by $\Delta(\sigma) = \delta(\sigma)\underline{\Delta}(\dot{\sigma})$ where δ is given by Lemma 2.4, and $\underline{\Delta}$ is a modular function for $\mathcal{G}$, and we may further assume that Δ and $\underline{\Delta}$ are homomorphisms.

We are going to realize $\underline{L}$ as the integrated form of a Borel $*$-functor $\hat{\pi} : \mathscr{B}(\mathscr{E}, \vartheta, \kappa) \to \mathrm{End}(\Sigma^{(0)} * \mathscr{H})$ as in [MW08a, Definition 4.5 and Proposition 4.10]. To start, consider $\sigma \in \Sigma$ and $h \in C_c(A(r(\sigma)))$. Define $\bar{\pi}_\sigma(h) : \mathcal{H}(s(\sigma)) \to \mathcal{H}(r(\sigma))$ by

$$\bar{\pi}_\sigma(h)\xi = \int_{\mathcal{A}} h(a) L_{a\sigma}(\xi) \delta(a)^{-\frac{1}{2}} \, d\beta^{r(\sigma)}(a).$$

It is routine to check that

$$(2.9) \qquad\qquad \bar{\pi}_{t\sigma}(h) = \bar{\pi}_\sigma(h\kappa(t)) \quad \text{for } (t, \sigma) \in \mathcal{A} * \Sigma.$$

Since

$$\big(\bar{\pi}_\sigma(h)\xi \mid \eta\big) = \int_{\mathcal{A}} h(a)\delta(a)^{-\frac{1}{2}}\big(L_{a\sigma}(\xi) \mid \eta\big)\, d\beta^{r(\sigma)},$$

it follows from the usual Cauchy–Schwarz estimate that

$$\big|\big(\bar{\pi}_\sigma(h)\xi \mid \eta\big)\big|^2 \leq \|\xi\|^2\|\eta\|^2\Big(\int_{\mathcal{A}} |h(a)|\delta(a)^{-\frac{1}{2}}\Big)^2 \leq \|\xi\|^2\|\eta\|^2\|h\|_{I,r}\|h\|_{I,s} \leq \|\xi\|^2\|\eta\|^2\|h\|_I^2.$$

Therefore $\bar{\pi}_\sigma$ extends to all of $C^*(A(r(\sigma)))$ and still satisfies (2.9). Hence we get a map $\pi : \mathscr{B}(\mathscr{E}, \vartheta, \kappa) \to \operatorname{End}(\Sigma^{(0)} * \mathscr{H})$ defined by

$$\pi([\sigma, s]) = \bar{\pi}_\sigma(s).$$

It is not hard to check that π determines a $*$-functor $\hat{\pi}(b) = (r(b), \pi(b), s(b))$. To see that $\hat{\pi}$ is Borel we need to see that

$$\dot\sigma \mapsto \big(\pi(\check{f}(\dot\sigma))(h(s(\sigma))) \mid k(r(\sigma))\big)$$

is Borel for all $\check{f} \in \Gamma(\mathcal{G}, \mathscr{B}(\mathscr{E}, \vartheta, \kappa)))$ and all Borel sections h and k of $\Sigma^{(0)} * \mathscr{H}$. By Lemma B.5, we can assume that $\check{f}$ is defined by $f \in C_c(\mathcal{G}, \Sigma, \mathscr{E}, \kappa)$ with $f(\sigma) \in C_c(A(r(\sigma)))$ for all $\sigma \in \Sigma$. Then

$$\big(\pi(\check{f}(\dot\sigma))(h(s(\sigma))) \mid k(r(\sigma))\big) = \big(\pi_\sigma(f(\sigma))(h(s(\sigma))) \mid k(r(\sigma))\big)$$
$$= \int_{\mathcal{A}} f(\sigma)(a)\big(L_{a\sigma}(h(s(\sigma))) \mid k(r(\sigma))\big)\delta(a)^{-\frac{1}{2}}\, d\beta^{r(\sigma)}(a),$$

which is Borel since L is. Let $\underline{L}$ be the integrated form of $\hat{\pi}$.

Take $f \in C_c(\Sigma)$ and $\xi \in L^2(\Sigma^{(0)} * \mathscr{H}, \mu)$. Then

$$L(f)\xi(u) = \int_\Sigma f(\sigma)L_\sigma\big(\xi(s(\sigma))\big)\Delta(\sigma)^{-\frac{1}{2}}\, d\lambda^u(\sigma)$$
$$= \int_{\mathcal{G}} \int_{\mathcal{A}} f(\sigma a)L_{\sigma a}\big(\xi(s(\sigma))\big)\Delta(\sigma a)^{-\frac{1}{2}}\, d\beta^{s(\sigma)}(a)\, d\alpha^u(\dot\sigma)$$
$$= \int_{\mathcal{G}} \delta(\sigma)\int_{\mathcal{A}} f(a\sigma)L_{a\sigma}\big(\xi(s(\sigma))\big)\Delta(a\sigma)^{-\frac{1}{2}}\, d\beta^u(a)\, d\alpha^u(\dot\sigma)$$
$$= \int_{\mathcal{G}} \delta(\sigma)^{\frac{1}{2}}\int_{\mathcal{A}} j(f)(\sigma)(a)L_{a\sigma}\big(\xi(s(\sigma))\big)\underline{\Delta}(\dot\sigma)^{-\frac{1}{2}}\delta(a\sigma)^{-\frac{1}{2}}\, d\beta^u(a)\, d\alpha^u(\dot\sigma)$$
$$= \int_{\mathcal{G}} \pi(j(f(\gamma)))(h(s(\gamma)))\underline{\Delta}(\gamma)^{-\frac{1}{2}}(\gamma)$$
$$= \underline{L}(j(f))\xi(u).$$

This shows that j is isometric.

It only remains to see that j is surjective. Let $\check{F} \in \Gamma_c(\mathcal{G}; \mathscr{B}(\mathscr{E}, \vartheta, \kappa))$ and $\epsilon > 0$. Given $\sigma \in \Sigma$, there exists $f_\sigma \in C_c(\Sigma)$ such that

$$\|j(f_\sigma)(\sigma) - F(\sigma)\| < \epsilon.$$

Since $r^*\mathscr{E}$ is an upper-semicontinuous Banach bundle and κ is unitary-valued, there is an open neighborhood, $V_{\dot\sigma}$ of $\dot\sigma$ in $\mathcal{G}$ such that

$$\|j(f_\sigma)(\tau) - F(\tau)\| < \epsilon \quad \text{for all } \dot\tau \in V_{\dot\sigma}.$$

Fix $\sigma_1, \ldots, \sigma_n \in \Sigma$ such that the $V_{\dot\sigma_k}$ cover $p(\operatorname{supp} F)$. Let $\{\phi_k\} \subset C_c^+(\mathcal{G})$ be such that $\operatorname{supp} \phi_k \subset V_{\dot\sigma_k}$ and

$$\sum_{k=1}^n \phi_k(\dot\tau) = 1$$

if $\tau \in p(\operatorname{supp} F)$ and bounded by 1 otherwise. Then

$$f(\sigma) = \sum_{k=1}^n \phi_k(\dot\sigma) f_{\sigma_k}(\sigma)$$

belongs to $C_c(\Sigma)$, and

$$\|j(f)(\sigma) - F(\sigma)\| \leq \sum_{k=1}^n \big\|\phi_k(\dot\sigma)\big(j(f_{\sigma_k})(\sigma) - F(\sigma)\big)\big\| < \epsilon \sum_{k=1}^n \phi_k(\dot\sigma) \leq \epsilon.$$

Now it is straightforward to see that $j(C_c(\Sigma))$ is dense in $\Gamma_c(\mathcal{G}; \mathscr{B}(\mathscr{E}, \vartheta, \kappa))$ in the inductive-limit topology. This finishes the proof that j extends to an isomorphism $j : C^*(\Sigma) \to C^*(\mathcal{G}, \Sigma, \mathscr{E}, \kappa)$.

It remains to prove that j is isometric for reduced norms when $\mathcal{A}$ is amenable. First, note that by [Hol17, Example 3.14] the space $C_c(\Sigma)$ completes to a $C^*(\Sigma)$–$C^*(\mathcal{A})$ correspondence $X(\Sigma)$ with actions given by convolution and inner product $\langle f, g \rangle_{C^*(\mathcal{A})} = (f^* * g)|_{\mathcal{A}}$. Let $\mathscr{B}$ denote the Fell bundle $\mathscr{B}(\mathscr{E}, \vartheta, \kappa)$ described above so that $C^*(\mathcal{G}, \Sigma, \mathscr{E}, \kappa)$ is the C^*-algebra $C^*(\mathcal{G}, \mathscr{B})$ of this bundle, and similarly for reduced C^*-algebras. Let $\mathcal{H}(\mathscr{B})$ be the right-Hilbert $C^*(\mathcal{A})$-module of Section 1.2, so that the left action of $C_c(\mathcal{G}; \mathscr{B})$ on $\mathcal{H}(\mathscr{B})$ determined by multiplication in $C_c(\mathcal{G}; \mathscr{B})$ is isometric for the reduced norm. By construction of these maps, the map $j : C_c(\Sigma) \to C^*(\mathcal{G}, \Sigma, \mathscr{E}, \kappa)$ extends to a right-$C^*(\mathcal{A})$-module isomorphism $\rho : X(\Sigma) \to \mathcal{H}(\mathscr{B})$, which satisfies

(2.10) $$\rho(f \cdot \xi) = j(f) \cdot \rho(\xi) \quad \text{for } f \in C_c(\Sigma) \text{ and } \xi \in X(\Sigma).$$

Let $\mathcal{H}(\Sigma)$ and $\mathcal{H}(\mathcal{A})$ be the Hilbert modules described in Section 1.2 that carry faithful representations of $C_r^*(\Sigma)$ and $C_r^*(\mathcal{A})$ respectively. We can form the modules $X(\Sigma) \otimes_{C^*(\mathcal{A})} \mathcal{H}(\mathcal{A})$ and $\mathcal{H}(\mathscr{B}) \otimes_{C^*(\mathcal{A})} \mathcal{H}(\mathcal{A})$, and the isomorphism $\rho : X(\Sigma) \to \mathcal{H}(\mathscr{B})$ defined above determines an isomorphism $\rho \otimes \operatorname{id} : X(\Sigma) \otimes_{C^*(\mathcal{A})} \mathcal{H}(\mathcal{A}) \to \mathcal{H}(\mathscr{B}) \otimes_{C^*(\mathcal{A})} \mathcal{H}(\mathcal{A})$ which again intertwines the left actions similarly to (2.10). Since the bundle $\mathcal{A}$ is an amenable groupoid, the left action of $C^*(\mathcal{A}) = C_r^*(\mathcal{A})$ on $\mathcal{H}(\mathcal{A})$ is faithful, and it follows that the map $T \mapsto T \otimes 1$ from $\mathcal{L}(\mathcal{H}(\mathscr{B}))$ to $\mathcal{L}(\mathcal{H}(\mathscr{B}) \otimes_{C^*(\mathcal{A})} \mathcal{H}(A))$ is isometric. So for $f \in C_c(\Sigma)$, we have

$$\|\rho(f) \otimes 1\|_{\mathcal{L}(\mathcal{H}(\mathscr{B}) \otimes_{C^*(\mathcal{A})} \mathcal{H}(A))} = \|\rho(f)\|_{C_r^*(\mathcal{G}, \Sigma, \mathscr{E}, \kappa)}.$$

We will show that the map $f \otimes a \mapsto f \cdot a$ extends to an isomorphism $X(\Sigma) \otimes_{C^*(\mathcal{A})} \mathcal{H}(A)$ to $\mathcal{H}(\Sigma)$. This will complete the proof since then

$$\|f\|_{C_r^*(\Sigma)} = \|f \otimes 1\|_{\mathcal{L}(X(\Sigma) \otimes_{C^*(\mathcal{A})} \mathcal{H}(A))} = \|\rho(f) \otimes 1\|_{\mathcal{L}(\mathcal{H}(\mathscr{B}) \otimes_{C^*(\mathcal{A})} \mathcal{H}(A))} = \|\rho(f)\|_{C_r^*(\mathcal{G}, \Sigma, \mathscr{E}, \kappa)}$$

for all $f \in C_c(\Sigma)$.

To see that $f \otimes a \mapsto f \cdot a$ extends to the desired isomorphism, we fix $f, g \in C_c(\Sigma)$ and $a, b \in C_c(\mathcal{A})$, and calculate:

$$\langle f \otimes a, g \otimes b \rangle_{C_0(\Sigma^{(0)})} = \left(a^* * (f^* * g)|_{\mathcal{A}} * b \right)\big|_{C_0(\Sigma^{(0)})} \quad \text{and}$$
$$\langle f \cdot a, g \cdot b \rangle_{C_0(\Sigma^{(0)})} = \left(a^* * (f^* * g) * b \right)\big|_{C_0(\Sigma^{(0)})}.$$

For any $h \in C_c(\Sigma)$, and any $x \in \Sigma^{(0)}$, we have

$$(a^* * h * b)(x) = \int_{\Sigma} \int_{\Sigma} a^*(\beta) h(\gamma) b((\beta\gamma)^{-1}x) \, d\lambda^x(\beta) \, d\lambda^{s(\beta)}(\gamma).$$

Since $a, b \in C_c(\mathcal{A})$, the integrand is nonzero only when $\beta, \beta^{-1}\gamma \in \mathcal{A}_x$, and this forces $\gamma \in \mathcal{A}_x$. So

$$\left(a^* * (f^* * g) * b \right)\big|_{\Sigma^{(0)}} = \left(a^* * (f^* * g)|_{\mathcal{A}} * b \right)\big|_{\Sigma^{(0)}}.$$

This completes the proof of Theorem 2.12. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

3. The Abelian Case

In this section, we specialize to the case where $\mathcal{A}$ is a normal *abelian* subgroup bundle of Σ with a Haar system β as above. Then $C^*(\mathcal{A})$ is a commutative C^*-algebra. If $\hat{\mathcal{A}}$ is the Gelfand dual space of nonzero complex homomorphisms $\chi : C^*(\mathcal{A}) \to \mathbf{C}$, then the Gelfand transform $\mathcal{F} : C^*(\mathcal{A}) \to C_0(\hat{\mathcal{A}})$ is an isomorphism of $C^*(\mathcal{A})$ onto $C_0(\hat{\mathcal{A}})$. As usual, we write $\hat{f} = \mathcal{F}(f)$ for all $f \in C^*(\mathcal{A})$. As shown in [MRW96, Corollary 3.4], the Gelfand dual $\hat{\mathcal{A}}$ is an abelian group bundle $\hat{\pi} : \hat{\mathcal{A}} \to \Sigma^{(0)}$ with fibres $(A(u))\hat{\ } := \hat{A}(u)$. If $\chi \in \hat{\mathcal{A}}$, the corresponding complex homomorphism on $C_c(\mathcal{A})$ is given by

$$(3.1) \qquad\qquad \chi(f) = \int_{A(\hat{\pi}(\chi))} f(a) \overline{\chi(a)} \, d\beta^{\hat{\pi}(\chi)}(a).$$

(The complex conjugate appearing on the right-hand side of (3.1) is included to match up with our prejudice for the form of the Fourier transform.) If $f \in C_c(\mathcal{A})$, then

$$\hat{f}(\chi) = \int_{A(\hat{\pi}(\chi))} f(a) \overline{\chi(a)} \, \beta^{\hat{\pi}(\chi)}(a) = \chi(f).$$

Since $\mathcal{A}$ is abelian in this section, $\delta(\sigma)$ depends only on $\dot{\sigma}$ and each β^u is bi-invariant. The right action of Σ on $\hat{\mathcal{A}}$ given by

$$\chi \cdot \sigma(a) := \chi(\sigma a \sigma^{-1})$$

factors through a right action of $\mathcal{G}$ on $\hat{\mathcal{A}}$. So we may form the action groupoid $\hat{\mathcal{A}} \rtimes \mathcal{G}$ for the action of $\mathcal{G}$ on the *space* $\hat{\mathcal{A}}$. We can equip $\hat{\mathcal{A}} \rtimes \mathcal{G}$ with the Haar system $\underline{\alpha} = \{\underline{\alpha}^\chi\}_{\chi \in \hat{\mathcal{A}}} = \{\delta_\chi \times \alpha^{\hat{\pi}(\chi)}\}$.

3.1. **The associated T-groupoid.** We want to build a **T**-groupoid associated to Σ just as in [MRW96, §4] except that there $\mathcal{G}$ was assumed to be principal. We start by defining

$$\mathcal{D} = \{\, (\chi, z, \sigma) \in \hat{A} \times \mathbf{T} \times \Sigma : \hat{\pi}(\chi) = r(\sigma) \,\}.$$

We can make $\mathcal{D}$ into a locally compact Hausdorff groupoid by identifying it with $\hat{A} \rtimes \Sigma \times \mathbf{T}$. Thus,

$$(\chi, z_1, \sigma_1)(\chi \cdot \sigma_1, z_2, \sigma_2) = (\chi, z_1 z_2, \sigma_1 \sigma_2) \quad \text{and} \quad (\chi, z, \sigma)^{-1} = (\chi \cdot \sigma, \bar{z}, \sigma^{-1}).$$

We can identify $\mathcal{D}^{(0)}$ with $\hat{A}$, and then

$$r(\chi, z, \sigma) = \chi \quad \text{and} \quad s(\chi, z, \sigma) = \chi \cdot \sigma.$$

Let H be the subgroupoid of $\mathcal{D}$ consisting of triples of the form $(\chi, \overline{\chi(a)}, a)$ for $a \in A(\hat{\pi}(\chi))$. Note that if $(\chi_n, \overline{\chi_n(a_n)}, a_n) \to (\chi, z, \sigma)$ in $\mathcal{D}$, then $a_n \to \sigma$ and $\sigma = a \in A(\hat{\pi}(\chi))$ since $\mathcal{A}$ is closed in Σ. Then $\chi_n(a_n) \to \chi(a)$ by [MRW96, Proposition 3.3]. Hence H is closed in $\mathcal{D}$ with $H^{(0)} = \mathcal{D}^{(0)}$. To see that H has an open range map, we use Fell's Criterion.[5] Since we will use this frequently, we recall the formal statement.

Remark 3.1 (Fell's Criterion). Recall from [FD88a, Proposition II.13.2] that a surjection $f : X \to Y$ is an open map if and only if for every $x \in X$ and every net y_α converging to $f(x)$ in Y there is a subnet y_{α_j} and a net x_{α_j} in Y such that $f(x_{\alpha_j}) = y_{\alpha_j}$ for all j, and $x_{\alpha_j} \to x$.

So suppose that $\chi_n \to \chi = r(\chi, \overline{\chi(a)}, a)$ in $\hat{A} \times \Sigma^{(0)}$. Since $p : \mathcal{A} \to \Sigma^{(0)}$ is open, we can pass to a subnet, relabel, and assume that there are $a_n \to a$ in $\mathcal{A}$ such that $p(a_n) = \hat{\pi}(\chi_n)$. Then $(\chi_n, \overline{\chi_n(a_n)}, a_n) \to (\chi, \overline{\chi(a)}, a)$ in $\mathcal{D}$.

We have now showed that H has open range and source maps. Hence the quotient map $q : D \to D/H$ is open. Furthermore, if $d \in \mathcal{D}$, then it is not hard to see that $dH = Hd$. Thus, as in Lemma 2.2, we can form the locally compact Hausdorff groupoid $\widehat{\Sigma} := \mathcal{D}/H$, and the elements of $\widehat{\Sigma}$ are given by triples $[\chi, z, \sigma]$ where $[\chi, z, a\sigma] = [\chi, \chi(a)z, \sigma]$ for all $a \in \mathcal{A}$. Thus we can define maps $i : (\hat{A} \times \mathbf{T}) \to \widehat{\Sigma}$ and $j : \widehat{\Sigma} \to \hat{A} \rtimes \mathcal{G}$ by

$$i(\chi, z) = [\chi, z, \hat{\pi}(\chi)] \quad \text{and} \quad j([\chi, z, \sigma]) = (\chi, \dot{\sigma}).$$

Proposition 3.2. *With respect to the maps i and j above, $\widehat{\Sigma}$ is a **T**-groupoid over $\hat{A} \rtimes \mathcal{G}$:*

(3.2)

Proof. The map i is clearly injective. Suppose that $i(\chi_n, z_n) = [\chi_n, z_n, \hat{\pi}(\chi_n)] \to [\chi, z, \sigma]$. Since $q : \mathcal{D} \to \widehat{\Sigma}$ is open, we may assume that there exist $a_n \to a$ in $\mathcal{A}$ such that $(\chi_n, \overline{\chi_n(a_n)}z_n, a_n) \to (\chi, z, \sigma)$. But then $\sigma = a$ and $z_n \to \chi(a)z$. Thus $[\chi, z, \sigma] = [\chi, z, a] =$

---

[5]Alternatively, we could use β to equip H with a Haar system.

$[\chi, \chi(a)z, \hat{\pi}(\chi)] \in i(\hat{\mathcal{A}} \times \mathbf{T})$ and i has closed range. Replacing σ by $\hat{\pi}(\chi)$ in the above shows that $z_n \to z$ and it follows that i is a homeomorphism onto its range as required.

If $j([\chi, z, \sigma])$ is a unit, then $\dot{\sigma} = \hat{\pi}(\chi)$ and $\sigma = a \in A(\hat{\pi}(\chi))$. But then

$$[\chi, z, a] = [\chi, \chi(a)z, \hat{\pi}(\chi)] \in i(\hat{\mathcal{A}} \times \mathbf{T}).$$

In other words, (3.2) is "exact" at $\hat{\Sigma}$.

We still need to check that j is open. Again, we employ Fell's criterion (see Remark 3.1). Suppose that $(\chi_n, \gamma_n) \to (\chi, \dot{\sigma}) = [\chi, z, \sigma]$ in $\hat{\mathcal{A}} \rtimes \mathcal{G}$. Since p is open, we can pass to a subnet, relabel, and assume that there are $\sigma_n \to \sigma$ such that $p(\sigma_n) = \gamma_n$. But then $(\chi_n, z, \sigma_n) \to (\chi, z, \sigma)$. This suffices. $\qquad\square$

Given our Haar system on $\hat{\mathcal{A}} \rtimes \mathcal{G}$, we can build the restricted C^*-algebra $C^*(\hat{\mathcal{A}} \rtimes \mathcal{G}; \hat{\Sigma})$. Recall that this C^*-algebra is built from functions $\tilde{F} \in C_c(\hat{\Sigma})$ such that

$$z'\tilde{F}([\chi, z, \sigma]) = \tilde{F}([\chi, z'z, \sigma]) \quad \text{for all } z' \in \mathbf{T}.$$

Also recall that we write $C_c(\hat{\mathcal{A}} \rtimes \mathcal{G}; \hat{\Sigma})$ for the space of all such functions. To make the notation easier to work with, notice that any $\tilde{F} \in C_c(\hat{\mathcal{A}} \rtimes \mathcal{G}; \hat{\Sigma})$ is determined by its values on classes of the form $[\chi, 1, \sigma]$. Hence we identify $C_c(\hat{\mathcal{A}} \rtimes \mathcal{G}; \hat{\Sigma})$ with the collection $C_c^{\hat{\Sigma}}(\hat{\mathcal{A}} \rtimes \Sigma)$ of continuous functions F on $\hat{\mathcal{A}} \rtimes \Sigma$ such that

$$F(\chi, a\sigma) = \chi(a)F(\chi, \sigma) \quad \text{for all } a \in A(r(\sigma))$$

and such that the support of F has compact image in $\hat{\mathcal{A}} \rtimes \mathcal{G}$.

If $F, G \in C_c^{\hat{\Sigma}}(\hat{\mathcal{A}} \rtimes \Sigma)$, then

$$F * G(\chi, \sigma) = \int_{\mathcal{G}} F(\chi, \tau)G(\chi \cdot \tau, \tau^{-1}\sigma) \, d\alpha^{r(\sigma)}(p(\tau))$$

and

$$F^*(\chi, \sigma) = \overline{F(\chi \cdot \sigma, \sigma^{-1})}.$$

3.2. **The isomorphism.** The Gelfand transform gives us an isomorphism of $C^*(\mathcal{A}) = \Gamma_0(\Sigma^{(0)}; \mathscr{E})$ onto $C_0(\hat{\mathcal{A}}) = \Gamma_0(\Sigma^{(0)}; \hat{\mathscr{E}})$ where $\hat{\mathscr{E}} = \coprod C_0(\hat{A}(u))$ is the bundle described just before (3.1).

Our constructions in the previous section give us a dynamical system $(\hat{\mathscr{E}}, \Sigma, \hat{\vartheta})$ where $\hat{\vartheta}_\sigma : C_0(\hat{A}(s(\sigma))) \to C_0(\hat{A}(r(\sigma)))$ is given by

$$\hat{\vartheta}_\sigma(\hat{h})(\chi) = \hat{h}(\chi \cdot \sigma).$$

The corresponding left-action of $\mathcal{A}$ on $r^*\hat{\mathscr{E}}$ is determined on $\hat{h} \in C_0(\hat{A}(u))$ by

$$\hat{\kappa}(t)\hat{h}(\chi) = \overline{\chi(t)}\hat{h}(\chi).$$

We form the Fell bundle $\hat{\mathscr{B}}(\hat{\mathscr{E}}, \hat{\vartheta}, \hat{\kappa}) = \mathcal{A} \backslash r^*\hat{\mathscr{E}}$ for the twist $\hat{\kappa}$ on $(\hat{\mathscr{E}}, \Sigma, \hat{\vartheta})$. Sections $\check{g} \in \Gamma_c(\mathcal{G}; \hat{\mathscr{B}}(\hat{\mathscr{E}}, \hat{\vartheta}, \hat{\kappa}))$ are determined by $g \in C_c(\mathcal{G}, \Sigma, \hat{\mathscr{E}}, \hat{\kappa})$ where $g : \Sigma \to \hat{\mathscr{E}}$ is continuous, and

satisfies

$$g(t\sigma)(\chi) = \chi(t)g(\sigma)(\chi)$$

for all $(t, \sigma) \in \mathcal{A} * \Sigma$, and has support with compact image in $\mathcal{G}$.

From Theorem 2.12, we get isomorphisms $C^*(\Sigma) \to C^*(\mathcal{G}; \hat{\mathscr{B}}(\hat{\mathscr{E}}, \hat{\vartheta}, \hat{\kappa}))$ and $C_r^*(\Sigma) \to C_r^*(\mathcal{G}; \hat{\mathscr{B}}(\hat{\mathscr{E}}, \hat{\vartheta}, \hat{\kappa}))$ that send $f \in C_c(\Sigma)$ to the section $\hat{j}(f)$ given by $\hat{j}(\sigma) = (j(f)(\sigma))\hat{}$. Hence

$$\hat{j}(f)(\sigma)(\chi) = \delta(\sigma)^{\frac{1}{2}} \int_{\mathcal{A}} f(a\sigma)\overline{\chi(a)} \, d\beta^{r(\sigma)}(a).$$

Let $C_\infty^{\hat{\Sigma}}(\hat{\mathcal{A}} \rtimes \Sigma)$ be the collection of all $f \in C_0(\hat{\mathcal{A}} *_r \Sigma)$ such that there is a compact set $K \subset \mathcal{G}$ such that $f(\chi, \sigma) = 0$ if $\dot{\sigma} \notin K$. Since $r^*(C_0(\hat{\mathcal{A}})) \cong C_0(\Sigma *_r \hat{\mathcal{A}})$, we obtain a one-to-one correspondence between $C_\infty^{\hat{\Sigma}}(\hat{\mathcal{A}} \rtimes \Sigma)$ and $C_c(\mathcal{G}, \Sigma, \hat{\mathscr{E}}, \hat{\kappa})$ that carries $f \in C_\infty^{\hat{\Sigma}}(\hat{\mathcal{A}} \rtimes \Sigma)$ to the element $F_f \in C_c(\mathcal{G}, \Sigma, \hat{\mathscr{E}}, \hat{\kappa})$ given by

$$F_f(\sigma)(\chi) = f(\chi, \sigma).$$

Proposition 3.3. *The map $f \mapsto F_f$ is a $*$-isomorphism of $C_c^{\hat{\Sigma}}(\hat{\mathcal{A}} \rtimes \Sigma)$ into $C_c(\mathcal{G}, \Sigma, \hat{\mathscr{E}}, \hat{\kappa})$ which extends to isomorphisms*

$$C^*(\hat{\mathcal{A}} \rtimes \mathcal{G}; \hat{\Sigma}) \cong C^*(\mathcal{G}, \Sigma, \hat{\mathscr{E}}, \hat{\kappa}) \quad and \quad C_r^*(\hat{\mathcal{A}} \rtimes \mathcal{G}; \hat{\Sigma}) \cong C_r^*(\mathcal{G}, \Sigma, \hat{\mathscr{E}}, \hat{\kappa}).$$

Proof. We first prove the isomorphism of full C^*-algebras. If $\chi \in \hat{A}(r(\sigma))$, then since evaluation at χ passes through the integral,

$$\begin{aligned}
F_f * F_g(\sigma)(\chi) &= \int_{\mathcal{G}} F_f(\tau)(\chi)\hat{\vartheta}_\tau(F_g(\tau^{-1}\sigma)(\chi) \, d\alpha^{r(\sigma)}(\dot{\tau}) \\
&= \int_{\mathcal{G}} F_f(\tau)(\chi)F_g(\tau^{-1}\sigma)(\chi \cdot \tau) \, d\alpha^{r(\sigma)}(\dot{\tau}) \\
&= \int_{\mathcal{G}} f(\chi, \tau)g(\chi \cdot \tau, \tau^{-1}\sigma) \, d\alpha^{r(\sigma)}(\dot{\tau}).
\end{aligned}$$

Hence $F_{f*g} = F_f * F_g$. A similar computation shows that $F_{f^*} = F_f^*$. Thus $f \mapsto F_f$ is a $*$-isomorphism onto its range.

The $\|\cdot\|_I$-norm on $C_c(\mathcal{G}, \Sigma, \hat{\mathscr{E}}, \hat{\kappa})$ is given by

$$\|F_f\|_I = \max\big\{\|F_f\|_{I,r}, \|F_f\|_{I,s}\big\},$$

where

$$\|F_f\|_{I,r} = \sup_{u \in \Sigma^{(0)}} \int_{\mathcal{G}} \|F_f(\sigma)\|_\infty \, d\alpha^u(\dot{\sigma}) \quad \text{and} \quad \|F_f\|_{I,s} = \sup_{u \in \Sigma^{(0)}} \int_{\mathcal{G}} \|F_f(\sigma)\|_\infty \, d\alpha_u(\dot{\sigma}).$$

The set $\{F_f : f \in C_c^{\hat{\Sigma}}(\hat{\mathcal{A}} \rtimes \Sigma)\}$ is clearly dense in $C_c(\mathcal{G}, \Sigma, \hat{\mathscr{E}}, \hat{\kappa})$ in this $\|\cdot\|_I$-norm.

There exists $\chi_\sigma \in \hat{\mathcal{A}}(r(\sigma))$ such that $\|F_f(\sigma)\|_\infty = |f(\chi_\sigma, \sigma)|$. Thus

$$
\begin{aligned}
\|F_f\|_{I,r} &= \sup_{u \in \Sigma^{(0)}} \int_{\mathcal{G}} |f(\chi_\sigma, \sigma)| \, d\alpha^u(\dot{\sigma}) \\
&= \sup_{\substack{u \in \Sigma^{(0)} \\ \hat{\pi}(\chi) = u}} \int_{\mathcal{G}} |f(\chi, \sigma)| \, d\alpha^u(\dot{\sigma}) \\
&= \sup_{\chi \in \hat{\mathcal{A}}} \int_{\mathcal{G}} |f(\chi, \sigma)| \, d\alpha^{\hat{\pi}(\chi)}(\dot{\sigma}) \\
&= \|f\|_{I,r}.
\end{aligned}
$$

Similarly, $\|F_f\|_{I,s} = \|f\|_{I,s}$, and $f \mapsto F_f$ is isometric for the respective I-norms. The isomorphism of full C^*-algebras follows.

For the isomorphism of reduced C^*-algebras, let $\mathcal{H}(\hat{\mathscr{B}})$ be the right-Hilbert $C_0(\hat{\mathcal{A}})$-module described in Section 1.2 for the Fell bundle $\hat{\mathscr{B}}$. Regard $\mathcal{H}(\hat{\mathscr{B}})$ as a right $C_0(\hat{\mathcal{A}})$-module by identifying $C_0(\mathcal{G}^{(0)}; \hat{\mathscr{e}})$ with $C_0(\hat{\mathcal{A}})$ via $f \mapsto \big(\chi \mapsto f(p(\chi))(\chi)\big)$. Then $C_c(\mathcal{G}, \hat{\mathscr{B}})$ acts on the left of $\mathcal{H}(\hat{\mathscr{B}})$ by convolution, and the map implementing this action is isometric for the reduced norm on $C_c(\mathcal{G}, \hat{\mathscr{B}})$. Now consider the Hilbert module $\mathcal{H}(\hat{\mathcal{A}} \rtimes \mathcal{G}; \hat{\Sigma})_{C_0(\hat{\mathcal{A}})}$ obtained from the twist $\hat{\Sigma}$ as described in Section 1.2, so that the left action of $C_c(\hat{\mathcal{A}} \rtimes \mathcal{G}; \hat{\Sigma})$ on $\mathcal{H}(\hat{\mathcal{A}} \rtimes \mathcal{G}; \hat{\Sigma})$ by convolution is isometric for the reduced norm on $C_c(\hat{\mathcal{A}} \rtimes \mathcal{G}; \hat{\Sigma})$. A straightforward calculation shows that $\langle f, g \rangle_{C_0(\hat{\mathcal{A}})} = \langle F_f, F_g \rangle_{C_0(\hat{\mathcal{A}})}$ and so $f \mapsto F_f$ extends to an isomorphism of Hilbert modules $\mathcal{H}(\hat{\mathcal{A}} \rtimes \mathcal{G}; \hat{\Sigma}) \cong \mathcal{H}(\hat{\mathscr{B}})$, which intertwines the left actions because $f \mapsto F_f$ is a homomorphism. Hence

$$
\|f\|_{C_r^*(\hat{\mathcal{A}} \rtimes \mathcal{G}; \hat{\Sigma})} = \|f\|_{\mathcal{L}(\mathcal{H}(\hat{\mathcal{A}} \rtimes \mathcal{G}; \hat{\Sigma}))} = \|F_f\|_{\mathcal{L}(\mathcal{H}(\hat{\mathcal{B}}))} = \|F_f\|_{C_r^*(\mathcal{G}, \Sigma, \hat{\mathscr{e}}, \hat{\kappa})}. \qquad \square
$$

Since we can identify $C_\infty^{\hat{\Sigma}}(\hat{\mathcal{A}} \rtimes \Sigma)$ and $C_c(\mathcal{G}, \Sigma, \hat{\mathscr{e}}, \hat{\kappa})$, we can view $C_\infty^{\hat{\Sigma}}(\hat{\mathcal{A}} \rtimes \Sigma)$ as a dense subalgebra of $C^*(\hat{\mathcal{A}} \rtimes \mathcal{G}; \hat{\Sigma})$. Furthermore, if $f \in C_c(\Sigma)$, then

$$
(3.3) \qquad \Phi(f)(\chi, \sigma) = \delta(\sigma)^{\frac{1}{2}} \int_{\mathcal{A}} f(a\sigma) \overline{\chi(a)} \, d\beta^{r(\sigma)}(a)
$$

defines an element of $C_\infty^{\hat{\Sigma}}(\hat{\mathcal{A}} \rtimes \Sigma)$. Noticing that $\Phi(f)(\chi, \sigma) = j(f)(\sigma)(\chi)$, we can combine Theorem 2.12 and Proposition 3.3 to obtain the main result in this section.

Theorem 3.4. *Let Σ be a second countable locally compact groupoid with a Haar system λ. Suppose that $\mathcal{A}$ is a closed abelian normal subgroup bundle with a Haar system β. Then the map $\Phi : C_c(\Sigma) \to C_\infty^{\hat{\Sigma}}(\hat{\mathcal{A}} \rtimes \Sigma)$ given in (3.3) extends to an isomorphism of $C^*(\Sigma)$ onto the restricted C^*-algebra $C^*(\hat{\mathcal{A}} \rtimes \mathcal{G}; \hat{\Sigma})$. Moreover, this isomorphism descends to an isomorphism $C_r^*(\Sigma) \cong C_r^*(\hat{\mathcal{A}} \rtimes \mathcal{G}; \hat{\Sigma})$.*

Remark 3.5. Theorem 3.4 generalizes [MRW96, Proposition 4.5] where it was assumed that $\mathcal{A} = \Sigma'$, that $\mathcal{G}$ was principal, and that $C^*(\Sigma)$ was CCR.

4. Examples: Pushouts of groupoid extensions and cocycles

To give some perspective on the sorts of groupoids to which Theorem 3.4 applies, we want to look at some pertinent examples. In particular, we want to highlight the construction of a *pushout of groupoids.* We show that the **T**-groupoid, $\widehat{\Sigma}$, defined in Section 3.1 is a particular case of this construction. Moreover, we describe pushout groupoids arising from 2-cocycles and prove that we recover the constructions used in [IKSW19] as particular examples.

In this section, we need a more formal perspective on the extensions we've been working with. We fix a locally compact Hausdorff groupoid $\mathcal{G}$. In our applications, $\mathcal{G}$ will have a Haar system $\{\alpha^u\}_{u \in \mathcal{G}^{(0)}}$, but this is not required in this section. Then we call a locally compact abelian group bundle $p_{\mathcal{A}} : \mathcal{A} \to \mathcal{G}^{(0)}$ a $\mathcal{G}$-bundle if $p_{\mathcal{A}}$ is open and $\mathcal{G}$ acts on the left of $\mathcal{A}$ by automorphisms. For compatibility with [IKSW19], and hence with some of the examples in this section, we will write the group operations in the fibres of such $\mathcal{A}$ additively. An extension Σ of $\mathcal{A}$ by $\mathcal{G}$ is determined by a diagram

(4.1)

$$\mathcal{A} \xrightarrow{\iota} \Sigma \xrightarrow{p} \mathcal{G}$$

where p is continuous and open, and ι is a homemorphism onto its (necessarily closed) range. We call (4.1) a *compatible extension* if the action of $\mathcal{G}$ on $\mathcal{A}$ induced by conjugation is the given $\mathcal{G}$-action on $\mathcal{A}$; that is, $\sigma\iota(a)\sigma^{-1} = \iota(\dot{\sigma} \cdot a)$ where as usual we have written $\dot{\sigma}$ in place of $p(\sigma)$.

Remark 4.1. If $\iota(\mathcal{A}) = \mathcal{A}$ and ι is the inclusion map, then we obtain an extension by a wide normal subgroup bundle as in (2.1) from Section 2. (Since p is open, we can identify $\Sigma/\iota(\mathcal{A})$ with $\mathcal{G}$.) However, as we shall see below, we can have $\iota(\mathcal{A}) \subset \Sigma$ without ι being the inclusion map. Then we formally obtain a different extension—see Example 4.8.

Definition 4.2. If Σ_1 and Σ_2 are compatible extensions of a locally compact abelian group $\mathcal{G}$-bundle $\mathcal{A}$, then we say that they are *properly isomorphic* if there is a groupoid isomorphism $f : \Sigma_1 \to \Sigma_2$ such that the diagram

(4.2)

$$\mathcal{A} \xrightarrow{\iota_1} \Sigma_1 \xrightarrow{p_1} \mathcal{G}, \quad f : \Sigma_1 \to \Sigma_2, \quad \mathcal{A} \xrightarrow{\iota_2} \Sigma_2 \xrightarrow{p_2} \mathcal{G}$$

commutes. We let $T_{\mathcal{G}}(\mathcal{A})$ be the collection of proper isomorphism classes of compatible extensions as in (4.1). We denote the equivalence class of an extension Σ as in diagram 4.1 by $[\Sigma]$.

Remark 4.3. These sorts of extensions were introduced in [Kum88, §2].[6] Tu denotes this set by $\text{ext}(\mathcal{G}, \mathcal{A})$ (see [Tu06, §5]). Note that if $[\Sigma] \in T_{\mathcal{G}}(\mathcal{A})$, then we can then equip Σ with

---

[6]Kumjian worked with étale groupoids (which he called "sheaf groupoids") in [Kum88].

a Haar system $\{\lambda^u\}_{u \in \mathcal{G}^{(0)}}$ using the right-hand side of (2.3) provided $\mathcal{G}$ has a Haar system α since $p_\mathcal{A}$ open implies that $\mathcal{A}$ has a Haar system β.

Example 4.4 (The Semidirect Product). We can build a fundamental compatible extension $\Sigma(\mathcal{A}, \mathcal{G})$ from the fibred product $\{a, \gamma) \in \mathcal{A} \times \mathcal{G} : p_\mathcal{A}(a) = r(\gamma)\}$ as in [Kum88, Definition 2.1]. We let $\Sigma(\mathcal{A}, \mathcal{G})^{(2)} = \{((a_1, \gamma_1), (a_2, \gamma_2)) : s(\gamma_1) = r(\gamma_2)\}$, and then define

$$(a_1, \gamma_1)(a_2, \gamma_2) = (a_1 + \gamma_1 \cdot a_2, \gamma_1 \gamma_2) \quad \text{and} \quad (a, \gamma)^{-1} = (-(\gamma^{-1} \cdot a), \gamma^{-1}).$$

Then we can identify the unit space of $\Sigma(\mathcal{A}, \mathcal{G})$ with $\mathcal{G}^{(0)}$ so that $r(a, \gamma) = r(\gamma)$ and $s(a, \gamma) = s(\gamma)$. We can then exhibit $\Sigma(\mathcal{A}, \mathcal{G})$ as an extension by letting $\iota(a) = (a, p_\mathcal{A}(a))$, and letting $p(a, \gamma) = \gamma$. Since

$$(a', \gamma)(a, p_\mathcal{A}(a))(-\gamma^{-1} \cdot a', \gamma^{-1}) = (\gamma \cdot a, p_\mathcal{A}(\gamma \cdot a)),$$

$\Sigma(\mathcal{A}, \mathcal{G})$ is a compatible extension as required.

Example 4.5. For $i = 1, 2$ let $\mathcal{A}_i$ be a locally compact abelian group $\mathcal{G}$-bundle. Note that $\mathcal{A}_1 * \mathcal{A}_2 = \{(a, a') : p_{\mathcal{A}_1}(a) = p_{\mathcal{A}_2}(a')\}$ is also a locally compact abelian group $\mathcal{G}$-bundle. Let Σ_i be a compatible groupoid extension of $\mathcal{G}$ by $\mathcal{A}_i$. Then as in [Kum88, §2], we may form the fibered product

$$\Sigma_1 *_\mathcal{G} \Sigma_2 := \{(\sigma_1, \sigma_2) \in \Sigma_1 \times \Sigma_2 \mid p_1(\sigma_1) = p_2(\sigma_2)\}.$$

Then it straightforward to see that $\Sigma_1 *_\mathcal{G} \Sigma_2$ is a compatible groupoid extension of $\mathcal{G}$ by $\mathcal{A}_1 * \mathcal{A}_2$.

Assume now that $\mathcal{B}$ is another abelian group $\mathcal{G}$-bundle, and that $f : \mathcal{A} \to \mathcal{B}$ is an $\mathcal{G}$-equivariant map. Following Proposition 2.6 of [Kum88], we prove that we can "pushout" Σ in a unique way to an extension of $\mathcal{G}$ by $\mathcal{B}$.

Proposition 4.6 (Pushout Construction). *Let $\mathcal{A}$ and $\mathcal{B}$ be locally compact abelian group $\mathcal{G}$-bundles. Let $f : \mathcal{A} \to \mathcal{B}$ be a continuous $\mathcal{G}$-equivariant map. Assume that Σ is an extension of $\mathcal{G}$ by $\mathcal{A}$ as in (4.1). Then there is an extension $f_*\Sigma$ of $\mathcal{G}$ by $\mathcal{B}$ and a homomorphism $f_* : \Sigma \to f_*\Sigma$ such that the following diagram commutes*

(4.3)
$$\begin{array}{ccc}
\mathcal{A} & \xrightarrow{\iota} & \Sigma \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f_*} \quad \searrow^{p} \\
\mathcal{B} & \xrightarrow{\iota_*} & f_*\Sigma \xrightarrow{p_*} \mathcal{G}.
\end{array}$$

Moreover, f_* and $f_*\Sigma$ are unique up to proper isomorphism in the sense that if Σ' is another extension such that the diagram

(4.4)
$$\begin{array}{ccc}
\mathcal{A} & \xrightarrow{\iota} & \Sigma \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} \quad \searrow^{p} \\
\mathcal{B} & \xrightarrow{\iota'} & \Sigma' \xrightarrow{p'} \mathcal{G}
\end{array}$$

commutes, then there is a proper isomorphism $g : f_*\Sigma \to \Sigma'$ such that $g \circ f_* = f'$.

Proof. We follow the proof of Proposition 2.6 of [Kum88]. Form the fibred-product groupoid

$$\mathcal{D} := (\mathcal{B} * \mathcal{G}) *_{\mathcal{G}} \Sigma = \{\, ((b, \gamma), \sigma) \in (\mathcal{B} * \mathcal{G}) \times \Sigma : p_{\mathcal{B}}(b) = r(\gamma) \text{ and } \dot{\sigma} = \gamma \,\}$$

as above. Define $\theta : \mathcal{A} \to \mathcal{D}$ via $\theta(a) = \big((-f(a), p_{\mathcal{A}}(a)), \iota(a)\big)$. Since ι is a homeomorphism onto its closed range, $\theta(\mathcal{A})$ is a closed wide subgroupoid of $\mathcal{D}$.

Let $d = ((b, \gamma), \sigma) \in \mathcal{D}$. We claim that $d\theta(\mathcal{A}) = \theta(\mathcal{A})d$. To see this, note that

$$
\begin{aligned}
d\theta(a) &= \big((b, \gamma), \sigma\big)((-f(a), p_{\mathcal{A}}(a)), \iota(a)) \\
&= \big((b - \gamma \cdot f(a), \gamma), \sigma\iota(a)\big) \\
&= \big((-f(\gamma \cdot a) + p_{\mathcal{A}}(\gamma \cdot a) \cdot b, \gamma), \iota(\dot{\sigma} \cdot a)\sigma\big)
\end{aligned}
$$

which, since $\dot{\sigma} = \gamma$, is

$$
\begin{aligned}
&= \big((-f(\gamma \cdot a), p_{\mathcal{A}}(\gamma \cdot a)), \iota(\gamma \cdot a)\big)(b, \gamma, \sigma) \\
&= \theta(\gamma \cdot a)d.
\end{aligned}
$$

Let $f_* \Sigma := \mathcal{D}/\theta(\mathcal{A})$. As usual, we denote the class of $((b, \sigma), \gamma)$ in $f_*\Sigma$ by $[(b, \sigma), \gamma]$. Then $[(b, \gamma), \iota(a)\sigma] = [(b + f(a), \gamma), \sigma]$. Since $j(\mathcal{A})$ has a Haar system by Remark 4.3, $f_*\Sigma$ is a locally compact Hausdorff groupoid by Lemma 2.2. The operations are given by

$$[(b_1, \gamma_1), \sigma_1][(b_2, \gamma_2), \sigma_2] = [(b_1 + \gamma_1 b_2, \gamma_1 \gamma_2), \sigma_1 \sigma_2] \quad \text{and}$$
$$[(b, \gamma), \sigma]^{-1} = [(-\gamma^{-1} \cdot b, \gamma^{-1}), \sigma^{-1}].$$

We can identify the unit space with $\mathcal{G}^{(0)}$ and then

$$r([(b, \gamma), \sigma]) = r(\gamma) \quad \text{and} \quad s([(b, \gamma), \sigma]) = s(\gamma).$$

To see that $f_*\Sigma$ is a compatible extension of $\mathcal{B}$, let

$$\iota_*(b) = [(b, p_{\mathcal{B}}(b)), p_{\mathcal{B}}(b)] \quad \text{and} \quad p_*([(b, \gamma), \sigma]) = \gamma.$$

It is not hard to verify the algebraic requirements for an extension. The most difficult one might be the inclusion $p_*^{-1}(\mathcal{G}^{(0)}) \subseteq \iota_*(\mathcal{B})$ for which we provide an outline of the proof: let $[(b, \gamma), \sigma] \in f_*\Sigma$ such that $p_*([(b, \gamma), \sigma]) = u \in \mathcal{G}^{(0)}$. It follows that $\gamma = u$ and, therefore, $\dot{\sigma} = u$. Since Σ is an extension, there exists $a \in A_u$ such that $\iota(a) = \sigma$. It follows that $[((b, u), \iota(a))] = [((b + f(a), u), u)] = \iota_*(b + f(a))$. It is easy to check that $b + f(a)$ is independent of the choice of the representative of $[(b, \gamma), \sigma]$.

Since ι_* and p_* are clearly continuous and since ι_* is easily seen to be a homeomorphism onto its range, we just need to see that p_* is open. For this, we again apply Fell's Criterion (see Remark 3.1). Suppose that $\gamma_n \to \gamma = p_*\big([(b, \sigma), \gamma]\big)$. Since $p : \Sigma \to \mathcal{G}$ is open, we can pass to a subnet, relabel, and assume that there are $\sigma_n \to \sigma$ in Σ such that $\dot{\sigma}_n = \gamma_n$. Since $p_{\mathcal{B}}$ is open, we can pass to subnet, relabel, and assume there are $b_n \to b$ in $\mathcal{B}$ such that $p_{\mathcal{B}}(b_n) = r(\gamma_n)$. Then $[(b_n, \gamma_n), \sigma_n] \to [(b, \gamma), \sigma]$ as required.

The map f_* is the composition of the embedding of Σ into $\mathcal{D}$ and the quotient map $\mathcal{D} \mapsto \mathcal{D}/i(\mathcal{A})$: $f_*(\sigma) = [((0_u, p(\sigma)), \sigma)]$, where $u = r(\sigma)$. Since f is $\mathcal{G}$-invariant, $p_{\mathcal{B}}(f(a)) = p_{\mathcal{A}}(a)$ and

$$f_*(\iota(a)) = [(0, p_{\mathcal{A}}(a)), p_{\mathcal{A}}(a)] = [(f(a), p_{\mathcal{B}}(f(a))), p_{\mathcal{B}}(f(a))] = \iota_*(\iota(a)),$$

and (4.3) commutes as required.

Consider now another extension Σ' as in (4.4). Define a map $\tilde{g} : \mathcal{D} \to \Sigma'$ by $\tilde{g}((b, \gamma), \sigma) = \iota'(b)f'(\sigma)$. Since

$$\iota'(b_1)f'(\sigma_1)\iota'(b_2)f'(\sigma_2) = \iota'(b_1)\iota'(f'(\sigma_1) \cdot b_2)f'(\sigma_1)f'(\sigma_2)$$

and since $p'(f'(\sigma_1)) = \dot{\sigma}_1$, it follows that $\tilde{g}$ is a groupoid homomorphism. On the other hand,

$$\tilde{g}(\theta(a)) = \tilde{g}((-f(a), p_{\mathcal{A}}(a)), \iota(a)) = \iota'(-f(a))f'(\iota(a)) = \iota'(-f(a))\iota'(f(a)) = \iota'(p_{\mathcal{A}}(a)).$$

Hence $\tilde{g}$ factors through a homomorphism $g : f_*\Sigma \to \Sigma'$. Clearly, $g(\iota_*(b)) = \iota'(b)$ and $p' \circ g = p_*$ so that g satisfies the analogue of (4.2) as required. Furthermore, $g \circ f_* = f'$ by construction.

To see that g is a proper isomorphism, we still need to see that g is an isomorphism with a continuous inverse.

To that end, let $\alpha \in \Sigma'$. Then there is a $\sigma \in \Sigma$ such that $p(\sigma) = p'(\alpha)$. Using (4.4), there is a $b \in \mathcal{B}$ such that $\alpha = \iota'(b)f'(\sigma)$ and $\tilde{g}$, and hence g, is onto.

Now suppose that $\iota'(b)f'(\sigma)$ is a unit. Then $f'(\sigma) = \iota'(-b)$. Hence $p'(f'(\sigma))$ is a unit, and $\sigma = \iota(a)$ for some $a \in \mathcal{A}$. But then $\iota'(-b) = f'(\sigma) = f'(\iota(a)) = \iota'(f(a))$. Hence, $b = -f(a)$. That is,

$$((b, p(\sigma)), \sigma) = ((-f(a), \pi(a)), \iota(a)) \in \theta(\mathcal{A}).$$

Thus g is injective.

To see that g is an isomorphism of topological groupoids, it suffices to see that g is open. We use Fell's criterion. So suppose that $g(\alpha_i) \to g(\alpha)$. Since $p' \circ g = p_*$, we have $p(\sigma_i) \to p(\sigma)$. Since p is open, we can pass to a subnet, relabel, and assume there are $a_i \in \mathcal{A}$ such that $\iota(a_i)\sigma_i \to \sigma$. But

$$\alpha_i = [(-f(a_i) + b_i), p(\sigma_i), \iota(a_i)\sigma_i],$$

and then

$$\iota'(-f(a_i) + b_i)f'(\iota(a_i)\sigma_i) \to \iota'(b)f'(\sigma).$$

It follows that

$$\iota'(-f(a_i) + b_i) \to \iota'(b).$$

Since ι' is a homeomorphism onto its range, $\alpha_i \to \alpha$ as required. $\square$

As in [Kum88, §2], we can use our pushout construction to introduce a binary operation on $T_{\mathcal{G}}(\mathcal{A})$. Suppose that $[\Sigma], [\Sigma'] \in T_{\mathcal{G}}(\mathcal{A})$. Let $\nabla^{\mathcal{A}} : \mathcal{A} * \mathcal{A} \to \mathcal{A}$ be given by $\nabla^{\mathcal{A}}(a, a') = a + a'$. It is not hard to see that the class of $\nabla^{\mathcal{A}}_*(\Sigma *_{\mathcal{G}} \Sigma')$ in $T_{\mathcal{G}}(\mathcal{A})$ depends only the classes of Σ and Σ'. Then we define

(4.5) $$[\Sigma] + [\Sigma'] = [\nabla^{\mathcal{A}}_*(\Sigma *_{\mathcal{G}} \Sigma')].$$

Example 4.7. Let $[\Sigma] \in T_{\mathcal{G}}(\mathcal{A})$. Recall the semidirect product $\Sigma(\mathcal{A}, \mathcal{G})$ of Example 4.4 We get a commutative diagram

$$
\begin{array}{ccc}
\mathcal{A} * \mathcal{A} & \xrightarrow{\iota * \iota} & \Sigma(\mathcal{A}, \mathcal{G}) *_{\mathcal{G}} \Sigma \\
\downarrow{\scriptstyle \nabla^{\mathcal{A}}} & & \downarrow{\scriptstyle g} \qquad\searrow^{\tilde{p}} \\
\mathcal{A} & \xrightarrow{\iota} & \Sigma \qquad \xrightarrow{p} \quad \mathcal{G}
\end{array}
$$

where $g((a, \dot{\sigma}), \sigma) = \iota(a)\sigma$. Then the uniqueness assertion in Proposition 4.6 implies that $\nabla_*^{\mathcal{A}}(\Sigma(\mathcal{A}, \mathcal{G}) *_{\mathcal{G}} \Sigma)$ is properly isomorphic to Σ. In other words, $[\Sigma(\mathcal{A}, \mathcal{G})] + [\Sigma] = [\Sigma]$.

Example 4.8. Let $\mathcal{A} \xrightarrow{\iota} \Sigma \xrightarrow{p} \mathcal{G}$ be a compatible extension. Then we obtain another compatible extension $\mathcal{A} \xrightarrow{\iota'} \Sigma \xrightarrow{p} \mathcal{G}$ by letting $\iota'(a) = \iota(-a) = \iota(a)^{-1}$. We will write Σ^{-1} for Σ viewed as this alternate extension. Let $\theta : \mathcal{A} \to \mathcal{A}$ be given by inversion: $\theta(a) = -a$. Then θ is $\mathcal{G}$-invariant. Since the diagram

$$
\begin{array}{ccc}
\mathcal{A} & \xrightarrow{\iota} & \Sigma \\
\downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle \mathrm{id}} \quad\searrow^{p} \\
\mathcal{A} & \xrightarrow{\iota'} & \Sigma^{-1} \qquad \xrightarrow{p} \quad \mathcal{G}
\end{array}
$$

commutes, we can identify $[\theta_* \Sigma]$ with $[\Sigma^{-1}]$ by Proposition 4.6.

Example 4.9. Take $[\Sigma] \in T_{\mathcal{G}}(\mathcal{A})$. Recall the semidirect product $\Sigma(\mathcal{A}, \mathcal{G})$ of Example 4.4. The map $g : \Sigma * \Sigma^{-1} \to \Sigma(\mathcal{A}, \mathcal{G})$ given by $g(\sigma, \tau) = (\iota^{-1}(\sigma\tau^{-1}), \dot{\sigma})$ is a homomorphism. Since the diagram

$$
\begin{array}{ccc}
\mathcal{A} * \mathcal{A} & \xrightarrow{\iota * \iota'} & \Sigma *_{\mathcal{G}} \Sigma^{-1} \\
\downarrow{\scriptstyle \nabla^{\mathcal{A}}} & & \downarrow{\scriptstyle g} \qquad\searrow^{\tilde{p}} \\
\mathcal{A} & \xrightarrow{\iota} & \Sigma(\mathcal{A}, \mathcal{G}) \qquad \xrightarrow{p} \quad \mathcal{G}
\end{array}
$$

commutes, we see that $[\Sigma] + [\Sigma^{-1}] = [\Sigma(\mathcal{A}, \mathcal{G})]$ for all $\Sigma \in T_{\mathcal{G}}(\mathcal{A})$.

Example 4.10. Take $[\Sigma], [\Sigma'] \in T_{\mathcal{G}}(\mathcal{A})$. Let $\tilde{f} : \Sigma *_{\mathcal{G}} \Sigma' \to \Sigma' *_{\mathcal{G}} \Sigma$ be the flip. Similarly, let $f : \mathcal{A} * \mathcal{A} \to \mathcal{A} * \mathcal{A}$ be given by $f(a, a') = (a', a)$. The diagram

$$
\begin{array}{ccc}
\mathcal{A} * \mathcal{A} & \xrightarrow{\iota * \iota'} & \Sigma *_{\mathcal{G}} \Sigma' \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \tilde{f}} \qquad\searrow^{\tilde{p}} \\
\mathcal{A} * \mathcal{A} & \xrightarrow{\iota' * \iota} & \Sigma' *_{\mathcal{G}} \Sigma \xrightarrow{\tilde{p}} \mathcal{G} \\
\downarrow{\scriptstyle \nabla^{\mathcal{A}}} & & \downarrow{\scriptstyle \nabla_*^{\mathcal{A}}} \qquad\nearrow_{p} \\
\mathcal{A} & \xrightarrow{i} & \nabla_*(\Sigma' *_{\mathcal{G}} \Sigma)
\end{array}
$$

commutes. Since $\nabla^{\mathcal{A}} \circ f = \nabla^{\mathcal{A}}$, it follows from Proposition 4.6 that $[\Sigma] + [\Sigma'] = [\Sigma'] + [\Sigma]$.

We have proved most of the following theorem which is based on [Kum88, Theorem 2.7].

Proposition 4.11. *Let $\mathcal{G}$ be a locally compact groupoid (with Haar system) and let $\mathcal{A}$ be a locally compact abelian group $\mathcal{G}$-bundle. Then the binary operation $([\Sigma_1], [\Sigma_2]) \mapsto [\nabla_*^{\mathcal{A}}(\Sigma_1 *_{\mathcal{G}} \Sigma_2)]$ of (4.5) makes $T_{\mathcal{G}}(\mathcal{A})$ into an abelian group with neutral element given by the class $[\Sigma(\mathcal{A}, \mathcal{G})]$ of the semidirect product of Example 4.4, and $[\Sigma]^{-1} = [\Sigma^{-1}]$ as in Example 4.8. Moreover, $T_{\mathcal{G}}$ is a functor from the category of $\mathcal{G}$-bundles to the category of abelian groups. Indeed, if $f : \mathcal{A} \to \mathcal{B}$ is a continuous $\mathcal{G}$-equivariant map of $\mathcal{G}$-bundles and $T_{\mathcal{G}}(f) : T_{\mathcal{G}}(\mathcal{A}) \to T_{\mathcal{G}}(\mathcal{B})$ is the induced map, then $Tg(f)[\Sigma] = [f_*\Sigma]$ and $T_{\mathcal{G}}([\Sigma(\mathcal{A}, \mathcal{G})]) = [\Sigma(\mathcal{B}, \mathcal{G})]$.*

Proof. By considering diagrams similar to that in Example 4.10, we see that the operation in (4.5) is well-defined and associative. We saw that $[\Sigma(\mathcal{A}, \mathcal{G})]$ acts as an identity in Example 4.7 and the statement about inverses follows from Example 4.9. Functoriality follows as in the proof of [Kum88, Theorem 2.7]. The remaining statements follow from functorality. $\square$

4.1. **Extensions by 2-cocycles.** Extensions associated to groupoid 2-cocycles are discussed in [IKSW19, Appendix A]. For convenience, we review the basics here. Assume that $\mathcal{A}$ is the trivial bundle $A \times \mathcal{G}^{(0)}$ for some locally compact abelian group A and that $\phi : \mathcal{G}^{(2)} \to A$ is a normalized 2-cocycle. Then the extension Σ_ϕ of $\mathcal{G}$ by A determined by ϕ is obtained by giving the product $A \times \mathcal{G}$ the groupoid structure where $(a_1, \gamma_1)(a_2, \gamma_2) = (a_1 + a_2 + \phi(\gamma_1, \gamma_2), \gamma_1\gamma_2)$ if $(\gamma_1, \gamma_2) \in \mathcal{G}^{(2)}$ and $(a, \gamma)^{-1} = (-a - \phi(\gamma^{-1}, \gamma), \gamma^{-1})$. We exhibit Σ_ϕ as an extension of $\mathcal{G}$ by $A \times \mathcal{G}^{(0)}$ via $i(a, u) = (a, u)$ and $p(a, \gamma) = \gamma$. Suppose that $f : A \to B$ is a continuous group homomorphism from A to a locally compact abelian group B. Then one can define a B-valued 2-cocycle $f_*(\phi) : \mathcal{G}^{(2)} \to B$ via $f_*(\phi)(\gamma_1, \gamma_2) = f(\phi(\gamma_1, \gamma_2))$.

Lemma 4.12. *Let $\Sigma_{f_*(\phi)}$ be the extension of $\mathcal{G}$ by B determined by $f_*(\phi)$. Then $f_*\Sigma_\phi$ is properly isomorphic to $\Sigma_{f_*(\phi)}$.*

Proof. Define $g : \Sigma_\phi \to \Sigma_{f*\phi}$ by $g(a, \gamma) = (f(a), \gamma)$. The diagram

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\ i\ } & \Sigma_\phi \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ \mathcal{B} & \xrightarrow{\ i\ } & \Sigma_{f_*(\phi)} \end{array} \quad \substack{\xrightarrow{\ p\ } \\ \ \\ \xrightarrow[\ p\ ]{}} \ \mathcal{G}$$

commutes. Therefore the lemma follows from Proposition 4.6. $\square$

4.2. **The T-groupoid of an extension.** The **T**-groupoid constructed in Section 3.1 associated to an extension Σ as in (2.1) is an example of the pushout construction of Proposition 4.6. As in Section 3.1, we assume that $\mathcal{A}$ is a wide normal abelian subgroup bundle of Σ equipped with a Haar system so that the Gelfand dual, $\hat{\mathcal{A}}$, of $C^*(\mathcal{A})$ is an abelian group bundle over $\mathcal{G}^{(0)}$. If we let $\mathcal{G} = \Sigma/\iota(\mathcal{A})$, then $\mathcal{A}$ is a $\mathcal{G}$-bundle. It follows from Proposition A.1 that $\hat{\mathcal{A}}$ has a Haar system, so we can view $\hat{\mathcal{A}}$ as a $\mathcal{G}$-bundle. Hence $\hat{\mathcal{A}} * \mathcal{A} = \{(\chi, a) : \hat{\pi}(\chi) = p_{\mathcal{A}}(a)\}$ is also a $\mathcal{G}$-bundle. Since $\mathcal{G}$ and Σ both act on $\hat{\mathcal{A}}$, regarded

as a topological space fibered over $\mathcal{G}^{(0)}$, we can form the action groupoids $\hat{\mathcal{A}} \rtimes \mathcal{G}$ and $\hat{\mathcal{A}} \rtimes \Sigma$ as well as the extension

$$\hat{\mathcal{A}} * \mathcal{A} \xhookrightarrow{\iota_*} \hat{\mathcal{A}} \rtimes \Sigma \xrightarrow{p_*} \hat{\mathcal{A}} \rtimes \mathcal{G}$$

where $\iota_*(\chi, a) = (\chi, a)$ and $p_*(\chi, \sigma) = (\chi, \dot{\sigma})$.

Proposition 4.13. *If $f : \hat{\mathcal{A}} * \mathcal{A} \to \hat{\mathcal{A}} \times \mathbf{T}$ is the canonical map given by*

$$f(\chi, a) = (\chi, \overline{\chi(a)}),$$

then $\widehat{\Sigma}$ is properly isomorphic to the pushout $f_*(\hat{\mathcal{A}} \rtimes \Sigma)$. Therefore, if $[\Sigma]$ is the semidirect product $[\Sigma(\mathcal{A}, \mathcal{G})]$, then $[\hat{\mathcal{A}} \rtimes \Sigma] = [\Sigma(\mathcal{A}, \hat{\mathcal{A}} \rtimes \mathcal{G})]$ and hence $[\widehat{\Sigma}]$ is trivial.

Proof. Proposition 4.6 implies that there is a unique extension $f_*(\hat{\mathcal{A}} \rtimes \Sigma)$ of $\hat{\mathcal{A}} \rtimes \mathcal{G}$ by $\hat{\mathcal{A}} \times \mathbf{T}$ and a twist morphism that is compatible with f. In particular, $f_*(\hat{\mathcal{A}} \rtimes \Sigma)$ is a $\mathbf{T}$-groupoid, also called a twist. If $\widehat{\Sigma} = \mathcal{D}/H$ is the $\mathbf{T}$-groupoid from Proposition 3.2, we get a natural map $g : \hat{\mathcal{A}} \rtimes \Sigma$ to $\widehat{\Sigma}$ given by $g(\chi, \sigma) = [\chi, 1, \sigma]$, and the diagram

$$
\begin{array}{ccc}
\hat{\mathcal{A}} * \mathcal{A} & \xrightarrow{\iota_*} & \hat{\mathcal{A}} \rtimes \Sigma \\
{\scriptstyle f} \downarrow & & {\scriptstyle g} \downarrow \quad \searrow^{p_*} \\
\hat{\mathcal{A}} \times \mathbf{T} & \xrightarrow{i} & \widehat{\Sigma} \quad \xrightarrow{j} \hat{\mathcal{A}} \rtimes \mathcal{G}
\end{array}
$$

commutes. $\square$

4.3. **The T-groupoid defined by a 2-cocycle.** We assume now the setting from Section 4.1: $\mathcal{A} = A \times \mathcal{G}^{(0)}$, $\phi : \mathcal{G}^{(2)} \to A$ is a 2-cocycle, and Σ_ϕ is the extension defined by ϕ. Then $\hat{\mathcal{A}} \cong \hat{A} \times \mathcal{G}^{(0)}$ and there is a 2-cocycle

$$\tilde{\phi} : (\hat{\mathcal{A}} \rtimes \mathcal{G})^{(2)} \to \mathcal{A} * \hat{\mathcal{A}} \simeq A \times \hat{A} \times \mathcal{G}^{(0)}$$

defined by

$$\tilde{\phi}\big((\chi, r(\gamma_1), \gamma_1), (\chi, r(\gamma_2), \gamma_2)\big) = \big(\phi(\gamma_1, \gamma_2), \chi, r(\gamma_1)\big)$$

if $(\gamma_1, \gamma_2) \in \mathcal{G}^{(2)}$. For brevity, we remove $r(\gamma)$ when writing the elements in $\hat{\mathcal{A}} \rtimes \mathcal{G}$ and $\hat{\mathcal{A}} \rtimes \Sigma_\phi$; that is, we write (χ, γ) instead of $(\chi, r(\gamma), \gamma)$.

We now claim that we can identify $\mathcal{A} \rtimes \Sigma_\phi$ with $\Sigma_{\tilde{\phi}}$, the extension of $\mathcal{A} \rtimes \mathcal{G}$ determined by $\tilde{\phi}$. To see this, notice that

$$\hat{\mathcal{A}} \rtimes \Sigma_\phi = \{\big(\chi, (a, \gamma)\big) : \chi \in \hat{A}, a \in A, \gamma \in \mathcal{G}\}$$

with operations

$$\big(\chi,(a_1,\gamma_1)\big)\big(\chi,(a_2,\gamma_2)\big) = \big(\chi,(a_1 + a_2 + \phi(\gamma_1,\gamma_2),\gamma_1\gamma_2)\big), \text{ and}$$

(4.6)
$$\big(\chi,(a,\gamma)\big)^{-1} = \big(\chi,(-a - \phi(\gamma^{-1},\gamma),\gamma^{-1})\big).$$

Also,

$$\Sigma_{\tilde{\phi}} = \{\big((a,\chi),(\chi,\gamma)\big) \,:\, a \in A, \chi \in \hat{A}, \gamma \in \mathcal{G}\}$$

with operations

(4.7) $$\big((a_1,\chi),(\chi,\gamma_1)\big)\big((a_2,\chi),(\chi,\gamma_2)\big) = \big((a_1 + a_2 + \phi(\gamma_1,\gamma_2),\chi),(\chi,\gamma_1\gamma_2)\big), \text{ and}$$

$$\big((a,\chi),(\chi,\gamma)\big)^{-1} = \big((-a - \phi(\gamma^{-1},\gamma),\chi),(\chi,\gamma^{-1})\big).$$

Lemma 4.14. *The map* $V : \hat{A} \rtimes \Sigma_\phi \to \Sigma_{\tilde{\phi}}$ *defined by*

$$V\big(\chi,(a,\gamma)\big) = \big((a,\chi),(\chi,\gamma)\big)$$

is a groupoid homomorphism.

Proof. This follows from direct comparison of the multiplication formulas (4.6) and (4.7). $\square$

Consider the 2-cocycle $\widehat{\phi} := f_*\tilde{\phi} : (A \rtimes \mathcal{G})^{(2)} \to \mathbf{T} \times \hat{A}$ defined via

$$\widehat{\phi}\big((\chi,\gamma_1),(\chi,\gamma_2)\big) = \big(\overline{\chi(\phi(\gamma_1,\gamma_2))},\chi\big).$$

Lemmas 4.12 and 4.14 imply that $\widehat{\Sigma}$ is isomorphic to the $\mathbf{T}$-groupoid defined by $\widehat{\phi}$.

Example 4.15. The following example was studied in [IKSW19]. Let X be a second-countable locally compact Hausdorff space, and G a second-countable locally compact abelian group. Let $\mathscr{G}$ denote the sheaf of germs of continuous G-valued functions on X, and let $c \in Z^2(\mathscr{U},\mathscr{G})$ be a normalized Čech two cocycle for some locally finite cover $\mathscr{U} = \{U_i\}_{i \in I}$ of X by precompact open sets. The blow-up groupoid $\mathcal{G}_{\mathscr{U}}$ with respect to the natural map from $\bigsqcup_i U_i$ into X is

$$\mathcal{G}_{\mathscr{U}} = \{(i,x,j) \,:\, x \in U_{ij} := U_i \cap U_j\}$$

with $(i,x,j)(j,x,k) = (i,x,k)$ and $(i,x,j)^{-1} = (j,x,i)$. As noted in [IKSW19, Remark 3.3], the Čech 2-cocycle c defines a groupoid 2-cocycle $\phi_c : \mathcal{G}_{\mathscr{U}}^{(2)} \to G$ via

$$\phi_c\big((i,x,j),(j,x,k)\big) = c_{ijk}(x).$$

Let Σ_c be the extension of $\mathcal{G}_{\mathscr{U}}$ by the 2-cocycle ϕ_c. Define

$$\widehat{\phi} : \Big((\hat{G} \times \bigsqcup_i U_i) \rtimes \mathcal{G}_{\mathscr{U}}\Big)^{(2)} \to \mathbf{T} \times \hat{G} \times \bigsqcup_i U_i$$

by

$$\widehat{\phi}\big((\tau,(i,x,j)),(\tau,(j,x,k))\big) = \big(\overline{\tau(c_{ijk}(x))},\tau\big).$$

Then $\widehat{\phi}$ is a groupoid 2-cocycle, and the pushout groupoid $\widehat{\Sigma}$ is isomorphic to the $\mathbf{T}$-groupoid that is the extension of $(\hat{G} \times \bigsqcup_i U_i) \rtimes \mathcal{G}_{\mathscr{U}}$ defined by $\widehat{\phi}$.

Let $\mathcal{V} = \{\hat{G} \times U_i\}_{i \in I}$, let $\mathcal{S}$ be the sheaf of germs of continuous $\mathbf{T}$-valued functions, and define $\nu^c \in Z^2(\mathcal{V}, \mathcal{S})$ by

$$\nu^c\big((\tau, (i, x, j)), (\tau, (j, x, k))\big) = \overline{\tau(c_{ijk}(x))}.$$

Then the 2-cocycle $\hat{\phi}$ is defined by the Čech 2-cocycle $\nu^c \in Z^2(\mathcal{V}, \mathcal{S})$.

That is, ν^c is the normalized 2-cocycle considered in [IKSW19, Equation (3.4)]. Hence the generalized Raeburn–Taylor C^*-algebra $A(\nu)$ studied in [IKSW19] is isomorphic to the restricted C^*-algebra of the $\mathbf{T}$-groupoid defined by the 2-cocycle ν^c.

By [IKSW19, Lemma 5.2], $A(\nu)$ is a continuous-trace C^*-algebra with spectrum $\hat{G} \times X$ with Dixmier–Douady invariant $\delta(A(\nu)) = [\nu^c]$. To get a concrete example, let $G = \mathbf{Z}$ and choose a Čech 2-cocycle c associated to any line bundle.

5. Examples and applications

5.1. Closed normal abelian subgroups.
Let G be a locally compact group, and let $H \le G$ be a closed normal abelian subgroup. Putting $\mathcal{A} := H$, $\Sigma := G$ and $\mathcal{G} := G/H$, we obtain an instance of the situation of Section 3, in which the groupoids involved have a single unit.

The group G acts on the left of H by conjugation, this descends to an action of G/H because H is abelian, and these left actions induce right actions of G and G/H on the space $\hat{H}$. So we obtain an extension of groupoids

$$\hat{H} \times H \longrightarrow \hat{H} \rtimes G \longrightarrow \hat{H} \rtimes (G/H)$$

with common unit space $\hat{H}$ (regarded as a topological space).

The groupoid $\mathcal{D}$ of Equation 3.1 is the cartesian product $(\hat{H} \rtimes G) \times \mathbf{T}$ of the transformation groupoid for the action of G on $\hat{H}$, with the circle group. The closed subgroupoid $\iota(\hat{H} \times H)$ is the set $\{((\chi, h), \overline{\chi(h)}) : (\chi, h) \in \hat{H} \times H\}$. So the associated $\mathbf{T}$-groupoid of Section 3.1 is the quotient

$$\hat{\Sigma} := \{[(\chi, g), z] : (\chi, g) \in \hat{H} \rtimes G, z \in \mathbf{T}\},$$

in which $[(\chi, g), z] = [(\chi, hg), \overline{\chi(h)}z]$ for any $h \in H$.

The inclusion $\iota : (\chi, h) \mapsto [(\chi, h), \overline{\chi(h)}]$ and the groupoid homomorphism $\pi : [(\chi, g), z] \mapsto (\chi, gH)$ yield the twist

$$\hat{H} \times \mathbf{T} \xrightarrow{\iota} \hat{\Sigma} \xrightarrow{\pi} \hat{H} \rtimes (G/H),$$

and Theorem 3.4 yields an isomorphism $C^*(G) \cong C^*(\hat{H} \rtimes (G/H); \hat{\Sigma})$.

Remark 5.1. An interesting special case of the construction described above occurs when the closed normal abelian subgroup H is a clopen subgroup as studied by Zeller-Meier [ZM68, 2.31]. This implies first that the quotient group G/H is a discrete group, and second that the group C^*-algebra $C^*(H) \cong C_0(\hat{H})$ is a subalgebra of $C^*(G)$ that contains an approximate identity for $C^*(G)$. Since G/H is discrete, there exists a continuous section $\hat{H} \rtimes (G/H) \to G$ for the quotient map. Therefore the twist $\hat{\Sigma}$ is topologically trivial, in the sense that it is determined by a continuous $\mathbf{T}$-valued 2-cocycle on $\hat{H} \rtimes (G/H)$.

Example 5.2. As an illustrative example, consider the integer Heisenberg group

$$G = \langle\, a,\, b,\, c \mid ab = cba,\, ca = ac,\, cb = bc \,\rangle.$$

(A similar analysis applies for the Heisenberg group of upper-triangular 3×3 real matrices with diagonal entries equal to 1, but the integer example is slightly easier to describe.) We consider two natural (clopen) normal abelian subgroups of G in the context of our result.

   (a) First consider the subgroup $H_c \cong \mathbf{Z}$ generated by the element c. This H_c is precisely the center $\mathcal{Z}(G)$, and so the action of $G/H_c \cong \mathbf{Z}^2$ on $\widehat{H}_c \cong \mathbf{T}$ is trivial. Hence the semidirect product $\widehat{H}_c \rtimes (G/H_c)$ is just the product $\widehat{H}_c \times (G/H_c) \cong \mathbf{T} \times \mathbf{Z}^2$ regarded as group bundle. As discussed in Remark 5.1, $C^*(H_c) \cong C(\mathbf{T})$ embeds as a unital subalgebra of $C^*(G)$ which is central because H_c is central, and therefore makes $C^*(G)$ into a $C(\mathbf{T})$-algebra. For $z \in \mathbf{T}$, the fibre of $\widehat{\Sigma}$ corresponding to the character $\chi_z : c \mapsto z$ of H_c is the extension of $\mathbf{Z}^2$ corresponding to the 2-cocycle $(m, n) \mapsto z^{m_2 n_1}$. The corresponding fibre of $C^*(H)$ is isomorphic to the rotation algebra A_θ where $z = e^{2\pi i \theta}$. So we recover, in this instance, Anderson and Paschke's description [AP89] of $C^*(G)$ as the section algebra of a field of rotation algebras.

   (b) Now consider the subgroup $H_{b,c} \cong \mathbf{Z}^2$ of G generated by b and c, so that $\widehat{H}_{b,c} \cong \mathbf{T}^2$. The quotient $G/H_{b,c}$ is isomorphic to $\mathbf{Z}$ via $aH_{b,c} \mapsto 1$, and acts on $\mathbf{T}^2$ by $1 \cdot (w, z) = (zw, z)$, inducing an action α of $\mathbf{Z}$ on $C(\mathbf{T}^2)$. The twist $\widehat{\Sigma}$ is the trivial twist over $\mathbf{T}^2 \rtimes \mathbf{Z}$, and so we recover the well-known description $C^*(G) \cong C^*(\widehat{\Sigma}; \mathbf{T}^2 \rtimes \mathbf{Z}) \cong C^*(\mathbf{T}^2 \rtimes \mathbf{Z})$ as the crossed-product algebra $C(\mathbf{T}^2) \rtimes_\alpha \mathbf{Z}$.

Remark 5.3. Williams proved in [Wil81, pp 357–358] that when G/H is abelian and G is a semidirect product, $G \cong H \rtimes (G/H)$, then $C^*(G) \cong C^*(\widehat{H} \rtimes (G/H))$. Note that the extension $\widehat{\Sigma}$ is trivial (see Proposition 4.13).

Remark 5.4. The situation described in Example 5.2(a) generalises as follows. If H is any closed subgroup of $\mathcal{Z}(G)$, then $C^*(H)$ includes in the centre of the multiplier algebra of $C^*(G)$, making $C^*(G)$ into a $C_0(\widehat{H})$-algebra. The transformation group $\widehat{H} \rtimes (G/H)$ coincides with the group bundle $\widehat{H} \times (G/H)$. Each $\chi \in \widehat{H}$ determines a central extension $\mathbf{T} \longrightarrow \widehat{\Sigma}_\chi \longrightarrow G/H$ in which $\widehat{\Sigma}_\chi$ coincides with the quotient group $(G \times \mathbf{T})/\{(h, \overline{\chi(h)}) : h \in H\}$. The fibre of $C^*(G)$ corresponding to a character χ of H is the twisted group C^*-algebra of this extension.

5.2. Extensions of effective étale groupoids.

In this section, we consider the situation where the groupoid $\mathcal{G}$ in the extension (2.1) is an effective étale groupoid; that is, $r, s : \mathcal{G} \to \mathcal{G}^{(0)}$ are local homeomorphisms, and the topological interior of the isotropy bundle of $\mathcal{G}$ is $\mathcal{G}^{(0)}$. Since our standing hypotheses include that $\mathcal{G}$ is second-countable and Hausdorff, the latter is equivalent to the condition that $\mathcal{G}$ is topologically principal in the sense that the set $\{x \in \mathcal{G}^{(0)} : \mathcal{G}_x^x = \{x\}\}$ is dense in $\mathcal{G}^{(0)}$ [Ren08, Proposition 3.6].

It follows that the semidirect product $\hat{A} \rtimes \mathcal{G}$ is also étale and effective; this is certainly well known but also easy to prove directly:

Lemma 5.5. *Let $\mathcal{G}$ be a locally compact, Hausdorff groupoid acting on the right of a locally compact Hausdorff space X. If $\mathcal{G}$ is étale (resp., effective) then so is $X \rtimes \mathcal{G}$.*

Proof. Suppose that $\mathcal{G}$ is étale and fix $(x, \gamma) \in X \rtimes \mathcal{G}$. Fix an open bisection neighbourhood U of γ in $\mathcal{G}$ and an open neighbourhood W of $x \in X$. Then $(W \times U) \cap (X \rtimes \mathcal{G}) = \{(y, \eta) \in W \times U \mid s(y) = r(\gamma)\}$ is an open bisection neighbourhood of (x, γ). Hence $X \rtimes \mathcal{G}$ is étale.

Now suppose that $\mathcal{G}$ is effective. Suppose that $U \subseteq X \rtimes \mathcal{G}$ is open and consists entirely of isotropy. Then it is a union of sets of the form $W * V$ where $W \subseteq X$ is open, and $V \subseteq \mathcal{G}$ is an open bisection consisting of isotropy. For any such set, since $\mathcal{G}$ is effective, we have $V \subseteq \mathcal{G}^{(0)}$, and so $W * V \subseteq X * \mathcal{G}^{(0)} = (X \rtimes \mathcal{G})^{(0)}$. $\qquad\square$

It follows from our main theorem that $C_r^*(\Sigma)$ is equal to the reduced C^*-algebra of a twist $\widehat{\Sigma}$ over the étale effective groupoid $\hat{A} \rtimes \mathcal{G}$, and so Renault's theory of Cartan subalgebras of C^*-algebras applies.

Theorem 5.6. *Suppose that Σ is an extension of a wide normal abelian subgroup bundle with a Haar system as in (2.1). Further assume that $\mathcal{G}$ is étale and effective. Then $C^*(\mathcal{A}) \cong C_0(\hat{A})$ is a Cartan subalgebra of $C_r^*(\Sigma)$. The Weyl twist of the Cartain pair $(C_r^*(\Sigma), C^*(\hat{A}))$ is $(\hat{A} \rtimes \mathcal{G}, \widehat{\Sigma})$.*

Proof. We apply [Ren08, Theorem 5.2] to the twist $\hat{A} \times \mathbf{T} \longrightarrow \widehat{\Sigma} \longrightarrow \hat{A} \rtimes \mathcal{G}$ to see that the pair $(C_r^*(\hat{A} \rtimes \mathcal{G}; \widehat{\Sigma}), C_0(\hat{A}))$ is a Cartan pair with Weyl twist $(\hat{A} \rtimes \mathcal{G}; \widehat{\Sigma})$. The result then follows from the final statement of Theorem 3.4. $\qquad\square$

Example 5.7. Let X be a locally compact Hausdorff space and let $n \mapsto \sigma_n$ be an action of $\mathbf{N}^k$ by local homeomorphisms of X. Let Σ be the associated Deaconu–Renault groupoid $\{(x, m-n, y) : \sigma^m(x) = \sigma^n(y)\}$. Let $\mathcal{A}$ denote the interior of the isotropy in Σ, and suppose that $\mathcal{A}$ is a clopen subset of Σ (this may seem like a restrictive hypothesis but it is automatic if Σ is minimal by [KPS16, Proposition 2.1] applied to the cocycle $c(x, m, y) = m$, in which case it is given by $\mathcal{A} = \{(x, m, x) : (y, m, y) \in \Sigma \text{ for all } y \in X\}$ [SW16, Proposition 3.10]). Then $\mathcal{A}$ is an abelian group bundle. Specifically, for each $x \in X$, the set

$$\mathrm{Stab}^{\mathrm{ess}}\{p - q : p, q \in \mathbf{Z}^k \text{ and } \sigma^p = \sigma^q \text{ on a neighbourhood of } x\}$$

is a subgroup of $\mathbf{Z}^k$, we have $\mathrm{Stab}_x^{\mathrm{ess}} = \mathrm{Stab}_y^{\mathrm{ess}}$ whenever $\Sigma_y^x \neq \emptyset$, and as a set, $\mathcal{A} = \bigsqcup_{x \in X}\{x\} \times \mathrm{Stab}_x^{\mathrm{ess}}$. Since $\mathcal{A}$ is clopen, we can form the quotient groupoid $\mathcal{G} = \Sigma/\mathcal{A}$, which is étale and effective. A quick calculation shows that the action of $\mathcal{G}$ on $\mathcal{A}$ is given by $\gamma \cdot (s(\gamma), m) = (r(\gamma), m)$, and so the induced action of $\mathcal{G}$ on $\hat{A}$ is given by $(r(\gamma), \chi) \cdot \gamma = (s(\gamma), \chi)$ for $\chi \in (\mathrm{Stab}_{r(\gamma)})\widehat{}$. So, as a groupoid, $\hat{A} \rtimes \mathcal{G}$ is just the fibred product $\hat{A} * \mathcal{G}$—for example if $\mathcal{G}$ is minimal, then $\hat{A} \rtimes \mathcal{G} \cong \hat{A}_x \times \mathcal{G}$ for any $x \in X$. Since $\mathcal{A}$ is open in Σ, the map $p_{\mathcal{A}}$ is open and Theorem 3.4 applies. Combining with [Ren08, Theorem 5.2], this implies that $(C^*(\hat{A} \rtimes \mathcal{G}; \widehat{\Sigma}), C_0(\hat{A}))$ is a Cartan pair with Weyl twist $(\hat{A} \rtimes \mathcal{G}; \widehat{\Sigma})$.

Example 5.8. As a particular case of the preceding example, let Λ be a row-finite k-graph with no sources, and let $\mathcal{G}_\Lambda$ be the associated infinite-path groupoid, which is the Deaconu–Renault groupoid for the action of $\mathcal{N}_\tau^k$ on Λ^∞ by shift maps [KP00]. Suppose that the interior $\mathcal{I}$ of the isotropy in $\mathcal{G}_\Lambda$ is closed—for example, this is automatic if Λ is cofinal, in which case $\mathcal{I} \cong \Lambda^\infty \times \mathrm{Per}(\Lambda)$ [KPS16, Corollary 2.2]. (See [BLY17, Theorem 4.4] for a characterisation of when $\mathcal{I}$ is closed in $\mathcal{G}_\Lambda$.) Then our main theorem implies that $C^*(\mathcal{I}) \cong C_0(\hat{\mathcal{I}})$ is a Cartan subalgebra of $C^*(\Lambda)$, recovering (c) $\implies$ (b) of [BNR$^+$16, Corollary 4.5], and also gives a recipe for describing the associated Weyl twist.

Appendix A. Dual Haar Systems

If A is an abelian group with Haar measure β, then the *dual Haar measure* $\widehat{\beta}$ on $\hat{A}$ is the choice of Haar measure on $\hat{A}$ such that the Fourier Transform induces an isomorphism from $L^2(A, \beta)$ to $L^2(\hat{A}, \widehat{\beta})$. The methods of proof in [MRW96, §4] used that if $p : \mathcal{A} \to T$ has a Haar system, then $\hat{\pi} : \hat{\mathcal{A}} \to T$ also has a Haar system and that to produce one it suffices to simply take the dual Haar measures on the fibres. Unfortunately, Henrik Kreidler pointed out that the proof of this assertion in [MRW96, Proposition 3.6] has a gap. While our proof of Theorem 3.4 does not require it, we nevertheless take this opportunity to provide a proper proof.

Proposition A.1 (Hilsum & Renault). *Suppose that $p : \mathcal{A} \to X$ is an abelian group bundle with a Haar system $\beta = \{\beta^u\}_{u \in X}$, and that $\hat{p} : \hat{\mathcal{A}} \to X$ is the dual group bundle where $\hat{\mathcal{A}}$ is the Gelfand dual of $C^*(\mathcal{A})$. Then $\hat{p}$ is open. Hence $\hat{\mathcal{A}}$ has a Haar system $\widehat{\beta} = \{\widehat{\beta}^u\}_{u \in X}$. If we wish, we can assume that $\widehat{\beta}^u$ is the Haar measure on $\hat{A}(u)$ dual to β^u on $A(u)$.*

Proof. Since abelian groups are amenable, it is not hard to check that the $C_0(X)$-algebra $C^*(\mathcal{A})$ is a continuous field over X with fibres $C^*(A(u)) \cong C_0(\hat{A}(u))$—for example, see [Wil19, Corollary 5.39]. Hence the induced map $\hat{p} : \hat{\mathcal{A}} \to X$ is open—see [Wil07, Proposition C.10].

To prove the assertion about the system of dual Haar measures, we need some notation and what amounts to an operator-valued weight. We let $\mathcal{F} : C^*(\mathcal{A}) \to C_0(\hat{\mathcal{A}})$ be the Gelfand Transform; so $\mathcal{F}(a) = \hat{a}$ for all $a \in C^*(\mathcal{A})$. We will use convolution notation for multiplication in $C^*(\mathcal{A})$ as we primarily work in $C_c(\mathcal{A})$. For each $u \in X$ we can define a trace $\tau_u : C^*(\mathcal{A})^+ \to [0, \infty]$ by

$$\text{(A.1)} \qquad \tau_u(a) = \int_{\hat{\mathcal{A}}} \hat{a}(\chi) \, d\widehat{\beta}^u(\chi).$$

Thus for each $a \in C^*(\mathcal{A})^+$, we obtain a $[0, \infty]$-valued function, $\tau(a)$, on X given by $u \mapsto \tau_u(a)$. Let

$$P_\tau = \{ a \in C^*(\mathcal{A})^+ : \tau(a) \in C_0(X) \} \quad \text{and} \quad \mathcal{N}_\tau = \{ a \in C^*(\mathcal{A}) : a^*a \in P_\tau \}.$$

Let

$$\mathcal{M}_\tau = \mathcal{N}_\tau^* \mathcal{N}_\tau = \text{span} \, P_\tau.$$

By linearity, (A.1) holds for all $a \in \mathcal{M}_\tau$ and $u \mapsto \tau_u(a)$ is in $C_0(X)$. If $a, b \in C^*(\mathcal{A})^+$, then

$$\tau_u(a * b) \le \|a\| \tau_u(b).$$

To see that $\mathcal{M}_\tau$ is not reduced to $\{0\}$, consider $f \in C_c(\mathcal{A})$ and let $b = f^* * f$. Since $\hat{b} = |\hat{f}|^2$, it follows from the definition of $\widehat{\beta}^u$ that

$$b(u) = \int_{\mathcal{A}} |f(s)|^2 \, d\beta^u(s) = \int_{\hat{\mathcal{A}}} |\hat{f}(\chi)|^2 \, d\widehat{\beta}^u(\chi) = \tau_u(b).$$

Since $u \mapsto b(u)$ is in $C_c(X)$, we have $b \in \mathcal{M}_\tau$. More generally we have the following.

Lemma A.2. *We have*

$$C_c(\mathcal{A})^2 := \mathrm{span}\{\, g^* * f : f, g \in C_c(\mathcal{A}) \,\} \subset \mathcal{M}_\tau,$$

and for each $b \in C_c(\mathcal{A})^2$ we have

$$\tau_u(b) = b(u).$$

In particular, $\mathcal{M}_\tau$ is dense in $C^*(\mathcal{A})$.

Proof. We use the usual polarization identity for complex sesquilinear forms ([Ped89, p. 80]):

$$b = g^* * f = \sum_{k=0}^{3} i^k f_k^* * f_k,$$

where $f_k := f + i^k g$. Now, as above,

$$b(u) = \sum_{k=0}^{3} i^k f_k^* * f_k(u) = \sum_{k=0}^{3} i^k \int_{\mathcal{A}} |f_k(s)|^2 \, d\beta^u(s)$$

$$= \sum_{k=0}^{3} i^k \int_{\hat{\mathcal{A}}} |\hat{f}_k(\chi)|^2 \, d\widehat{\beta}^u(\chi) = \int_{\hat{\mathcal{A}}} \hat{b}(\chi) \, d\widehat{\beta}^u(\chi) = \tau_u(b). \qquad \square$$

Lemma A.3. *$\mathcal{M}_\tau$ is a (not necessarily closed) ideal in $C^*(\mathcal{A})$.*

Proof. Suppose $a \in \mathcal{M}_\tau$ and $b \in C^*(\mathcal{A})$ we need to see that $ab \in \mathcal{M}_\tau$. By linearity, it will suffice to consider only $a, b \in C^*(\mathcal{A})^+$.

Suppose that $b \in C_c(\mathcal{A})^2 \cap C^*(\mathcal{A})^+$. Since $a = c^* * c$, there are $\{\, f_i \,\}$ in $C_c(\mathcal{A})$ such that $a_i := f_i^* * f_i \to a$ in $C^*(\mathcal{A})^+$. Note that $a_i * b \in C_c(\mathcal{A})^2 \cap C^*(\mathcal{A})^+ \subset \mathcal{M}_\tau$. We have

$$|\tau_u(a * b) - \tau_u(a_i * b)| = \left| \int_{\hat{\mathcal{A}}} (a - a_i)^\wedge(\chi)\hat{b}(\chi) \, d\widehat{\beta}^u(\chi) \right| \leq \|a - a_i\|\tau_u(b).$$

Thus $u \mapsto \tau_u(a_i * b)$ is in $C_c(X)$ by Lemma A.2, and these functions converge uniformly to $u \mapsto \tau_u(a * b)$. Hence the latter is in $C_0(X)$.

Now suppose that $b \in C^*(\mathcal{A})^+$. As above, we can find $b_i \to b$ with $b_i \in C_c(\mathcal{A})^2 \cap C^*(\mathcal{A})^+$. Then

$$|\tau_u(a * b) - \tau_u(a * b_i)| \leq \tau_u(a)\|b - b_i\|.$$

By the above argument, $u \mapsto \tau_u(a * b_i)$ is in $C_0(X)$, and again these functions converge uniformly to $u \mapsto \tau_u(a * b)$. Hence $a * b \in \mathcal{M}_\tau$. Therefore $\mathcal{M}_\tau$ is an ideal. $\square$

Since $\mathcal{M}_\tau$ is a dense ideal in $C^*(\mathcal{A})$, it contains the Pedersen ideal of $C^*(\mathcal{A})$. Since the Pedersen ideal of $C_0(\hat{\mathcal{A}})$ is $C_c(\hat{\mathcal{A}})$, we deduce that $\mathcal{F}^{-1}(C_c(\hat{\mathcal{A}})) \subset \mathcal{M}_\tau$. Thus for $f \in C_c(\hat{\mathcal{A}})$,

$$u \mapsto \tau_u(\mathcal{F}^{-1}(f))$$

is continuous. But

$$\int_{\hat{\mathcal{A}}} f(\chi) \, d\widehat{\beta}^u(\chi) = \tau_u(\mathcal{F}^{-1}(f)).$$

Thus $\widehat{\beta}$ is a Haar system on $\widehat{\mathcal{A}}$.   □

Remark A.4. Proposition A.1 can be reformulated without any reference to the dual group bundle. This new formulation is valid for non-abelian locally compact group bundles. It hinges on a notion of operator-valued weight, which is familiar in the theory of von Neumann algebras but not so well developed for C^*-algebras. Let us first consider the commutative case. Suppose that $q : Z \to X$ is a continuous, surjective and open map, where X and Z are locally compact spaces. Given a q-system of measures $\tau = (\tau_x)_{x \in X}$, we define $T(f)(x) = \int f d\tau_x$ for suitable functions f on Z. One can show that the usual continuity requirement for τ, namely $T(f)$ is continuous for all $f \in C_c(Z)$, is equivalent to the fact that $T(f)$ is lower semicontinuous for all $f \in C_0^+(Z)$. Given a $C_0(X)$-algebra A, we define a $C_0(X)$-weight as an additive and $C_0^+(X)$-homogeneous map $T : A^+ \to \mathrm{LSC}^+(X)$, where $\mathrm{LSC}^+(X)$ is the convex cone of lower semicontinuous functions $f : X \to [0, \infty]$. The $C_0(X)$-weight T is called densely defined if its domain $\{\, a \in A^+ : T(a) \in C_0^+(X) \,\}$ is dense in A^+. It is called lower semicontinuous if $T(a) \le \liminf T(a_n)$ whenever (a_n) converges to a in A^+. It can be shown that, in the above commutative example, the map $\tau \mapsto T$ is a bijection between continuous q-systems of measures and densely defined and lower semicontinuous $C_0(X)$-weights on $C_0(Z)$. In particular, Proposition A.1 can be expressed as the existence of a densely defined and lower semicontinuous $C_0(X)$-weight $T : C^*(\mathcal{A})^+ \to \mathrm{LSC}^+(X)$ such that $T(f) = f|_X$ for all $f \in C_c^+(\mathcal{A})$. We call it the canonical $C_0(X)$-weight of the C^*-algebra of the group bundle $\mathcal{A} \to X$. As said earlier, this formulation makes sense for a non-abelian locally compact group bundle $G \to X$ endowed with a Haar system λ and it remains true. However, its proof has to be modified. Here is a quick sketch. One defines T by $T(a)(x) = \varphi_x(L_x(a_x))$, where $a \in C^*(G)^+$ is viewed as a section of the field of C^*-algebras $x \mapsto C^*(G_x)$, L_x is the regular representation and φ_x is is the canonical weight on the von Neumann algebra of G_x as defined in [Haa78]. In order to show that $T(a)$ is lower semicontinuous, one uses a bounded approximate identity in $C_c^+(G)$.

Appendix B. Comments on Quotient Banach Bundles

This material is based on [KMRW98, §2] with appropriate modifications for upper semicontinuous Banach bundles.

We let G be a second countable locally compact Hausdorff groupoid with a Haar system λ. We suppose that $p : \mathscr{E} \to G^{(0)}$ is an upper semicontinuous Banach bundle admitting a continuous G-action $\gamma \cdot a = \alpha_\gamma(a)$ where $\alpha_\gamma : E(s(\gamma)) \to E(r(\gamma))$ is an *isometric* Banach space isomorphism of the fibres of $\mathscr{E}$.

Now let X be a *free* and proper left G-space. We can form the pullback $X * \mathscr{E} = \{\, (x, a) \in X \times \mathscr{E} : r(x) = p(a) \,\}$. Then $X * \mathscr{E}$ is an upper semicontinuous Banach bundle over X which is a continuous Banach bundle if $\mathscr{E}$ is. We get a left G-action on $X * \mathscr{E}$ given by

$$\gamma \cdot (x, a) = (\gamma \cdot x, \alpha_\gamma(a)).$$

We will write $\mathscr{E}^X$ for the orbit space $G \backslash (X * \mathscr{E})$ with its quotient topology, and write $[x, a]$ for the orbit of (x, a).

Lemma B.1. *The map $p^X : \mathscr{E}^X \to G \backslash X$ given by $p^X([x, a]) = G \cdot x$ is a continuous open surjection.*

Proof. Continuity is clear. To see that p^X is open, we use Fell's criterion (see Remark 3.1). Suppose that $G \cdot x_i \to G \cdot x = p^X([x, a])$. It suffices to lift a subnet to $\mathscr{E}^X$. After passing to a subnet, and relabeling, we can assume that $x_i \to x$. Since $r(x_i) \to r(x) = p(a)$ in $\Sigma^{(0)}$ and since p is open, we can pass to another subnet, relabel, and assume that there are $a_i \to a$ in $\mathscr{E}$ such that $p(a_i) = r(x_i)$. But then $[x_i, a_i] \to [x, a]$ as required. $\qquad\square$

To see that $\mathscr{E}^X$ is an upper-semicontinuous Banach bundle, we first have to make the fibres $(p^X)^{-1}(G \cdot x)$ into Banach spaces. The map $a \mapsto [x, a]$ is a bijection of $E(r(x))$ onto $(p^X)^{-1}(G \cdot x)$. Thus we can define $\big\| [x, a] \big\| = \|a\|$, $[x, a] + [x, b] = [x, a + b]$, and $\lambda[x, a] = [x, \lambda a]$.[7] This norm is independent of our choice of representative $x \in G \cdot x$ and makes $(p^X)^{-1}(G \cdot x)$ into a Banach space. Note that $E^X(G \cdot x) := (p^X)^{-1}(G \cdot x)$ is isomorphic to $E(r(x))$; however, this identification is non-canonical as it depends on the choice of representative for $G \cdot x$.

Lemma B.2 ([KMRW98, Proposition 2.15]). *Suppose that $p : \mathscr{E} \to \Sigma^{(0)}$ is an upper-semicontinuous (continuous) Banach bundle on which G acts by isometric isomorphisms $\alpha_\gamma : E(s(\gamma)) \to E(r(\gamma))$. Let $p^X : \mathscr{E}^X \to G\backslash X$ be the quotient bundle described above with the given Banach space structure on the fibres $E^X(G \cdot x)$. Then $\mathscr{E}^X$ is an upper-semicontinuous (continuous) Banach bundle over $G\backslash X$. In particular, if $\mathscr{E}$ is a C^*-bundle and G acts by $*$-isomorphisms, then $\mathscr{E}^X$ is a C^*-bundle.*

Proof. The proof proceeds by checking that the axioms for a Banach bundle hold just as in [KMRW98, Proposition 2.15]. The only "upgrade" from [KMRW98] is to include the possibility that p is merely upper-semicontinuous. But if $[x_i, a_i] \to [x, a]$ in $\mathscr{E}^X$ with $\|a_i\| \geq \epsilon > 0$ for all i, then we can pass to a subnet, relabel, and assume that $(\gamma_i x_i, \alpha_{\gamma_i}(a_i)) \to (x, a)$. Since each α_{γ_i} is isometric, $\|a\| \geq \epsilon$. $\qquad\square$

Sections. Given a Banach bundle $p : \mathscr{E} \to X$, the space $E = \Gamma_0(X; \mathscr{E})$ s a Banach space under pointwise operations and the supremum norm. However, it is not immediately clear that E consists of more than just the zero section. We would like to know that there are "sufficiently many sections" in that given $a \in E(x)$ there exists $f \in \Gamma(X; \mathscr{E})$ such that $f(x) = a$. Then, for example, if $\mathscr{E}$ is a C^*-bundle, it is routine to show that $A = \Gamma_0(X; \mathscr{E})$ is a $C_0(X)$-algebra with fibres $A(x)$—see [Wil07, Proposition C.23]. A remarkable result due to Douady and del Soglio-Hérault implies that any continuous Banach bundle has sufficiently many sections [DdSH]. While the authors of [DdSH] never published their proof, Fell and Doran give their proof in [FD88a, Appendix C]. There they also point out that the "same proof" gives the same result for upper-semicontinuous Banach bundles and they attribute this observation to Hofmann—see [FD88a, Remark C.18] and [Hof77, Proposition 3.4]. However, Hofmann's proof is buried in some unpublished notes [Hof74]. Normally, we can ignore this abstract approach as our bundles come from Banach spaces—or better yet $C_0(X)$-algebras—and the existence of nice sections is obvious. However, in this section, we constructed our quotient bundles as an orbit spaces. Hence at first blush, the elements of $A^X = \Gamma_0(G\backslash X; \mathscr{E}^X)$ are a bit mysterious. But we can give a fairly

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[7]More formally, we should observe that $\big\| [x, a] \big\|$ does not depend on the choice of x since G acts isometrically. Moreover $[x, a] + [\gamma \cdot x, b] = [x, a] + [x, \alpha_\gamma^{-1}(b)] = [x, a + \alpha_\gamma^{-1}(b)]$, etc.

concrete description using the freeness of the G-action and assuming our original bundles have sufficiently many sections.

Note that if $\check{f} \in \Gamma(G \backslash X; \mathscr{E}^X)$, then since the G-action on X is free, there is a function $f : X \to \mathscr{E}$ such that $f(x) \in E(r(x))$ and

$$\check{f}(G \cdot x) = [x, f(x)].$$

Furthermore,

$$f(\gamma \cdot x) = \alpha_\gamma(f(x)).$$

Now suppose that $x_i \to x$. Then

$$\check{f}(G \cdot x_i) = [x, f(x_i)] \to \check{f}(G \cdot x) = [x, f(x)].$$

Since p^X is open, we can pass to a subnet, relabel, and assume that there are γ_i such that

$$\bigl(\gamma_i \cdot x_i, \alpha_{\gamma_i}(f(x_i))\bigr) \to \bigl(x, f(x)\bigr) \quad \text{in } X * \mathscr{E}.$$

Since the G-action on X is free and proper, we can assume that $\gamma_i \to s(x)$. But then since the G action on $X * \mathscr{E}$ is continuous, $(x_i, f(x_i)) \to (x, f(x))$, and in particular, $f(x_i) \to f(x)$. Since we can repeat this argument for any subnet of $\{f(x_i)\}$ it follows that the original net converges to $f(x)$ and $f : X \to \mathscr{E}$ is continuous. As a consequence, we have the following where, as is standard, we have identified sections $f \in \Gamma(X, X * \mathscr{E})$ of the pullback with continuous functions $f : X \to \mathscr{E}$ such that $p(f(x)) = r(x)$.

Proposition B.3. *The sections in $\check{f} \in \Gamma(G \backslash X; \mathscr{E}^X)$ are in one-to-one correspondence with sections $f \in \Gamma(X; X * \mathscr{E})$ such that*

(B.1)
$$f(\gamma \cdot x) = \alpha_\gamma\bigl(f(x)\bigr) \quad \text{for all } (\gamma, x) \in G * X.$$

We have

$$\check{f}(G \cdot x) = [x, f(x)].$$

In particular, we can identify $\Gamma_c(G \backslash X, \mathscr{E}^X)$ with the space $C_c(G \backslash X, X, \mathscr{E}, \alpha)$ of continuous functions $f : X \to \mathscr{E}$ which transform according to (B.1), and such that the support of f has compact image in $G \backslash X$.

Recall that we say a net $\{\check{f}_i\}$ in $\Gamma_c(G \backslash X, \mathscr{E}^X)$ converges to $\check{f}$ in the inductive-limit topology if $\check{f}_i \to \check{f}$ uniformly and the supports of the $\check{f}_i$ are all contained in a fixed compact set. This is equivalent to saying that the corresponding functions $f_i \in C_c(G \backslash X, X, \mathscr{E}, \alpha)$ converge uniformly to f with supports all contained in a set with compact image in $G \backslash X$.

Since the G-action on X is free and proper, it is easy to explicitly produce sections satisfying (B.1) in abundance. Let $h \in \Gamma_c(X; X * \mathscr{E})$. Let

$$\Theta_h(x) = \int_G \alpha_\gamma\bigl(h(\gamma^{-1} \cdot x)\bigr) \, d\lambda^{r(x)}(\gamma).$$

Then $\Theta_h \in C_c(G \backslash X, X, \mathscr{E}, \alpha)$. Then it is not hard to verify that if $a \in E(r(x))$ for all $n \geq 1$ we can find h_n such that $\|\Theta_{h_n}(x) - a\| < 1/n$. With a little more work, we can show that there are sufficiently many sections in $\Gamma_c(G \backslash X; \mathscr{E}^X)$.

Lemma B.4. *Let $a \in E(r(x))$. Then there exists $h \in \Gamma_c(X, X * \mathscr{E})$ such that $\Theta_h(x) = a$.*

Proof. Since G acts freely and properly on X, the orbit $G \cdot x$ is closed in X and $\gamma \mapsto \gamma^{-1} \cdot x$ is a homeomorphism of $G^{r(x)}$ onto $G \cdot x$. Therefore $\gamma^{-1} \cdot x \mapsto \alpha_\gamma^{-1}(a)$ is in $\Gamma(G \cdot x, X * \mathscr{E})$. Let V be a compact neighborhood of $r(x)$ in G. Then we can find $h_0 \in \Gamma_c(X, X * \mathscr{E})$ such that

(B.2)
$$h_0(\gamma^{-1} \cdot x) = \alpha_\gamma^{-1}(x) \quad \text{for all } \gamma \in V.$$

By the vector-valued Tietze Extension Theorem for upper-semicontinuous Banach bundles (as in [MW08a, Proposition A.5]),[8] we may as well assume that $h_0 \in \Gamma_c(X, X * \mathscr{E})$ and still satisfies (B.2).

Take $f \in C_c^+(X)$ such that $\gamma \mapsto f(\gamma^{-1} \cdot x)$ has support in V and

$$\int_G f(\gamma^{-1} \cdot x) \, d\lambda^{r(x)}(\gamma) = 1.$$

Let $h(x) = f(x)h_0(x)$. Then

$$\Theta_h(x) - a = \int_G f(\gamma^{-1} \cdot x)\alpha_\gamma\big(h_0(\gamma^{-1} \cdot x)\big) - a) \, d\lambda^{r(x)}(\gamma) = 0. \quad \square$$

This explicit construction of sections is also useful to make the following observation.

Lemma B.5. *Suppose that $\mathscr{E}_0$ is a sub-bundle of $\mathscr{E}$ that is invariant under the G-action and such that $E_0(u)$ is dense in $E(u)$ for all $u \in G^{(0)}$. Then for each $a \in E(r(x))$ and each $\epsilon > 0$, there is a $h_0 \in \Gamma(X, X * \mathscr{E}_0)$ such that $\|\Theta_{h_0}(x) - a\| < \epsilon$. Hence the sections corresponding to $C_c(G\backslash X, X, \mathscr{E}_0, \alpha)$ are dense in $\Gamma_c(G\backslash X, \mathscr{E}^X)$ in the inductive-limit topology.*

Appendix C. The Fell Bundle of a Twisted Dynamical System

As in the main body of the paper, we let $\mathcal{A}$ be a normal subgroup bundle of Σ. As in Section 2, we fix Haar systems β on $\mathcal{A}$ and λ on Σ. We also let α be the Haar system on $\mathcal{G} := \Sigma/\mathcal{A}$ obtained from Lemma 2.7. Recall that we usually write $\dot{\sigma}$ in place of $\sigma\mathcal{A}$.

Let $p_\mathscr{E} : \mathscr{E} \to \Sigma^{(0)}$ be an upper-semicontinuous C^*-bundle with fibres $E(u) = p_\mathscr{E}^{-1}(u)$, and suppose that $(\mathscr{E}, \Sigma, \vartheta)$ is a groupoid dynamical system. Let $U(E(u)) \subset M(E(u))$ be the unitary group of $E(u)$, and let $\coprod_{u \in \Sigma^{(0)}} U(E(u))$ be the corresponding (algebraic) group bundle over $\Sigma^{(0)}$. Then a (Green–Renault) *twisting map* for ϑ is a unit-space fixing homomorphism $\kappa : \mathcal{A} \to \coprod_{u \in \Sigma^{(0)}} U(E(u))$ that induces a continuous action[9] of $\mathcal{A}$ on $\mathscr{E}$ by isometric (Banach space) isomorphisms via $a \cdot e = \kappa(a)e$ such that

$$\vartheta_a(e) = \kappa(a)e\kappa(a)^* \quad \text{for all } (a, e) \in \mathcal{A} * \mathscr{E}, \text{ and}$$
$$\kappa(\sigma a \sigma^{-1}) = \bar{\vartheta}_\sigma(\kappa(a)) \quad \text{for all } (\sigma, a) \in \Sigma * \mathcal{A}.$$

Since $e \mapsto e^*$ is continuous, so is $(a, e) \mapsto e\kappa(a)^* = (\kappa(a)e^*)^*$.

Remark C.1. If Σ is a group G and $\mathcal{A} \leq G$ a normal subgroup N, then κ is a twisting map for (E, G, ϑ) as in [Gre78]. The extension of twists to groupoid dynamical systems comes from [Ren87, §3] where it was assumed that $\mathcal{A}$ was an abelian group bundle.

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[8]Hence we are assuming $X * \mathscr{E}$ has sufficiently many sections.

[9]That is, the map $(a, e) \mapsto \kappa(a)e$ is continuous from $\mathcal{A} * \mathscr{E} \to \mathscr{E}$.

We want to associate a Fell bundle over $\mathcal{G}$ to a twisted dynamical system $(\mathscr{E}, \mathcal{G}, \Sigma, \vartheta, \kappa)$ as defined above. This approach follows [MW08a, §2]. Define a left $\mathcal{A}$-action by isometric isomorphisms on $r^*\mathscr{E} = \Sigma * \mathscr{E} = \{\, (\sigma, e) \in \Sigma \times \mathscr{E} : r(\sigma) = p_{\mathscr{E}}(e) \,\}$ by

$$a \cdot (\sigma, e) = (a\sigma, e\kappa(a)^*).$$

As in Appendix B, we can form a Banach bundle $\mathscr{B} = \mathscr{B}(\mathscr{E}, \vartheta, \kappa) = \mathscr{E}^\Sigma$ over $\mathcal{G} = \mathcal{A}\backslash\Sigma$ by forming the quotient $\mathcal{A}\backslash r^*\mathscr{E}$. We will write $[\sigma, e]$ for the orbit of (σ, e) in $\mathscr{B}$. Thus $p_{\mathscr{B}} : \mathscr{B} \to \mathcal{G}$ is given by $p_{\mathscr{B}}([\sigma, e]) = \dot\sigma$.

If $(\dot\sigma, \dot\tau) \in \mathcal{G}^{(2)}$, then we can define

(C.1) $$(\sigma, e)(\tau, f) = \big(\sigma\tau, e\vartheta_\sigma(f)\big),$$

and compute that

$$(a\sigma, e\kappa(a)^*)(b\tau, f\kappa(b)^*) = (a\sigma b\sigma^{-1}) \cdot \big(\sigma\tau, e\vartheta_\sigma(f)\big).$$

Hence (C.1) is well defined on elements of $\mathscr{B}$ and defines a "multiplication map" on $\mathscr{B}^{(2)} = \{\, ([\sigma, e], [\tau, f]) \in \mathscr{B} \times \mathscr{B} : (\dot\sigma, \dot\tau) \in \mathcal{G}^{(2)} \,\}$ given by

(C.2) $$[\sigma, e][\tau, f] := [\sigma\tau, e\vartheta_\sigma(f)].$$

Similarly, we get a well-defined involution on $\mathscr{B}$ defined by

(C.3) $$[\sigma, e]^* := [\sigma^{-1}, \vartheta_\sigma^{-1}(e^*)].$$

Lemma C.2. *With multiplication and involution defined by* (C.2) *and* (C.3), *respectively,* $\mathscr{B}(\mathscr{E}, \vartheta, \kappa)$ *is a Fell bundle over* $\mathcal{G}$ *as defined in* [MW08a, Definition 1.1].

Proof. The continuity of multiplication and the involution follows from the openness of the projection $p_{\mathscr{B}} : \mathscr{B} \to \mathcal{G}$ and the continuity of the action of Σ on $\mathscr{E}$. The algebraic properties of multiplication are routine to verify as are axioms (a), (b), and (c) of [MW08a, Definition 1.1].

For (d) and (e), recall that the fibre $B(\dot\sigma)$ of $\mathscr{B}$ over $\dot\sigma \in \mathcal{G}$ is $\{\, [\sigma, e] \,\}$ equipped with the Banach-space structure induced by the map $e \mapsto [\sigma, e]$ for $e \in E(r(\sigma))$. So for $u \in \Sigma^{(0)}$, the space $B(u)$ is a C^*-algebra isomorphic to $E(u)$ with the induced multiplication and involution coming from $\mathscr{B}$. This is (d).

For (e), we must show that $B(\dot\sigma)$ is a $B(r(\sigma)) - B(s(\sigma))$-imprimitivity bimodule under the actions and inner products induced by $\mathscr{B}$. These actions and inner products are exactly those coming from viewing $E(r(\sigma))$ as a $E(r(\sigma)) - E(s(\sigma))$-imprimitivity bimodule using the isomorphism $\vartheta_\sigma^{-1} : E(r(\sigma)) \to E(s(\sigma))$ (see [MW08a, Example 2.1]). $\square$

Given $\mathscr{B} = \mathscr{B}(\mathscr{E}, \vartheta, \kappa)$, we can form the associated Fell bundle C^*-algebra. First it is convenient to recall, that as described in Appendix B, sections $\check{f} \in \Gamma(\mathcal{G}, \mathscr{B})$ are determined by continuous functions $f : \Sigma \to \mathscr{E}$ such that

(C.4) $$f(a\sigma) = f(\sigma)\kappa(a)^* \quad \text{for all } (a, \sigma) \in \mathcal{A} * \mathscr{E}.$$

Then

$$\check{f}(\dot\sigma) = [\sigma, f(\sigma)].$$

Thus $\Gamma_c(\mathcal{G}, \mathscr{B})$ is isomorphic to the space $C_c(\mathcal{G}, \Sigma, \mathscr{E}, \kappa)$ of continuous functions $f : \Sigma \to \mathscr{E}$ that satisfy (C.4) and whose support has compact image in $\mathcal{G}$, with operations

$$(C.5) \qquad f * g(\sigma) = \int_{\mathcal{G}} f(\tau) \vartheta_\tau(g(\tau^{-1}\sigma) \, d\alpha^{r(\sigma)}(\dot{\tau}) \quad \text{and} \quad f^*(\sigma) = \vartheta_\sigma(f(\sigma^{-1})^*).$$

For example,

$$\check{f} * \check{g}(\dot{\sigma}) = \int_{\mathcal{G}} \check{f}(\dot{\tau}) \check{g}(\dot{\tau}^{-1}\dot{\sigma}) \, d\alpha^{r(\sigma)}(\dot{\tau}) = \int_{\mathcal{G}} [\sigma, f(\tau)\vartheta_\tau(g(\tau^{-1}\sigma))] \, d\alpha^{r(\sigma)}(\dot{\tau}),$$

and since $(\sigma, a) \mapsto [\sigma, a]$ is an isomorphism of Banach spaces, this gives

$$\check{f} * \check{g}(\dot{\sigma}) = \Big[\sigma, \int_{\mathcal{G}} f(\tau)\vartheta_\tau(g(\tau^{-1}\sigma) \, d\alpha^{r(\sigma)}(\dot{\tau})\Big].$$

The I-norm on $C_c(\mathcal{G}, \Sigma, \mathscr{E}, \kappa)$ is given by

$$\|f\|_I = \max\Big\{ \sup_{u \in \Sigma^{(0)}} \int_{\mathcal{G}} \|f(\sigma)\| \, d\alpha^u(\dot{\sigma}), \ \sup_{u \in \Sigma^{(0)}} \int_{\mathcal{G}} \|f(\sigma)\| \, d\alpha_u(\dot{\sigma})\Big\}.$$

The Fell bundle C^*-algebra $C^*(\mathcal{G}; \mathscr{B}(\mathscr{E}, \vartheta, \kappa))$ is the completion of $C_c(\mathcal{G}, \Sigma, \mathscr{E}, \kappa)$ universal for I-norm decreasing representations of $C_c(\mathcal{G}, \Sigma, \mathscr{E}, \kappa)$.

Remark C.3. Using (C.5), it follows that the Fell bundle C^*-algebra $C^*(\mathcal{G}; \mathscr{B}(\mathscr{E}, \vartheta, \kappa))$ is the same as Renault's twisted crossed product $C^*(\mathcal{G}, \Sigma, C^*(\mathcal{A}), \alpha)$ from [Ren87].

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