

# SMOOTH CARTAN TRIPLES AND LIE TWISTS OVER HAUSDORFF ÉTALE LIE GROUPOIDS

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**ABSTRACT.** We characterise when a smooth structure on the unit space of a Hausdorff étale groupoid can be extended to a Lie-groupoid structure on the whole groupoid. We introduce Lie twists over Hausdorff Lie groupoids, building on Kumjian’s notion of a twist over a topological groupoid. We establish necessary and sufficient conditions on a family of sections of a twist over a Lie groupoid under which the twist can be made into a Lie twist so that all the specified sections are smooth. We obtain conditions on a twist over an étale groupoid whose unit space is a smooth manifold and a family of sections of the twist that characterise when the pair can be made into a Lie twist for which the given sections are smooth. We use these results to describe conditions on a Cartan pair of  $C^*$ -algebras and a family of normalisers of the subalgebra, under which Renault’s Weyl twist for the pair can be made into a Lie twist for which the given normalisers correspond to smooth sections.

## 1. INTRODUCTION

The motivating idea of this paper is that Connes’ reconstruction theorem for manifolds [6] and Kumjian–Renault theory for twists over étale groupoids [14, 19] can be employed in tandem to understand the extent to which Lie-groupoid structures on the components of topological twists over étale groupoids can be recovered from spectral and functional-analytic data.

Connes’ theorem says that a spectral triple  $(\mathcal{A}, \mathcal{H}, D)$  that satisfies slightly strengthened versions of the first five axioms laid out in [5] and in which the algebra  $\mathcal{A}$  is unital and commutative, determines a manifold  $M$  that realises  $\mathcal{A}$  as the algebra  $C^\infty(M)$  of smooth functions. This can be regarded as a deep geometric strengthening of Gelfand duality for commutative  $C^*$ -algebras: the Gelfand–Naimark Theorem says that every unital commutative  $C^*$ -algebra  $A$  is the algebra of continuous functions on a compact Hausdorff space  $X$ ; and, very roughly speaking, Connes’ theorem describes when this  $X$  can be endowed with the structure of a manifold in terms of data that specifies a subalgebra  $\mathcal{A}$  of  $A$  that is to consist of smooth functions and an unbounded operator associated to  $A$  that is to be a Dirac-type differential operator.

Kumjian’s and Renault’s theorems can also be regarded as a strengthening of the Gelfand–Naimark Theorem. They describe under what circumstances a given commutative subalgebra  $B$  of a  $C^*$ -algebra  $A$  “coordinatises  $A$ ” (to borrow a phrase of Muhly’s) in the sense that there is a topological *twist*  $\mathbb{T} \times G^{(0)} \xrightarrow{\iota} E \xrightarrow{\pi} G$  (often called the *Weyl twist*) over an effective Hausdorff étale groupoid  $G$  (called the *Weyl groupoid*) with unit space the Gelfand spectrum  $\widehat{B}$  of  $B$  such that the Gelfand isomorphism  $B \cong C_0(\widehat{B})$  extends to an isomorphism of  $A$  onto the reduced twisted groupoid  $C^*$ -algebra  $C_r^*(G; E)$ . The characterisation is in terms of *normalisers* of  $B$  in  $A$ : elements  $n \in A$  satisfying  $n^*Bn \cup nBn^* \subseteq B$ . The point is that the partial-isometric factors of polar decompositions of normalisers determine partial homeomorphisms of  $\widehat{B}$ , which can be stitched together into the desired groupoid  $G$ ; the twist  $E$  then measures the phase-differences between the partial-isometric factors of pairs of normalisers that determine the same partial homeomorphisms. Proposition 4.1 of [10] shows that any family of normalisers that densely spans  $A$  is “big enough” to recover  $G$  and  $E$ .

Taken together, the results described in the preceding two paragraphs suggest a natural question. Suppose that we are given a twist  $E$  over a Hausdorff effective étale groupoid  $G$ , and hence a corresponding Cartan pair  $B \subseteq A$  of  $C^*$ -algebras. Suppose in addition that  $G$  and  $E$  carry smooth structures under which they are Lie groupoids and such that the bundle map  $\pi: E \rightarrow G$  is a submersion and the inclusion  $\iota: \mathbb{T} \times G^{(0)} \rightarrow E$  is smooth—that is, the pair constitutes an  $S^1$ -central extension in the sense of

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[1, Definition 4.1], with the additional property that the reduction of the extension to the unit space is a trivial smooth  $\mathbb{T}$ -bundle. There is then, in addition to the smooth subalgebra  $B^\infty = C_0^\infty(G^{(0)})$  of  $B$ , a natural collection of “smooth normalisers” that densely span  $C_r^*(G; E)$ , namely the smooth  $\mathbb{T}$ -equivariant functions on  $E$  whose open supports are preimages of open bisections. The natural question, which we answer here, is, “to what extent is this process reversible?” That is, given a Cartan pair  $B \subseteq A$  and a family  $\mathcal{N}$  of normalisers that densely spans  $A$  and for which  $B^\infty := \mathcal{N} \cap B$  is a dense  $*$ -subalgebra of  $B$ , what properties characterise the existence of a Lie-twist structure on the Weyl twist for which  $\mathcal{N}$  is a family of smooth sections and  $B^\infty$  is the algebra of smooth functions on  $G^{(0)}$ ? Connes’ theorem does the heavy lifting of deciding which pairs  $B, B^\infty$  can appear, so it is reasonable to take as given that  $\widehat{B}$  comes already endowed with a smooth-manifold structure and that  $B^\infty = C_0^\infty(\widehat{B})$  and ask what is required of  $\mathcal{N}$ .

There are two parts to our answer to this question. The first part deals with whether the Weyl groupoid  $G$  can be made into a Lie groupoid. This is reasonably straightforward. We prove in Section 3 that if  $G$  is an étale groupoid and  $G^{(0)}$  is a manifold, then there is a unique smooth structure on  $G$  with respect to which the range map is smooth. The source map is also smooth with respect to this structure if and only if the partial homeomorphisms  $r(\gamma) \mapsto s(\gamma)$ ,  $\gamma \in U$ , determined by open bisections  $U$  of  $G$  are in fact diffeomorphisms, and in this case  $G$  is a Lie groupoid. Translating via Renault’s theory into statements about normalisers, this is equivalent to the condition that for each normaliser of the Cartan subalgebra  $B$ , the partial homeomorphism on  $\widehat{B}$  determined by the partial-isometric factor in its polar decomposition is a partial *diffeomorphism*, and we prove that this is equivalent to the existence of a family  $\mathcal{N}$  of normalisers  $n$  that densely span  $A$  and satisfy  $nB^\infty n^* \cup n^* B^\infty n \subseteq B^\infty$ .

The second part is more subtle and deals with the existence of a smooth structure on the twist  $E$ . We begin by analysing in Section 4 what families  $\{\sigma_\alpha\}_\alpha$  of local sections of a twist  $\mathbb{T} \times G^{(0)} \xrightarrow{\iota} E \xrightarrow{\pi} G$  over a Lie groupoid  $G$  can be the smooth sections for a Lie-twist structure on  $E$ . Since our definition of a Lie twist requires that  $\iota$  be smooth, we require that whenever the domain  $U_\alpha$  of  $\sigma_\alpha$  intersects  $G^{(0)}$ , the restriction of  $\sigma_\alpha$  to  $U_\alpha \cap G^{(0)}$  is the image under  $\iota$  of a smooth section of  $\mathbb{T} \times G^{(0)}$ ; we call this condition  $(U^\infty)$ , the letter  $U$  standing for “units.” This provides a means of evaluating whether the difference between various combinations of the  $\sigma_\alpha$  are smooth, and the remaining conditions exploit this. We require that on intersections  $U_\alpha \cap U'_\alpha$  the difference  $\sigma_\alpha(\gamma)\sigma_{\alpha'}(\gamma)^{-1}$  is the image under  $\iota$  of a smooth function (Condition  $(S^\infty)$ , for “smooth”); that when  $U_{\alpha_1}U_{\alpha_2}$  intersects  $U_\alpha$ , the difference  $\sigma_{\alpha_1}(\gamma_1)\sigma_{\alpha_2}(\gamma_2)\sigma_\alpha(\gamma_1\gamma_2)^{-1}$  is the image of a smooth function on its domain (Condition  $(M^\infty)$ , for “multiplication”); and that where  $U_\alpha \cap U_{\alpha'}^{-1}$  is nonempty, the product  $\sigma_\alpha(\gamma)\sigma_{\alpha'}(\gamma^{-1})$  is the image of a smooth function (Condition  $(I^\infty)$ , for “inverses”). Our first main theorem says, roughly speaking, that Conditions  $(U^\infty)$  and  $(S^\infty)$  guarantee that there is a unique smooth structure on  $E$  under which it is a smooth principal  $\mathbb{T}$ -bundle and the  $\sigma_\alpha$  are all smooth; and that, for this smooth structure, the maps  $\iota, \pi$  in the twist and the subspaces  $E^{(0)}$  and  $\pi^{-1}(G^{(0)})$  all satisfy suitable smoothness conditions. We then prove that  $E$  is a Lie twist over  $G$  with this smooth structure precisely if, in addition, Conditions  $(M^\infty)$  and  $(I^\infty)$  are satisfied. In Subsection 4.1, we then specialise to the situation where  $G$  is an étale Lie groupoid. We use the étale property to re-cast conditions  $(U^\infty)$ – $(I^\infty)$  in terms of smoothness of maps from open subsets of  $G^{(0)}$  into  $\mathbb{T}$ .

To translate this into properties of families of normalisers, we start Section 5 by making explicit the relationship between sections of the line bundle  $L$  of the twist  $E$ , which Renault uses to define the associated  $C^*$ -algebra, and sections of the twist itself. Specifically, the phase  $\text{Ph}(\mathfrak{s})$  of a section  $\mathfrak{s}$  of  $L$  determines a section  $\sigma_\mathfrak{s}$  of  $E$ , and  $\sigma_\mathfrak{s} = \sigma_{\mathfrak{s}'}$  precisely if  $\mathfrak{s}$  and  $\mathfrak{s}'$  differ by a positive continuous function on  $G^{(0)}$ . Combining this with the faithful conditional expectation  $P: C_r^*(G; E) \rightarrow C_0(G^{(0)})$  extending restriction of functions, we identify conditions on a family  $\{\mathfrak{s}_\alpha\}_\alpha$  of normalisers, phrased in terms of membership of images under  $P$  of combinations of the  $\mathfrak{s}_\alpha$  in algebras of smooth bounded functions on open subsets of  $G^{(0)}$ . We show further that smoothness of these images can be characterised by their behaving as local multipliers of  $C_0^\infty(G^{(0)})$  in an appropriate sense.

With all this in hand, we are able to define what we call a *smooth Cartan triple*: a triple  $(A, B, \mathcal{N})$  such that  $B \subseteq A$  is a Cartan pair,  $\widehat{B}$  is a smooth manifold,  $B^\infty := \mathcal{N} \cap B$  is equal to  $C_0^\infty(\widehat{B})$ , and  $\mathcal{N}$  is a family of normalisers of  $B$  that also normalise  $B^\infty$  and have the property that  $\text{Ph}(P(n)), \text{Ph}(P(mk^*)), \text{Ph}(P(nmk^*))$  and  $\text{Ph}(P(nm))$  are all local multipliers of  $B^\infty$  for all  $k, m, n \in \mathcal{N}$ . Our culminating theorem says that if  $\mathbb{T} \times G^{(0)} \xrightarrow{\iota} E \xrightarrow{\pi} G$  is a Lie twist over an effective Lie groupoid, and  $\mathcal{N}$  is a family of smooth sections, supported on bisections, of its line bundle that densely span  $C_r^*(G; E)$  and contain  $C_0^\infty(G^{(0)})$ , then  $(C_r^*(G; E), C_0(G^{(0)}), \mathcal{N})$  is a smooth Cartan triple; and conversely that if  $(A, B, \mathcal{N})$  is a smooth Cartan triple, then  $\mathcal{N}$  determines a smooth structure on the Weyl twist of  $B \subseteq A$ , compatible with the manifold structure on  $\widehat{B}$ , under which it becomes a Lie twist. We finish with some brief remarks: one on how our main theorem can be combined with Connes’ reconstruction theorem to obtain

a correspondence between Lie twists and purely functional-analytic data; and one on the effect of the choice of the family  $\mathcal{N}$  on the resulting smooth structure on  $E$ .

We include an appendix on differential geometry where we establish our notation and conventions, and collect a number of standard results that we need along the way. It is organised into two sections: one about smooth manifolds and their submanifolds in general; and one specifically relating to smooth principal bundles.

## 2. PRELIMINARIES: LIE GROUPOIDS

In this brief section, we introduce some preliminary material on the notions of topological groupoids and Lie groupoids and establish a few of their basic properties. In particular, when the groupoid in question is an étale Lie groupoid, we give an explicit description of an atlas for the natural smooth structure on its space of composable pairs.

We will work with the following definition of a topological groupoid; we will frequently drop the adjective and just say *groupoid*. In this paper, groupoids are **always** locally compact Hausdorff topological groupoids.

**Definition 2.1** (cf. [22, Remark 8.1.5]). A (*Hausdorff*) *topological groupoid* consists of

- (G1) two locally compact Hausdorff spaces  $G$  and  $G^{(0)}$ ,
- (G2) an injection  $G^{(0)} \hookrightarrow G$  that is a homeomorphism onto its range (under the subspace topology), so that we may regard  $G^{(0)}$  as a subspace, called the *unit space*, of  $G$ ,
- (G3) two continuous surjections  $s, r: G \rightarrow G^{(0)}$  called the *source map* and the *range map*,
- (G4) a continuous *composition* map  $G^{(2)} = G * G \rightarrow G$ ,  $(\gamma, \eta) \mapsto \gamma\eta$ , and
- (G5) a continuous *inversion* map  $G \rightarrow G$ ,  $\gamma \mapsto \gamma^{-1}$ ,

such that

- (G6)  $s(\gamma\eta) = s(\eta)$  and  $r(\gamma\eta) = r(\gamma)$  for every  $(\gamma, \eta) \in G^{(2)}$ ,
- (G7) composition is associative,
- (G8)  $s(x) = r(x) = x$  for every  $x \in G^{(0)}$  and  $\gamma s(\gamma) = \gamma$  and  $r(\gamma)\gamma = \gamma$  for every  $\gamma \in G$ , and
- (G9)  $\gamma^{-1}\gamma = s(\gamma)$  and  $\gamma\gamma^{-1} = r(\gamma)$  for every  $\gamma \in G$ .

A groupoid is called *étale* if its range and source maps are local homeomorphisms.

Our notation for groupoids follows [18, 22]. We will typically denote groupoids by  $G$ , except for groupoids that are twists over another base groupoid, which we denote by  $E$ .

Given sets  $X, Y, Z$  and functions  $f: X \rightarrow Z$  and  $g: Y \rightarrow Z$ , we write  $X \underset{g}{*}_f Y$  for the fibred product

$$X \underset{g}{*}_f Y := \{(x, y) \in X \times Y : f(x) = g(y)\}.$$

If  $X, Y$  are topological spaces, we give  $X \underset{g}{*}_f Y$  the relative topology inherited from the product space. In particular, if  $G$  is a groupoid, then for subsets  $U, V \subseteq G$ , the collection of composable pairs in  $U \times V$  is the fibred product over the range and source maps. That is,

$$U \underset{s}{*}_r V = (U \times V) \cap G^{(2)}.$$

In this specific instance, we will drop the  $s$  and  $r$  in the notation, and just write  $U * V := U \underset{s}{*}_r V$ ; we will always retain the subscripts for fibred products that are not of this specific form.

We write  $UV$  for the collection of products of pairs in  $U * V$ :

$$UV := \{\gamma\eta : (\gamma, \eta) \in U * V\}.$$

If  $G$  is an étale groupoid, then it admits a base of open sets  $U \subseteq G$  such that  $r|_U$  and  $s|_U$  are homeomorphisms onto open subsets of its unit space  $G^{(0)}$ . Such sets  $U$  are called *open bisections* (or sometimes just *bisections*). By [12, Proposition 3.8], if  $U$  and  $V$  are open bisections, then  $UV$  is also a (possibly empty) open bisection.

We will be particularly interested in topological groupoids that come with additional smooth structure. In the following, we will refer to a topological manifold with a smooth structure simply as a “manifold” rather than as a “smooth manifold.” Our terminology and notation for manifolds follow the conventions of [15], and are detailed in Appendix A for ease of reference.

**Definition 2.2** ([9, Definitions 7.1.1 and 7.1.6]). A *Lie groupoid* is a topological groupoid  $G$  such that

- (L1)  $G$  and  $G^{(0)}$  are smooth manifolds (under their given topological structures),
- (L2) the inclusion  $G^{(0)} \hookrightarrow G$  is an embedding of manifolds (that is, in addition to (G2) above, it is a smooth immersion),
- (L3) the maps  $s, r: G \rightarrow G^{(0)}$  are submersions,

- (L4) the composition map  $G^{(2)} \rightarrow G, (\gamma, \eta) \mapsto \gamma\eta$ , is smooth, and
- (L5) the inversion map  $G \rightarrow G, \gamma \mapsto \gamma^{-1}$ , is smooth.

A Lie groupoid is called *étale* if its range and source maps are local diffeomorphisms.

The above definition makes sense: Since the fibred product  $M_{f^*g}N$  of two manifolds along transverse maps  $f: M \rightarrow K$  and  $g: N \rightarrow K$  is itself a manifold (Proposition A.12) and since a submersion of manifolds is transverse to any map (Remark A.10), the subspace  $G^{(2)}$  of  $G \times G$  is indeed a manifold in its own right.

*Remark 2.3.* For étale Lie groupoids, the manifold dimension of the unit space  $G^{(0)}$  and of the morphism space  $G$  are identical. Consequently, this dimension also coincides with the manifold dimension of the space  $G^{(2)}$  of composable pairs: given smooth functions  $f: M \rightarrow K$  and  $g: N \rightarrow K$ , the dimension of the fibred product  $M_{f^*g}N$  is  $\dim(M) + \dim(N) - \dim(K)$ ; see Proposition A.12 for a proof. In particular, for étale  $G$ , we have  $\dim(G^{(2)}) = 2 \dim(G) - \dim(G^{(0)}) = \dim(G)$ .

For practical reasons, we will now consider objects that are more general than Lie groupoids. We will always assume that smooth structures on these objects are compatible with the given underlying topologies where applicable.

**Definition 2.4** (cf. [9, p. 207]). Suppose that  $G$  is a topological groupoid and that both  $G$  and  $G^{(0)}$  are manifolds. An open bisection  $B$  of  $G$  is called a *smooth* bisection if  $r|_B: B \rightarrow r(B)$  and  $s|_B: B \rightarrow s(B)$  are diffeomorphisms.

Suppose that  $G$  is an étale groupoid and that both  $G$  and  $G^{(0)}$  are manifolds. Assume that both  $r$  and  $s$  are local diffeomorphisms (in particular, they are submersions by Lemma A.13), so that  $G^{(2)}$  is an embedded submanifold of  $G \times G$  by Proposition A.12. We will now describe the charts of  $G^{(2)}$ .

Let  $\mathcal{U}$  denote the set of smooth bisections of  $G$ . For a fixed  $(\gamma_1, \gamma_2) \in G^{(2)}$ , fix  $B'_i \in \mathcal{U}$  containing  $\gamma_i$ , and let  $W := s(B'_1) \cap r(B'_2)$ . Notice that

$$B_1 := B'_1 W = B'_1 \cap s^{-1}(W) \quad \text{and} \quad B_2 := W B'_2 = r^{-1}(W) \cap B'_2$$

are open, since  $r, s$  are continuous open maps ([23, Corollary 11.6]). As subsets of smooth bisections, they are therefore likewise smooth bisections, and we clearly have  $s(B_1) = W = r(B_2)$ .

Now fix a chart  $\varphi_1$  around  $\gamma_1$  in  $G$ ; by potentially shrinking sets, we can assume without loss of generality that the domain of  $\varphi_1$  is exactly  $B_1$ . Writing  $n$  for the manifold dimension of  $G$ , we define

$$\varphi_2 := \varphi_1 \circ (s|_{B_1})^{-1} \circ r|_{B_2}: B_2 \approx \mathbb{R}^n,$$

Since  $s$  and  $r$  are local diffeomorphisms, the map  $\varphi := \varphi_1 \times \varphi_2$  is a smooth chart around  $(\gamma_1, \gamma_2)$ , and it is easy to check that the set  $\varphi(G^{(2)} \cap (B_1 \times B_2)) = \varphi(B_1 * B_2)$  is exactly the diagonal  $\{(\vec{v}, \vec{v}) : \vec{v} \in \varphi(B_1)\}$  in  $\mathbb{R}^n \times \mathbb{R}^n$ . Thus, composing  $\varphi$  with the diffeomorphism  $\mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}, (\vec{x}, \vec{y}) \mapsto (\vec{x}, \vec{x} - \vec{y})$ , yields a chart that realises  $B_1 * B_2$  as an  $n$ -dimensional embedded submanifold of  $B_1 \times B_2$ .

Consequently, if  $\text{pr}_1$  denotes the projection onto the first component of  $G \times G$ , then  $(\varphi_1 \circ \text{pr}_1, B_1 * B_2)$  is a chart for  $G^{(2)}$ , and the collection of all such charts forms a smooth atlas. Since the smooth structures of  $G$  and  $G^{(0)}$  are compatible (meaning, we can turn one into the other by use of  $r$  or  $s$ ), we can instead also describe an atlas on  $G^{(2)}$  as follows.

**Lemma 2.5.** *Suppose that  $G$  is a topological groupoid and that both  $G$  and  $G^{(0)}$  are manifolds. Let  $\mathcal{A} = \{(W_\alpha, \psi_\alpha)\}_{\alpha \in \mathfrak{A}}$  be a maximal smooth atlas of  $G^{(0)}$ . Assume that both  $r$  and  $s$  are local diffeomorphisms and let  $\mathcal{U}$  denote the set of smooth bisections of  $G$ . Let*

$$\mathfrak{C} := \{(\alpha, B_1, B_2) \in \mathfrak{A} \times \mathcal{U} \times \mathcal{U} : s(B_1) = W_\alpha = r(B_2)\}.$$

*For  $\chi = (\alpha, B_1, B_2) \in \mathfrak{C}$ , let  $V_\chi := B_1 * B_2$  and define  $\Phi_\chi: V_\chi \rightarrow \mathbb{R}^n$  by  $\Phi_\chi(\gamma_1, \gamma_2) := \psi_\alpha(s(\gamma_1)) = \psi_\alpha(r(\gamma_2))$ . Then the collection  $\mathcal{C} = \{(V_\chi, \Phi_\chi)\}_{\chi \in \mathfrak{C}}$  is a smooth atlas for the standard manifold structure of  $G^{(2)}$ .*

### 3. ÉTALE GROUPOIDS WITH UNIT SPACE A MANIFOLD

The goal of this section is to prove that if  $G$  is an étale groupoid in which the unit space  $G^{(0)}$  is a manifold, then there is a smooth structure on  $G$  making it into a Lie groupoid if and only if the open bisections of  $G$  induce smooth maps between open subsets of  $G^{(0)}$ ; and moreover that this smooth structure is unique and admits an explicit atlas easily described in terms of the range map and a given atlas for  $G^{(0)}$ .

**Proposition 3.1.** *Suppose that  $G$  is an étale groupoid and that  $G^{(0)}$  is a smooth manifold. For a smooth atlas  $\{(W_\alpha, \psi_\alpha) : \alpha \in \mathfrak{A}\}$  for  $G^{(0)}$ , let*

$$\mathfrak{R} := \{(B, \alpha) : B \text{ is an open bisection of } G \text{ such that } r(B) \subseteq W_\alpha\}$$

and for each  $\rho = (B, \alpha) \in \mathfrak{R}$  define  $U_\rho = B$  and

$$\varphi_\rho : U_\rho \rightarrow \mathbb{R}^n, \quad \varphi_\rho(\gamma) := \psi_\alpha(r(\gamma)).$$

Then  $\mathcal{R} := \{(U_\rho, \varphi_\rho) : \rho \in \mathfrak{R}\}$  is an atlas for a smooth structure on  $G$ . Moreover, any other smooth structure  $\mathcal{B}$  on  $G$  with respect to which  $r : (G, \mathcal{B}) \rightarrow G^{(0)}$  is a local diffeomorphism is compatible with  $\mathcal{R}$ .

*Proof.* We claim that  $\mathcal{R}$  is a  $C^\infty$ -atlas for  $G$  in the sense of Definition A.2. In what follows, we will use the standard convention that for  $\rho, \rho' \in \mathfrak{R}$ , we write  $U_{\rho, \rho'}$  for the intersection  $U_\rho \cap U_{\rho'}$ .

To check (A1), fix  $\gamma \in G$ . Since  $G$  is étale, it has a base of open bisections, so we may pick a bisection  $B'$  that contains  $\gamma$ . Since  $\{W_\alpha\}_\alpha$  covers  $G^{(0)}$ , we may pick  $\alpha$  such that  $r(\gamma) \in W_\alpha$ . Now let  $B := B' \cap r^{-1}(W_\alpha)$ . The set  $B$  is open since  $W_\alpha$  and  $B'$  are open and since  $r$  is continuous; it is nonempty as it contains  $\gamma$ ; it is a bisection because  $B'$  is a bisection. Since  $r(B) \subseteq W_\alpha$ , we have  $(B, \alpha) \in \mathfrak{R}$ , and so  $U_{(B, \alpha)} = B$  contains  $\gamma$ , which proves that  $\{U_\rho\}_{\rho \in \mathfrak{R}}$  covers  $G$ .

To check (A2), note that by construction,  $\varphi_\rho$  is the composition of the homeomorphism  $r|_B : B \rightarrow W_\alpha$  and the homeomorphism  $\psi_\alpha : W_\alpha \rightarrow \psi_\alpha(W_\alpha) \subseteq \mathbb{R}^n$ ; hence  $\varphi_\rho$  is a homeomorphism onto its image.

For (A3), fix  $\rho = (B, \alpha)$  and  $\rho' = (B', \alpha') \in \mathfrak{R}$  such that  $U_{\rho, \rho'} \neq \emptyset$ . We have

$$\begin{aligned} \varphi_\rho|_{U_{\rho, \rho'}} \circ (\varphi_{\rho'}|_{U_{\rho, \rho'}})^{-1} &= (\psi_\alpha \circ r|_{B \cap B'}) \circ (\psi_{\alpha'} \circ r|_{B \cap B'})^{-1} \\ &= \psi_\alpha \circ (r|_{B \cap B'} \circ r_{B \cap B'}^{-1}) \circ \psi_{\alpha'}^{-1} = \psi_\alpha \circ \psi_{\alpha'}^{-1}, \end{aligned}$$

which is smooth by assumption on the atlas on  $G^{(0)}$ .

To see that  $r$  is a local diffeomorphism, we must check that, for any given open bisection  $B$ , the map  $r|_B$  and its inverse are not only continuous but also smooth. By replacing  $B$  by a smaller neighbourhood around any given point in  $B$ , we may assume without loss of generality that there exists  $\alpha \in \mathfrak{A}$  such that  $(B, \alpha)$  is an element of  $\mathfrak{R}$ . Then the map  $\varphi_\rho \circ r|_B \circ \psi_\alpha^{-1}$ , defined between open subsets of  $\mathbb{R}^n$ , is given by

$$\psi_\alpha \circ r|_B \circ \varphi_\rho^{-1} = \psi_\alpha \circ r|_B \circ (\psi_\alpha \circ r|_B)^{-1} = \text{id}.$$

Similarly,

$$\varphi_\rho \circ r|_B^{-1} \circ \psi_\alpha^{-1} = (\psi_\alpha \circ r|_B) \circ r|_B^{-1} \circ \psi_\alpha^{-1} = \text{id}.$$

Since it suffices to prove smoothness for *one* set of charts per point, we deduce that  $r|_B$  and its inverse are smooth.

Lastly, suppose that  $\mathcal{B} = \{(V_\beta, \phi_\beta)\}_{\beta \in \mathfrak{B}}$  is another smooth structure on  $G$  with respect to which  $r$  is a local diffeomorphism; we claim that  $\mathcal{B}$  and  $\mathcal{R}$  are compatible, meaning that their union is again a smooth atlas. By assumption on  $\mathcal{B}$ , whenever  $\rho = (B, \alpha) \in \mathfrak{R}$  and  $\beta \in \mathfrak{B}$  satisfy  $V_\beta \subseteq r(B)$ , the map  $\psi_\alpha \circ r|_B \circ \phi_\beta^{-1}$  is a smooth function between open subsets of  $\mathbb{R}^n$ . That is,

$$\varphi_\rho \circ \phi_\beta^{-1} = (\psi_\alpha \circ r|_B) \circ \phi_\beta^{-1}$$

is smooth. So the two atlases are compatible.  $\square$

*Remark 3.2.* Analogously, we could have asked for the source map  $s$  to become a local diffeomorphism. We do this by defining

$$\mathfrak{S} := \{(B, \alpha) : B \text{ open bisection of } G \text{ such that } s(B) \subseteq W_\alpha\}.$$

For each  $\sigma = (B, \alpha)$ , we let  $V_\sigma := B$  and define

$$\phi_\sigma : V_\sigma \rightarrow \mathbb{R}^n, \quad \phi_\sigma(\gamma) := \psi_\alpha(s(\gamma)).$$

Then by symmetry, the argument for  $\mathcal{R}$  yields that  $\mathcal{S} := \{(U_\sigma, \phi_\sigma)\}_{\sigma \in \mathfrak{S}}$  is likewise a  $C^\infty$ -atlas for  $G$  (but *a priori* for a different smooth structure). In this case,  $s$  is a local diffeomorphism.

If  $G$  is an étale *Lie* groupoid, then for every smooth open bisection  $B$  of  $G$  in the sense of Definition 2.4, the composition  $s|_B \circ r|_B^{-1}$  is a diffeomorphism between open subsets of  $G^{(0)}$ . The definition of a smooth bisection, of course, only makes sense when  $G$  carries a smooth structure, but the condition just described does not; so it is a necessary condition for the existence of a smooth structure on  $G$  making it into a Lie groupoid. We make this formal with the following definition.

**Definition 3.3.** Suppose that  $G$  is an étale groupoid such that  $G^{(0)}$  is a smooth manifold. We say that an open bisection  $B \subseteq G$  acts smoothly on  $G^{(0)}$  if the map

$$s|_B \circ r|_B^{-1} : r(B) \rightarrow s(B)$$

is a diffeomorphism. We say that  $G$  acts smoothly on  $G^{(0)}$  if it admits a cover by open bisections that act smoothly on  $G^{(0)}$ .

*Remark 3.4.* If  $G$  is an étale groupoid such that  $G^{(0)}$  is a smooth manifold and  $G$  acts smoothly on  $G^{(0)}$ , then we can choose a base  $\mathfrak{U}$  of open bisections  $B$  for which  $s|_B \circ r|_B^{-1}$  is a diffeomorphism in such a way that each of  $r(\mathfrak{U})$  and  $s(\mathfrak{U})$  is an atlas for the unit space.

**Lemma 3.5.** *Suppose that  $G$  is an étale groupoid and that  $G^{(0)}$  is a smooth manifold. Then  $G$  acts smoothly on  $G^{(0)}$  if and only if every open bisection  $B \subseteq G$  acts smoothly on  $G^{(0)}$ .*

*Proof.* Clearly if every bisection acts smoothly, then the collection of all open bisections is a cover of  $G$  by open bisections that act smoothly. Conversely, assume that  $\mathcal{B}$  is a cover of  $G$  consisting of open bisections of  $G$  that act smoothly on  $G^{(0)}$ . Let  $U$  be any open bisection of  $G$ , and fix  $\gamma \in U$ . Since  $\mathcal{B}$  covers  $G$ , there exists  $B \in \mathcal{B}$  with  $\gamma \in B$ . Let  $V := U \cap B$ . Then  $V$  is a neighbourhood of  $\gamma$ . The homeomorphism  $s|_U \circ r|_U^{-1}$  agrees with the diffeomorphism  $s|_B \circ r|_B^{-1}$  on the open neighbourhood  $r(V)$  of  $r(\gamma)$ , so it is differentiable at  $r(\gamma)$ ; similarly its inverse is differentiable at  $s(\gamma)$ .  $\square$

**Lemma 3.6.** *Suppose that  $G$  is an étale groupoid and that  $G^{(0)}$  is a smooth manifold. If  $G$  acts smoothly on  $G^{(0)}$ , then the atlases  $\mathcal{R}$  of Proposition 3.1 and  $\mathcal{S}$  of Remark 3.2 are compatible.*

*Proof.* Take  $\rho = (B, \alpha) \in \mathfrak{R}$  and  $\sigma = (B', \beta) \in \mathfrak{S}$  with  $U := B \cap B' \neq \emptyset$ . Then

$$\begin{aligned} \varphi_\rho|_U \circ (\phi_\sigma|_U)^{-1} &= (\psi_\alpha \circ r|_U) \circ (\psi_\beta \circ s|_U)^{-1} \\ &= \psi_\alpha \circ (r|_U \circ s|_U^{-1}) \circ \psi_\beta^{-1}. \end{aligned}$$

This is a diffeomorphism since  $(r|_U \circ s|_U^{-1}) = (s|_U \circ r|_U^{-1})^{-1}$  is a diffeomorphism by assumption and since the maps  $\psi_\alpha, \psi_\beta$  are smooth charts of  $G^{(0)}$ . Analogously, the composition of  $\phi_\sigma$  with  $\varphi_\rho^{-1}$  is a diffeomorphism.  $\square$

Our next goal is to verify that  $G$  is a Lie groupoid with respect to the smooth structures we have considered.

**Proposition 3.7.** *Suppose that  $G$  is an étale groupoid and that  $G^{(0)}$  is a smooth manifold. Then the following are equivalent:*

- (1)  $G$  acts smoothly on  $G^{(0)}$ ;
- (2) there is a manifold structure on  $G$  with respect to which it is an étale Lie groupoid; and
- (3) the manifold structure on  $G$  obtained from Proposition 3.1 is the unique manifold structure on  $G$  with respect to which it is an étale Lie groupoid.

*Proof.* Clearly (3) implies (2). That (2) implies (1) is immediate: If  $G$  is an étale Lie groupoid, then  $r$  and  $s$  are local diffeomorphisms, meaning that every point of  $G$  has a neighbourhood  $U$  such that  $U$  is diffeomorphic to  $r(U)$  via  $r$  and to  $s(U)$  via  $s$ . So the collection of such bisections covers  $G$ , and hence  $G$  acts smoothly on  $G^{(0)}$ .

For (1) implies (3), suppose that  $G$  acts smoothly on  $G^{(0)}$ . We first argue that the manifold structure on  $G$  obtained from Proposition 3.1 makes it into an étale Lie groupoid. We will argue uniqueness at the end. We have already verified Condition (L1) of Definition 2.2. It remains to show the conditions revolving around differentiability, so we start with a smooth atlas  $\mathcal{A} = \{(W_\alpha, \psi_\alpha)\}_{\alpha \in \mathfrak{A}}$  of  $G^{(0)}$ , and we equip  $G$  with the smooth structure described in Proposition 3.1.

For (L2), fix  $x \in G^{(0)}$  and let  $(W_\alpha, \psi_\alpha)$  be a chart around  $x$ . If  $B'$  is any open bisection around  $x$ , then  $B := B' \cap G^{(0)} \cap W_\alpha$  is also an open bisection around  $x$ , because  $G$  is étale and so  $G^{(0)}$  is open in  $G$ . Since  $B \subseteq G^{(0)}$ , the map  $r|_B$  is the identity on  $B$ ; in particular,  $r(B) = B \subseteq W_\alpha$ , and so  $(B, \alpha) \in \mathfrak{R}$ . It follows that  $\varphi_{(B, \alpha)} = \psi_\alpha$ , and

$$\varphi_{(B, \alpha)} \circ i \circ \psi_\alpha^{-1}$$

is the identity map on the open set  $\psi_\alpha(B) \subseteq \mathbb{R}^n$ . Thus  $i$  is smooth and its differential is everywhere injective, so  $i$  is an immersion.

For (L3), note that the range and source maps are surjective by assumption on the topological groupoid  $G$ . We have constructed the smooth structure on  $G$  exactly so that  $r$  and  $s$  are local diffeomorphisms, and so they are submersions by Lemma A.13. (This also explains why, once we have proved that  $G$  satisfies (L4) and (L5), it will follow that it is an étale Lie groupoid.)

For (L4), fix  $(\gamma_1, \gamma_2) \in G^{(2)}$ . Let  $(W, \psi)$  be a chart around  $s(\gamma_1) = r(\gamma_2)$ . By shrinking  $W$ , we may assume without loss of generality that there exist smooth open bisections  $B_i$  around  $\gamma_i$  such that  $s(B_1) = W = r(B_2)$ . Since we have already shown that  $s$  and  $r$  are local diffeomorphisms, Lemma 2.5 describes an atlas for  $G^{(2)}$ , and in particular shows that  $\Phi: B_1 * B_2 \rightarrow \mathbb{R}^n$  defined by  $\Phi(\eta_1, \eta_2) := \psi(s(\eta_1)) = \psi(r(\eta_2))$  is a chart around  $(\gamma_1, \gamma_2)$ .

Let  $(W', \psi')$  be any chart around  $s(\gamma_2)$ . By shrinking its domain and by shrinking  $B_2$ , we can assume without loss of generality that  $\text{dom}(\psi') = s(B_2) = s(B_1 B_2)$ .<sup>1</sup> Since the set  $B_1 B_2$  is also an open bisection of  $G$  [12, Proposition 3.8], the map  $\varphi: B_1 B_2 \rightarrow \mathbb{R}^m$  given by  $\gamma \mapsto \psi'(s(\gamma))$ , is a smooth chart around  $\gamma_1 \gamma_2$  by Remark 3.2. (This does not require that  $B_1 B_2$  is a smooth bisection.)

Now, to see that the multiplication map  $M: B_1 * B_2 \rightarrow B_1 B_2$  is smooth at  $(\gamma_1, \gamma_2)$ , consider the composition

$$\begin{aligned} \varphi \circ M \circ \Phi^{-1}: \quad \mathbb{R}^n &\xrightarrow{\Phi^{-1}} B_1 * B_2 \xrightarrow{M} B_1 B_2 \xrightarrow{\varphi} \mathbb{R}^m \\ \psi(r(\eta_2)) &\longmapsto (\eta_1, \eta_2) \longmapsto \eta_1 \eta_2 \longmapsto \psi'(s(\eta_1 \eta_2)) = \psi'(s(\eta_2)). \end{aligned}$$

In short, this is the map from  $\psi(r(B_2))$  to  $\psi'(s(B_2))$  satisfying  $\psi(r(\eta_2)) \mapsto \psi'(s(\eta_2))$  for all  $\eta_2 \in B_2$ . By assumption,  $G$  acts smoothly on  $G^{(0)}$ , so by Lemma 3.5, the map

$$s|_{B_2} \circ r|_{B_2}^{-1}: r(B_2) \rightarrow s(B_2)$$

is a diffeomorphism. Since  $\psi$  and  $\psi'$  are smooth by assumption, we conclude that the composition  $\varphi \circ M \circ \Phi^{-1}$  is a smooth map. Since  $\Phi$  is a chart around  $(\gamma_1, \gamma_2)$  and  $\varphi$  is a chart around  $\gamma_1 \gamma_2 = M(\gamma_1, \gamma_2)$ , this proves that  $M$  is a smooth map.

For (L5), fix  $\gamma \in G$ , take  $(B, \alpha) \in \mathfrak{R}$  with  $\gamma \in B$ , and let  $(B', \alpha')$  be an element of  $\mathfrak{S}$  such that  $\gamma^{-1} \in B'$ ; we will make use of the fact that the atlases  $\mathcal{R} = \{(U_\alpha, \varphi_\alpha)\}_\alpha$  and  $\mathcal{S} = \{(U_\alpha, \phi_\alpha)\}_\alpha$  for  $G$  are compatible. Let  $I: G \rightarrow G$  be the inversion map. By replacing  $B$  with  $B \cap I^{-1}(B')$ , we may assume without loss of generality that  $I(B) \subseteq B'$ . We compute, using at the penultimate step that  $I(B) \subseteq B'$ :

$$\begin{aligned} \phi_{(B', \alpha')} \circ I \circ \varphi_{(B, \alpha)}^{-1} &= (\psi_{\alpha'} \circ s|_{B'}) \circ I \circ (\psi_\alpha \circ r|_B)^{-1} \\ &= \psi_{\alpha'} \circ (s|_{B'} \circ I \circ r|_B^{-1}) \circ \psi_\alpha^{-1} \\ &= \psi_{\alpha'} \circ (r|_B \circ r|_B^{-1}) \circ \psi_\alpha^{-1} \\ &= \psi_{\alpha'} \circ \psi_\alpha^{-1}, \end{aligned}$$

which is smooth by assumption on the atlas  $\mathcal{A}$  of  $G^{(0)}$ .

For uniqueness of this smooth structure, observe that any smooth structure on  $G$  under which it becomes an étale Lie groupoid is in particular a smooth structure on  $G$  for which  $r: G \rightarrow G^{(0)}$  is a local diffeomorphism; so the uniqueness follows from Proposition 3.1.  $\square$

*Remark 3.8.* If  $G$  is an étale Lie groupoid, then clearly it admits a cover by smooth bisections; and then since smoothness of maps between open submanifolds of  $G^{(0)}$  is a local property, every open bisection of  $G$  is a smooth bisection. Proposition 3.7 implies that, conversely, if  $G$  is a smooth manifold and admits a cover by smooth bisections, then it is a Lie groupoid. To see this, first note that if an open bisection  $B$  is smooth, then the smooth structure on it induced by the range map as in Proposition 3.1 is the same as its given smooth structure ( $r|_B$  is a diffeomorphism with respect to both). Since these bisections cover  $G$ , we deduce that the original smooth structure on  $G$  as a whole is the same as the one obtained from Proposition 3.1; and Proposition 3.7 implies that it is a Lie groupoid with this smooth structure.

**Lemma 3.9.** *Suppose that  $G$  is an étale Lie groupoid. Then every open bisection of  $G$  is a smooth bisection with respect to this smooth structure. In particular, if  $B_1$  and  $B_2$  are smooth bisections such that  $s(B_1) \cap r(B_2) \neq \emptyset$ , then  $B_1 B_2$  is also a smooth bisection.*

*Proof.* Fix an open bisection  $B$ . We must show that  $r|_B: B \rightarrow r(B)$  is smooth with smooth inverse. It then follows immediately that, for each  $B$ ,  $s|_B: B \rightarrow s(B)$  is smooth with smooth inverse since  $I(B) = B^{-1}$  is also an open bisection and since  $s|_B = r|_{B^{-1}} \circ I$  and  $(s|_B)^{-1} = I \circ (r|_{B^{-1}})^{-1}$  are compositions of smooth maps.

Take any  $\gamma \in B$ , and let  $(W, \psi)$  be a chart around  $r(\gamma)$ ; we can assume without loss of generality that  $W \subseteq r(B)$ , and we let  $B' := r^{-1}(W) \cap B$ , which is an open bisection around  $\gamma$ . By Proposition 3.7,

<sup>1</sup>To maintain the equality  $s(B_1) = W = r(B_2)$ , we must likewise shrink  $B_1$  and  $W$ , and we adjust  $\Phi$  accordingly. None of this changes that  $B_i$  is an open bisection around  $\gamma_i$ : For  $i = 2$ , this is by construction, and for  $i = 1$ , since  $\gamma_2 \in B_2$ , we have  $r(\gamma_2) \in r(B_2) = s(B_1)$ , so  $\gamma_1$  remains an element of the (potentially shrunken)  $B_1$ .

(2)  $\implies$  (3), the map  $\varphi: B' \rightarrow \mathbb{R}^n$  given by  $\eta \mapsto \psi(r(\eta))$  is a chart around  $\gamma$ . Consider the composition

$$\begin{array}{ccccccc} \psi \circ r|_{B'} \circ \varphi^{-1}: \mathbb{R}^n \supseteq \varphi(B') & \xrightarrow{\varphi^{-1}} & B' & \xrightarrow{r|_{B'}} & r(B') \subseteq W & \xrightarrow{\psi} & \mathbb{R}^n \\ & & \psi(r(\eta)) & \longmapsto & \eta & \longmapsto & r(\eta) & \longmapsto & \psi(r(\eta)). \end{array}$$

This is just the identity map, which is trivially smooth. We have shown that, for each point  $\gamma$  in the domain of  $r|_B$ , there exists a neighbourhood  $B'$  of  $\gamma$  and smooth charts  $\psi$  around  $r(\gamma)$  and  $\varphi$  around  $\gamma$  such that  $\psi \circ r|_{B'} \circ \varphi^{-1}$  is smooth; so by definition,  $r|_B$  is a smooth map between manifolds.

Since  $B$  is a bisection, we can play the same game “backwards”: start with an arbitrary  $u \in r(B)$  with chart  $(W, \psi)$ , lift  $u$  to a unique  $\gamma \in B$ , let  $W' := r(B) \cap W$ , and then consider the composition

$$\begin{array}{ccccccc} \varphi \circ r|_B^{-1} \circ \psi^{-1}: \mathbb{R}^n & \xleftarrow{\varphi} & B & \xleftarrow{r|_B^{-1}} & r(B) \supset W' & \xleftarrow{\psi^{-1}} & \psi(W') \subseteq \mathbb{R}^n \\ & & \psi(r(\eta)) & \longleftarrow & h & \longleftarrow & r(\eta) & \longleftarrow & \psi(r(\eta)) \end{array}$$

This shows that  $r|_B^{-1}$  is smooth.

The final claim follows since products of open bisections are open bisections [12, Proposition 3.8].  $\square$

#### 4. TWISTS OVER LIE GROUPOIDS

This section contains the bulk of the technical work of the paper. We consider topological twists  $\mathbb{T} \times G^{(0)} \xrightarrow{\iota} E \xrightarrow{\pi} G$  in which  $G$  is a Lie groupoid. Although our primary application is when  $G$  is étale, we do not require this hypothesis for most of the results in this section. We define what it means for the twist to be a *Lie twist* and establish a number of structural consequences of the definition. We show, as a reality check, that the topologically-trivial twist over a Lie groupoid determined by a smooth normalised  $\mathbb{T}$ -valued 2-cocycle is a Lie twist.<sup>2</sup>

We identify four conditions ( $U^\infty$ ), ( $S^\infty$ ), ( $M^\infty$ ) and ( $I^\infty$ ) satisfied by any collection of local smooth sections of a Lie twist (see Definition 4.8). Our main result, Theorem 4.15, shows that, conversely, any family of local sections satisfying the first two of these conditions and with support covering  $G$  uniquely determines a smooth structure on the twist; and Proposition 4.20 shows that the remaining two conditions characterise when it is a Lie twist over  $G$  under this smooth structure. In a brief final subsection, Section 4.1, we restrict attention to the situation where  $G$  is an étale Lie groupoid, and reinterpret Conditions ( $S^\infty$ ), ( $M^\infty$ ), ( $I^\infty$ ) in terms of smoothness of functions defined on  $G^{(0)}$  when the local sections in question are all supported on bisections.

**Definition 4.1.** Let  $G$  be a locally compact Hausdorff groupoid, and regard  $\mathbb{T} \times G^{(0)}$  as a trivial group bundle with fibres  $\mathbb{T}$ . A *twist* over  $G$  consists of a locally compact Hausdorff groupoid  $E$  and groupoid homomorphisms  $\iota, \pi$  such that

$$\mathbb{T} \times G^{(0)} \xrightarrow{\iota} E \xrightarrow{\pi} G$$

is a central groupoid extension, which means that

- (T1)  $\iota: \mathbb{T} \times G^{(0)} \rightarrow \pi^{-1}(G^{(0)})$  is a homeomorphism that satisfies  $\iota(1, \pi(y)) = y$  for all  $y \in E^{(0)}$ , where  $\pi^{-1}(G^{(0)})$  has the subspace topology from  $E$ ;
- (T2)  $\pi$  is a continuous, open surjection; and
- (T3)  $\iota(z, \pi(r(e)))e = e\iota(z, \pi(s(e)))$  for all  $e \in E$  and  $z \in \mathbb{T}$ .

We remind the reader that the above definition implies that  $\pi$  restricts to a homeomorphism of  $E^{(0)}$  onto  $G^{(0)}$  with inverse given by  $\iota|_{\{1\} \times G^{(0)}}$ . It is well known that  $\pi$  is a proper map and a topological principal  $\mathbb{T}$ -bundle map in the sense of Definition A.14; see the discussion in [7]. The existence of local trivialisations around all points of  $G$  implies that *any* continuous section of  $\pi$  gives rise to a topological local trivialisaton, as explained in Lemma A.18.

We will be interested in understanding what we call Lie twists over Lie groupoids. The fundamental idea is that a Lie twist should be to its underlying Lie groupoid as a topological twist is to its underlying topological groupoid. So, roughly speaking, a Lie twist should be a topological twist in which both groupoids are Lie groupoids, and the maps respect their smooth structures. Our definition is related to, but slightly more rigid than, [1, Definition 4.1] as discussed below.

<sup>2</sup>We remind the reader that the group  $\mathbb{T}$  admits a unique smooth structure making it a Lie group, namely the one determined by local logarithm functions (see, for example [15, Exercise 20-11(c)]); we always regard it as carrying that smooth structure.



**Definition 4.2.** Let  $G$  be a Lie groupoid. Regard  $\mathbb{T} \times G^{(0)}$  as a trivial smooth group bundle with fibres  $\mathbb{T}$ . A *Lie twist*  $\mathbb{T} \times G^{(0)} \xrightarrow{\iota} E \xrightarrow{\pi} G$  over  $G$  consists of a Lie groupoid  $E$  and groupoid homomorphisms  $\iota: \mathbb{T} \times G^{(0)} \rightarrow E$  and  $\pi: E \rightarrow G$  under which  $E$  is a topological twist as above and, in addition,

- (LT1) the map  $\pi: E \rightarrow G$  is a submersion, and
- (LT2) the map  $\iota: \mathbb{T} \times G^{(0)} \rightarrow E$  is smooth.

Our definition is more restrictive than the notion of an  $S^1$ -central extension in [1, Definition 4.1] because our definition implies that  $\iota: \mathbb{T} \times G^{(0)} \rightarrow \pi^{-1}(G^{(0)})$  is a diffeomorphism, so  $\pi: \pi^{-1}(G^{(0)}) \rightarrow G^{(0)}$  is the *trivial* principal  $\mathbb{T}$ -bundle over  $G^{(0)}$  (see Lemma 4.4 below). There are also cosmetic differences; for example, we do not insist that  $G^{(0)}$  and  $E^{(0)}$  are equal, but identify them via  $\pi$ .

Our definition does not explicitly insist that  $\pi^{-1}(G^{(0)})$  is an embedded submanifold, that  $\iota$  is a diffeomorphism onto this embedded submanifold or that  $\pi: E \rightarrow G$  is a smooth principal  $\mathbb{T}$ -bundle with respect to the natural  $\mathbb{T}$ -action, but this follows as we demonstrate in the next two lemmas. It then also follows that  $E^{(0)}$  is an embedded submanifold of  $E$  because  $G^{(0)}$  is an embedded manifold of  $\mathbb{T} \times G^{(0)}$ ; see Theorem 4.15(3).

**Lemma 4.3.** *Suppose that  $\mathbb{T} \times G^{(0)} \xrightarrow{\iota} E \xrightarrow{\pi} G$  is a topological twist over a groupoid  $G$ , and that  $E$ ,  $G^{(0)}$ , and  $G$  are manifolds (not necessarily Lie groupoids). Suppose that the left  $\mathbb{T}$ -action on  $E$  given by  $z \cdot e = \iota(z, r(e))e$  is smooth and that  $\pi$  is a submersion. Then  $\pi: E \rightarrow G$  is a smooth principal  $\mathbb{T}$ -bundle.*

*Proof.* By the Quotient Manifold Theorem [15, Theorem 21.10], since  $\mathbb{T}$  is compact and acts smoothly and freely on  $E$ , the quotient space  $\mathbb{T} \backslash E$  has a smooth manifold structure with respect to which  $q: E \rightarrow \mathbb{T} \backslash E$  is a smooth principal  $\mathbb{T}$ -bundle. Moreover, the smooth structure on  $\mathbb{T} \backslash E$  is the *unique* smooth structure on  $\mathbb{T} \backslash E$  for which the quotient map  $q: E \rightarrow \mathbb{T} \backslash E$  is a submersion.

The map  $\pi$  induces a homeomorphism  $\tilde{\pi}: \mathbb{T} \backslash E \approx G$  given by  $\tilde{\pi}(\mathbb{T} \cdot e) = \pi(e)$ . Let  $M$  denote the set  $\mathbb{T} \backslash E$  when equipped with the manifold structure that the homeomorphism  $\tilde{\pi}$  induces from  $G$ , i.e.,  $\tilde{\pi}: M \rightarrow G$  is a diffeomorphism by construction. Consider the following commutative diagram:

$$\begin{array}{ccc} & E & \\ \tilde{\pi}^{-1} \circ \pi \swarrow & & \searrow \pi \text{ submersion} \\ M & \xleftarrow[\text{diffeo}]{\tilde{\pi}} & G \end{array}$$

By definition of the smooth structure on  $M$ , the map  $\tilde{\pi}^{-1} \circ \pi$  is then also a submersion. By definition of  $\tilde{\pi}$ , we have  $(\tilde{\pi}^{-1} \circ \pi)(e) = \mathbb{T} \cdot e$ . That is,  $\tilde{\pi}^{-1} \circ \pi = q$  is the quotient map. By uniqueness of the smooth structure on  $\mathbb{T} \backslash E$ , we conclude that  $\mathbb{T} \backslash E = M$ , i.e.,  $\tilde{\pi}: \mathbb{T} \backslash E \approx G$  is a diffeomorphism, not just a homeomorphism. Since  $q: E \rightarrow \mathbb{T} \backslash E$  is a smooth principal bundle, it follows that  $\pi: E \rightarrow G$  is a smooth principal bundle.  $\square$

**Lemma 4.4.** *Suppose that  $\mathbb{T} \times G^{(0)} \xrightarrow{\iota} E \xrightarrow{\pi} G$  is a Lie twist. Then the following hold:*

- (LT3) *The  $\mathbb{T}$ -action on  $E$  given by  $z \cdot e = \iota(z, r(e))e$  is smooth;*
- (LT4)  *$\pi: E \rightarrow G$  is a smooth principal  $\mathbb{T}$ -bundle with respect to this action;*
- (LT5)  *$\pi^{-1}(G^{(0)})$  is an embedded submanifold of  $E$ ; and*
- (LT6)  *$\iota$  is a diffeomorphism of the trivial principal  $\mathbb{T}$ -bundle onto this embedded submanifold.*

*Proof.* Since  $E$  is a Lie groupoid, multiplication in  $E$  and the range map are smooth. Since  $\iota$  is smooth by assumption, the  $\mathbb{T}$ -action is a composition of smooth maps, and hence itself smooth. Since  $\pi$  is a submersion by assumption, it then follows from Lemma 4.3 that  $\pi: E \rightarrow G$  is a smooth principal bundle. Since  $G^{(0)} \subseteq G$  is an embedded submanifold, it follows from [15, Corollary 6.31] that  $\pi^{-1}(G^{(0)})$  is an embedded submanifold. By [15, Theorem 5.29], the map  $\iota$  is smooth from  $\mathbb{T} \times G^{(0)}$  to  $\pi^{-1}(G^{(0)})$ . Since  $\iota$  is also a homeomorphism between these principle  $\mathbb{T}$ -bundles, Lemma A.16 implies that it is a diffeomorphism.  $\square$

The following lemma and corollary serve as a “reality-check” that the definition of a Lie twist, and the assumptions of Lemma 4.3, are reasonable, and in particular that smooth 2-cocycles on Lie groupoids give rise to Lie twists.

Recall that a *continuous 2-cocycle* on a topological groupoid  $G$  is a continuous function  $\mathbf{c}: G^{(2)} \rightarrow \mathbb{T}$  such that  $\mathbf{c}(\alpha, \beta)\mathbf{c}(\alpha\beta, \gamma) = \mathbf{c}(\alpha, \beta\gamma)\mathbf{c}(\beta, \gamma)$  for all composable triples  $(\alpha, \beta, \gamma)$ . The cocycle  $\mathbf{c}$  is *normalised* if  $\mathbf{c}(r(\gamma), \gamma) = 1 = \mathbf{c}(\gamma, s(\gamma))$  for all  $\gamma$ .

**Lemma 4.5.** *Suppose that  $G$  is a topological groupoid that is a manifold (but not necessarily a Lie groupoid), that  $G^{(0)}$  is an embedded submanifold, and that  $\mathbf{c}: G^{(2)} \rightarrow \mathbb{T}$  is a normalised 2-cocycle. Let*

$E_{\mathbf{c}} := \mathbb{T} \times G$  be the groupoid with multiplication defined by

$$(z_1, \gamma_1)(z_2, \gamma_2) = (\mathbf{c}(\gamma_1, \gamma_2)z_1z_2, \gamma_1\gamma_2) \text{ for } (\gamma_1, \gamma_2) \in G^{(2)}$$

and inversion by

$$(z, \gamma)^{-1} = (\overline{\mathbf{c}(\gamma, \gamma^{-1})z}, \gamma^{-1}).$$

- (1) If  $E_{\mathbf{c}}$  is equipped with the product-manifold structure, then its left  $\mathbb{T}$ -action defined by  $z \cdot (w, \gamma) = (zw, \gamma)$  is smooth and the natural projection map  $\pi: E_{\mathbf{c}} \rightarrow G$  is a submersion.
- (2) If  $G$  is a Lie groupoid and  $\mathbf{c}$  is smooth, then  $E_{\mathbf{c}}$  is also a Lie groupoid.

*Remark 4.6.* It is important that  $\mathbf{c}$  is normalised. Otherwise  $(1, r(\gamma))(1, \gamma) = (\mathbf{c}(r(\gamma), \gamma), \gamma) \neq (1, \gamma)$ , so the inclusion  $\iota: \mathbb{T} \times G^{(0)} \rightarrow E_{\mathbf{c}}$  does not carry units to units.

Applying Lemma 4.3, we get:

**Corollary 4.7.** *Suppose that  $G$  is a topological groupoid that is a manifold, that  $G^{(0)}$  is an embedded submanifold, and that  $\mathbf{c}: G^{(2)} \rightarrow \mathbb{T}$  is a normalised continuous 2-cocycle.*

- (1)  $\pi: E_{\mathbf{c}} \rightarrow G$  is a smooth principal  $\mathbb{T}$ -bundle.
- (2) If  $G$  is a Lie groupoid and  $\mathbf{c}$  is smooth, then  $\mathbb{T} \times G^{(0)} \xrightarrow{\iota} E_{\mathbf{c}} \xrightarrow{\pi} G$  is a Lie twist.

*Proof of Lemma 4.5.* As a manifold,  $E_{\mathbf{c}}$  is the Cartesian-product manifold, so it is a (trivial) smooth principal  $\mathbb{T}$ -bundle for the action  $z \cdot (w, g) = (zw, g)$  irrespective of the properties of  $\mathbf{c}$ . In particular,  $\pi$  is a smooth fibration and the action is smooth; and  $\pi$  is a submersion since, in charts, it is just projection onto the first  $m - 1$  coordinates in Euclidean space.

It is well-known that  $E_{\mathbf{c}}$  is a topological groupoid, so we only need to check the conditions about differentiability. Since the Cartesian product of two Lie groupoids is again a Lie groupoid, we know that the inclusion of  $\{1\} \times G^{(0)} = (\mathbb{T} \times G)^{(0)}$  into  $\mathbb{T} \times G$  is an immersion and that the source map  $\mathbb{T} \times G \rightarrow G^{(0)}$ ,  $(z, \gamma) \mapsto s(\gamma)$ , is a submersion, just like the range map. Since none of these maps distinguish between  $\mathbb{T} \times G$  and  $E_{\mathbf{c}}$ , we conclude that  $E_{\mathbf{c}}$  satisfies Conditions (L1), (L2), and (L3) of Definition 2.2.

The composition map  $E_{\mathbf{c}}^{(2)} \rightarrow E_{\mathbf{c}}$  is smooth because it is built out of smooth functions: rearranging coordinates, multiplication in  $G$ , applying the smooth 2-cocycle  $\mathbf{c}$ , and multiplication in  $\mathbb{T}$ . This proves Condition (L4). A similar argument proves smoothness of inversion (Condition (L5)).  $\square$

Given a family  $\{\sigma_{\alpha}: U_{\alpha} \rightarrow \pi^{-1}(U_{\alpha})\}_{\alpha \in \mathfrak{A}}$  of sections of a map  $\pi$ , we will often write  $\{(U_{\alpha}, \sigma_{\alpha})\}_{\alpha}$  for short. If there is no ambiguity regarding the sets  $U_{\alpha}$ , we will even write  $\{\sigma_{\alpha}\}_{\alpha}$ .

The following is our key technical definition. As we shall see in Theorem 4.15 and Proposition 4.20, it lays out conditions on a family of sections of a twist over a Lie groupoid under which they determine a compatible Lie-groupoid structure on the twist.

**Definition 4.8.** Suppose that  $\mathbb{T} \times G^{(0)} \xrightarrow{\iota} E \xrightarrow{\pi} G$  is a topological twist over a Lie groupoid  $G$ , that  $\{U_{\alpha}\}_{\alpha \in \mathfrak{A}}$  is a collection of open subsets of  $G$ , and that  $\{\sigma_{\alpha}: U_{\alpha} \rightarrow \pi^{-1}(U_{\alpha})\}_{\alpha \in \mathfrak{A}}$  is a family of continuous sections of  $\pi$ . We define the following properties for the family  $\{\sigma_{\alpha}\}_{\alpha \in \mathfrak{A}}$ .

- ( $U^{\infty}$ ) For each  $\alpha$ , there exists a smooth map  $k_{\alpha}: U_{\alpha} \cap G^{(0)} \rightarrow \mathbb{T}$  such that  $\sigma_{\alpha}(x) = \iota(k_{\alpha}(x), x)$  for all  $x \in U_{\alpha} \cap G^{(0)}$ .
- ( $S^{\infty}$ ) For each  $\alpha, \alpha'$ , the map  $U_{\alpha} \cap U_{\alpha'} \rightarrow \mathbb{T} \times G^{(0)}$  given by  $\gamma \mapsto \iota^{-1}(\sigma_{\alpha}(\gamma)\sigma_{\alpha'}(\gamma)^{-1})$  is smooth.
- ( $M^{\infty}$ ) For each  $\alpha, \alpha_1, \alpha_2 \in \mathfrak{A}$ , the map  $\{(\gamma_1, \gamma_2) \in U_{\alpha_1} * U_{\alpha_2} : \gamma_1\gamma_2 \in U_{\alpha}\} \rightarrow \mathbb{T} \times G^{(0)}$  given by  $(\gamma_1, \gamma_2) \mapsto \iota^{-1}(\sigma_{\alpha_1}(\gamma_1)\sigma_{\alpha_2}(\gamma_2)\sigma_{\alpha}(\gamma_1\gamma_2)^{-1})$  is smooth.
- ( $I^{\infty}$ ) For each  $\alpha, \alpha' \in \mathfrak{A}$ , the map  $U_{\alpha} \cap U_{\alpha'} \rightarrow \mathbb{T} \times G^{(0)}$  given by  $\gamma \mapsto \iota^{-1}(\sigma_{\alpha}(\gamma)\sigma_{\alpha'}(\gamma^{-1}))$  is smooth.

*Remark 4.9.* Resume the notation of Definition 4.8. Consider the  $\mathbb{T}$ -valued functions  $f_{\alpha, \alpha'}: U_{\alpha} \cap U_{\alpha'} \rightarrow \mathbb{T}$ ,  $g_{\alpha, \alpha_1, \alpha_2}: \{(\gamma_1, \gamma_2) \in U_{\alpha_1} * U_{\alpha_2} : \gamma_1\gamma_2 \in U_{\alpha}\} \rightarrow \mathbb{T}$ , and  $h_{\alpha, \alpha'}: U_{\alpha} \cap U_{\alpha'} \rightarrow \mathbb{T}$  defined implicitly by

$$\begin{aligned} \iota(f_{\alpha, \alpha'}(\gamma), r(\gamma)) &= \sigma_{\alpha}(\gamma)\sigma_{\alpha'}(\gamma)^{-1}, \\ \iota(g_{\alpha, \alpha_1, \alpha_2}(\gamma_1, \gamma_2), r(\gamma_1)) &= \sigma_{\alpha_1}(\gamma_1)\sigma_{\alpha_2}(\gamma_2)\sigma_{\alpha}(\gamma_1\gamma_2)^{-1}, \text{ and} \\ \iota(h_{\alpha, \alpha'}(\gamma), r(\gamma)) &= \sigma_{\alpha}(\gamma)\sigma_{\alpha'}(\gamma^{-1}). \end{aligned}$$

Then the map of ( $S^{\infty}$ ) (respectively ( $M^{\infty}$ ), respectively ( $I^{\infty}$ )) is smooth if and only if  $f_{\alpha, \alpha'}$  (respectively  $g_{\alpha, \alpha_1, \alpha_2}$ , respectively  $h_{\alpha, \alpha'}$ ) is smooth. In particular, ( $S^{\infty}$ ) (respectively ( $M^{\infty}$ ), respectively ( $I^{\infty}$ )) holds if and only if condition ( $S_{\mathbb{T}}^{\infty}$ ) (respectively ( $M_{\mathbb{T}}^{\infty}$ ), respectively ( $I_{\mathbb{T}}^{\infty}$ )) below holds:

- ( $S_{\mathbb{T}}^{\infty}$ ) For all  $\alpha, \alpha'$ , the map  $f_{\alpha, \alpha'}$  is smooth.
- ( $M_{\mathbb{T}}^{\infty}$ ) For all  $\alpha, \alpha_1, \alpha_2 \in \mathfrak{A}$ , the map  $g_{\alpha, \alpha_1, \alpha_2}$  is smooth.
- ( $I_{\mathbb{T}}^{\infty}$ ) For all  $\alpha, \alpha'$ , the map  $h_{\alpha, \alpha'}$  is smooth.

Since Condition  $(U^\infty)$  is already given in terms of  $\mathbb{T}$ -valued functions, any reasonable definition of a Condition  $(U_{\mathbb{T}}^\infty)$  would coincide with  $(U^\infty)$ .

*Remark 4.10.* Intuitively, each of the four conditions in Definition 4.8 represents a property that the  $\sigma_\alpha$  must satisfy if  $E$  is a Lie twist and the  $\sigma_\alpha$  are all smooth. Firstly,  $(U^\infty)$  corresponds to the requirement that  $\iota$  is a diffeomorphism onto  $\pi^{-1}(G^{(0)})$ : it says that if we use  $\iota$  to identify  $\pi^{-1}(G^{(0)})$  with  $\mathbb{T} \times G^{(0)}$  and hence sections  $G^{(0)} \rightarrow \pi^{-1}(G^{(0)})$  with functions  $G^{(0)} \rightarrow \mathbb{T}$ , then where they are supported on  $G^{(0)}$ , the  $\sigma_\alpha$  are smooth functions into  $\mathbb{T}$ . The letter  $U$  stands for “units.” With  $(U^\infty)$  in hand, we can measure relative smoothness of sections when the products of their values land in  $\pi^{-1}(G^{(0)})$ , and this motivates the remaining properties:

Condition  $(S^\infty)$  corresponds to compatibility of the sections  $\sigma_\alpha$ : if  $\sigma_\alpha$  and  $\sigma_{\alpha'}$  are both smooth, then, in the Lie groupoid  $E$ , their difference  $\sigma_\alpha \sigma_{\alpha'}^{-1}$  must be smooth where defined; the  $S$  stands for “smooth.” Similarly, in the situation of  $(M^\infty)$ , if  $\sigma_{\alpha_1}, \sigma_{\alpha_2}$  and  $\sigma_\alpha$  are smooth, then  $(\gamma_1, \gamma_2) \mapsto \sigma_\alpha(\gamma_1 \gamma_2)$  and  $(\gamma_1, \gamma_2) \mapsto \sigma_{\alpha_1}(\gamma_1) \sigma_{\alpha_2}(\gamma_2)$  are smooth maps; the  $M$  stands for “multiplication.” Finally, if  $\sigma_\alpha$  and  $\sigma_{\alpha'}$  are smooth, then in the situation of  $(I^\infty)$ ,  $\gamma \mapsto \sigma_\alpha(\gamma)$  and  $\gamma \mapsto \sigma_{\alpha'}(\gamma^{-1})^{-1}$  define smooth maps and hence, using smoothness of the multiplication in  $E$ , so is their product; the  $I$  stands for “inverses.”

We shall see later not only that these conditions are necessary, but also that they are sufficient in the sense that any such family of sections determines a unique smooth structure on  $E$ , and with respect to this structure it becomes a Lie twist over  $G$  in the sense of Definition 4.2.

Condition  $(M^\infty)$  looks a little unwieldy because the domain  $\{(\gamma_1, \gamma_2) \in U_{\alpha_1} * U_{\alpha_2} : \gamma_1 \gamma_2 \in U_\alpha\}$  of the map in question is not easily expressed in terms of the sets  $U_\alpha$ ; the point is that the inverse image in  $G^{(2)}$  of a given subset of  $G$  under the multiplication map is hard to picture. Importantly, though, since multiplication is continuous on the fibred product  $U_{\alpha_1} * U_{\alpha_2}$ , the domain of the map is an open subset of  $U_{\alpha_1} * U_{\alpha_2}$  and hence of  $G^{(2)}$  (see the proof of Lemma 4.11 below). Typically we will be interested in the situation where  $\{U_\alpha\}_{\alpha \in \mathfrak{A}}$  is a base for the topology on  $G$ , and in this case the condition can be rephrased more nicely in terms of the  $U_\alpha$  as follows.

**Lemma 4.11.** *Suppose that  $\mathbb{T} \times G^{(0)} \xrightarrow{\iota} E \xrightarrow{\pi} G$  is a topological twist over a Lie groupoid  $G$  and that  $\{U_\alpha\}_{\alpha \in \mathfrak{A}}$  is a collection of open subsets of  $G$ . If a given family  $\{\sigma_\alpha : U_\alpha \rightarrow \pi^{-1}(U_\alpha)\}_{\alpha \in \mathfrak{A}}$  of continuous sections satisfies  $(M^\infty)$ , then it also satisfies:*

$(M_B^\infty)$  *For all  $\alpha, \alpha_1, \alpha_2 \in \mathfrak{A}$ , the map  $U_{\alpha_1} * U_{\alpha_2} \rightarrow \mathbb{T} \times G^{(0)}$ ,  $(\gamma_1, \gamma_2) \mapsto \iota^{-1}(\sigma_{\alpha_1}(\gamma_1) \sigma_{\alpha_2}(\gamma_2) \sigma_\alpha(\gamma_1 \gamma_2)^{-1})$ , is smooth.*

*If  $\{U_\alpha\}_{\alpha \in \mathfrak{A}}$  is a base for the topology of  $G$  and the  $\sigma_\alpha$  satisfy  $(S^\infty)$  and  $(M_B^\infty)$ , then they satisfy  $(M^\infty)$ .*

*Proof.* If  $\alpha_1, \alpha_2, \alpha$  satisfy  $U_{\alpha_1} U_{\alpha_2} \subseteq U_\alpha$ , then  $\{(\gamma_1, \gamma_2) \in U_{\alpha_1} * U_{\alpha_2} : \gamma_1 \gamma_2 \in U_\alpha\} = U_{\alpha_1} * U_{\alpha_2}$ . So  $(M^\infty)$  implies  $(M_B^\infty)$ .

Now suppose that  $\{U_\alpha\}_{\alpha \in \mathfrak{A}}$  is a base for the topology of  $G$  and that the  $\sigma_\alpha$  satisfy  $(S^\infty)$  and  $(M_B^\infty)$ . Fix  $\alpha, \alpha_1, \alpha_2$  such that  $U_{\alpha_1} U_{\alpha_2} \cap U_\alpha \neq \emptyset$ . Fix a point  $(\gamma_1, \gamma_2) \in U_{\alpha_1} * U_{\alpha_2}$  such that  $\gamma_1 \gamma_2 \in U_\alpha$ . Since the multiplication map  $M : G^{(2)} \rightarrow G$  is continuous, the set  $M^{-1}(U_\alpha)$  is open. The set  $U_{\alpha_1} * U_{\alpha_2}$  is open by definition of the topology on  $G^{(2)}$ , so the intersection

$$M^{-1}(U_\alpha) \cap (U_{\alpha_1} * U_{\alpha_2}) = \{(\gamma_1, \gamma_2) \in U_{\alpha_1} * U_{\alpha_2} : \gamma_1 \gamma_2 \in U_\alpha\}$$

is open. Since the  $U_\alpha$  are a base for the topology, there exist  $\beta_1, \beta_2 \in \mathfrak{A}$  such that  $U_{\beta_i} \subseteq U_{\alpha_i}$  and  $(\gamma_1, \gamma_2) \in U_{\beta_1} * U_{\beta_2} \subseteq M^{-1}(U_\alpha) \cap (U_{\alpha_1} * U_{\alpha_2})$ . In particular,  $U_{\beta_1} U_{\beta_2} \subseteq U_\alpha$ . By  $(S^\infty)$ , for  $i = 1, 2$ , the map  $\omega_i : \eta \mapsto \iota^{-1}(\sigma_{\alpha_i}(\eta) \sigma_{\beta_i}(\eta)^{-1})$  is smooth from  $U_{\beta_i}$  to  $\mathbb{T} \times G^{(0)}$ . By definition, this map satisfies  $\omega_i(\eta) \cdot \sigma_{\beta_i}(\eta) = \sigma_{\alpha_i}(\eta)$  for all  $\eta \in U_{\beta_i}$ . By  $(M_B^\infty)$ , the map  $(\eta_1, \eta_2) \mapsto \iota^{-1}(\sigma_{\beta_1}(\eta_1) \sigma_{\beta_2}(\eta_2) \sigma_\alpha(\eta_1 \eta_2)^{-1})$  is smooth from  $U_{\beta_1} * U_{\beta_2}$  to  $\mathbb{T} \times G^{(0)}$ . On this same domain, using centrality of the  $\mathbb{T}$ -action, we see that

$$\begin{aligned} \iota^{-1}(\sigma_{\alpha_1}(\eta_1) \sigma_{\alpha_2}(\eta_2) \sigma_\alpha(\eta_1 \eta_2)^{-1}) &= \iota^{-1}((\omega_1(\eta_1) \cdot \sigma_{\beta_1}(\eta_1)) (\omega_2(\eta_2) \cdot \sigma_{\beta_2}(\eta_2)) \sigma_\alpha(\eta_1 \eta_2)^{-1}) \\ &= (\omega_1(\eta_1) \omega_2(\eta_2), r(\eta_1)) \iota^{-1}(\sigma_{\beta_1}(\eta_1) \sigma_{\beta_2}(\eta_2) \sigma_\alpha(\eta_1 \eta_2)^{-1}). \end{aligned}$$

Since multiplication in  $\mathbb{T} \times G^{(0)}$  is smooth, it follows that  $(\eta_1, \eta_2) \mapsto \iota^{-1}(\sigma_{\alpha_1}(\eta_1) \sigma_{\alpha_2}(\eta_2) \sigma_\alpha(\eta_1 \eta_2)^{-1})$  is smooth on the open neighbourhood  $U_{\beta_1} * U_{\beta_2}$  of  $(\gamma_1, \gamma_2)$  in  $M^{-1}(U_\alpha) \cap U_{\alpha_1} * U_{\alpha_2}$ . Since the point  $(\gamma_1, \gamma_2) \in M^{-1}(U_\alpha) \cap (U_{\alpha_1} * U_{\alpha_2})$  was arbitrary, the result follows.  $\square$

**Lemma 4.12** (Motivation). *Suppose that  $\mathbb{T} \times G^{(0)} \xrightarrow{\iota} E \xrightarrow{\pi} G$  is a Lie twist over a Lie groupoid  $G$ . Suppose that the left  $\mathbb{T}$ -action on  $E$  given by  $z \cdot e = \iota(z, r(e))e$  is smooth and that  $\pi$  is a submersion. If  $\iota : \mathbb{T} \times G^{(0)} \rightarrow \pi^{-1}(G^{(0)})$  is a diffeomorphism, then there exists a base  $\mathcal{B} = \{U_\alpha\}_{\alpha \in \mathfrak{A}}$  for the topology on  $G$  and a family  $\{\sigma_\alpha : U_\alpha \rightarrow \pi^{-1}(U_\alpha)\}_{\alpha \in \mathfrak{A}}$  of sections that satisfies  $(U^\infty)$ ,  $(S^\infty)$ ,  $(M^\infty)$ , and  $(I^\infty)$ . If  $G$  is étale, then  $\mathcal{B}$  can be chosen to consist of open bisections.*

*Proof.* By Condition (LT3),  $\pi: E \rightarrow G$  is a *smooth* principal bundle. Therefore, for every  $\gamma \in G$ , there exists a neighbourhood  $V_\gamma$  and a smooth map  $\psi_\gamma: V_\gamma \rightarrow E$  such that  $\pi \circ \psi_\gamma = \text{id}_{V_\gamma}$ . Fix a base  $\mathcal{B}_0$  for the topology on  $G$  (to prove the final statement when  $G$  is étale, take  $\mathcal{B}_0$  to be the collection of all open bisections of  $G$ ). Let

$$\mathfrak{A} := \{(W, \gamma) : W \in \mathcal{B}_0 \text{ and } W \subseteq V_\gamma\},$$

and for each  $(W, \gamma) \in \mathfrak{A}$  let  $U_{(W, \gamma)} := W$ . If  $U$  is any open subset of  $G$  and  $\gamma \in U$ , then since  $\mathcal{B}_0$  is a base, there exists  $W \in \mathcal{B}_0$  with  $\gamma \in W \subseteq V_\gamma \cap U$ , so  $W = U_{(W, \gamma)}$  is an element of  $\mathcal{B} := \{U_\alpha\}_{\alpha \in \mathfrak{A}}$ . This proves that  $\mathcal{B}$  is also a base for the topology of  $G$ .

By definition of  $\mathcal{B}$ , we can choose a function  $\Gamma: \mathfrak{A} \rightarrow G$  such that  $B_\alpha \subseteq V_{\Gamma(\alpha)}$  for all  $\alpha \in \mathfrak{A}$ . For each  $\alpha \in \mathfrak{A}$ , define  $\sigma_\alpha := \psi_{\Gamma(\alpha)}|_{U_\alpha}$ . Since  $G$  and  $E$  are Lie groupoids, multiplication and inversion on  $G$  and  $E$  are smooth. Since each  $\psi_\gamma$  is smooth and each  $U_\alpha$  is open, and using Lemma 4.11, it follows that for all  $\alpha, \alpha', \alpha_i \in \mathfrak{A}$  with  $U_{\alpha_1} U_{\alpha_2} \subseteq U_{\alpha_3}$ , the following maps are smooth:

$$\begin{aligned} U_\alpha \cap U_{\alpha'} &\rightarrow E, & \eta &\mapsto \sigma_\alpha(\eta)\sigma_{\alpha'}(\eta)^{-1}, \\ U_{\alpha_1} * U_{\alpha_2} &\rightarrow E, & (\eta_1, \eta_2) &\mapsto \sigma_{\alpha_1}(\eta_1)\sigma_{\alpha_2}(\eta_2)\sigma_{\alpha_3}(\eta_1\eta_2)^{-1}, \text{ and} \\ U_\alpha \cap U_{\alpha'}^{-1} &\rightarrow E, & \eta &\mapsto \sigma_\alpha(\eta)\sigma_{\alpha'}(\eta^{-1}). \end{aligned}$$

Since  $\iota: \mathbb{T} \times G^{(0)} \rightarrow \pi^{-1}(G^{(0)})$  is a diffeomorphism, composing any of the above smooth maps with  $\iota^{-1}: \pi^{-1}(G^{(0)}) \rightarrow \mathbb{T} \times G^{(0)}$  yields a smooth map, giving  $(S^\infty)$ ,  $(M^\infty)$ , and  $(I^\infty)$ . Also, again since  $\iota: \mathbb{T} \times G^{(0)} \rightarrow \pi^{-1}(G^{(0)})$  is a diffeomorphism, for each  $\alpha \in \mathfrak{A}$  such that  $U_\alpha \cap G^{(0)} \neq \emptyset$ , the map  $\iota^{-1} \circ \sigma_\alpha|_{U_\alpha \cap G^{(0)}}$  is smooth, and so composing with the smooth projection map  $\mathbb{T} \times G^{(0)} \rightarrow \mathbb{T}$  gives a smooth map  $k_\alpha$  as required by  $(U^\infty)$ .  $\square$

In Theorem 4.15, we prove that, given a *topological* twist  $\mathbb{T} \times G^{(0)} \xrightarrow{\iota} E \xrightarrow{\pi} G$  over a Lie groupoid  $G$  and a suitably large collection of sections that satisfies  $(U^\infty)$  and  $(S^\infty)$ , then we can lift the smooth structure of  $G$  to one on  $E$ . In order for us to be able to replace the collection with a slightly different family of sections, we require the following lemma.

**Lemma 4.13.** *Suppose that  $\mathbb{T} \times G^{(0)} \xrightarrow{\iota} E \xrightarrow{\pi} G$  is a topological twist over a Lie groupoid  $G$ , that  $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in \mathfrak{A}}$  is an atlas of  $G$ , and that  $\{\sigma_\alpha: U_\alpha \rightarrow \pi^{-1}(U_\alpha)\}_{\alpha \in \mathfrak{A}}$  is a family of continuous sections. Suppose further that we are given an open refinement  $\{V_\beta\}_{\beta \in \mathfrak{B}}$  of  $\{U_\alpha\}_\alpha$ , i.e., an open cover of  $G$  with a map  $F: \mathfrak{B} \rightarrow \mathfrak{A}$  such that for all  $\beta \in \mathfrak{B}$ , we have  $V_\beta \subseteq U_{F(\beta)}$ . For each  $\beta \in \mathfrak{B}$ , let  $\kappa_\beta := \sigma_{F(\beta)}|_{V_\beta}: V_\beta \rightarrow \pi^{-1}(V_\beta)$ .*

- (1) *If  $\{(U_\alpha, \sigma_\alpha)\}_\alpha$  satisfies  $(U^\infty)$  (respectively  $(S^\infty)$  respectively  $(M^\infty)$  respectively  $(I^\infty)$ ), then so does  $\{(V_\beta, \kappa_\beta)\}_\beta$ .*
- (2) *Suppose that  $E$  is a manifold and that  $\sigma_\alpha$  is smooth for each  $\alpha$ . Then  $\kappa_\beta$  is smooth for each  $\beta$ .*
- (3) *Suppose that  $\{(U_\alpha, \sigma_\alpha)\}_\alpha$  satisfies  $(S^\infty)$ . Suppose further that  $E$  is a manifold and that, with respect to that smooth structure, its multiplication is smooth, the map  $\iota: \mathbb{T} \times G^{(0)} \rightarrow \pi^{-1}(G^{(0)})$  is a diffeomorphism, and  $\kappa_\beta$  is smooth for each  $\beta$ . Then  $\sigma_\alpha$  is smooth for each  $\alpha$ .*

*Proof.* For (1), fix  $\beta, \beta', \beta_i \in \mathfrak{B}$  with  $V_{\beta_1} V_{\beta_2} \subseteq V_{\beta_3}$  and let  $\alpha = F(\beta)$ ,  $\alpha' = F(\beta')$  and  $\alpha_i = F(\beta_i)$ . Then the maps

$$\begin{aligned} V_\beta \cap G^{(0)} &\rightarrow \mathbb{T}, & x &\mapsto k_\alpha(x), \\ V_\beta \cap V_{\beta'} &\rightarrow E, & h &\mapsto \kappa_\beta(h)\kappa_{\beta'}(h)^{-1}, \\ V_{\beta_1} * V_{\beta_2} &\rightarrow E, & (h_1, h_2) &\mapsto \kappa_{\beta_1}(h_1)\kappa_{\beta_2}(h_2)\kappa_{\beta_3}(h_1h_2)^{-1}, \text{ and} \\ V_\beta \cap V_{\beta'}^{-1} &\rightarrow E, & h &\mapsto \kappa_\beta(h)\kappa_{\beta'}(h^{-1}) \end{aligned}$$

are the restrictions of the corresponding maps  $U_\alpha \cap G^{(0)} \rightarrow \mathbb{T}$ ,  $U_\alpha \cap U_{\alpha'} \rightarrow E$ ,  $U_{\alpha_1} * U_{\alpha_2} \rightarrow E$ , and  $U_\alpha \cap U_{\alpha'}^{-1} \rightarrow E$  determined by  $\{\sigma_\alpha\}_\alpha$ , which are all smooth by assumption. Since multiplication in  $G$  is open (Exercise 1.2.6 and its solution (p. 353) of [24]) and since restriction of a smooth map to an open set is again smooth, we obtain (1).

For (2), take  $\beta \in \mathfrak{B}$  and let  $\alpha = F(\beta)$ . Then  $\sigma_\alpha: U_\alpha \rightarrow \pi^{-1}(U_\alpha)$  is smooth by assumption. Since  $V_\beta$  is an open subset of  $U_\alpha$ , it follows that  $\sigma_\alpha|_{V_\beta}: V_\beta \rightarrow \pi^{-1}(U_\alpha)$  is likewise smooth. As  $\pi^{-1}(V_\beta)$  is an embedded submanifold of  $\pi^{-1}(U_\alpha)$  and as the image of  $\sigma_\alpha|_{V_\beta}$  is contained in  $\pi^{-1}(V_\beta)$ , this implies that  $\kappa_\beta = \sigma_\alpha|_{V_\beta}: V_\beta \rightarrow \pi^{-1}(V_\beta)$  is smooth.

For (3), pick  $\alpha \in \mathfrak{A}$ . To see that  $\sigma_\alpha$  is smooth, fix  $\gamma \in U_\alpha$ ; it suffices to check that  $\gamma$  has a neighbourhood  $U$  such that  $\sigma_\alpha|_U$  is smooth. Since  $\{V_\beta\}_\beta$  is a cover of  $G$ , there exists  $\beta \in \mathfrak{B}$  such that  $\gamma \in U_\alpha \cap V_\beta$ . Let  $U := U_\alpha \cap V_\beta$ . By definition of  $F$ , we have  $\gamma \in U \subseteq U_\alpha \cap U_{F(\beta)}$ . By  $(S^\infty)$  applied to  $U_\alpha, U_{F(\beta)}$  and by definition of  $\kappa_\beta$ , the map  $U \rightarrow \mathbb{T} \times G^{(0)}$  given by  $\eta \mapsto \iota^{-1}(\sigma_\alpha(\eta)\kappa_\beta(\eta)^{-1})$  is

smooth. Since  $\iota$  is a diffeomorphism onto an open subset of  $E$ , it follows that the map  $U \rightarrow E$  given by  $\eta \mapsto \sigma_\alpha(\eta)\kappa_\beta(\eta)^{-1}$  is smooth. Now,  $\kappa_\beta$  and multiplication on  $E$  are smooth by assumption. Hence the map  $\sigma_\alpha|_U: U \rightarrow E$  given by  $\eta \mapsto (\sigma_\alpha(\eta)\kappa_\beta(\eta)^{-1})\kappa_\beta(\eta)$  is also smooth. Since  $\gamma$  was arbitrary, this proves that  $\sigma_\alpha$  is smooth.  $\square$

**Lemma 4.14.** *Suppose that  $\mathbb{T} \times G^{(0)} \xrightarrow{\iota} E \xrightarrow{\pi} G$  is a topological twist over a Lie groupoid  $G$ , that  $E$  is a manifold, and that the  $\mathbb{T}$ -action on  $E$  is smooth. Suppose that  $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in \mathfrak{A}}$  is an atlas of  $G$ , and that  $\{\sigma_\alpha: U_\alpha \rightarrow \pi^{-1}(U_\alpha)\}_{\alpha \in \mathfrak{A}}$  is a family of continuous sections. Suppose that  $\mathfrak{B}$  is a family of triples*

$$\mathfrak{B} \subseteq \{(\alpha, U, \omega) : \alpha \in \mathfrak{A}, U \subseteq U_\alpha \text{ is open, and } \omega \in C^\infty(U, \mathbb{T})\}$$

such that  $(\alpha, U_\alpha, 1_{U_\alpha}) \in \mathfrak{B}$  for all  $\alpha \in \mathfrak{A}$ . Define  $F: \mathfrak{B} \rightarrow \mathfrak{A}$  by  $F(\alpha, U, \omega) = \alpha$ . For each  $\beta = (\alpha, U, \omega) \in \mathfrak{B}$ , let  $V_\beta := U$ , let  $\omega_\beta := \omega$ , and define  $\kappa_\beta: V_\beta \rightarrow E$  by  $\kappa_\beta(\gamma) = \omega(\gamma) \cdot \sigma_\alpha(\gamma)$ .

- (1) If  $\{(U_\alpha, \sigma_\alpha)\}_\alpha$  satisfies  $(U^\infty)$  (respectively  $(S^\infty)$  respectively  $(M^\infty)$  respectively  $(I^\infty)$ ), then so does  $\{(V_\beta, \kappa_\beta)\}_\beta$ .
- (2) For  $\alpha \in \mathfrak{A}$ , the section  $\sigma_\alpha$  is smooth if and only if the sections  $\{\kappa_\beta : F(\beta) = \alpha\}$  are all smooth.

*Proof.* (1) For  $(U^\infty)$ , fix  $\beta = (\alpha, U, \omega) \in \mathfrak{B}$ . Since  $\{(U_\alpha, \sigma_\alpha)\}_\alpha$  satisfies  $(U^\infty)$ , there exists a smooth function  $k_\alpha: U_\alpha \cap G^{(0)} \rightarrow \mathbb{T}$  such that  $\sigma_\alpha(x) = \iota(k_\alpha(x), x)$  for each  $x \in U_\alpha \cap G^{(0)}$ . Since  $k_\alpha$  is smooth, so is its restriction  $l_\beta$  to the open set  $V_\beta \cap G^{(0)}$ . Since  $\omega_\beta$  is smooth, so is its restriction to  $V_\beta \cap G^{(0)}$ . Thus the pointwise product  $\omega_\beta l_\beta$  is smooth. We have  $\kappa_\beta(x) = \iota(\omega_\beta(x)l_\beta(x), x)$  by definition of  $\kappa_\beta$ .

For  $(S^\infty)$ , fix  $\beta, \beta' \in \mathfrak{B}$ , and let  $\alpha = F(\beta)$  and  $\alpha' = F(\beta')$ . For  $\gamma \in V_\beta \cap V_{\beta'} \subseteq U_{F(\beta)} \cap U_{F(\beta')}$ , we have

$$\iota^{-1}(\kappa_\beta(\gamma)\kappa_{\beta'}(\gamma)^{-1}) = \iota^{-1}\left(\omega_\beta(\gamma) \cdot \sigma_\alpha(\gamma) (\overline{\omega_{\beta'}(\gamma)}, r(\gamma))^{-1}\right) = (\omega_\beta(\gamma)\overline{\omega_{\beta'}(\gamma)}, r(\gamma)) \iota^{-1}(\sigma_\alpha(\gamma)\sigma_{\alpha'}(\gamma)^{-1})$$

because the  $\mathbb{T}$ -action is central and  $\iota$  intertwines the action of  $\mathbb{T}$  on  $\mathbb{T} \times G^{(0)}$  by multiplication in the second coordinate with the  $\mathbb{T}$ -action on  $E$ . Since  $\omega_\beta$  and  $\omega_{\beta'}$  are smooth, so is their product, and since  $\{(U_\alpha, \sigma_\alpha)\}_\alpha$  satisfies  $(S^\infty)$ , the map  $\gamma \mapsto \sigma_\alpha(\gamma)\sigma_{\alpha'}(\gamma)^{-1}$  is smooth on the open subset  $V_\beta \cap V_{\beta'}$  of  $U_\alpha \cap U_{\alpha'}$ . Since multiplication in  $\mathbb{T} \times G^{(0)}$  is smooth, it follows that  $\gamma \mapsto \iota^{-1}(\kappa_\beta(\gamma)\kappa_{\beta'}(\gamma)^{-1})$  is smooth.

For  $(M^\infty)$ , fix  $\beta_i \in \mathfrak{B}$  such that  $V_{\beta_1}V_{\beta_2} \cap V_{\beta_3} \neq \emptyset$ , and let  $\alpha_i := F(\beta_i)$  for each  $i$ . Fix  $\gamma_i \in V_{\beta_i}$  such that  $\gamma_1\gamma_2 = \gamma_3$  as above. Since the  $\mathbb{T}$ -action is central, and since  $\iota$  intertwines the action of  $\mathbb{T}$  on  $G^{(0)} \times \mathbb{T}$  (by multiplication in the second coordinate) with the  $\mathbb{T}$  action on  $E$ , we have

$$\iota^{-1}(\kappa_{\beta_1}(\gamma_1)\kappa_{\beta_2}(\gamma_2)\kappa_{\beta_3}(\gamma_3)^{-1}) = (\omega_{\beta_1}(\gamma_1)\omega_{\beta_2}(\gamma_2)\overline{\omega_{\beta_3}(\gamma_1\gamma_2)}, s(\gamma_2)) \iota^{-1}(\sigma_{\alpha_1}(\gamma_1)\sigma_{\alpha_2}(\gamma_2)\sigma_{\alpha_3}(\gamma_3)^{-1}).$$

Since multiplication in  $G$  is smooth and  $\omega_{\beta_3}$  is smooth, the map  $(\gamma_1, \gamma_2) \mapsto \overline{\omega_{\beta_3}(\gamma_1\gamma_2)}$  is smooth on the open subset  $V_{\beta_1} * V_{\beta_2}$  of  $U_{\alpha_1} * U_{\alpha_2}$ . Since  $\omega_{\beta_1}$  and  $\omega_{\beta_2}$  are also smooth, it follows that  $(\gamma_1, \gamma_2) \mapsto \omega_{\beta_1}(\gamma_1)\omega_{\beta_2}(\gamma_2)\overline{\omega_{\beta_3}(\gamma_1\gamma_2)}$  is smooth. Similarly, that  $\{(U_\alpha, \sigma_\alpha)\}_\alpha$  satisfies  $(M^\infty)$  ensures that the map  $(\gamma_1, \gamma_2) \mapsto \iota^{-1}(\sigma_{\alpha_1}(\gamma_1)\sigma_{\alpha_2}(\gamma_2)\sigma_{\alpha_3}(\gamma_3)^{-1})$  is smooth on the open subset of  $V_{\beta_1} * V_{\beta_2}$  where it is defined, and so  $(M^\infty)$  again follows from smoothness of multiplication in  $\mathbb{T} \times G^{(0)}$ .

For  $(I^\infty)$ , fix  $\beta, \beta' \in \mathfrak{B}$  such that  $V_\beta^{-1} \cap V_{\beta'} \neq \emptyset$ , and put  $\alpha = F(\beta)$  and  $\alpha' = F(\beta')$ . Then  $V_\beta \cap V_{\beta'}^{-1} \subseteq U_\alpha \cap U_{\alpha'}^{-1}$  so that the latter is nonempty. For  $\gamma \in V_\beta \cap V_{\beta'}^{-1}$ ,

$$\iota^{-1}(\kappa_\beta(\gamma)\kappa_{\beta'}(\gamma^{-1})) = (\omega_\beta(\gamma)\omega_{\beta'}(\gamma^{-1})) \iota^{-1}(\sigma_\alpha(\gamma)\sigma_{\alpha'}(\gamma^{-1}), s(\gamma)).$$

Since  $\omega_\beta$  and  $\omega_{\beta'}$  are smooth and since inversion is smooth in  $G$ , the map  $\gamma \mapsto \omega_\beta(\gamma)\omega_{\beta'}(\gamma^{-1})$  is smooth. Since  $\{(U_\alpha, \sigma_\alpha)\}_\alpha$  satisfies  $(I^\infty)$ , the map  $\gamma \mapsto \iota^{-1}(\sigma_\alpha(\gamma)\sigma_{\alpha'}(\gamma^{-1}))$  is smooth. So  $(I^\infty)$  once again follows from smoothness of multiplication in  $\mathbb{T} \times G^{(0)}$ .

(2) Since  $\mathfrak{B}$  contains each triple  $(\alpha, U_\alpha, 1_{U_\alpha})$ , the “if” implication is trivial. For the “only if” implication, suppose that  $\sigma_\alpha$  is smooth, and fix  $\beta \in \mathfrak{B}$  with  $F(\beta) = \alpha$ . Then by definition, we have  $V_\beta \subseteq U_\alpha$  and on  $V_\beta$  we have  $\kappa_\beta(\gamma) = \omega_\beta(\gamma) \cdot \sigma_\alpha(\gamma)$ . Since  $\omega_\beta$  and  $\sigma_\alpha$  are smooth and the  $\mathbb{T}$ -action on  $E$  is smooth, it follows that  $\kappa_\beta$  is smooth.  $\square$

**Theorem 4.15.** *Suppose that  $\mathbb{T} \times G^{(0)} \xrightarrow{\iota} E \xrightarrow{\pi} G$  is a topological twist over a Lie groupoid  $G$ . Suppose that  $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in \mathfrak{A}}$  is an atlas of  $G$  and that  $\{\sigma_\alpha: U_\alpha \rightarrow \pi^{-1}(U_\alpha)\}_{\alpha \in \mathfrak{A}}$  is a family of continuous sections. If  $\{\sigma_\alpha\}_\alpha$  satisfies  $(U^\infty)$  and  $(S^\infty)$ , then there is a unique smooth structure on the topological space  $E$  with respect to which  $\pi: E \rightarrow G$  is a smooth principal  $\mathbb{T}$ -bundle and all the  $\sigma_\alpha$  are smooth. With respect to this structure, the following hold:*

- (1) If  $p: S \rightarrow G$  is another smooth principal  $\mathbb{T}$ -bundle and if  $f: E \rightarrow S$  is a continuous bundle map that is  $\mathbb{T}$ -equivariant, then  $f$  is smooth if and only if  $f \circ \sigma_\alpha: U_\alpha \rightarrow S$  is smooth for each  $\alpha$ .
- (2) For each  $x \in G^{(0)}$ , there exists an open neighbourhood  $U$  of  $x$  in  $G$  and a smooth section  $\sigma: U \rightarrow \pi^{-1}(U)$  such that  $\sigma(y) = \iota(1, y)$  for all  $y \in U \cap G^{(0)}$ .
- (3)  $E^{(0)}$  and  $\pi^{-1}(G^{(0)})$  are embedded submanifolds of  $E$ .

- (4)  $\iota: G^{(0)} \times \mathbb{T} \rightarrow \pi^{-1}(G^{(0)})$  is a diffeomorphism.  
(5) The range and source maps  $r, s: E \rightarrow E^{(0)}$  of  $E$  and the projection map  $\pi: E \rightarrow G$  are submersions.

*Proof.* By Lemma A.18, each  $\sigma_\alpha$  defines a topological trivialisation, namely the homeomorphism  $\mathbb{T} \times U_\alpha \rightarrow \pi^{-1}(U_\alpha)$  given by  $(z, \gamma) \mapsto z \cdot \sigma_\alpha(\gamma) = \iota(z, r(\gamma))\sigma_\alpha(\gamma)$ . We want to define an atlas for  $E$  using these local trivialisation maps. To be precise, we will define charts that map onto (open subsets of)  $\mathbb{R}^n \times \mathbb{T}$ , and we will leave it to the reader to modify these charts to map onto (open subsets of)  $\mathbb{R}^{n+1}$ .

For each  $U_\alpha$ , define  $W_\alpha := \pi^{-1}(U_\alpha)$  and  $\Phi_\alpha: W_\alpha \rightarrow \mathbb{R}^n \times \mathbb{T}$  by  $z \cdot \sigma_\alpha(\gamma) \mapsto (z, \varphi_\alpha(\gamma))$ . To see that the collection  $\{(W_\alpha, \Phi_\alpha)\}_{\alpha \in \mathfrak{A}}$  is an atlas for a smooth structure on  $E$ , note first that the  $W_\alpha$  cover  $E$  since the  $U_\alpha$  cover  $G$ . Moreover, each  $\Phi_\alpha$  is a homeomorphism onto its image by construction, so we already have Conditions (A1) and (A2) of Definition A.2. It remains to check (A3). So suppose that  $\alpha' \in \mathfrak{A}$  satisfies  $U_\alpha \cap U_{\alpha'} \neq \emptyset$ . We must show that

$$\Phi_{\alpha'} \circ \Phi_\alpha^{-1}: \Phi_\alpha(U_\alpha \cap U_{\alpha'}) \rightarrow \Phi_{\alpha'}(U_\alpha \cap U_{\alpha'})$$

is a diffeomorphism between open subsets of  $\mathbb{R}^n \times \mathbb{T}$ . By  $(S^\infty)$ , there exists a smooth map  $f := f_{\alpha, \alpha'}: U_\alpha \cap U_{\alpha'} \rightarrow \mathbb{T}$  such that the map

$$F: U_\alpha \cap U_{\alpha'} \rightarrow \mathbb{T} \times G^{(0)}, \quad \gamma \mapsto \iota^{-1}(\sigma_\alpha(\gamma)\sigma_{\alpha'}(\gamma)^{-1}),$$

can be written as

$$F(\gamma) = (f(\gamma), r(\gamma)).$$

For  $(z, \varphi_\alpha(\gamma)) \in \Phi_\alpha(U_\alpha \cap U_{\alpha'})$ , we compute

$$\begin{aligned} \Phi_{\alpha'}^{-1}(z, \varphi_\alpha(\gamma)) &= \iota(z, r(\gamma))\sigma_\alpha(\gamma) = \iota(z, r(\gamma))\sigma_\alpha(\gamma)\sigma_{\alpha'}(\gamma)^{-1}\sigma_{\alpha'}(\gamma) \\ &= \iota(z, r(\gamma))\iota(F(\gamma))\sigma_{\alpha'}(\gamma) = \iota(zf(\gamma), r(\gamma))\sigma_{\alpha'}(\gamma). \end{aligned}$$

Hence

$$(\Phi_{\alpha'} \circ \Phi_\alpha^{-1})(z, \varphi_\alpha(\gamma)) = (zf(\gamma), \varphi_{\alpha'}(\gamma)).$$

Writing  $\varphi_\alpha(\gamma) = \vec{x}$ , we obtain

$$(\Phi_{\alpha'} \circ \Phi_\alpha^{-1})(z, \vec{x}) = (z(f \circ \varphi_\alpha^{-1})(\vec{x}), (\varphi_{\alpha'} \circ \varphi_\alpha^{-1})(\vec{x})).$$

Since  $\varphi_{\alpha'} \circ \varphi_\alpha^{-1}$ ,  $f \circ \varphi_\alpha^{-1}$ , and multiplication on  $\mathbb{T}$  are smooth, we conclude that  $\Phi_{\alpha'} \circ \Phi_\alpha^{-1}$  is smooth. It is now clear that the sections we started with are all smooth: The map  $\Phi_\alpha \circ \sigma_\alpha \circ \varphi_\alpha^{-1}: \mathbb{R}^n \rightarrow \mathbb{T} \times \mathbb{R}^n$  is given by  $\vec{x} \rightarrow (1, \vec{x})$  and hence smooth, which means that  $\sigma_\alpha: U_\alpha \rightarrow E$  is smooth.

It will be helpful to know what other smooth sections this manifold structure on  $E$  allows. By Lemma 4.14, Part (1), the collection

$$(4.1) \quad \Theta := \{\omega \cdot (\sigma_\alpha|_U) : \alpha \in \mathfrak{A}, U \subseteq U_\alpha \text{ open}, \omega \in C^\infty(U, \mathbb{T})\}$$

also satisfies  $(S^\infty)$ . Since the above procedure did not make use of  $(U^\infty)$ , we may apply the procedure to  $\Theta$  to generate another atlas for a smooth structure on  $E$  with respect to which each element of  $\Theta$  is smooth, including each  $\sigma_\alpha$ . By construction, this new atlas *contains all the charts*  $\Phi_\alpha$  from the previously constructed atlas, and therefore generates the same smooth structure as the one that we constructed from only  $\{\sigma_\alpha\}_\alpha$ . To sum up, each element of  $\Theta$  is smooth with respect to the smooth structure with which we have equipped  $E$ .

To see that  $\pi: E \rightarrow G$  is a smooth principal  $\mathbb{T}$ -bundle, we will apply Lemma 4.3. The  $\mathbb{T}$ -action  $\mu: \mathbb{T} \times E \rightarrow E$  is smooth, since, in charts, it is just given by multiplication in  $\mathbb{T}$ . To be more precise, given any  $(w, e) \in \mathbb{T} \times E$ , choose any  $\alpha \in \mathfrak{A}$  with  $\pi(e) \in U_\alpha$ . Then

$$\Phi_\alpha \circ \mu \circ (\text{id}_{\mathbb{T}} \times \Phi_\alpha)^{-1}: \mathbb{T} \times (\mathbb{T} \times \mathbb{R}^n) \rightarrow \mathbb{T} \times \mathbb{R}^n, \quad (w, (z, \vec{x})) \mapsto (wz, \vec{x}),$$

is a smooth map, which proves that  $\mu$  is smooth around  $(w, e)$ . Next,  $\pi$  is smooth, since

$$\varphi_\alpha \circ \pi \circ \Phi_\alpha^{-1}: \mathbb{T} \times \mathbb{R}^n \rightarrow \mathbb{R}^n, \quad (z, \vec{x}) \mapsto \vec{x},$$

is clearly smooth. Using these local charts, we also see that  $\pi$  is a submersion, since  $\varphi_\alpha \circ \pi \circ \Phi_\alpha^{-1}$ , being a linear map, is its own derivative and clearly surjective. We have checked that all the assumptions of Lemma 4.3 hold, and we conclude that  $\pi: E \rightarrow G$  is indeed a smooth principal  $\mathbb{T}$ -bundle.

We can now invoke Lemma A.18 to make our lives easier. For *any* section  $\sigma: U \rightarrow \pi^{-1}(U)$  of  $\pi$  that is smooth with respect to the smooth structure that we just constructed on  $E$ , the map

$$(4.2) \quad \psi_\sigma: \pi^{-1}(U) \rightarrow \mathbb{T} \times U, \quad z \cdot \sigma(\gamma) \mapsto (z, \gamma),$$

is a smooth local trivialisation of  $E$ . Thus, if  $(U, \varphi)$  is a smooth chart of  $G$ , then  $\psi_\sigma$  determines a smooth chart

$$(4.3) \quad \Phi_{\sigma, \varphi}: \pi^{-1}(U) \rightarrow \mathbb{T} \times \mathbb{R}^n, \quad z \cdot \sigma(\gamma) \mapsto (z, \varphi(\gamma)),$$

of  $E$ . Note that  $\Phi_\alpha = \Phi_{\sigma_\alpha, \varphi_\alpha}$ .

Before we can prove that the smooth structure on  $E$  is unique, we first need to prove (1), so suppose that  $p: S \rightarrow G$  is a smooth principal  $\mathbb{T}$ -bundle and  $f: E \rightarrow S$  is a continuous  $\mathbb{T}$ -equivariant bundle map. For the forward implication, assume that  $f$  is smooth and fix  $\alpha \in \mathfrak{A}$  and  $\gamma \in U_\alpha$ . The set  $\pi^{-1}(\{\gamma\}) \cap \Phi_\alpha^{-1}(\{1\} \times \mathbb{R}^n)$  contains a unique element, namely  $e := \sigma_\alpha(\gamma)$ . Smoothness and  $\mathbb{T}$ -equivariance of  $f$  imply that, for every chart  $(V, \Phi^S)$  around  $f(e)$  in  $S$ , the map

$$\Phi^S \circ f \circ \Phi_\alpha^{-1}: \Phi_\alpha(W_\alpha \cap f^{-1}(V)) \rightarrow \Phi^S(V), \quad (z, \varphi_\alpha(\eta)) \mapsto \Phi^S(f(z \cdot \sigma_\alpha(\eta))) = \Phi^S(z \cdot f(\sigma_\alpha(\eta))),$$

is smooth as a map between neighbourhoods in Euclidean space. In other words,

$$\Phi^S \circ f \circ \Phi_\alpha^{-1}: (z, \vec{x}) \mapsto \Phi^S(z \cdot f(\sigma_\alpha(\varphi_\alpha^{-1}(\vec{x}))))$$

is smooth. Since  $\mathbb{R}^n \rightarrow \mathbb{T} \times \mathbb{R}^n, \vec{x} \mapsto (1, \vec{x})$ , is smooth, it follows that the map

$$\vec{x} \mapsto (\Phi^S \circ [f \circ \sigma_\alpha] \circ \varphi_\alpha^{-1})(\vec{x})$$

is smooth, defined on the open subset  $U' := \text{pr}_{\mathbb{R}^n}(\Phi_\alpha(W_\alpha \cap f^{-1}(V)))$  of  $\mathbb{R}^n$ . Note that the open set  $U := \varphi_\alpha^{-1}(U')$  in  $G$  contains  $\gamma$ : since  $f(e) \in V$  by choice of  $V$  and since  $\pi(e) = \gamma \in U_\alpha$ , we have  $e \in f^{-1}(V) \cap \pi^{-1}(U_\alpha) = f^{-1}(V) \cap W_\alpha$ . Since  $e = \sigma_\alpha(\gamma)$ , the definition of  $\Phi_\alpha$  shows that  $\Phi_\alpha(e) = (1, \varphi_\alpha(\gamma))$  and thus  $\varphi_\alpha(\gamma) \in U'$ ; that is,  $\gamma \in U$  as claimed. Moreover, since  $\Phi^S$  was chosen as a map around  $f(e) = (f \circ \sigma_\alpha)(\gamma)$ , the proof that  $\Phi^S \circ [f \circ \sigma_\alpha] \circ \varphi_\alpha^{-1}$  is smooth implies that  $f \circ \sigma_\alpha$  is smooth around  $\gamma$ . As  $\gamma$  was arbitrary, it follows that  $f \circ \sigma_\alpha$  is smooth on all of  $U_\alpha$ .

For the backwards implication of (1), assume that  $f \circ \sigma_\alpha$  is smooth for every  $\alpha \in \mathfrak{A}$ . In order to prove that  $f$  is smooth, fix  $e \in E$  and  $\alpha \in \mathfrak{A}$  such that  $\gamma := \pi(e) \in U_\alpha$ . Then there exists a unique  $z_0 \in \mathbb{T}$  such that  $e = z_0 \cdot \sigma_\alpha(\gamma)$ . Since the  $\mathbb{T}$ -action on  $S$  and the function  $f \circ \sigma_\alpha$  are smooth, the map  $\tau: \mathbb{T} \times U_\alpha \rightarrow S, (z, \eta) \mapsto z \cdot f(\sigma_\alpha(\eta))$ , is smooth. In particular, for any neighbourhood  $U$  of  $\gamma$  in  $U_\alpha$  and any smooth chart  $(V, \Phi^S)$  around  $e = z \cdot f(\sigma_\alpha(\gamma))$  in  $S$  with  $z \cdot f(\sigma_\alpha(U)) \subseteq V$ , the map

$$\Phi^S \circ \tau \circ (\text{id}_{\mathbb{T}} \times \varphi_\alpha)^{-1}: \mathbb{T} \times \varphi_\alpha(U) \rightarrow \Phi^S(V)$$

is smooth. Now,  $\mathbb{T}$ -equivariance of  $f$  and the definition of  $\Phi_\alpha$  show that for any  $(z, \eta) \in \mathbb{T} \times U_\alpha$ ,

$$\tau(z, \eta) = f(z \cdot \sigma_\alpha(\eta)) = (f \circ \Phi_\alpha^{-1})(z, \varphi_\alpha(\eta)).$$

Hence for any  $(z, \vec{x}) \in \mathbb{T} \times \varphi_\alpha(U)$ , we have

$$[\Phi^S \circ \tau \circ (\text{id}_{\mathbb{T}} \times \varphi_\alpha)^{-1}](z, \vec{x}) = (\Phi^S \circ f \circ \Phi_\alpha^{-1})(z, \vec{x}).$$

We have shown that the map

$$\Phi^S \circ f \circ \Phi_\alpha^{-1}: \mathbb{T} \times \varphi_\alpha(U) \rightarrow \Phi^S(V)$$

is smooth. As  $\gamma \in U$ , we have  $(z_0, \varphi_\alpha(\gamma)) \in \mathbb{T} \times \varphi_\alpha(U)$ . Thus

$$e = z_0 \cdot \sigma_\alpha(\gamma) \in \Phi_\alpha(\mathbb{T} \times \varphi_\alpha(U))$$

and  $f$  is smooth around  $e \in E$ . Since  $e$  was arbitrary, this proves that  $f$  is smooth.

We can now prove that the smooth structure is unique. Let  $\tilde{E}$  be a copy of  $E$  carrying a smooth structure such that

( $\tilde{E}1$ )  $\pi: \tilde{E} \rightarrow G$  is a smooth principal  $\mathbb{T}$ -bundle, and

( $\tilde{E}2$ ) for each  $\alpha \in \mathfrak{A}$ , the map  $\tilde{\sigma}_\alpha: U_\alpha \rightarrow \tilde{E}, \gamma \mapsto \sigma_\alpha(\gamma)$ , is smooth.

We show that  $f: E \rightarrow \tilde{E}$ , defined as the identity map on the underlying topological space, is a diffeomorphism, so the smooth structure on  $\tilde{E}$  coincides with the structure we gave  $E$  by hand. Clearly,  $f$  is a continuous,  $\mathbb{T}$ -equivariant bundle map. By assumption ( $\tilde{E}2$ ), each  $\tilde{\sigma}_\alpha = f \circ \sigma_\alpha$  is smooth. Because of (1) combined with Assumption ( $\tilde{E}1$ ), this implies that  $f$  is smooth. An application of Lemma A.16 now shows that  $f$  is a diffeomorphism.

Next, we prove (2). To this end, fix  $x_0 \in G^{(0)}$  and  $\alpha \in \mathfrak{A}$  such that  $x_0 \in U_\alpha$ . Let  $z = -k_\alpha(x_0) \in \mathbb{T}$  (the point antipodal to  $k_\alpha(x_0)$ ), and consider  $W := k_\alpha^{-1}(\mathbb{T} \setminus \{z\})$ . Since  $W$  is open in  $G^{(0)}$  and  $G^{(0)}$  has the subspace topology, there exists an open  $U \subseteq G$  such that  $W = U \cap G^{(0)}$ . Since  $W \subseteq U_\alpha \cap G^{(0)}$ , we can assume without loss of generality that  $U \subseteq U_\alpha$ . Let  $\psi: \mathbb{T} \setminus \{z\} \approx \mathbb{R}$  be a smooth chart for this subset of  $\mathbb{T}$ . Since  $k_\alpha$  maps  $W$  to  $\mathbb{T} \setminus \{z\}$ , we may consider  $\psi \circ k_\alpha|_W: W \rightarrow \mathbb{R}$ .

Since  $G^{(0)}$  is properly embedded in  $G$  by [15, Proposition 5.5], it follows that  $W$  is properly embedded in  $U$  (see Lemma A.7). So [15, Lemma 5.34(b)] yields a smooth function  $\tilde{\omega}: U \rightarrow \mathbb{R}$  such that  $\tilde{\omega}|_W \equiv$

$\psi \circ k_\alpha|_W$ ; let  $\omega := \psi^{-1} \circ \tilde{\omega}: U \rightarrow \mathbb{T} \setminus \{z\}$ . Since  $\bar{\omega}: \gamma \mapsto \overline{\omega(\gamma)}$  is an element of  $C^\infty(U, \mathbb{T})$ , we may consider the section  $\sigma := \bar{\omega} \cdot (\sigma_\alpha|_U)$ , an element of the set  $\Theta$  as defined in Equation (4.1). For  $x \in U \cap G^{(0)} = W$ , it follows from  $\tilde{\omega}|_W \equiv \psi \circ k_\alpha|_W$  that

$$(4.4) \quad \omega(x) = (\psi^{-1} \circ \tilde{\omega})(x) = k_\alpha(x).$$

Hence

$$\sigma(x) = \overline{\omega(x)} \cdot \sigma_\alpha(x) = \overline{k_\alpha(x)} \cdot \iota(x, k_\alpha(x)) = \iota(1, x)$$

as claimed.

Our next goal is to verify (3), namely that  $\pi^{-1}(G^{(0)})$  and  $E^{(0)}$  are embedded submanifolds of  $E$ . The former follows from Remark A.10 and Theorem A.11(1), since  $G^{(0)}$  is an embedded submanifold of the Lie groupoid  $G$  and since  $\pi$  is a submersion. To see that the unit space is an embedded submanifold, fix  $y_0 \in E^{(0)}$ , and let  $x_0 := \pi(y_0) \in G^{(0)}$ . Let  $m$  be the manifold dimension of  $G^{(0)}$ ; note that, since  $G$  is not necessarily étale,  $m$  may be strictly smaller than the manifold dimension  $n$  of  $G$ . As  $G^{(0)}$  is an embedded submanifold of  $G$ , we can pick a chart  $(U, \varphi)$  of  $G$  around  $x_0$  such that

$$(4.5) \quad \varphi(U \cap G^{(0)}) = \varphi(U) \cap (\mathbb{R}^m \times \{0\}^{n-m}).$$

By potentially shrinking  $U$ , we may assume by (2) that there exists a section  $\sigma: U \rightarrow \pi^{-1}(U)$  such that  $\sigma(x) = \iota(1, x)$  for all  $x \in U \cap G^{(0)}$ . We claim that the smooth chart  $\Phi_{\sigma, \varphi}$  of  $E$  maps  $\pi^{-1}(U) \cap E^{(0)}$  onto  $\{1\} \times \mathbb{R}^m \times \{0\}^{n-m}$ :

Given  $y \in \pi^{-1}(U) \cap E^{(0)}$ , we have  $\pi(y) \in U \cap G^{(0)}$  and thus  $y = \iota(1, \pi(y)) = 1 \cdot \sigma(\pi(y))$  by choice of  $\sigma$ . Thus,  $\Phi_{\sigma, \varphi}(y) = (1, \varphi(\pi(y)))$  by definition of  $\Phi_{\sigma, \varphi}$  (see (4.3)). Then since  $\pi(y) \in \pi(\pi^{-1}(U) \cap E^{(0)}) = U \cap G^{(0)}$ , it follows from the choice of  $\varphi$  (Equation (4.5)) that  $\Phi_{\sigma, \varphi}(y) \in \{1\} \times \mathbb{R}^m \times \{0\}^{n-m}$ .

Now suppose that  $e \in \pi^{-1}(U)$  satisfies  $\Phi_{\sigma, \varphi}(e) = (1, \vec{x}, 0) \in \{1\} \times \mathbb{R}^m \times \{0\}^{n-m}$ ; we must argue that  $e \in E^{(0)}$ . As  $\pi(e) \in U$ , there exists a unique  $z \in \mathbb{T}$  such that  $e = z \cdot \sigma(\pi(e))$ , so that

$$(1, \vec{x}, 0) = \Phi_{\sigma, \varphi}(e) \stackrel{(4.3)}{=} (z, \varphi(\pi(e))).$$

By Equation (4.5),  $(\vec{x}, 0) = \varphi(\pi(e))$  implies that  $\pi(e) = x \in G^{(0)}$ . The above computation further implies that  $z = 1$ , so that  $e = \sigma(x) = \iota(1, x) \in E^{(0)}$ , as claimed. This finishes the proof that  $E^{(0)}$  is an embedded submanifold of  $E$ .

For (4), we must show that  $\iota: \mathbb{T} \times G^{(0)} \rightarrow \pi^{-1}(G^{(0)})$  is a diffeomorphism. We already know from the theory of topological twists that  $\iota$  is a homeomorphism. To see that  $\iota$  is smooth, fix  $(z, x_0) \in \mathbb{T} \times G^{(0)}$ . By (2), there is an open neighbourhood  $U$  of  $x_0$  in  $G$  and a smooth section  $\sigma: U \rightarrow \pi^{-1}(U)$  such that  $\sigma(x) = \iota(1, x)$  for all  $x \in U_0 := U \cap G^{(0)}$ . It suffices to show that  $\iota$  is smooth on the open neighbourhood  $\mathbb{T} \times U_0$  of  $(z, x_0)$  in  $\mathbb{T} \times G^{(0)}$ . As explained earlier, the map  $\psi_\sigma$  in (4.2) is a diffeomorphism of  $\pi^{-1}(U)$  onto  $\mathbb{T} \times U$ . For  $x \in U_0$ , we have  $\psi_\sigma \circ \iota(z, x) = \psi_\sigma(z \cdot \iota(1, x)) = \psi_\sigma(z \cdot \sigma(x)) = (z, x)$ , so  $\psi_\sigma \circ \iota$  is the identity on, and in particular a diffeomorphism of,  $\mathbb{T} \times U_0$ . Hence  $\iota|_{\mathbb{T} \times U_0} = \psi_\sigma^{-1} \circ (\psi_\sigma \circ \iota|_{\mathbb{T} \times U_0})$  is smooth. Now,  $\pi: \pi^{-1}(G^{(0)}) \rightarrow G^{(0)}$  is a principal  $\mathbb{T}$ -bundle (see Remark A.15) and, by construction of the  $\mathbb{T}$ -action on  $E$ ,  $\iota$  is a  $\mathbb{T}$ -equivariant bundle map. It now follows from Lemma A.16 that the smooth homeomorphism  $\iota$  is a diffeomorphism onto  $\pi^{-1}(G^{(0)})$ .

Finally, for (5), note that we already know that  $\pi$  is a submersion because  $\pi: E \rightarrow G$  is a smooth principal bundle. We prove that  $r_E$  is a submersion; a similar argument shows that  $s_E$  is a submersion. To see that  $r_E$  is smooth, note that it is given by

$$r_E(e) = \iota(1, r_G(\pi(e))).$$

Since the maps  $\pi: E \rightarrow G$ ,  $r_G: G \rightarrow G^{(0)}$ ,  $\iota: \mathbb{T} \times G^{(0)} \rightarrow E$ , and  $G^{(0)} \rightarrow \mathbb{T} \times G^{(0)}$ ,  $x \mapsto (1, x)$ , are all smooth, it follows that  $r_E: E \rightarrow E$  is smooth with image contained in  $E^{(0)}$ . Since we have shown in (3) that  $E^{(0)}$  is an embedded submanifold of  $E$ , it now follows from [15, Corollary 5.30] that  $r_E$  is also smooth as a map  $E \rightarrow E^{(0)}$ , as claimed.

To see that  $r_E$  is a submersion, fix  $e$  in  $E$ . We have

$$dr_E|_e = d(r_G \circ \pi)|_e = dr_G|_{\pi(e)} \circ d\pi|_e.$$

Hence if  $\pi(e) \in U_\alpha$ , then

$$dr_E|_e \circ d(\sigma_\alpha|_{\pi(e)}) = dr_G|_{\pi(e)} \circ d\pi|_e \circ d(\sigma_\alpha|_{\pi(e)}) = d(r_G \circ \pi \circ \sigma_\alpha)|_{\pi(e)} = dr_G|_{\pi(e)}.$$

This is surjective because  $G$  is a Lie groupoid, so we conclude that  $dr_E|_e$  is surjective as well.  $\square$

*Remark 4.16.* Condition  $(U^\infty)$  in Theorem 4.15 is critical and cannot be removed. To see why, observe that since the topological group  $\mathbb{T}$  admits only one nontrivial continuous automorphism, namely  $z \mapsto \bar{z}$ , if  $G^{(0)}$  is connected, then there are just two possible continuous injective homomorphisms  $\iota: \mathbb{T} \times G^{(0)} \rightarrow E$



that satisfy  $\pi \circ \iota = \text{id}_{G^{(0)}}$ . Now suppose that  $G = G^{(0)}$  is a (connected) manifold viewed as a groupoid consisting entirely of units, and that  $E = \mathbb{T} \times G$  viewed as a (topologically) trivial group bundle over  $G$ , with  $\pi: E \rightarrow G$  the projection map. Then  $\iota: \mathbb{T} \times G^{(0)} \rightarrow E$  is either  $\iota(z, x) = (z, x)$  or  $\iota(z, x) = (\bar{z}, x)$ . Let  $f: G \rightarrow \mathbb{T}$  be a continuous function that is not differentiable. Fix a maximal atlas  $\mathcal{A} = \{(U_\alpha, \varphi_\alpha) : \alpha \in \mathfrak{A}\}$  for  $G$ ; so the  $U_\alpha$  are a base for the topology on  $G$ . Then  $\mathcal{A}$  is a base of bisections of  $G$ . For each  $\alpha \in \mathfrak{A}$ , define  $\sigma_\alpha: U_\alpha \rightarrow E$  by  $\sigma_\alpha(x) := (f(x), x)$ . Then  $\{\sigma_\alpha\}_{\alpha \in \mathfrak{A}}$  trivially satisfies  $(M^\infty)$  and  $(I^\infty)$  because  $x \mapsto (x, x)$  and  $(x, x) \mapsto x^2 = x$  are mutually inverse diffeomorphisms between  $G^{(2)}$  and  $G$ , and  $x \mapsto x^{-1}$  is the identity map on  $G$ . It also satisfies  $(S^\infty)$ , because for any  $\alpha, \alpha' \in \mathfrak{A}$  and any  $x \in U_\alpha \cap U_{\alpha'}$ , we have  $\sigma_\alpha(x)\sigma_{\alpha'}(x)^{-1} = (f(x), x)(f(x), x)^{-1} = (1, x) = \iota(1, x)$ . As in Theorem 4.15, there is a unique smooth structure on  $E$  that makes the maps  $\sigma_\alpha$  smooth. Specifically, we identify  $E$  with  $\mathbb{T} \times G$  by the map  $h: \mathbb{T} \times G \rightarrow E$  defined by  $h(z, x) := (zf(x), x)$ , and then charts  $\psi_\alpha: U_\alpha \times \mathbb{T} \rightarrow \pi^{-1}(U_\alpha)$  are given by  $\psi_\alpha(z, x) = h(z, x) = (zf(x), x)$ . In particular, we have  $\sigma_\alpha(x) = \iota(f(x), x)$  or  $\iota(\overline{f(x)}, x)$  (depending on the choice of  $\iota$  above) for each  $\alpha \in \mathfrak{A}$  and  $x \in U_\alpha$ . That is, the map  $k_\alpha$  in  $(U^\infty)$  is either  $f$  or  $\bar{f}$ , so is not smooth. Hence  $\mathcal{A}$  does not satisfy  $(U^\infty)$ . Multiplication and inversion in  $E$  are smooth with respect to the smooth structure induced by  $\mathcal{A}$ , and each  $\sigma_\alpha: U_\alpha \rightarrow E$  is by definition smooth in this smooth structure. So since the composition of the map  $x \mapsto \iota(1, x)^{-1}\sigma_\alpha(x)$  with the projection from  $\mathbb{T} \times G$  onto  $\mathbb{T}$  is either  $f$  or  $\bar{f}$ , neither of which is smooth, we deduce that  $\iota|_{\{1\} \times G}$  is not smooth as a map into  $E$  with the smooth structure  $\mathcal{A}$ .

That is, in order for the smooth structure on  $E$  induced by  $\mathcal{A}$  to be compatible through  $\iota$  with the canonical differential structure on  $\mathbb{T} \times G$ , we *must* assume  $(U^\infty)$ .

**Corollary 4.17.** *Suppose that  $\mathbb{T} \times G^{(0)} \xrightarrow{\iota} E \xrightarrow{\pi} G$  is a Lie twist. Suppose that  $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in \mathfrak{A}}$  is an atlas of  $G$  and that  $\{\sigma_\alpha: U_\alpha \rightarrow \pi^{-1}(U_\alpha)\}_{\alpha \in \mathfrak{A}}$  is a family of continuous sections satisfying  $(U^\infty)$  and  $(S^\infty)$ . Let  $\tilde{E}$  be the topological space  $E$  equipped with the smooth structure induced by  $\{\sigma_\alpha\}_\alpha$  as in Theorem 4.15. Then  $E = \tilde{E}$  if and only if the all maps  $\sigma_\alpha: U_\alpha \rightarrow E$  are smooth.*

*Proof.* Since  $E$  is a Lie twist,  $\pi: E \rightarrow G$  is a smooth principal  $\mathbb{T}$ -bundle by Lemma 4.4, (LT4). With respect to the same  $\mathbb{T}$ -action and the same projection map,  $\pi: \tilde{E} \rightarrow G$  is also a smooth principal  $\mathbb{T}$ -bundle. In particular,  $f: \tilde{E} \rightarrow E$ , defined as the identity map on the underlying topological space, is a  $\mathbb{T}$ -equivariant bundle map. Since  $E$  and  $\tilde{E}$  have the same topology by assumption,  $f$  is a homeomorphism. Thus, by Lemma A.16, we have that  $E = \tilde{E}$  if and only if  $f$  is smooth. By Theorem 4.15, Part (1), this is equivalent to  $f \circ \sigma_\alpha: U_\alpha \rightarrow E$  being smooth for all  $\alpha$ , as claimed.  $\square$

While we do not have an immediate use for the following corollary, it seems worth recording a checkable condition under which two families of sections as in Theorem 4.15 determine the same smooth structure.

**Corollary 4.18.** *Suppose that  $\mathbb{T} \times G^{(0)} \xrightarrow{\iota} E \xrightarrow{\pi} G$  is a topological twist over an étale Lie groupoid. Suppose that  $\{U_\alpha\}_{\alpha \in \mathfrak{A}}$  and  $\{V_\beta\}_{\beta \in \mathfrak{B}}$  are bases of open bisections for  $G$  and that  $\{\sigma_\alpha: U_\alpha \rightarrow \pi^{-1}(U_\alpha)\}_{\alpha \in \mathfrak{A}}$  and  $\{\tau_\beta: V_\beta \rightarrow \pi^{-1}(V_\beta)\}_{\beta \in \mathfrak{B}}$  are families of continuous sections that both satisfy  $(U^\infty)$  and  $(S^\infty)$ . Suppose that there is a common refinement  $\mathcal{R} = \{W_\delta: \delta \in \mathfrak{D}\}$  of  $\{U_\alpha\}_\alpha$  and  $\{V_\beta\}_\beta$  (with refinement maps  $\mathfrak{a}: \mathfrak{D} \rightarrow \mathfrak{A}$  and  $\mathfrak{b}: \mathfrak{D} \rightarrow \mathfrak{B}$ ) that is a cover of  $G$ , such that for each  $\delta \in \mathfrak{D}$  the map  $\iota^{-1}(\sigma_{\mathfrak{a}(\delta)}(\gamma)\tau_{\mathfrak{b}(\delta)}(\gamma)^{-1})$  is smooth on  $W_\delta$ . Then the unique smooth structures on  $E$ , obtained from Theorem 4.15, for which the  $\sigma_\alpha$  and the  $\tau_\beta$  are smooth coincide.*

*Proof.* By Lemma 4.13, the system  $\{(W_\delta, \sigma_{\mathfrak{a}(\delta)})\}_\delta$  yields the same smooth structure as  $\{(U_\alpha, \sigma_\alpha)\}_\alpha$  and similarly  $\{(W_\delta, \tau_{\mathfrak{b}(\delta)})\}_\delta$  yields the same smooth structure as  $\{(V_\beta, \tau_\beta)\}_\beta$ . For each  $\delta \in \mathfrak{D}$ , let  $\omega_\delta: W_\delta \rightarrow \mathbb{T}$  be the function such that  $\iota^{-1}(\sigma_{\mathfrak{a}(\delta)}(\gamma)\tau_{\mathfrak{b}(\delta)}(\gamma)^{-1}) = \omega_\delta(\gamma)$  for all  $\gamma \in W_\delta$ . So  $\omega_\delta$  is smooth by assumption. Now Lemma 4.14 applied to the system  $\{(W_\delta, \tau_{\mathfrak{b}(\delta)})\}_\delta$  and with the indexing set

$$\{(\delta, W_\delta, 1_{W_\delta}), (\delta, W_\delta, \omega_\delta) : \delta \in \mathfrak{D}\}$$

shows that the system  $\{(W_\delta, \eta_{\delta, i}) : (\delta, i) \in \mathfrak{D} \times \{0, 1\}\}$  given by  $\eta_{\delta, 0} = \tau_{\mathfrak{b}(\delta)}$  and  $\eta_{\delta, 1} = \sigma_{\mathfrak{a}(\delta)}$  determines the same smooth structure as  $\{(W_\delta, \tau_{\mathfrak{b}(\delta)})\}_\delta$ . Lemma 4.14, this time applied to the system  $\{(W_\delta, \sigma_{\mathfrak{a}(\delta)})\}_\delta$  and with the indexing set

$$\{(\delta, W_\delta, 1_{W_\delta}), (\delta, W_\delta, \overline{\omega_\delta}) : \delta \in \mathfrak{D}\},$$

shows that the same system  $\{(W_\delta, \eta_{\delta, i})\}_{(\delta, i)}$  gives the same smooth structure as  $\{(W_\delta, \sigma_{\mathfrak{a}(\delta)})\}_\delta$ , and we are done.  $\square$

*Remark 4.19.* The previous result is not an “if and only if” statement: to conclude from coincidence of the smooth structure on  $E$  coming from the  $\sigma_\alpha$  and the  $\tau_\beta$  that functions of the form  $\iota^{-1}(\sigma_{\mathfrak{a}(\delta)}(\gamma)\tau_{\mathfrak{b}(\delta)}(\gamma)^{-1})$  are smooth requires that multiplication and inversion in  $E$  are smooth maps; so if either of the families

$\{\sigma_\alpha\}_\alpha$  or  $\{\tau_\beta\}_\beta$  satisfy  $(M^\infty)$  and  $(I^\infty)$ , then the statement becomes an “if and only if” statement. The correct general “if and only if” statement is that two families  $\sigma_\alpha$  and  $\tau_\beta$  give the same smooth structure if and only if there are smooth functions  $\omega_{\alpha,\beta}: U_\alpha \cap V_\beta \rightarrow \mathbb{T}$  such that  $\omega_{\alpha,\beta}(\gamma) \cdot \sigma_\alpha(\gamma) = \tau_\beta(\gamma)$ ; but this is an instance of a general statement about smooth principle  $\Gamma$ -bundles (it could be deduced, for example, using Lemma A.18) rather than about twists over Lie groupoids.

The following companion result to Theorem 4.15 shows that conditions  $(M^\infty)$  and  $(I^\infty)$  are the additional conditions required to ensure that the twist  $E$  is a Lie groupoid under the smooth structure it obtains from the theorem.

**Proposition 4.20.** *Suppose that  $\mathbb{T} \times G^{(0)} \xrightarrow{\iota} E \xrightarrow{\pi} G$  is a topological twist over a Lie groupoid. Suppose that  $\{U_\alpha\}_{\alpha \in \mathfrak{A}}$  is a base of open bisections for  $G$  and that  $\{\sigma_\alpha: U_\alpha \rightarrow \pi^{-1}(U_\alpha)\}_{\alpha \in \mathfrak{A}}$  is a family of continuous sections that satisfies  $(U^\infty)$  and  $(S^\infty)$ . Equip  $E$  with the smooth structure from Theorem 4.15.*

- (1) *If  $\{\sigma_\alpha\}_\alpha$  satisfies  $(M^\infty)$ , then the groupoid multiplication of  $E$  is smooth.*
- (2) *If  $\{\sigma_\alpha\}_\alpha$  satisfies  $(I^\infty)$ , then the groupoid inversion of  $E$  is smooth.*
- (3) *If  $\{\sigma_\alpha\}_\alpha$  satisfies both  $(M^\infty)$  and  $(I^\infty)$ , then  $E$  is a Lie groupoid.*

*Proof.* First assume that  $(M^\infty)$  holds. To check that multiplication is smooth, fix a point  $(e_1, e_2) \in E^{(2)}$ . Choose  $\alpha \in \mathfrak{A}$  such that  $\pi(e_1 e_2) \in U_\alpha$ . Since we have a base for the topology on  $G$  and multiplication is continuous, there exist  $\alpha_1, \alpha_2 \in \mathfrak{A}$  such that  $\pi(e_i) \in U_i := U_{\alpha_i}$  and  $U_1 U_2 \subseteq U_\alpha$ . Since  $\pi^{-1}(U_1) * \pi^{-1}(U_2)$  is a neighbourhood of  $(e_1, e_2)$  in  $E^{(2)}$ , it suffices to show that multiplication is smooth on  $\pi^{-1}(U_1) * \pi^{-1}(U_2)$ .

As before, each  $\beta \in \mathfrak{A}$  gives rise to a diffeomorphism  $\psi_\beta: \pi^{-1}(U_\beta) \rightarrow \mathbb{T} \times U_\beta$  such that  $z \cdot \sigma_\beta(\gamma) \mapsto (z, \gamma)$ ; we write  $\psi_i := \psi_{\alpha_i}$  for  $i = 1, 2$ . The map  $\psi_1 \times \psi_2: \pi^{-1}(U_1) \times \pi^{-1}(U_2) \rightarrow \mathbb{T} \times U_1 \times \mathbb{T} \times U_2$  is a diffeomorphism. Let  $\psi: \pi^{-1}(U_1) \times \pi^{-1}(U_2) \rightarrow \mathbb{T}^2 \times U_1 \times U_2$  be the diffeomorphism obtained by composing the map  $(z_1, \gamma_1, z_2, \gamma_2) \mapsto (z_1, z_2, \gamma_1, \gamma_2)$  with  $\psi_1 \times \psi_2$ . Then  $\psi$  carries  $\pi^{-1}(U_1) * \pi^{-1}(U_2)$  to  $\mathbb{T}^2 \times (U_1 * U_2)$ . Since  $r, s: E \rightarrow E^{(0)}$  are submersions (Theorem 4.15, (5)), the subset  $\pi^{-1}(U_1) * \pi^{-1}(U_2)$  is an embedded submanifold of  $\pi^{-1}(U_1) \times \pi^{-1}(U_2)$  (Proposition A.12). Likewise  $U_1 * U_2$  is an embedded submanifold of  $U_1 \times U_2$  and hence  $\mathbb{T}^2 \times (U_1 * U_2)$  is an embedded submanifold of  $\mathbb{T}^2 \times U_1 \times U_2$ . By [15, Theorem 5.27 and Corollary 5.30],  $\psi$  therefore restricts to a diffeomorphism  $\psi: \pi^{-1}(U_1) * \pi^{-1}(U_2) \rightarrow \mathbb{T}^2 \times (U_1 * U_2)$ . The trivialisation map  $\psi_\alpha: \pi^{-1}(U_\alpha) \rightarrow \mathbb{T} \times U_\alpha$  restricts to a trivialisation, also denoted  $\psi_\alpha$ , of the open  $\mathbb{T}$ -invariant subset  $\pi^{-1}(U_1 U_2)$ . Let  $M: E^{(2)} \rightarrow E$  be the multiplication map. Using that the  $\mathbb{T}$ -action is central at the second step, we calculate:

$$\psi_\alpha(M(\psi^{-1}(z_1, z_2, \gamma_1, \gamma_2))) = \psi_\alpha(M(\psi_1^{-1}(z_1, \gamma_1), \psi_2^{-1}(z_2, \gamma_2))) = \psi_\alpha(z_1 z_2 \cdot (\sigma_{\alpha_1}(\gamma_1) \sigma_{\alpha_2}(\gamma_2))).$$

By  $(M_\mathbb{T}^\infty)$ , the function  $g = g_{\alpha, \alpha_1, \alpha_2}: U_1 * U_2 \rightarrow \mathbb{T}$  determined by  $\sigma_{\alpha_1}(\gamma_1) \sigma_{\alpha_2}(\gamma_2) = g(\gamma_1, \gamma_2) \cdot \sigma_\alpha(\gamma_1 \gamma_2)$  is smooth. We have

$$\psi_\alpha(z_1 z_2 \cdot (\sigma_{\alpha_1}(\gamma_1) \sigma_{\alpha_2}(\gamma_2))) = \psi_\alpha(z_1 z_2 \cdot (g(\gamma_1, \gamma_2) \cdot \sigma_\alpha(\gamma_1 \gamma_2))) = (z_1 z_2 g(\gamma_1, \gamma_2), \gamma_1 \gamma_2).$$

Since the (iterated) multiplication map  $\mathbb{T}^3 \rightarrow \mathbb{T}$ , the map  $g$  and the multiplication map from  $G^{(2)}$  to  $G$  are all smooth, we deduce that multiplication in  $E$  is smooth.

Next assume that  $(I^\infty)$  holds. To check that the inversion map  $I: E \rightarrow E$  is smooth, fix  $e \in E$ . Choose  $\alpha, \alpha' \in \mathfrak{A}$  such that  $\pi(e) \in U_\alpha$  and  $\pi(e^{-1}) \in U_{\alpha'}$ . Let  $W := U_\alpha \cap U_{\alpha'}^{-1}$ . Then  $\pi^{-1}(W)$  is an open neighbourhood of  $e$ , so it suffices to show that  $I$  is smooth on  $\pi^{-1}(W)$ . Observe that  $\psi_\alpha$  restricts to a diffeomorphism from  $\pi^{-1}(W)$  to  $\mathbb{T} \times W$ , and  $\psi_{\alpha'}$  restricts to a diffeomorphism from  $\pi^{-1}(W^{-1}) = \pi^{-1}(W)^{-1}$  to  $\mathbb{T} \times W^{-1}$ . For  $\gamma \in W$  we have  $\gamma^{-1} \in U_{\alpha'}$ , and we calculate

$$(\psi_\alpha \circ I \circ \psi_{\alpha'})(z, \gamma^{-1}) = \psi_\alpha((z \cdot \sigma_{\alpha'}(\gamma^{-1}))^{-1}) = \psi_\alpha(\overline{z} \cdot \sigma_{\alpha'}(\gamma^{-1})^{-1}).$$

By  $(I^\infty)$ , the map  $h = h_{\alpha, \alpha'}: W \rightarrow \mathbb{T}$  such that  $h(\gamma) \cdot \sigma_{\alpha'}(\gamma^{-1})^{-1} = \sigma_\alpha(\gamma)$  is smooth. We have

$$\psi_\alpha(\overline{z} \cdot \sigma_{\alpha'}(\gamma^{-1})^{-1}) = \psi_\alpha(\overline{z} \cdot (\overline{h(\gamma)} \cdot \sigma_\alpha(\gamma))) = (z \overline{h(\gamma)}, \gamma).$$

Since multiplication and inversion in  $\mathbb{T}$ , the map  $h$ , and inversion in the Lie groupoid  $G$  are all smooth, we deduce that inversion in  $E$  is smooth.

The twist  $E$  is by definition a topological groupoid. By Theorem 4.15, it has the structure of a smooth manifold with respect to which  $r$  and  $s$  are submersions and  $E^{(0)}$  is an embedded submanifold; this takes care of (L1), (L2), and (L3). We have further proved above that multiplication and inversion in  $E$  are smooth maps by  $(M^\infty)$  and  $(I^\infty)$ . So (L4) and (L5) hold.  $\square$

For our results on  $C^*$ -algebras in Section 5, it will be convenient to be able to move back and forth between sections of a Lie twist over a Lie groupoid and sections of the associated line bundle. The following technical lemma shows that we can do so.

Given a continuous vector-valued function  $f$  on a topological space  $X$ , we write  $\text{supp}^\circ(f)$  for the open support of  $f$ ; that is

$$\text{supp}^\circ(f) = \{x \in X : f(x) \neq 0\}.$$

**Lemma 4.21.** *Suppose that  $\mathbb{T} \times G^{(0)} \xrightarrow{\iota} E \xrightarrow{\pi} G$  is a topological twist over a groupoid  $G$ . Let  $L = \mathbb{T} \backslash (\mathbb{C} \times E)$  be the line bundle associated to  $E$ , and let  $E' := \{[\lambda, e] \in L : |\lambda| = 1\} \subseteq L$ . Define  $\rho: E \rightarrow E'$  by  $\rho(e) = [1, e]$ . Then  $\rho$  is an isomorphism of topological principal  $\mathbb{T}$ -bundles. Moreover, the following hold.*

- (1) *Any continuous section  $\mathfrak{s} \in \Gamma_0(G; L)$  gives rise to a continuous section  $\sigma_{\mathfrak{s}}: \text{supp}^\circ(\mathfrak{s}) \rightarrow E$  of  $\pi$  given by*

$$\sigma_{\mathfrak{s}}(\gamma) = \rho^{-1} \left( \frac{\mathfrak{s}(\gamma)}{|\mathfrak{s}(\gamma)|} \right).$$

- (2) *If  $U$  is an open bisection of  $G$ , then any continuous section  $\sigma: U \rightarrow E$  of  $\pi$  gives rise to a section  $\tilde{\sigma} := \rho \circ \sigma: U \rightarrow E' \subseteq L$  of the line bundle. Moreover, the map*

$$\begin{aligned} \{f \in C_0(r(U), \mathbb{R}^{>0}) : \text{supp}^\circ(f) = r(U)\} &\rightarrow \{\mathfrak{s} \in \Gamma_0(G; L) : \sigma_{\mathfrak{s}} = \sigma\}, \\ f &\mapsto f \cdot \tilde{\sigma} = [\gamma \mapsto f(r(\gamma)) \cdot \rho(\sigma(\gamma))], \end{aligned}$$

*is a bijection and its inverse carries  $\mathfrak{s}$  to the function  $x \mapsto |(\mathfrak{s} \circ r|_U^{-1})(x)|$ .*

Now suppose in addition that  $\mathbb{T} \times G^{(0)} \xrightarrow{\iota} E \xrightarrow{\pi} G$  is a Lie twist. Then  $\mathbb{T}$  acts smoothly, freely, and properly on the product manifold  $\mathbb{C} \times E$ . With respect to the quotient-manifold structure on  $L = \mathbb{T} \backslash (\mathbb{C} \times E)$ ,  $\rho$  is an isomorphism of smooth principal  $\mathbb{T}$ -bundles, and the following hold.

- (3) *The correspondence in (1) maps smooth sections of  $L$  to smooth sections of  $\pi$ .*  
(4) *If  $\sigma$  is smooth, then the bijection in (2) restricts to a bijection between smooth functions  $f$  and smooth sections  $\mathfrak{s}$  of  $L$ .*

*Proof.* As discussed on [19, p. 39], it is standard that  $L$  is a Fell line bundle (see also [8, Example 5.5]), and that the unitary bundle  $E'$  of  $L$  yields a principal  $\mathbb{T}$ -bundle  $\pi': E' \rightarrow G$  in the quotient topology, given by  $\pi'([z, e]) = \pi(e)$ . The map  $\rho$  is bijective because the map  $[\lambda, e] \mapsto \lambda \cdot e$  is an inverse for it (that this formula is well-defined and that it is an algebraic inverse for  $\rho$  follows from the observation that  $[\lambda, e] = [1, \lambda \cdot e]$  for all  $\lambda \in \mathbb{T}$  and  $e \in E$ ). This  $\rho$  intertwines  $\mathbb{T}$ -actions by definition of the equivalence relation on  $\mathbb{C} \times E$  defining  $L$ . It is continuous because the map  $e \mapsto (\lambda, e)$  is continuous from  $E$  into  $\mathbb{C} \times E$ . To see that it is open, observe that the map  $(\lambda, e) \mapsto \lambda \cdot e$  is continuous on  $\mathbb{T} \times E$ , so it descends to a continuous map on the quotient by definition of the quotient topology.

(1) The map  $(z, e) \mapsto |z|$  is continuous on  $\mathbb{C} \times E$  and descends to the map  $\lambda \mapsto |\lambda|$  on  $L$ , so the latter is continuous. Thus if  $\mathfrak{s}$  is a continuous section of  $L$ , then the formula  $\gamma \mapsto \mathfrak{s}(\gamma)/|\mathfrak{s}(\gamma)|$  defines a continuous section on the open set  $\text{supp}^\circ(\mathfrak{s})$ , and so composing with the isomorphism  $\rho^{-1}$  yields a continuous section. It is bounded because, by construction,  $|\sigma_{\mathfrak{s}}(\gamma)| = 1$  for all  $\gamma \in \text{supp}^\circ(\mathfrak{s})$ .

(2) The first part is a direct consequence of the fact that  $\rho$  is an isomorphism. For the second statement, observe that  $f \mapsto f \cdot \tilde{\sigma}$  is certainly a map from  $\{f \in C_0(r(U), \mathbb{R}^{>0}) : \text{supp}^\circ(f) = r(U)\}$  to  $\{\mathfrak{s} \in \Gamma_0(G; L) : \sigma_{\mathfrak{s}} = \sigma\}$ . It is injective because  $f(\gamma) = |(f \cdot \tilde{\sigma})(\gamma)|$  for all  $\gamma \in U$  and  $f \in C_0(r(U), \mathbb{R}^{>0})$ . It is surjective because if  $\sigma_{\mathfrak{s}} = \sigma$ , then by definition  $\rho(\sigma(\gamma)) = \mathfrak{s}(\gamma)/|\mathfrak{s}(\gamma)|$  for all  $\gamma \in \text{supp}^\circ(\mathfrak{s})$ ; rearranging gives  $\mathfrak{s}(\gamma) = |\mathfrak{s}(\gamma)| \cdot \rho(\sigma(\gamma))$  for all  $\gamma$ , so the continuous function  $f: r(U) \rightarrow \mathbb{R}^{>0}$  given by  $f(r(\gamma)) = |\mathfrak{s}(\gamma)|$  satisfies  $\mathfrak{s} = f \cdot \tilde{\sigma}$ . The formula for the inverse follows immediately from the formula  $f(r(\gamma)) = |\mathfrak{s}(\gamma)|$ .

Now assume that  $\mathbb{T} \times G^{(0)} \xrightarrow{\iota} E \xrightarrow{\pi} G$  is a Lie twist. Let  $q: \mathbb{C} \times E \rightarrow L$  be the quotient map. The map  $\pi'$  is smooth by definition of the quotient manifold structure. It is a submersion because the composition  $\pi' \circ q \circ (e \mapsto (1, e))$  is precisely  $\pi$ , which is a submersion by assumption, and the differential of a composite is the composite of the differentials. The left  $\mathbb{T}$ -action on  $E'$  is smooth because the left  $\mathbb{T}$ -action on  $\mathbb{C} \times E$  by multiplication in the first coordinate is smooth. So Lemma 4.3 shows that  $\pi': E' \rightarrow G$  is a smooth principal  $\mathbb{T}$ -bundle. Let  $\sigma: U \rightarrow \pi^{-1}(U)$  be a smooth local section for  $\pi$  on an open subset  $U \subseteq G$ . Then  $\tilde{\sigma} := \rho \circ \sigma$  is given by  $\tilde{\sigma}(\gamma) = [1, \sigma(\gamma)]$ , which by definition of the smooth structure on  $L = \mathbb{T} \backslash (\mathbb{C} \times E)$  is a smooth section for  $\pi'$ . So Lemma A.17 shows that  $\rho$  induces a diffeomorphism between a local trivialisation of  $\pi^{-1}(U)$  and a local trivialisation of  $(\pi')^{-1}(U)$ . Since this applies to any local trivialisation of  $E$ , it follows that  $\rho$  is a diffeomorphism, and hence an isomorphism of smooth principal  $\mathbb{T}$ -bundles.

(3) If  $\mathfrak{s}$  is a smooth section of  $L$ , then the map  $\gamma \mapsto 1/|\mathfrak{s}(\gamma)|$  is smooth on  $\text{supp}^\circ(\mathfrak{s})$ , and so  $\sigma_{\mathfrak{s}}$  is smooth as product and concatenation of smooth functions.

(4) Suppose that  $\sigma: U \rightarrow E$  is a smooth section of  $\pi$ . Then  $\tilde{\sigma} = \rho \circ \sigma$  is smooth. Consequently, if  $f: r(U) = \text{supp}^\circ(f) \rightarrow \mathbb{R}^{>0}$  is smooth, then  $f \cdot \tilde{\sigma}$  is smooth, since the  $\mathbb{T}$ -action on  $L$  is smooth. It follows that the bijection in (2) maps smooth functions to smooth sections. Conversely, given  $\mathfrak{s} \in \Gamma_0^\infty(G; L)$  with  $\sigma_{\mathfrak{s}} = \sigma$ , we have seen above that  $\mathfrak{s}$  is the image of  $f: x \mapsto |(\mathfrak{s} \circ r|_U^{-1})(x)|$  under the bijection in (2).

Since Lemma 3.9 implies that  $U$  is a smooth bisection,  $f$  is a composition of smooth maps and hence smooth.  $\square$

**4.1. Lie twists over étale groupoids.** In this subsection, we revisit Conditions  $(S^\infty)$ – $(I^\infty)$  in the situation of a twist  $E$  over an étale Lie groupoid  $G$ . The point is that while Condition  $(U^\infty)$  is phrased purely in terms of the manifold structure on  $G^{(0)}$ , the remaining conditions depend on the manifold structure of  $G$  as a whole. We know from Section 3 that for étale groupoids  $G$ , the manifold structure of  $G$  is completely determined by that of  $G^{(0)}$ ; and critically, this is the data that is encoded by a Cartan pair  $B \subseteq A$  of  $C^*$ -algebras together with a smooth subalgebra  $B^\infty$  of  $B$ . So our work in this section, specifically Corollary 4.25, is the articulation point between Conditions  $(U^\infty)$ – $(I^\infty)$  and algebraic conditions on a Cartan pair of  $C^*$ -algebras (see Section 5).

**Definition 4.22.** Suppose that  $\mathbb{T} \times G^{(0)} \xrightarrow{\iota} E \xrightarrow{\pi} G$  is a topological twist over a groupoid  $G$ , that  $G^{(0)}$  is a manifold, that  $\{U_\alpha\}_{\alpha \in \mathfrak{A}}$  is a collection of open subsets of  $G$ , and that  $\{\sigma_\alpha: U_\alpha \rightarrow \pi^{-1}(U_\alpha)\}_{\alpha \in \mathfrak{A}}$  is a family of continuous sections of  $\pi$ . We define the following conditions for the family  $\{\sigma_\alpha\}_{\alpha \in \mathfrak{A}}$ .

- $(S_{(0)}^\infty)$  For all  $\alpha, \alpha' \in \mathfrak{A}$ , the map  $r(U_\alpha \cap U_{\alpha'}) \rightarrow \mathbb{T} \times G^{(0)}$ ,  $r(\gamma) \mapsto \iota^{-1}(\sigma_\alpha(\gamma)\sigma_{\alpha'}(\gamma)^{-1})$  for  $\gamma \in U_\alpha \cap U_{\alpha'}$ , is smooth;
- $(M_{(0)}^\infty)$  For all  $\alpha, \alpha_1, \alpha_2 \in \mathfrak{A}$ , the map  $r(U_{\alpha_1}U_{\alpha_2} \cap U_\alpha^{-1}) \rightarrow \mathbb{T} \times G^{(0)}$ ,  $r(\gamma_1) \mapsto \iota^{-1}(\sigma_{\alpha_1}(\gamma_1)\sigma_{\alpha_2}(\gamma_2)\sigma_\alpha(\gamma_1\gamma_2)^{-1})$  for all  $(\gamma_1, \gamma_2) \in U_{\alpha_1} * U_{\alpha_2}$  with  $\gamma_1\gamma_2 \in U_\alpha$ , is smooth; and
- $(I_{(0)}^\infty)$  For all  $\alpha, \alpha' \in \mathfrak{A}$ , the map  $r(U_\alpha \cap U_{\alpha'}^{-1}) \rightarrow \mathbb{T} \times G^{(0)}$ ,  $r(\gamma) \mapsto \iota^{-1}(\sigma_\alpha(\gamma)\sigma_{\alpha'}(\gamma^{-1}))$  for  $\gamma \in U_\alpha \cap U_{\alpha'}^{-1}$ , is smooth.

*Remark 4.23.* Condition  $(U^\infty)$  is already given in terms of functions defined on  $G^{(0)}$ . So similarly to  $(U_{\mathbb{T}}^\infty)$ , any reasonable definition of a Condition  $(U_{(0)}^\infty)$  would coincide with  $(U^\infty)$ .

The above definition makes no reference to a smooth structure on  $G$ . The smooth structure that is induced on an étale  $G$  from  $G^{(0)}$  is built in a way that the above conditions translate to Conditions  $(S^\infty)$ ,  $(M^\infty)$ ,  $(I^\infty)$  in Definition 4.8. To be precise:

**Lemma 4.24.** *Let  $G$  be an étale groupoid such that  $G^{(0)}$  is a manifold and such that  $G$  acts smoothly on  $G^{(0)}$ . Suppose that  $\mathbb{T} \times G^{(0)} \xrightarrow{\iota} E \xrightarrow{\pi} G$  is a topological twist. Let  $\{U_\alpha\}_{\alpha \in \mathfrak{A}}$  be an open cover of  $G$ , consisting of bisections, and for each  $\alpha \in \mathfrak{A}$ , let  $\sigma_\alpha: U_\alpha \rightarrow \pi^{-1}(U_\alpha)$  be a continuous section. Suppose that  $\{\sigma_\alpha\}_\alpha$  satisfies  $(U^\infty)$ . With respect to the Lie-groupoid structure on  $G$  obtained from Proposition 3.7, the following hold:*

- (1)  $\{\sigma_\alpha\}_\alpha$  satisfies  $(S^\infty)$  if and only if it satisfies  $(S_{(0)}^\infty)$ ;
- (2)  $\{\sigma_\alpha\}_\alpha$  satisfies  $(M^\infty)$  if and only if it satisfies  $(M_{(0)}^\infty)$ ; and
- (3)  $\{\sigma_\alpha\}_\alpha$  satisfies  $(I^\infty)$  if and only if it satisfies  $(I_{(0)}^\infty)$ .

*Proof.* By Lemma 3.9, every bisection of  $G$  is a smooth bisection. In particular, for all  $\alpha \in \mathfrak{A}$ ,  $r|_{U_\alpha}: U_\alpha \rightarrow r(U_\alpha)$  is a diffeomorphism. The result follows.  $\square$

We can now combine all of our previous results.

**Corollary 4.25.** *Let  $G$  be an étale groupoid such that  $G^{(0)}$  is a manifold and let  $\mathbb{T} \times G^{(0)} \xrightarrow{\iota} E \xrightarrow{\pi} G$  be a topological twist. Let  $\{U_\alpha\}_{\alpha \in \mathfrak{A}}$  be a base for the topology on  $G$  consisting of bisections, and for each  $\alpha \in \mathfrak{A}$ , let  $\sigma_\alpha: U_\alpha \rightarrow \pi^{-1}(U_\alpha)$  be a continuous section. Suppose that  $\{\sigma_\alpha\}_\alpha$  satisfies  $(U^\infty)$ ,  $(S_{(0)}^\infty)$ ,  $(M_{(0)}^\infty)$ , and  $(I_{(0)}^\infty)$ . If  $G$  acts smoothly on  $G^{(0)}$ , then there are unique smooth structures on  $E$  and  $G$  with respect to which the  $\sigma_\alpha$  are all smooth,  $\pi: E \rightarrow G$  is a smooth principal  $\mathbb{T}$ -bundle, and  $G$  is an étale Lie groupoid. With respect to these smooth structures,  $\mathbb{T} \times G^{(0)} \xrightarrow{\iota} E \xrightarrow{\pi} G$  is a Lie twist.*

*Proof.* By Proposition 3.7, since  $G$  acts smoothly on  $G^{(0)}$ ,  $G$  has a unique smooth structure with respect to which it is an étale Lie groupoid. Moreover, a chart around any  $\gamma \in G$  is given by  $(V, \varphi \circ r|_V)$ , where  $V$  is a (sufficiently small) open bisection containing  $\gamma$  and  $(r(V), \varphi)$  is a chart around  $r(\gamma)$  in  $G^{(0)}$ ; see Proposition 3.1. Using that  $\{U_\alpha\}_\alpha$  is a cover of open bisections, we may therefore let  $\{(V_\beta, \psi_\beta)\}_{\beta \in \mathfrak{B}}$  be an atlas of  $G$  for which there exists a refinement map  $F: \mathfrak{B} \rightarrow \mathfrak{A}$  such that  $V_\beta \subseteq U_{F(\beta)}$  for each  $\beta$ .

Lemma 4.24 states that the assumed properties on  $\sigma_\alpha$  translate to the four properties  $(U^\infty)$ – $(I^\infty)$  with respect to the Lie-groupoid structure on  $G$ . Thus, by Lemma 4.13(1), the sections  $\kappa_\beta := \sigma_{F(\beta)}|_{V_\beta}$  also satisfy the four properties  $(U^\infty)$ – $(I^\infty)$ . Since the domains of these ‘refined’ sections coincide with the domains of the charts of an atlas for  $G$ , the result now follows from Theorem 4.15.  $\square$

*Remark 4.26.* If we start with a Lie twist  $\mathbb{T} \times G^{(0)} \xrightarrow{\iota} E \xrightarrow{\pi} G$  where  $G$  is étale, then we can choose a family  $\{(U_\alpha, \sigma_\alpha)\}_\alpha$  of smooth local sections of  $\pi$  supported on bisections; see Lemma 4.12. These then satisfy the conditions of Lemma 4.24, and the uniqueness assertion in Corollary 4.25 implies that the resulting smooth structure is the same as the one we started with.

## 5. SMOOTH CARTAN TRIPLES

In this section we prove our main  $C^*$ -algebraic theorem. Given a Cartan pair  $B \subseteq A$  in which  $\widehat{B}$  is a manifold, we describe algebraic conditions—phrased purely in terms of the smooth subalgebra  $B^\infty$  of  $B$ , and  $C^*$ -algebraic data intrinsic to the pair  $B \subseteq A$ —that are equivalent to the corresponding Weyl groupoid acting smoothly on its unit space, and the sections of the Weyl twist corresponding to the given normalisers satisfying  $(U^\infty)$ – $(I^\infty)$ . We incorporate all of this data into our definition of a smooth Cartan triple  $(A, B, \mathcal{N})$ . Our main theorem says, roughly speaking, that every smooth Cartan triple arises from a Lie twist over an effective étale Lie groupoid, and conversely that every such twist gives rise to a smooth Cartan triple. We briefly discuss, in two concluding remarks, how our result can be combined with Connes' reconstruction theorem, and also the dependence of the smooth structure on the Weyl twist coming from our theorem on the choice of a family  $\mathcal{N}$  of normalisers in a smooth Cartan triple.

We first need to recall a little background about Renault's construction. A *Cartan subalgebra*  $B$  of a  $C^*$ -algebra  $A$  is a maximal abelian subalgebra such that there is a faithful conditional expectation  $P: A \rightarrow B$  and such that the set  $N(B) := \{n \in A : nBn^* \cup n^*Bn \subseteq B\}$  of normalisers of  $B$  densely spans  $A$  [19, Definition 5.1]. (Renault's definition requires, in addition, that  $B$  contains an approximate unit for  $A$ , but Pitts proved [17, Theorem 2.6] that this is a consequence of the remaining conditions.) Let  $\widehat{B}$  be the spectrum of  $B$ , so that we may identify  $B$  with  $C_0(\widehat{B})$  via the Gelfand representation. Recall that  $\text{supp}^\circ(k)$  denotes the *open support*  $\{x \in \widehat{B} \mid k(x) \neq 0\}$  of  $k \in C_0(\widehat{B})$ . We will explain how  $B \subseteq A$  gives rise to an effective groupoid  $G_{A,B}$  and twist  $E_{A,B}$  (cf. [14, p. 1.6] or [19, Proposition 4.7]).

If  $n \in N(B)$ , then  $n^*n, nn^* \in B$ , and there exists a unique  $*$ -isomorphism

$$(5.1) \quad \theta_n: C_0(\text{supp}^\circ(n^*n)) \rightarrow C_0(\text{supp}^\circ(nn^*)) \quad \text{that satisfies} \quad nfn^* = \theta_n(f)nn^*$$

for all  $f \in C_0(\widehat{B})$ . Recall that, if  $n, m \in N(B)$ , then  $\theta_n \circ \theta_m = \theta_{nm}$  and  $\theta_{n^*} = \theta_n^{-1}$  (cf. [14, Corollary 1.7]). We point out that, in the literature, the focus is usually on the homeomorphism

$$(5.2) \quad \widehat{\theta}_n: \text{supp}^\circ(n^*n) \rightarrow \text{supp}^\circ(nn^*) \quad \text{determined by} \quad \theta_n(f) = f \circ \widehat{\theta}_n^{-1}.$$

The *Weyl groupoid*  $G_{A,B}$  is the quotient of

$$\{(n, x) : n \in N(B), x \in \text{supp}^\circ(n^*n)\}$$

by the equivalence relation

$$(n, x) \sim (m, y) \iff$$

$$x = y \text{ and } \theta_n|_{C_0(U)} = \theta_m|_{C_0(U)} \text{ for an open neighbourhood } U \subseteq \widehat{B} \text{ of } x.$$

This groupoid has unit space  $\{[k, x] : k \in B, k(x) \neq 0\}$  homeomorphic to  $\widehat{B}$  via  $[k, x] \mapsto x$ , and groupoid operations  $r([n, x]) = \widehat{\theta}_n(x)$ ,  $s([n, x]) = x$ ,  $[n, \widehat{\theta}_m(x)][m, x] = [nm, x]$  and  $[n, x]^{-1} = [n^*, \widehat{\theta}_n(x)]$ . The sets  $\{[n, x] : (n^*n)(x) \neq 0\}$  indexed by  $n \in N(B)$  constitute a base of open bisections for the topology on  $G_{A,B}$ .

Let  $E_{A,B}$  denote the quotient of the same set  $\{(n, x) : n \in N(B), x \in \text{supp}^\circ(n^*n)\}$  by the more stringent equivalence relation  $\approx$  given by

$$(n, x) \approx (m, y) \iff (n, x) \sim (m, y) \text{ and } P(n^*m)(x) > 0.$$

We write  $\llbracket n, x \rrbracket$  for the equivalence class of  $(n, x)$  under  $\approx$ . Then the quotient

$$E_{A,B} = \{\llbracket n, x \rrbracket : n \in N(B), x \in \text{supp}^\circ(n^*n)\} / \approx$$

is a groupoid with structure maps identical to the ones for  $G_{A,B}$  above, with  $[\cdot, \cdot]$  replaced with  $\llbracket \cdot, \cdot \rrbracket$ . The sets  $\{\llbracket zn, x \rrbracket : (n^*n)(x) \neq 0, z \in U\}$  indexed by normalisers  $n \in N(B)$  and open sets  $U \subseteq \mathbb{T}$  constitute a base for the topology on  $E_{A,B}$ . There is a map  $\pi: E_{A,B} \rightarrow G_{A,B}$  given by  $\pi(\llbracket n, x \rrbracket) = [n, x]$ , and there is an inclusion  $i: \mathbb{T} \times G_{A,B}^{(0)} \rightarrow E_{A,B}$  given by  $i([z, x]) = \llbracket k, x \rrbracket$ , where  $k \in B$  is any element satisfying  $k(x) = z$ . The sequence

$$\mathbb{T} \times G_{A,B}^{(0)} \xrightarrow{i} E_{A,B} \xrightarrow{\pi} G_{A,B}$$

is a twist, called the *Weyl twist* of  $B \subseteq A$ . Renault proves in [19] that  $A$  is canonically isomorphic to the reduced twisted groupoid  $C^*_r(G_{A,B}; E_{A,B})$  in a way that carries the Cartan subalgebra  $B$  to  $C_0(G_{A,B}^{(0)})$ ; in particular, the latter is a Cartan subalgebra of  $C^*_r(G_{A,B}; E_{A,B})$ .

By [19, Proposition 4.8.] (see [11, Theorem 1.1] for further detail), every section of the line bundle  $L_{A,B}$  of  $E_{A,B}$  described in Lemma 4.21 whose support is a bisection is a normaliser of  $C_0(G_{A,B}^{(0)}) \cong B$

in  $C_r^*(G_{A,B}; E_{A,B}) \cong A$ , and every normaliser has this form. Indeed, under Renault's isomorphism  $A \cong C_r^*(G_{A,B}; E_{A,B})$ , it is routine to verify that  $n \in A$  maps to the section  $\mathfrak{s}_n: G_{A,B} \rightarrow L_{A,B}$  given by

$$(5.3) \quad \mathfrak{s}_n([m, x]) = \begin{cases} [(n^*n)^{1/2}(x), [n, x]] & \text{if } P(n^*m)(x) \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

In particular, the corresponding section  $\sigma_n := \sigma_{\mathfrak{s}_n}: \text{supp}^\circ(\mathfrak{s}_n) \rightarrow E_{A,B}$  of Lemma 4.21 is given by

$$(5.4) \quad \sigma_n([m, x]) = \begin{cases} [n, x] & \text{if } P(n^*m)(x) \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Our primary application for the following lemma is when  $G$  is effective, or equivalently  $C_0(G^{(0)})$  is a masa in  $C_r^*(G; E)$ <sup>3</sup>. However, since the same proof goes through for  $G$  not effective, we leave the hypothesis out.

**Lemma 5.1.** *Suppose that  $G$  is an étale groupoid,  $\mathbb{T} \times G^{(0)} \xrightarrow{\iota} E \xrightarrow{\pi} G$  is a topological twist, and  $G^{(0)}$  is a manifold. Let  $A = C_r^*(G; E)$ , and let  $B = C_0(G^{(0)}) \subseteq A$ . Let  $P: A \rightarrow B$  denote the conditional expectation of [19, Proposition 4.3]. Then the following are equivalent:*

- (1)  $G$  acts smoothly on  $G^{(0)}$ ;
- (2) for every normaliser  $n \in N(B)$ , the isomorphism  $\theta_n: C_0(\text{supp}^\circ(n^*n)) \rightarrow C_0(\text{supp}^\circ(nn^*))$  of (5.1) restricts to a  $*$ -isomorphism from  $C_0^\infty(\text{supp}^\circ(n^*n))$  to  $C_0^\infty(\text{supp}^\circ(nn^*))$ ; and
- (3) there is a family  $\mathcal{N} \subseteq N(B)$  of normalisers of  $B$  that densely span  $A$  such that for each  $n \in \mathcal{N}$ , the isomorphism  $\theta_n: C_0(\text{supp}^\circ(n^*n)) \rightarrow C_0(\text{supp}^\circ(nn^*))$  of (5.1) restricts to a  $*$ -isomorphism from  $C_0^\infty(\text{supp}^\circ(n^*n))$  to  $C_0^\infty(\text{supp}^\circ(nn^*))$ .

*Proof.* If  $n$  is a normaliser, then by [19, Proposition 4.8.],  $U_n := \text{supp}^\circ(n)$  is a bisection of  $G$  and  $\widehat{\theta}_n = r \circ (s|_{U_n})^{-1}$ . Moreover, any bisection of  $G$  arises in this fashion.

For (1)  $\implies$  (2), suppose that  $G$  acts smoothly on  $G^{(0)}$ . Then, by Lemma 3.5, for every open bisection  $U$  of  $G$ , the map  $s|_U \circ (r|_U)^{-1}$  is a diffeomorphism. So (2) follows from the observation of the first paragraph.

(2)  $\implies$  (3) is trivial: every compactly supported section of  $L$  whose support is a bisection is a normaliser of  $B$ . A partition-of-unity argument shows that such elements span  $\Gamma_c(G; L)$ . So they densely span  $C_r^*(G; E)$ .

For (3)  $\implies$  (1), note that since the elements of  $\mathcal{N}$  densely span  $A$ , their supports cover  $G$ . So the observation of the first paragraph shows that  $G$  is covered by open bisections that act smoothly on  $G^{(0)}$ ; that is,  $G$  acts smoothly on  $G^{(0)}$ .  $\square$

**Definition 5.2.** For  $X$  a topological space and  $f \in C_0(X)$ , let  $\text{Ph}(f) \in C_b(\text{supp}^\circ(f))$  be the unique function such that  $f = \text{Ph}(f)|f|$  on  $\text{supp}^\circ(f)$ .

**Lemma 5.3.** *Suppose that  $G$  is an étale groupoid,  $\mathbb{T} \times G^{(0)} \xrightarrow{\iota} E \xrightarrow{\pi} G$  is a topological twist, and  $G^{(0)}$  is a manifold. Let  $A = C_r^*(G; E)$ , and let  $B = C_0(G^{(0)}) \subseteq A$ . Let  $P: A \rightarrow B$  denote the conditional expectation of [19, Proposition 4.3]. Suppose that  $G$  acts smoothly on  $G^{(0)}$  and equip  $G$  with the manifold structure constructed in Proposition 3.7. Suppose that  $\{\mathfrak{s}_\alpha: U_\alpha \rightarrow L\}_{\alpha \in \mathfrak{A}} \subseteq \Gamma_0(G; L) \subseteq A$  is a family of  $C_0$ -sections of the line bundle  $L$  associated to  $E$  such that each  $U_\alpha$  is a bisection of  $G$ . For each  $\alpha \in \mathfrak{A}$ , define a continuous section of  $\pi$  by  $\sigma_\alpha := \sigma_{\mathfrak{s}_\alpha}: U_\alpha \rightarrow E$ . Then the following dictionary holds.*

- (1)  $\{\sigma_\alpha\}_\alpha$  satisfies  $(U^\infty)$  if and only if for all  $\alpha \in \mathfrak{A}$ , the function  $\text{Ph}(P(\mathfrak{s}_\alpha))$  is smooth.
- (2)  $\{\sigma_\alpha\}_\alpha$  satisfies  $(S^\infty)$  if and only if for all  $\alpha, \alpha' \in \mathfrak{A}$ , the function  $\text{Ph}(P(\mathfrak{s}_\alpha \mathfrak{s}_{\alpha'}^*))$  is smooth.
- (3)  $\{\sigma_\alpha\}_\alpha$  satisfies  $(M^\infty)$  if and only if for all  $\alpha, \alpha_1, \alpha_2 \in \mathfrak{A}$ , the function  $\text{Ph}(P(\mathfrak{s}_{\alpha_1} \mathfrak{s}_{\alpha_2} \mathfrak{s}_\alpha^*))$  is smooth.
- (4)  $\{\sigma_\alpha\}_\alpha$  satisfies  $(I^\infty)$  if and only if for all  $\alpha, \alpha' \in \mathfrak{A}$ , the function  $\text{Ph}(P(\mathfrak{s}_\alpha \mathfrak{s}_{\alpha'}^*))$  is smooth.

*Proof.* For (1), fix  $\alpha \in \mathfrak{A}$ . By definition of  $\sigma_\alpha$ ,

$$(5.5) \quad \mathfrak{s}_\alpha(\gamma) = [|\mathfrak{s}_\alpha(\gamma)|, \sigma_\alpha(\gamma)],$$

so defining  $k_\alpha: U_\alpha \cap G^{(0)} \rightarrow \mathbb{T}$  by  $\sigma_\alpha(x) = \iota(k_\alpha(x), x)$  for each  $x \in U_\alpha \cap G^{(0)}$ , we have

$$\mathfrak{s}_\alpha(x) = [k_\alpha(x) |\mathfrak{s}_\alpha(x)|, x] \in L_x.$$

<sup>3</sup>If  $G$  is not effective, then it admits an open bisection  $U \subseteq G \setminus G^{(0)}$  consisting of isotropy, and any nonzero  $f \in C_c(U)$  belongs to  $B' \setminus B$ , so  $B$  is not a masa. If  $G$  is effective, then, for example [3, Corollary 5.3] shows that  $B$  is maximal abelian.

If we think of  $\mathfrak{s}_\alpha|_{G^{(0)}} = P(\mathfrak{s}_\alpha)$  as a function with values in  $\mathbb{C}$ , then this reads as  $\mathfrak{s}_\alpha(x) = k_\alpha(x) |\mathfrak{s}_\alpha(x)|$ . That is,

$$P(\mathfrak{s}_\alpha) = k_\alpha |P(\mathfrak{s}_\alpha)| \text{ on } U_\alpha \cap G^{(0)}.$$

As  $U_\alpha \cap G^{(0)} = \text{supp}^\circ(P(\mathfrak{s}_\alpha))$ , we see that  $k_\alpha \equiv \text{Ph}(P(\mathfrak{s}_\alpha))$ . Since  $\{\sigma_\alpha\}_\alpha$  satisfies  $(U^\infty)$  if and only if each  $k_\alpha$  is smooth, the claim follows.

For (2), recall from Remark 4.9 that  $\{\sigma_\alpha\}_\alpha$  satisfies  $(S^\infty)$  if and only if, for all  $\alpha, \alpha'$ , the map  $f := f_{\alpha, \alpha'} : U_\alpha \cap U_{\alpha'} = \text{supp}^\circ(P(\mathfrak{s}_\alpha \mathfrak{s}_{\alpha'}^*) \circ r_G) \rightarrow \mathbb{T}$  determined by

$$\sigma_\alpha(\gamma) \sigma_{\alpha'}(\gamma)^{-1} = \iota(f(\gamma), r(\gamma))$$

is smooth. We will prove that  $f$  is smooth if and only if  $\text{Ph}(P(\mathfrak{s}_\alpha \mathfrak{s}_{\alpha'}^*))$  is smooth. For  $x \in G^{(0)}$ , assume that  $\gamma \in U_\alpha \cap U_{\alpha'}$  with  $r(\gamma) = x$ , then

$$\begin{aligned} (\mathfrak{s}_\alpha \mathfrak{s}_{\alpha'}^*)(x) &= \sum_{\eta \in G^x} \mathfrak{s}_\alpha(\eta) \overline{\mathfrak{s}_{\alpha'}(\eta^{-1})} \\ &= \mathfrak{s}_\alpha(\gamma) \overline{\mathfrak{s}_{\alpha'}(\gamma)} \\ &= [|\mathfrak{s}_\alpha(\gamma)|, \sigma_\alpha(\gamma)] [|\mathfrak{s}_{\alpha'}(\gamma)|, \sigma_{\alpha'}(\gamma)^{-1}] && \text{by Eq. (5.5) and [2, Eq. (2.3)]} \\ &= [|\mathfrak{s}_\alpha(\gamma)| |\mathfrak{s}_{\alpha'}(\gamma)|, \sigma_\alpha(\gamma) \sigma_{\alpha'}(\gamma)^{-1}] && \text{by definition of multiplication in } L \\ &= [|\mathfrak{s}_\alpha(\gamma)| |\mathfrak{s}_{\alpha'}(\gamma)| f(\gamma), \iota(1, x)]. \end{aligned}$$

Since  $r_G(\gamma) = x$  and  $x$  was arbitrary, this proves that

$$f = \text{Ph}(P(\mathfrak{s}_\alpha \mathfrak{s}_{\alpha'}^*)) \circ r_G \quad \text{on the domain of } f.$$

Since  $U_\alpha \cap U_{\alpha'}$  is a smooth bisection by Lemma 3.5, we conclude that  $\text{Ph}(P(\mathfrak{s}_\alpha \mathfrak{s}_{\alpha'}^*))$  is smooth if and only if  $f$  is smooth, as claimed.

For (3), recall from Remark 4.9 that  $\{\sigma_\alpha\}_\alpha$  satisfies  $(M^\infty)$  if and only if, for all  $\alpha, \alpha_1, \alpha_2$ , the map  $g := g_{\alpha, \alpha_1, \alpha_2} : \{(\gamma_1, \gamma_2) \in U_{\alpha_1} * U_{\alpha_2} : \gamma_1 \gamma_2 \in U_\alpha\} = \text{supp}^\circ(P(\mathfrak{s}_{\alpha_1} \mathfrak{s}_{\alpha_2} \mathfrak{s}_\alpha^*) \circ r_G) \rightarrow \mathbb{T}$  determined by

$$\sigma_{\alpha_1}(\gamma_1) \sigma_{\alpha_2}(\gamma_2) \sigma_\alpha(\gamma_1 \gamma_2)^{-1} = \iota(g(\gamma_1, \gamma_2), r(\gamma_1))$$

is smooth. We will prove that  $g$  is smooth if and only if  $\text{Ph}(P(\mathfrak{s}_{\alpha_1} \mathfrak{s}_{\alpha_2} \mathfrak{s}_\alpha^*))$  is smooth. For  $x \in G^{(0)}$ , if we assume that  $(\gamma_1, \gamma_2)$  is in the domain of  $g$  and that  $r(\gamma_1) = x$ , then similarly to above, we compute

$$\begin{aligned} (\mathfrak{s}_{\alpha_1} \mathfrak{s}_{\alpha_2} \mathfrak{s}_\alpha^*)(x) &= \sum_{\eta_1 \in G^x} \sum_{\eta_2 \in G^{s(\eta_1)}} \mathfrak{s}_{\alpha_1}(\eta_1) \mathfrak{s}_{\alpha_2}(\eta_2) \overline{\mathfrak{s}_\alpha(\eta_1 \eta_2)} \\ &= \mathfrak{s}_{\alpha_1}(\gamma_1) \mathfrak{s}_{\alpha_2}(\gamma_2) \overline{\mathfrak{s}_\alpha(\gamma_1 \gamma_2)} \\ &= [|\mathfrak{s}_{\alpha_1}(\gamma_1)| |\mathfrak{s}_{\alpha_2}(\gamma_2)| |\mathfrak{s}_\alpha(\gamma_1 \gamma_2)|, \sigma_{\alpha_1}(\gamma_1) \sigma_{\alpha_2}(\gamma_2) \sigma_\alpha(\gamma_1 \gamma_2)^{-1}] && \text{by Eq. (5.5)} \\ &= [|\mathfrak{s}_{\alpha_1}(\gamma_1)| |\mathfrak{s}_{\alpha_2}(\gamma_2)| |\mathfrak{s}_\alpha(\gamma_1 \gamma_2)| g(\gamma_1, \gamma_2), \iota(1, x)]. \end{aligned}$$

Since  $r_G(\gamma_1) = x$  and  $x$  was arbitrary, this proves that

$$g = \text{Ph}(P(\mathfrak{s}_{\alpha_1} \mathfrak{s}_{\alpha_2} \mathfrak{s}_\alpha^*)) \circ r_G \quad \text{on the domain of } g.$$

This proves that  $\text{Ph}(P(\mathfrak{s}_{\alpha_1} \mathfrak{s}_{\alpha_2} \mathfrak{s}_\alpha^*))$  is smooth if and only if  $g$  is smooth.

For (4), recall from Remark 4.9 that  $\{\sigma_\alpha\}_\alpha$  satisfies  $(I^\infty)$  if and only if, for all  $\alpha, \alpha'$ , the map  $h := h_{\alpha, \alpha'} : U_\alpha \cap U_{\alpha'}^{-1} = \text{supp}^\circ(P(\mathfrak{s}_\alpha \mathfrak{s}_{\alpha'}) \circ r_G) \rightarrow \mathbb{T}$  determined by

$$\sigma_\alpha(\gamma) \sigma_{\alpha'}(\gamma^{-1}) = \iota(h(\gamma), r(\gamma))$$

is smooth. We will prove that  $h$  is smooth if and only if  $\text{Ph}(P(\mathfrak{s}_\alpha \mathfrak{s}_{\alpha'}))$  is smooth. For  $x \in G^{(0)}$ , assume that  $\gamma \in U_\alpha \cap U_{\alpha'}^{-1}$  with  $r(\gamma) = x$ , so that  $\gamma^{-1} \in U_{\alpha'}$ , then a computation similar to the ones above shows

$$(\mathfrak{s}_\alpha \mathfrak{s}_{\alpha'})(x) = \sum_{\eta \in G^x} \mathfrak{s}_\alpha(\eta) \mathfrak{s}_{\alpha'}(\eta^{-1}) = \mathfrak{s}_\alpha(\gamma) \mathfrak{s}_{\alpha'}(\gamma^{-1}) = [|\mathfrak{s}_\alpha(\gamma)| |\mathfrak{s}_{\alpha'}(\gamma)| h(\gamma), \iota(1, r(\gamma))].$$

Since  $r_G(\gamma) = x$  and  $x$  was arbitrary, this proves that

$$h = \text{Ph}(P(\mathfrak{s}_\alpha \mathfrak{s}_{\alpha'})) \circ r_G \quad \text{on the domain of } h.$$

Again, we conclude that  $\text{Ph}(P(\mathfrak{s}_\alpha \mathfrak{s}_{\alpha'}))$  is smooth if and only if  $h$  is smooth.  $\square$

**Definition 5.4.** Let  $B$  be a commutative  $C^*$ -algebra, and let  $B^\infty$  be a subalgebra of  $B$ . Let  $U \subseteq \widehat{B}$  be an open set. Let  $I_U \cong C_0(U)$  be the corresponding ideal of  $B$ , and let  $I_U^\infty := B^\infty \cap I_U$ . We say that  $f \in C_b(U)$  is a *local multiplier* of  $B^\infty$  if  $f \cdot I_U^\infty := \{fg : g \in I_U^\infty\} \subseteq I_U^\infty$ .

The definition of a local multiplier of  $B^\infty$  is motivated by the following simple lemma.

**Lemma 5.5.** *Let  $M$  be a manifold, let  $B := C_0(M)$  and let  $B^\infty := C_0^\infty(M) \subseteq B$ . Fix an open subset  $U \subseteq M$  and a function  $f \in C_b(U)$ . Then  $f$  is smooth if and only if  $f$  is a local multiplier of  $B^\infty$ .*

*Proof.* If  $f$  is smooth, then for any  $g \in C^\infty(U)$ , the product  $fg$  is smooth. In particular, if  $g \in I_U^\infty$ , then  $fg$  is smooth. It belongs to  $C_0(U)$  because  $f$  is bounded. So  $f$  is a local multiplier of  $B^\infty$ . Conversely, suppose that  $f$  is a local multiplier of  $B^\infty$ . Fix a point  $x \in U$ ; we must show that  $f$  is smooth at  $x$ . Since  $M$  is a normal topological space, there exists an open set  $V$  such that  $x \in V \subseteq \overline{V} \subseteq U$ , and then by the extension lemma [15, Lemma 2.26] for smooth functions, there is a smooth function  $g: M \rightarrow \mathbb{R}$  such that  $g|_{\overline{V}} \equiv 1$  and  $\text{supp}(g) \subseteq U$ . Since  $f$  is a local multiplier of  $B^\infty$ , we have  $fg \in B^\infty$ . Since  $fg \equiv f$  on  $V$  we deduce that  $f$  is smooth on the neighbourhood  $V$  of  $x$ , and we are done.  $\square$

If  $B$  is a commutative  $C^*$ -algebra with  $\widehat{B}$  a smooth manifold, we write  $B^\infty$  for the image of  $C_0^\infty(\widehat{B})$  under the Gelfand transform. If  $B$  is a Cartan subalgebra of  $A$ , then for  $n \in N(B)$ , we will further write  $B_n := C_0(\text{supp}^\circ(n^*n))$  and  $B_n^\infty := B^\infty \cap B_n$ .

**Definition 5.6.** Suppose that  $B \subseteq A$  is a Cartan pair with conditional expectation  $P: A \rightarrow B$ , and that  $\widehat{B}$  is a smooth manifold. Let  $\mathcal{N}$  be a family of normalisers of  $B$  that densely spans  $A$ . We say that  $(A, B, \mathcal{N})$  is a *smooth Cartan triple* if  $\mathcal{N} \cap B = B^\infty$ , and the family  $\mathcal{N}$  satisfies the following.

- (N\*) Each  $n \in \mathcal{N}$  normalises  $B^\infty$ , i.e.,  $nB^\infty n^* \subseteq B^\infty$  and  $n^*B^\infty n \subseteq B^\infty$ .
- (U\*) For all  $n \in \mathcal{N}$ ,  $\text{Ph}(P(n))$  is a local multiplier of  $B^\infty$ .
- (S\*) For all  $m, k \in \mathcal{N}$ ,  $\text{Ph}(P(mk^*))$  is a local multiplier of  $B^\infty$ .
- (M\*) For all  $n, m, k \in \mathcal{N}$ ,  $\text{Ph}(P(nmk^*))$  is a local multiplier of  $B^\infty$ .
- (I\*) For all  $n, m \in \mathcal{N}$ ,  $\text{Ph}(P(nm))$  is a local multiplier of  $B^\infty$ .

As seen in Lemma 5.3, Conditions (U\*)–(I\*) in the definition of a smooth Cartan triple correspond to Conditions (U $^\infty$ )–(I $^\infty$ ) in Definition 4.8. Condition (N\*), on the other hand, is motivated by the following lemma and its corollary.

**Lemma 5.7.** *Let  $B \subseteq A$  be a Cartan pair and suppose that  $\widehat{B}$  is a smooth manifold. For a normaliser  $n$  of  $B$ , the following conditions (1) and (2) are equivalent.*

- (1) *The homeomorphism  $\widehat{\theta}_n: \text{supp}^\circ(n^*n) \rightarrow \text{supp}^\circ(nn^*)$  of (5.2) has a smooth inverse.*
- (2) *The isomorphism  $\theta_n: C_0(\text{supp}^\circ(n^*n)) \rightarrow C_0(\text{supp}^\circ(nn^*))$  of (5.1) maps  $B_n^\infty$  into  $B^\infty$ .*

*The condition*

- (3)  $nB^\infty n^* \subseteq B^\infty$

*implies both (1) and (2), and also that  $nn^* \in B^\infty$ . If  $n^*n \in B^\infty$ , then (1), (2) and (3) are equivalent.*

*Proof.* The implication (1)  $\implies$  (2) is trivial, since the composition of smooth functions is smooth. For (2)  $\implies$  (1), let  $y \in \text{supp}^\circ(nn^*)$  be arbitrary and let  $(V, \psi)$  be a smooth chart around  $x = \widehat{\theta}_n^{-1}(y)$ . Write  $\psi = (\psi_1, \dots, \psi_k)$  where  $k = \dim(\widehat{B})$  and  $\psi_i: V \rightarrow \mathbb{R}$ . Now choose a smaller, precompact, open neighbourhood  $V'$  of  $x$  with  $\overline{V'} \subseteq V \cap \text{supp}^\circ(n^*n)$  and, using the smooth Urysohn Lemma [15, Proposition 2.25], choose  $h \in C_0^\infty(V \cap \text{supp}^\circ(n^*n))$  such that  $h|_{V'} \equiv 1$ . Then  $h\psi_i \in B_n^\infty$ , so by (2), we have that  $\theta_n(h\psi_i): \text{supp}^\circ(nn^*) \rightarrow \mathbb{R}$  is smooth. By definition of  $\theta_n$ , this implies that  $(h\psi_i) \circ \widehat{\theta}_n^{-1}: \widehat{\theta}_n(V') \rightarrow \mathbb{R}$  is smooth. By choice of  $h$ , we have  $(h\psi_i) \circ \widehat{\theta}_n^{-1} = \psi_i \circ \widehat{\theta}_n^{-1}$  on  $\widehat{\theta}_n(V')$ , so we have proved that each of the coordinates of  $\psi \circ \widehat{\theta}_n^{-1}$  is smooth around  $y$ , meaning that  $\widehat{\theta}_n^{-1}$  is smooth.

For (3)  $\implies$  (2), fix  $f \in B_n^\infty$  and a point  $x \in \text{supp}^\circ(nfn^*)$ . It suffices to show that  $\theta_n(f)$  is smooth at  $x$ . Choose a precompact open neighbourhood  $V \subseteq \text{supp}^\circ(n^*n)$  of  $\widehat{\theta}_n^{-1}(x)$  and use the smooth Urysohn Lemma [15, Proposition 2.25] to find  $h \in B^\infty$  that is constant 1 on  $\overline{V}$ . Then  $nhn^*, nfn^* \in B^\infty$  by assumption. Equation (5.1) shows that  $nhn^* = \theta_n(h)nn^*$  and  $nfn^* = \theta_n(f)nn^*$ . We have  $\theta_n(h) = h \circ \widehat{\theta}_n^{-1}$ , which is identically 1 on  $W := \widehat{\theta}_n(V)$ . Hence  $\theta_n(h)nn^*$  is smooth and agrees with  $nn^*$  on the neighbourhood  $W$  of  $x$ . In particular,  $nn^*$  is smooth at  $x$  and  $1/(nn^*)$  is smooth on  $W$ . Hence  $\theta_n(f)|_W = (\theta_n(f)nn^*)/(nn^*)$  is smooth on  $W$ . In particular,  $\theta_n(f)$  is smooth at  $x \in W$  as required.

Lastly, assume that  $n^*n \in B^\infty$ ; it remains to show (2)  $\implies$  (3). By definition of  $\theta_n$  (cf. proof of [14, 6 $^\circ$  Proposition]), (2) implies that the partial isometry  $v$  in the polar decomposition  $n = v\sqrt{n^*n}$  of  $n$  in  $A^{**}$  satisfies  $vB_n^\infty v^* = B_n^\infty$ . If  $h \in B^\infty$ , then  $hn^*n \in B_n^\infty$  since  $B^\infty$  is closed under pointwise multiplication. In particular,  $v(hn^*n)v^* \in B^\infty$ . Since  $B$  is abelian, we have  $nhn^* = v\sqrt{n^*nh}\sqrt{n^*nv^*} = v(hn^*n)v^* \in B^\infty$  as required.  $\square$

**Example 5.8.** It may seem at first glance that, in the setting of Lemma 5.7, if  $nB^\infty n^* \subseteq B^\infty$  (and in particular  $nn^*$  is smooth), then  $n^*n$  should be smooth too. This is not the case. Define  $h: \mathbb{T} \rightarrow \mathbb{T}$  by  $h(e^{\pi it}) = e^{\pi it^3}$  for  $-1 \leq t \leq 1$ . Then  $h$  is a homeomorphism and is smooth, but it is not a diffeomorphism



since the cubed-root function is not differentiable at 0. Let  $G$  be the groupoid equal as a topological space to  $\{0, 1\}^2 \times \mathbb{T}$  with structure maps

$$r((i, j), x) = ((i, i), h^{i-j}(x)) \quad s((i, j), x) = ((j, j), x) \quad \text{and} \quad ((i, j), h^{j-k}(x))(j, k), x) = ((i, k), x).$$

Then  $C_r^*(G) \cong M_2(C(\mathbb{T}))$  and  $C_0(G^{(0)}) \cong C(\mathbb{T}) \oplus C(\mathbb{T})$  embedded as diagonal matrices. So  $\widehat{B} = \mathbb{T} \sqcup \mathbb{T}$  is a smooth manifold, and  $B^\infty = C^\infty(\mathbb{T}) \oplus C^\infty(\mathbb{T})$ . Let  $n_0 \in C_r^*(G)$  be the indicator function of  $\{(0, 1)\} \times \mathbb{T}$ . One can compute that

$$(5.6) \quad (n_0 \varphi n_0^*)((i, j), x) = \begin{cases} 0 & \text{if } ij \neq 0, \\ \varphi((1, 1), h(x)) & \text{if } i = j = 0. \end{cases}$$

and

$$(5.7) \quad (n_0^* \varphi n_0)((i, j), x) = \begin{cases} 0 & \text{if } ij \neq 1, \\ \varphi((0, 0), h^{-1}(x)) & \text{if } i = j = 1. \end{cases}$$

Thus,  $n_0 n_0^*$  is the indicator function of  $\{(0, 0)\} \times \mathbb{T} \subseteq G^{(0)}$ , while  $n_0^* n_0$  is the indicator function of  $\{(1, 1)\} \times \mathbb{T} \subseteq G^{(0)}$ . It further follows from (5.2) that  $\widehat{\theta}_{n_0}^{-1} = h$  is smooth, but  $\widehat{\theta}_{n_0} = h^{-1}$  is not.

Fix  $f_0 \in C^\infty(\mathbb{T})^+$  such that  $f_0 \circ \widehat{\theta}_{n_0} \notin C^\infty(\mathbb{T})$ . Let  $f = f_0 \oplus 0$ , the copy of  $f_0$  supported on  $\{(0, 0)\} \times \mathbb{T} \subseteq G^{(0)}$ , and define  $n := \sqrt{f} n_0$ . Then for  $g = g_0 \oplus g_1 \in B^\infty$ , we have

$$n g n^* = \sqrt{f} n_0 g n_0^* \sqrt{f} \stackrel{(5.6)}{=} \sqrt{f} (g_1 \circ h) 1_{(0,0) \times \mathbb{T}} \sqrt{f} = [f_0(g_1 \circ h)] \oplus 0 \in B^\infty.$$

So  $n B^\infty n^* \subseteq B^\infty$ , and in particular  $n n^* \in B^\infty$ . However,

$$n^* n = n_0^* f n_0 \stackrel{(5.7)}{=} 0 \oplus [f_0 \circ h^{-1}] \notin B^\infty.$$

**Corollary 5.9.** *Let  $B \subseteq A$  be a Cartan pair and suppose that  $\widehat{B}$  is a smooth manifold. For a normaliser  $n$  of  $B$ , the following are equivalent.*

- (1) *One of  $n^* n$  and  $n n^*$  is smooth and  $\widehat{\theta}_n$  is a diffeomorphism.*
- (2) *Both  $n^* n$  and  $n n^*$  are smooth and  $\widehat{\theta}_n$  is a diffeomorphism.*
- (3) *One of  $n^* n$  and  $n n^*$  is smooth and  $\theta_n$  restricts to a  $*$ -isomorphism from  $B_n^\infty$  to  $B_{n^*}^\infty$ .*
- (4) *Both  $n^* n$  and  $n n^*$  are smooth and  $\theta_n$  restricts to a  $*$ -isomorphism from  $B_n^\infty$  to  $B_{n^*}^\infty$ .*
- (5)  *$n$  normalizes  $B^\infty$ , i.e.,  $n B^\infty n^* \cup n^* B^\infty n \subseteq B^\infty$ .*

*Proof.* If  $\widehat{\theta}_n$  is a diffeomorphism, then both  $\widehat{\theta}_n^{-1}$  and  $\widehat{\theta}_n = \widehat{\theta}_{n^*}^{-1}$  are smooth, so  $n$  and  $n^*$  both satisfy (1) of Lemma 5.7. Similarly, if  $\theta_n$  is an isomorphism from  $B_n^\infty$  to  $B_{n^*}^\infty$ , then both  $n$  and  $n^*$  satisfy (2) of Lemma 5.7. In either case, we then have that  $n n^* = \theta_n(n^* n)$  is smooth if  $n^* n$  is smooth, and vice versa. Finally, if  $n$  normalises  $B^\infty$ , then both  $n$  and  $n^*$  satisfy (3) of Lemma 5.7. So the result follows directly from Lemma 5.7.  $\square$

**Theorem 5.10.** (1) *Let  $\mathbb{T} \times G^{(0)} \xrightarrow{\iota} E \xrightarrow{\pi} G$  be a Lie twist over an effective étale Lie groupoid. Let  $A := C_r^*(G; E)$  and  $B := C_0(G^{(0)}) \subseteq A$ , and suppose that  $\mathcal{N} \subseteq \Gamma_0^\infty(G; L)$  is a set of smooth sections of the line bundle  $L$  of  $E$ , supported on bisections. If  $\mathcal{N}$  densely spans  $A$  and contains  $C_0^\infty(G^{(0)})$ , then  $(A, B, \mathcal{N})$  is a smooth Cartan triple.*

- (2) *Suppose that  $(A, B, \mathcal{N})$  is a smooth Cartan triple. Let  $\mathbb{T} \times G^{(0)} \xrightarrow{\iota} E \xrightarrow{\pi} G$  be the Weyl twist associated to  $B \subseteq A$ . For each  $n \in \mathcal{N}$ , let  $\sigma_n: \text{supp}^\circ(n) \subseteq G \rightarrow E$  be the associated section described at (5.4). Then there is a unique smooth structure on  $G$  extending that on  $G^{(0)} \cong \widehat{B}$  under which it is a Lie groupoid, there is a unique smooth structure on  $E$  under which the  $\sigma_n$  are all smooth and  $\pi: E \rightarrow G$  is a smooth principal  $\mathbb{T}$ -bundle, and with respect to these smooth structures,  $\mathbb{T} \times G^{(0)} \xrightarrow{\iota} E \xrightarrow{\pi} G$  is a Lie twist, and the elements of  $\mathcal{N}$  determine smooth sections of the associated line bundle via the formula (5.3).*

*Proof.* (1) By [19, Proposition 4.8], the normalisers of  $B$  in  $A$  are precisely the elements that, when viewed as sections of  $L = \mathbb{T} \backslash (\mathbb{C} \times E)$  using [18, p. II.4.2], are supported on bisections. Since  $\mathcal{N}$  consists of smooth sections of  $L$ , the elements of  $\mathcal{N}$  satisfy Condition (N $^*$ ) of Definition 5.6. Lemma 5.3 shows that, for all  $n, n_i \in \mathcal{N}$ , the phases of the functions  $P(n)$ ,  $P(n_1 n_2^*)$ ,  $P(n_1 n_2 n^*)$ , and  $P(n_1 n_2)$  are smooth and bounded (with modulus 1), so for each such phase  $f$  and each  $g \in C_0^\infty(\text{supp}^\circ(f))$ , the product  $f g$  is smooth and belongs to  $C^\infty(\text{supp}^\circ(f))$ ; that is, they are all local multipliers of  $B^\infty$ . So  $(A, B, \mathcal{N})$  is a smooth Cartan triple.

(2) Corollary 5.9, (5)  $\implies$  (3), shows that for  $n \in \mathcal{N}$  we have  $n^* n \in \mathcal{N} \cap B = B^\infty$  and that  $\theta_n$  restricts to an isomorphism  $B_n^\infty \rightarrow B_{n^*}^\infty$ . There is an explicit isomorphism  $A \cong C_r^*(G; E)$  which maps

$B$  to  $C_0(G^{(0)})$ . Give  $G^{(0)}$  the smooth structure that makes that homeomorphism a diffeomorphism; in particular,  $B^\infty$  is identified with  $C_0^\infty(G^{(0)})$ .

Each  $n \in \mathcal{N}$  normalises  $B^\infty$  by Condition (N\*), so it follows from Corollary 5.9, (5)  $\implies$  (4), that  $\theta_n: B_n^\infty \rightarrow B_{n^*}^\infty$  is a  $*$ -isomorphism. Since  $\mathcal{N}$  densely spans  $A$ , Lemma 5.1, (3)  $\implies$  (1) therefore implies that  $G$  acts smoothly on  $G^{(0)}$ . By Proposition 3.7, (1)  $\implies$  (2), there is a unique étale Lie-groupoid structure on  $G$  for which  $G^{(0)}$  is an embedded submanifold. Since the phases of the functions  $P(n)$ ,  $P(n_1 n_2^*)$ ,  $P(n_1 n_2 n^*)$ , and  $P(n_1 n_2)$  are local multipliers of  $B^\infty$  for all  $n, n_i \in \mathcal{N}$  by the remaining four assumptions (U\*)–(I\*) on  $\mathcal{N}$ , Lemma 5.5 shows that they are all smooth functions into  $\mathbb{T}$ . Hence, Lemma 5.3 shows that the family  $\{\sigma_n: \text{supp}^\circ(n) \rightarrow E\}_{n \in \mathcal{N}}$  of sections of  $\pi$  that is induced by the given family  $\mathcal{N}$  satisfies (U $^\infty$ )–(I $^\infty$ ) from Definition 4.8 with respect to the given smooth structure on  $G$ . Thus Theorem 4.15 gives the desired smooth structure on  $E$  with respect to which  $\pi$  is a smooth principal  $\mathbb{T}$ -bundle and each  $\sigma_n$  is smooth. To see that the elements of  $\mathcal{N}$  are smooth, fix  $n \in \mathcal{N}$ . Then  $n = \sqrt{n^* n} \cdot (\rho \circ \sigma_n)$ , so the preimage of  $n$  under the bijection in Lemma 4.21(2) is the function  $\sqrt{n^* n}$ . Since  $n^* n$  and hence  $\sqrt{n^* n}$  is smooth, it follows from Lemma 4.21(4) that  $n$  is smooth.  $\square$

*Remark 5.11.* Our main theorem is phrased so as to provide a correspondence between Lie twists over effective étale Lie groupoids and triples  $(A, B, \mathcal{N})$  in which we are *given* a manifold structure on  $\widehat{B}$  for which  $\mathcal{N}$  consists of smooth sections and  $B^\infty = \mathcal{N} \cap B$  is the algebra of smooth  $C_0$ -functions. We can, of course, combine this with Connes' reconstruction theorem [6, Theorem 1.1] (see also [20, 21]): the algebra of smooth functions on a compact oriented smooth manifold is characterised by the functional-analytic data of a spectral triple. Putting Theorem 5.10 together with Connes' result, we obtain a correspondence between Lie twists over étale Lie groupoids whose unit spaces are compact oriented smooth manifolds, and purely functional-analytic tuples  $(A, B, \mathcal{N}, \mathcal{H}, D)$  such that, putting  $B^\infty := \mathcal{N} \cap B$ ,

- the triple  $(B^\infty, \mathcal{H}, D)$  is a spectral triple satisfying Axioms (1)–(5) of [5],
- $B$  is the  $C^*$ -completion of  $B^\infty$ , and
- the triple  $(A, B, \mathcal{N})$  is a smooth Cartan triple.

However, since we are using Connes' formidable result here as a black box, we have chosen to emphasise that once the smooth structure on  $\widehat{B}$  is given, the remaining smooth structure on  $G$  and  $E$  can be recovered from operator-algebraic information.

*Remark 5.12.* Given a Cartan pair  $B \subseteq A$  and the algebra  $B^\infty = C_0^\infty(\widehat{B})$  of smooth functions for a manifold structure on  $\widehat{B}$ , it is an interesting question to what extent the smooth structure on the associated Weyl twist  $E$  is unique. The point is that the smooth structure on  $E$  is determined by a *choice* of a family  $\mathcal{N}$  extending  $B^\infty$  and satisfying the conditions in Definition 5.6. To see why this is non-unique, consider the situation where  $G = X \rtimes \mathbb{Z}/2\mathbb{Z}$  is the transformation groupoid for a fixed-point-free order-2 diffeomorphism of a compact manifold  $X$  and  $E = \mathbb{T} \times G$  is the trivial twist. Then continuous functions from  $G$  to  $\mathbb{T}$  can be identified with sections of  $E$  via  $f \mapsto [\gamma \mapsto (f(\gamma), \gamma)]$ . We can then obtain families  $\mathcal{N}$  as in Definition 5.6 as follows: Let  $n_0: G^{(0)} = X \times \{0\} \rightarrow \mathbb{T}$  be the constant function  $n_0(x, 0) = 1$ ; this is the identity element of  $C_r^*(G; E)$  regarded as a normaliser of  $C_0(G^{(0)})$ . Fix any *continuous* function  $f: X \rightarrow \mathbb{T}$ , and let  $n_1: X \times \{1\} \rightarrow \mathbb{T}$  be the function  $n_1(x, 1) = f(x)$ , again regarded as a normaliser. Now take  $\mathcal{N} = \{hn_0, hn_1 : h \in C^\infty(X)\}$ . It is routine to check that these normalisers have the required properties because the continuous phase  $f$  cancels with its own conjugate wherever it appears in the conditions in Definition 5.6. Since  $n_1$  is, by definition, smooth with respect to the smooth structure on  $E$  obtained from Theorem 5.10 for  $\mathcal{N}$ , this smooth structure only agrees with the standard smooth structure on  $\mathbb{T} \times G$  if  $f$  is itself smooth. So the smooth structure is not unique.

It should be possible to parameterise the possible choices of smooth structure in terms of cohomological data. Given any two families  $\mathcal{M}$  and  $\mathcal{N}$  as in Definition 5.6, by passing to a common refinement and making appropriate use of Lemma 4.13, we can assume that the corresponding families of sections  $\{\sigma_m : m \in \mathcal{M}\}$  and  $\{\tau_n : n \in \mathcal{N}\}$  have the same supports. More precisely we may assume without loss of generality that  $\mathcal{M} = \{m_\alpha : \alpha \in \mathfrak{A}\}$  and  $\mathcal{N} = \{n_\alpha : \alpha \in \mathfrak{A}\}$  and that for each  $\alpha$ , the corresponding sections  $\sigma_\alpha := \sigma_{m_\alpha}$  and  $\tau_\alpha := \tau_{n_\alpha}$  coming from Lemma 4.21 have the same support  $U_\alpha$ . For each  $\alpha \in U$  we therefore obtain a continuous function  $c_\alpha: U_\alpha \rightarrow \mathbb{T}$  such that  $\sigma_\alpha(\gamma)\tau_\alpha(\gamma)^{-1} = i(c_\alpha(\gamma), r(\gamma))$  for all  $\gamma \in U_\alpha$ . On double overlaps  $U_{\alpha, \beta} := U_\alpha \cap U_\beta$ , the difference  $c_\alpha \bar{c}_\beta$  is a smooth function by (S $^\infty$ ). So the  $c_\alpha$  should determine a continuous Čech 0-cocycle on the *space*  $G$  with smooth transition functions. The smooth structures corresponding to  $\mathcal{M}$  and  $\mathcal{N}$  should coincide exactly when the functions  $c_\alpha$  are themselves all smooth. So the possible distinct smooth structures should be parameterised by the group of continuous Čech cohomology classes of 0-cocycles with smooth transition functions modulo the subgroup of cohomology classes of smooth Čech 0-cocycles.

As a final note, however, we point out that there is, up to *equivalence* only one possible smooth structure on  $E$  that can arise from Theorem 5.10. Specifically, [16, Proposition I.13] shows that the smooth structure on a given topological principal  $\mathbb{T}$ -bundle over a manifold is unique *up to equivalence* and so the smooth structures on  $E$  obtained from different choices of  $\mathcal{N}$  all coincide up to equivalence. We have not investigated whether the diffeomorphisms implementing these equivalences can be expected to be groupoid homomorphisms and/or morphisms of topological twists, but this seems related to the fact that the space  $\mathbb{T}$  admits many smooth structures, all of which are equivalent, but only one for which it is a Lie group.

## APPENDIX A. DIFFERENTIAL GEOMETRY

Since a significant proportion of the imagined audience of this paper—and all of its authorship—consists of  $C^*$ -algebraists who may not have at their mental fingertips all of the differential-geometric concepts and terminology used throughout, we have included this brief appendix collecting the key relevant information. Our primary reference is [15], and we recommend it to those, like us, trying to find the fundamentals of differential geometry that they need all in one well-organised place.

**A.1. Manifolds and submanifolds.** To understand and prove the (seemingly well-known) fact that the composable pairs in a Lie groupoid form an embedded submanifold (see the discussion following Definition 2.2 on page 4), we need some definitions and lemmas. Our notation follows that used in [4].

**Definition A.1** ([15, p. 2ff]). Suppose that  $M$  is a topological space. We say that  $M$  is a *topological manifold of dimension  $m$*  if  $M$  is Hausdorff, second-countable, and *locally Euclidean of dimension  $m$* , meaning that each point of  $M$  has a neighbourhood that is homeomorphic to an open subset of  $\mathbb{R}^m$ . A *topological chart* of  $M$  is a pair  $(U, \varphi)$ , where  $U$  is an open subset of  $M$  and  $\varphi: U \rightarrow \varphi(U)$  is a homeomorphism onto an open subset  $\varphi(U)$  of  $\mathbb{R}^m$ .

**Definition A.2.** Suppose that  $M$  is a topological manifold of dimension  $m$ . A *topological atlas* of  $M$  is a collection  $\mathcal{A} = \{(W_\alpha, \psi_\alpha)\}_{\alpha \in \mathfrak{A}}$  where

- (A1)  $\{W_\alpha\}_{\alpha \in \mathfrak{A}}$  is an open cover of  $M$  and
- (A2) each  $\psi_\alpha: W_\alpha \rightarrow \mathbb{R}^m$  is a topological chart.

We call  $\mathcal{A}$  a *smooth atlas* (or just *atlas*) if we furthermore have

- (A3) for all  $\alpha, \alpha' \in \mathfrak{A}$ , either the set  $W_{\alpha, \alpha'} := W_\alpha \cap W_{\alpha'}$  is empty or the map

$$\psi_{\alpha, \alpha'} := \psi_\alpha|_{W_{\alpha, \alpha'}} \circ (\psi_{\alpha'}|_{W_{\alpha, \alpha'}})^{-1}: \psi_{\alpha'}(W_{\alpha, \alpha'}) \rightarrow \psi_\alpha(W_{\alpha, \alpha'})$$

is a diffeomorphism (between open subsets of  $\mathbb{R}^m$ ).

In this case, we will call one of the topological charts  $\psi_\alpha: W_\alpha \rightarrow \mathbb{R}^m$  a *smooth chart* (or just *chart*).

We say that two atlases are *compatible* if their union is another atlas. An atlas is called *maximal* if it is not properly contained in any larger smooth atlas. A topological manifold  $M$  is called a *smooth manifold* (or just *manifold*) if it comes with a chosen maximal smooth atlas  $\mathcal{A}$ . We will often refer to a choice of an atlas as a *smooth structure on  $M$* . Two smooth structures  $\mathcal{A}, \mathcal{B}$  on  $M$  are called *equivalent* if there exists a diffeomorphism  $f: (M, \mathcal{A}) \rightarrow (M, \mathcal{B})$ .

*Remark A.3.* Recall that every topological manifold  $M$  that admits a smooth structure  $\mathcal{A}$ , admits uncountably many incompatible smooth structures  $\mathcal{A}_t, t \in \mathbb{R}$ , that are built on top of the same, fixed topology on  $M$  [15, Problem 1-6]. In other words, the identity map  $\text{id}_M$  is not a diffeomorphism between  $(M, \mathcal{A}_t)$  and  $(M, \mathcal{A}_s)$  for any two distinct real numbers  $s, t$ . However, there are many manifolds  $M$  for which any two smooth structures are equivalent, i.e., the smooth structure is *unique up to diffeomorphism*. An example of such a manifold is the circle  $\mathbb{T}$  (see [15, Exercise 15-13]).

**Definition A.4** ([15, p. 77]). Given two (smooth) manifolds  $M, N$ , a smooth map  $f: M \rightarrow N$  is

- a *submersion* if its differential  $df_x$  is surjective for each  $x \in M$  (i.e.,  $f$  has constant rank  $\dim(N)$ ),
- an *immersion* if its differential  $df_x$  is injective for each  $x \in M$  (i.e.,  $f$  has constant rank  $\dim(M)$ ), and
- an *embedding* if it is an immersion and a homeomorphism onto its image.

**Definition A.5** ([15, p. 98ff]). Suppose that  $M$  is a smooth manifold. An *embedded submanifold* of  $M$  is a subset  $K \subseteq M$  that is a manifold in the subspace topology, endowed with a smooth structure with respect to which the inclusion map  $K \hookrightarrow M$  is a smooth embedding. Embedded submanifolds are also called *regular submanifolds* by some authors.

We will often think of embedded submanifolds in terms of the *local slice condition*:

**Lemma A.6** ([15, Theorem 5.8]). *Suppose that  $M$  is a manifold of dimension  $m$ . A subset  $K$  is an embedded submanifold of dimension  $k$  if and only if it satisfies the local  $k$ -slice condition: for every  $y \in K$ , there exists a chart  $(U, \varphi)$  around  $y$  in  $M$  such that  $\varphi(U \cap K) = \varphi(U) \cap \mathbb{R}^k$ , where  $\mathbb{R}^k \subseteq \mathbb{R}^m$  is (the image of) the standard inclusion.*

We were unable to locate an explicit reference for the following standard fact, so we provide a brief proof.

**Lemma A.7.** *If  $S$  is an embedded submanifold of  $M$  and  $U \subseteq M$  is open, then  $S \cap U$  is an embedded submanifold of  $U$ . If  $S$  is properly embedded in  $M$ , then  $S \cap U$  is properly embedded in  $U$ .*

*Proof.* The first statement follows almost immediately from the definition of an embedded submanifold: Since  $U \cap S$  is open in  $M$ , it is an open subset of  $S$  in the subspace topology, and hence an open submanifold. Since  $i$  restricts to a smooth embedding  $i: U \cap S \rightarrow M$ , we deduce that  $U \cap S$  is an embedded submanifold.

For the second statement, if  $S$  is properly embedded in  $M$ , then it is closed in  $M$  by [15, Proposition 5.5]. Hence  $S \cap U$  is closed in the subspace topology on  $U$ . So  $S \cap U$  is properly embedded in  $U$  by a second application of [15, Proposition 5.5].  $\square$

**Lemma A.8** ([15, Propositions 5.4 and 5.7]). *Suppose that  $M$  and  $N$  are smooth manifolds. Let  $f: M \rightarrow N$  be a smooth map. Then its graph  $\Gamma_f \subseteq M \times N$  is a properly embedded submanifold of the same dimension as  $M$ .*

*Proof.* Let  $m$  and  $n$  denote the manifold dimensions of  $M$  and  $N$ . Fix  $(x, f(x))$  in  $\Gamma_f$ , and let  $\psi: V \approx \mathbb{R}^n$  be a chart around  $f(x)$  in  $N$ . Since  $f$  is continuous,  $f^{-1}(V)$  is a neighbourhood around  $x$ . So there is a chart  $\varphi: U \approx \mathbb{R}^m$  around  $x$  in  $M$  with  $U \subseteq f^{-1}(V)$ . Consider the smooth bijective map

$$\Omega: U \times \mathbb{R}^n \rightarrow U \times \mathbb{R}^n, (\vec{v}, \vec{w}) \mapsto \left( \vec{v}, \vec{w} - \psi\left(f(\varphi^{-1}(\vec{v}))\right) \right).$$

Let  $\Lambda := \Omega \circ (\varphi \times \psi)$ . Then  $(U \times V, \Lambda)$  is a chart around  $(x, f(x))$  in  $M \times N$ , and

$$\Lambda(U \times V \cap \Gamma_f) = \Omega\left\{(\varphi(y), \psi(f(y))) : y \in U\right\} = \left\{(\varphi(y), \vec{0}) : y \in U\right\} = \mathbb{R}^m \times \{\vec{0}\}.$$

This proves that  $\Gamma_f$  is a submanifold of dimension  $m$ .  $\square$

**Definition A.9** ([15, p. 143]). Two embedded submanifolds  $K_1, K_2 \subseteq M$  are said to be *transverse* if for each  $x \in K_1 \cap K_2$ , the tangent spaces  $T_x K_1$  and  $T_x K_2$  together span  $T_x M$ . If  $f: M \rightarrow N$  is a smooth map and  $K \subseteq N$  is an embedded submanifold, we say that  $f$  is *transverse to  $K$*  if for every  $x \in f^{-1}(K)$ , the spaces  $T_{f(x)} K$  and  $\text{im}(df_x)$  together span  $T_{f(x)} N$ . A pair of maps  $f: M \rightarrow K$  and  $g: N \rightarrow K$  are *transverse* if whenever  $f(x) = g(y) = z \in K$ , the images  $\text{im}(df_x)$  and  $\text{im}(dg_y)$  together span  $T_z(K)$ .

*Remark A.10.* If  $f: M \rightarrow N$  is a submersion, then it is transverse to *any* embedded submanifold of  $N$ , since  $\text{im}(df_x)$  alone already spans all of  $T_p N$ .

**Theorem A.11** ([15, Theorem 6.30]). *Suppose that  $M$  and  $N$  are smooth manifolds, that  $K$  is an embedded submanifold of  $N$  of dimension  $k$ , and that  $f: M \rightarrow N$  is smooth.*

- (1) *If  $f$  is transverse to  $K$ , then  $f^{-1}(K)$  is a submanifold of  $M$  whose codimension in  $M$  coincides with the codimension of  $K$  in  $N$ .*
- (2) *If  $K'$  is an embedded submanifold of  $N$  that is transverse to  $K$ , then  $K \cap K'$  is also a submanifold and its codimension is the sum of the codimensions of  $K$  and  $K'$ .*

The following two results are well-known, but since we were unable to track a published references for them, we provide proofs.

**Proposition A.12.** *Suppose that  $M, N, K$  are smooth manifolds of dimensions  $m, n, k$  respectively. If  $f: M \rightarrow K, g: N \rightarrow K$  are transverse, then the fibred product  $M \times_{f, g} N$  is a submanifold of  $M \times N$  of dimension  $m + n - k$ .*

*Proof.* We follow the idea of the proof of [13, Proposition 3.3]. By Lemma A.8, the graphs  $\Gamma_f \subseteq M \times K$  and  $\Gamma_g \subseteq N \times K$  are submanifolds of dimension  $m$  and  $n$ . We claim that the submanifolds  $\Gamma_f \times \Gamma_g$  and  $H := \{(x, k, y, k) : x \in M, k \in K, y \in N\}$  of  $(M \times K) \times (N \times K)$  are transverse. So for any  $p = (x, f(x), y, g(y)) = (x, k, y, k) \in (\Gamma_f \times \Gamma_g) \cap H$ , we must show that

$$T_p(\Gamma_f \times \Gamma_g) + T_p H = T_p((M \times K) \times (N \times K)).$$

Pick a tangent vector at  $(M \times K) \times (N \times K)$  at  $p$ , say  $(a, b, c, d)$ . Since  $f$  and  $g$  are transverse and since  $f(x) = k = g(y)$ , there exist  $\eta \in T_x M$  and  $\mu \in T_y N$  such that the element  $d - b$  of  $T_k K$  can be written as  $df_x(\eta) + dg_y(\mu)$ . Thus,  $d = b + df_x(\eta) + dg_y(\mu)$ , and hence

$$\begin{aligned} (a, b, c, d) &= (a + \eta, b + df_x(\eta), c - \mu, b + df_x(\eta)) \\ &\quad + (-\eta, -df_x(\eta), 0, 0) \\ &\quad + (0, 0, \mu, dg_y(\mu)). \end{aligned}$$

The first line is an element of  $T_p H$ , the second line is an element of  $T_{(x, f(x))} \Gamma_f$ , and the third line is an element of  $T_{(y, g(y))} \Gamma_g$ . Hence their sum is an element of  $T_p(\Gamma_f \times \Gamma_g)$ . This proves our claim.

It now follows from Theorem A.11(2) that the intersection  $L$  of  $\Gamma_f \times \Gamma_g$  and  $H$  is also a submanifold, namely of dimension

$$\begin{aligned} \dim(L) &= \dim(\Gamma_f \times \Gamma_g) + \dim H - \dim((M \times K) \times (N \times K)) \\ &= [m + n] + [m + k + n] - [m + k + n + k] = m + n - k. \end{aligned}$$

The map  $\iota: L \rightarrow M \times N$  given by  $\iota(x, f(x), y, g(y)) = (x, y)$ , has image exactly  $M \times N$  and has constant rank  $\dim(L)$ . Since  $\iota$  is a topological homeomorphism onto its image, it is an embedding by [15, Theorem 4.14(b)]. By [23, Theorem 11.13], this implies that  $\iota(L) = M \times N$  is a submanifold of  $M \times N$  of dimension  $\dim(L)$ , which we computed earlier as  $m + n - k$ .  $\square$

**Lemma A.13.** *A local diffeomorphism  $f: M \rightarrow N$  between two smooth manifolds is a submersion.*

*Proof.* We need to check that each differential  $df_p: T_p M \rightarrow T_{f(p)} N$  is surjective, so fix  $p \in M$  and let  $D \in T_{f(p)} N$  be an arbitrary derivative of  $\mathfrak{G}_{f(p)}$ .

For any fixed germ  $[h]_p$  at  $p$ , let  $h: U \rightarrow \mathbb{R}$  be a representative of the germ. By assumption on  $f$ , there is a neighbourhood  $V$  of  $p$  such that  $f(V)$  is open in  $N$  and  $f|_V: V \rightarrow f(V)$  is a diffeomorphism. We may shrink  $V$  to an even smaller neighbourhood  $W$  of  $p$  such that  $W \subseteq U$ . Since the map  $h \circ (f|_W)^{-1}: f(W) \rightarrow \mathbb{R}$  is a smooth function defined around  $f(p)$ , we can evaluate  $D$  at  $[h \circ (f|_W)^{-1}]_{f(p)}$ . The resulting value does not depend on the choices of  $W$  or  $h$ : if  $W' \subseteq U$  is another open neighbourhood of  $p$  and  $h'$  is another representative of  $[h]_p$ , then there is an open neighbourhood  $W''$  of  $p$  such that  $h|_{W''} = h'|_{W''}$ , and then  $f(W \cap W' \cap W'')$  is an open neighbourhood of  $f(p)$  on which  $h \circ (f|_W)^{-1}$  and  $h' \circ (f|_{W'})^{-1}$  coincide. Hence

$$[h \circ (f|_W)^{-1}]_{f(p)} = [h' \circ (f|_{W'})^{-1}]_{f(p)}.$$

As the value of  $D$  at  $[h \circ (f|_W)^{-1}]_{f(p)}$  depends neither on the choice of  $W$  nor of  $h$ , we may define

$$\tilde{D}[h]_p := D[h \circ (f|_W)^{-1}]_{f(p)}$$

To see that  $\tilde{D}$  is a derivative of  $\mathfrak{G}_p$ , we calculate:

$$\begin{aligned} \tilde{D}([h_1]_p [h_2]_p) &= \tilde{D}([h_1 h_2]_p) \\ &= D[(h_1 h_2) \circ (f|_W)^{-1}]_{f(p)} \\ &= D[(h_1 \circ (f|_W)^{-1})(h_2 \circ (f|_W)^{-1})]_{f(p)} \\ &= D[(h_1 \circ (f|_W)^{-1})]_{f(p)} e_{f(p)}(h_2 \circ (f|_W)^{-1}) + e_{f(p)}(h_1 \circ (f|_W)^{-1}) D[(h_2 \circ (f|_W)^{-1})]_{f(p)} \\ &= \tilde{D}([h_1]_p) e_p([h_2]_p) + e_p([h_1]_p) \tilde{D}([h_2]_p). \end{aligned}$$

For any germ  $[g]_{f(p)}$  of  $f(p)$ , we have

$$df_p(\tilde{D})[g]_{f(p)} = \tilde{D}[g \circ f]_p = D[(g \circ f) \circ (f|_W)^{-1}]_{f(p)} = D[g]_{f(p)},$$

proving that  $df_p(\tilde{D}) = D$ , as claimed.  $\square$

**A.2. Principal bundles.** Here we collect the facts about principal bundles, and in particular smooth principal bundles, that we use throughout the paper, for ease of reference.

**Definition A.14** ([4, Definition 11.2.2]). Let  $M$  and  $P$  be topological spaces,  $\Gamma$  a topological group,  $\pi: P \rightarrow M$  a continuous map, and  $\Gamma \times P \rightarrow P$ ,  $(s, x) \mapsto s \cdot x$ , a continuous left action that is free and transitive on each fibre  $\pi^{-1}(m)$ . Define a left action of  $\Gamma$  on  $\Gamma \times P$  by

$$\begin{aligned} \Gamma \times (\Gamma \times U_\alpha) &\rightarrow \Gamma \times U_\alpha \\ (s, (t, x)) &\mapsto (st, x). \end{aligned}$$

Suppose that  $\{U_\alpha\}_{\alpha \in \mathfrak{A}}$  is an open cover of  $M$  and that for each  $\alpha$ , the map  $\psi_\alpha: \pi^{-1}(U_\alpha) \rightarrow \Gamma \times U_\alpha$  is a  $\Gamma$ -equivariant homeomorphism such that the diagram

$$\begin{array}{ccc} \pi^{-1}(U_\alpha) & \xrightarrow{\psi_\alpha} & \Gamma \times U_\alpha \\ & \searrow \pi & \swarrow \text{pr}_2 \\ & & U_\alpha \end{array}$$

commutes. Then,  $\pi: P \rightarrow M$ , together with the given  $\Gamma$ -action, is called a *topological principal  $\Gamma$ -bundle over  $M$* ,  $\Gamma$  is called the *structure group* of the bundle, and each  $(U_\alpha, \psi_\alpha)$  is called a *topological local trivialisation*.

We call  $\pi: P \rightarrow M$  a *smooth principal bundle* if  $M, P$  are manifolds,  $\Gamma$  is a Lie group,  $\pi$  is smooth, the  $\Gamma$ -action on  $P$  is smooth, and the  $\psi_\alpha$  can be chosen to be diffeomorphisms, in which case we call them *smooth local trivialisations*.

As with manifolds, if the term ‘principal bundle’ appears without an adjective, we will implicitly mean that it is smooth.

*Remark A.15.* If  $\pi: P \rightarrow M$  is a smooth principal  $\Gamma$ -bundle, then  $\pi$  is a submersion [15, Exercise 10-19]. If  $N \subseteq M$  is an embedded submanifold, then  $P_N := \pi^{-1}(N)$  is therefore an embedded submanifold of  $P$  [15, Corollary 6.31]. One can further show that  $\pi|_{P_N}: P_N \rightarrow N$  is a smooth principal  $\Gamma$ -bundle and that the inclusion  $P_N \hookrightarrow P$  is a principal- $\Gamma$ -bundle morphism, covering the inclusion  $N \hookrightarrow M$ .

**Lemma A.16.** *Suppose that  $\pi_i: P_i \rightarrow M$ ,  $i = 1, 2$  are smooth principal  $\Gamma$ -bundles and that  $\Psi: P_1 \rightarrow P_2$  is a  $\Gamma$ -equivariant bundle map. If  $\Psi$  is a smooth homeomorphism, then  $\Psi$  is a diffeomorphism.*

*Proof.* First consider the case in which  $P_1 = P_2$  as topological spaces (not necessarily as manifolds) and that  $\Psi = \text{id}_{P_1 \rightarrow P_2}$  is the identity map; in particular, since we are asking for this map to be a bundle map, we must have  $\pi_1 = \pi_2$ . Fix an arbitrary  $e_2 \in P_2$ , and write  $e_1$  for the same point regarded as an element of  $P_1$ . Recall that  $\pi_i$  is an open map, so for  $i = 1, 2$ , there is an open neighbourhood  $V_i$  of  $\pi_i(e_i)$  such that there is a  $\Gamma$ -equivariant diffeomorphism  $\psi_i$  making the diagram

$$\begin{array}{ccc} P_i \supseteq \pi_i^{-1}(V_i) & \xrightarrow{\psi_i} & \Gamma \times V_i \subseteq \Gamma \times M \\ & \searrow \pi_i & \swarrow \text{pr}_2 \\ & & V_i \end{array}$$

commute. Now,  $\pi_1(e_1) = \pi_2(e_2)$ , so  $V := V_1 \cap V_2$  is an open neighbourhood of  $\pi_i(e)$  for each  $i$ . Consider

$$\Gamma \times V \xleftarrow[\psi_1]{\psi_1^{-1}} \pi_1^{-1}(V) \xleftarrow{\text{id}} \pi_2^{-1}(V) \xleftarrow[\psi_2^{-1}]{\psi_2} \Gamma \times V.$$

Since  $\text{pr}_2 \circ \psi_i = \pi_i$ , there is a unique map  $T: \Gamma \times V \rightarrow \Gamma$  such that

$$(\psi_2 \circ \psi_1^{-1})(\gamma, m) = (T(\gamma, m), m).$$

Since  $\psi_i$  and  $\Psi$  are  $\Gamma$ -equivariant, it follows that

$$T(\gamma, m) = \gamma T(1_\Gamma, m),$$

so the map  $t: V \rightarrow \Gamma$  given by  $t(m) = T(1_\Gamma, m)$  satisfies

$$(\psi_2 \circ \psi_1^{-1})(\gamma, m) = (\gamma t(m), m).$$

An easy computation now shows that interchanging the roles of  $\psi_1$  and  $\psi_2$  amounts to composing  $t$  with the inversion map in  $\Gamma$ :

$$(\psi_1 \circ \psi_2^{-1})(\gamma, m) = (\psi_2 \circ \psi_1^{-1})^{-1}(\gamma, m) = (\gamma t(m)^{-1}, m).$$

By assumption, the map  $\text{id}: \pi_1^{-1}(V) \rightarrow \pi_2^{-1}(V)$  is smooth, so  $\psi_2 \circ \psi_1^{-1}$  is smooth, and thus so are  $T$  and  $t$ . Since  $\Gamma$  is a Lie group, the map  $\Gamma \times V \rightarrow \Gamma$  given by  $(\gamma, m) \mapsto \gamma t(m)^{-1}$  is smooth. In particular,  $\psi_1 \circ \psi_2^{-1}$  is smooth. Hence  $\text{id}: \pi_2^{-1}(V) \rightarrow \pi_1^{-1}(V)$  is smooth around  $e_2$ . In other words,  $\text{id}: P_1 \rightarrow P_2$  is a diffeomorphism.

We now deal with the general situation; for reasons that will become clear in a moment, we stray from the notation in the lemma. Fix given smooth principal  $\Gamma$ -bundles  $\pi_1: P_1 \rightarrow M$  and  $\pi_2: P_2 \rightarrow M$  and a  $\Gamma$ -equivariant bundle map  $\Psi: P_1 \rightarrow P_2$  which is a smooth homeomorphism. We show that  $\Psi^{-1}$  is smooth.

Since  $\Psi$  is a *homeomorphism*, we may let  $P_2$  be the topological space  $P_1$  endowed with the smooth structure for which  $\Psi$  is a diffeomorphism. In other words, if we again write  $e_1$  for the point  $e_2 \in P_2$  but

regarded as a point of  $P_1$ , then the map  $\Psi_2: P_2 \rightarrow Q, e_2 \mapsto \Psi(e_1)$ , is a smooth map between manifolds with smooth inverse. We equip  $P_2$  with the projection map  $\pi_2: P_2 \rightarrow M$  given by  $e_2 \mapsto \pi^Q(\Psi_2(e_2))$  and with the  $\Gamma$ -action  $\gamma e_2 := \Psi_2^{-1}(\gamma \cdot \Psi_2(e_2))$ . Since  $\pi^Q: Q \rightarrow M$  is a smooth principal  $\Gamma$ -bundle, so is  $\pi_2: P_2 \rightarrow M$ , and by construction,  $\Psi_2$  is an isomorphism of smooth principal bundles:

$$\begin{array}{ccc} P_2 & \xrightarrow{\Psi_2, \cong} & Q \\ \downarrow \pi_2 & \circlearrowleft & \swarrow \pi^Q \\ M & & \end{array}$$

Since  $\Psi$  is a bundle map,

$$\pi_2(e_2) = \pi^Q(\Psi_2(e_2)) = \pi^Q(\Psi(e_1)) = \pi_1(e_1).$$

Moreover, since  $\Psi$  is  $\Gamma$ -equivariant,

$$\gamma \cdot e_2 = \Psi_2^{-1}(\gamma \cdot \Psi_2(e_2)) = \Psi_2^{-1}(\gamma \cdot \Psi(e_1)) = \Psi_2^{-1}(\Psi(\gamma \cdot e_1)) = \gamma \cdot e_1.$$

In other words, the diagram

$$\begin{array}{ccccc} & & \Psi & & \\ & & \circlearrowleft & & \\ P_1 & \xrightarrow{\text{id}} & P_2 & \xrightarrow{\Psi_2} & Q \\ & \searrow \pi_1 & \downarrow \pi_2 & \swarrow \pi^Q & \\ & & M & & \end{array}$$

of principal  $\Gamma$ -bundles commutes. By construction of the smooth structure and by the assumption that  $\Psi$  is smooth, the map  $\text{id}_{P_1 \rightarrow P_2} = \Psi_2^{-1} \circ \Psi$  is smooth. We are now in the setting of the first paragraph, and conclude that  $\text{id}$  is a diffeomorphism. Thus,  $\Psi^{-1} = \Psi_2 \circ \text{id}_{P_2 \rightarrow P_1}$  is smooth, as claimed.  $\square$

**Lemma A.17.** *Suppose that  $\pi_1: P_1 \rightarrow M$  is a smooth principal  $\Gamma$ -bundle, that  $\pi_2: P_2 \rightarrow M$  is a topological principal  $\Gamma$ -bundle, and that  $\Psi: P_1 \rightarrow P_2$  is a  $\Gamma$ -equivariant bundle map. If  $\Psi$  is a homeomorphism and  $P_2$  is given the unique smooth structure that makes  $\Psi$  a diffeomorphism, then  $\pi_2$  is a smooth principal bundle.*

*Proof.* Since  $\Gamma$  acts smoothly on  $P_1$  and  $\Psi$  is  $\Gamma$ -equivariant, the action on  $P_2$  is automatically smooth. Since  $\pi_1$  is smooth and  $\Psi$  is a bundle map,  $\pi_2$  is smooth, and since  $\pi_1$  allows smooth trivialisations, so does  $\pi_2$ .  $\square$

**Lemma A.18.** *Suppose that  $\pi: P \rightarrow M$  is a topological (respectively smooth) principal  $\Gamma$ -bundle for which the  $\Gamma$ -action is proper. Then for any  $U \subseteq M$  open and any continuous (respectively smooth) section  $\sigma: U \rightarrow \pi^{-1}(U)$ , the map*

$$\Gamma \times U \rightarrow \pi^{-1}(U), \quad (\gamma, m) \mapsto \gamma \cdot \sigma(m),$$

*is a homeomorphism (respectively diffeomorphism) and thus defines a topological (respectively smooth) local trivialisation  $\psi_\sigma: \pi^{-1}(U) \rightarrow \Gamma \times U$  of  $P$ .*

*Proof.* For this proof, let  $f: \Gamma \times U \rightarrow \pi^{-1}(U)$  be the map in the displayed equation; once we have shown that  $f$  is invertible, we will care more about its inverse  $\psi_\sigma$  and thus write  $f = \psi_\sigma^{-1}$ .

Recall that  $P_U := \pi^{-1}(U)$  is an embedded submanifold of  $P$  and  $\pi: P_U \rightarrow U$  is itself a topological (respectively smooth) principal  $\Gamma$ -bundle (Remark A.15). Likewise, with respect to the projection  $q$  onto the second coordinate,  $q: \Gamma \times U \rightarrow U$  is a topological (respectively smooth) principal  $\Gamma$ -bundle. Note that  $f$  is equivariant by construction and satisfies  $\text{pr}_2 \circ f = \pi \circ f$ . Thus, in the topological case, we only need to show that  $f$  is a homeomorphism; in the smooth case, we further need to check that  $f$  is smooth so that we can invoke Lemma A.16

The map  $f$  is injective since the  $\Gamma$ -action on  $P$  is free and since  $\sigma$  is injective, being a section, and  $f$  is surjective since the  $\Gamma$ -action is transitive. To see that  $f$  is open, we invoke [24, Proposition 1.1], so suppose that  $\{e_\lambda\}_\lambda$  is a net in  $\pi^{-1}(U)$  such that  $e_\lambda \rightarrow \gamma \cdot \sigma(m)$  for some  $\gamma \in \Gamma$  and  $m \in U$ ; we must show that (a subnet of)  $\{e_\lambda\}_\lambda$  can be lifted to elements in  $\Gamma \times U$  which converge to  $(\gamma, m)$ . Let  $m_\lambda := \pi(e_\lambda)$ ; since  $\pi$  is continuous and  $M$  is Hausdorff,  $m_\lambda \rightarrow m$ . For each  $\lambda$ , we have  $m_\lambda = \pi(\sigma(m_\lambda))$  since  $\sigma$  is a section. Hence transitivity and freeness of the  $\Gamma$ -action implies that there exists a unique  $\gamma_\lambda \in \Gamma$  such that  $e_\lambda = \gamma_\lambda \cdot \sigma(m_\lambda)$ . It remains to check that (a subnet of)  $\{\gamma_\lambda\}_\lambda$  converges to  $\gamma$ . This follows immediately from properness of the  $\Gamma$ -action: both  $\gamma_\lambda \cdot \sigma(m_\lambda)$  and  $\sigma(m_\lambda)$  converge.

In the smooth case,  $f$  is smooth as a map  $\Gamma \times U \rightarrow P$  since  $\sigma$  is smooth and since the  $\Gamma$ -action on  $P$  is smooth. It follows that  $f$  is also smooth as a map  $\Gamma \times U \rightarrow P_U$  [15, Corollary 5.30]. By Lemma A.16, it now follows that  $f$  is a diffeomorphism.  $\square$

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