NORM UPPER-SEMICONTINUITY OF FUNCTIONS SUPPORTED ON OPEN ABELIAN ISOTROPY IN ÉTALE GROUPOIDS (A CORRIGENDUM TO *RECONSTRUCTION OF GROUPOIDS AND C*-RIGIDITY OF DYNAMICAL SYSTEMS*, ADV. MATH 390 (2021), 107923)

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Abstract. We consider étale Hausdorff groupoids in which the interior of the isotropy is abelian. We prove that the norms of the images under regular representations, of elements of the reduced groupoid C^*-algebra whose supports are contained in the interior of the isotropy vary upper semicontinuously. This corrects an error in [2].

Introduction

The purpose of this note is to correct an error in the proof of [2, Lemma 5.2].

The error. The error occurs in the final paragraph of the proof of [2, Lemma 5.2]. The fourth line in that paragraph deals with an element a of the C^*-subalgebra

$$A := \{a \in C^*_r(G) : f_a|_{G\setminus\mathcal{I}} = 0\} \subseteq C^*_r(G), \tag{0.1}$$

and asserts that

$$\text{“since } f_a|_{\mathrm{Iso}(G)^\circ_u} = 0, \text{ we have } \|\pi_u(a)\| = 0.\text{”} \tag{0.2}$$

However, π_u has domain $C^*(\mathrm{Iso}(G)^\circ)$, not A. Indeed, [2, Lemma 5.2] is later applied to prove [2, Corollary 5.3], which asserts that $A = C^*(\mathrm{Iso}(G)^\circ)$. So the application of π_u to an element of A is circular reasoning.

Consequences of the error. The error invalidates the proof of [2, Corollary 5.3], which underpins the main results of [2]: it is used in the paragraph immediately subsequent to [2, Proposition 6.5] to identify $\big(C_0(G^{(0)})'_{C^*_{c^{-1}(\mathrm{id}_\Gamma)}}\big)_u$ with $\big(C^*_r(\mathrm{Iso}(c^{-1}(\mathrm{id}_\Gamma))^\circ)\big)_u$. This in turn is used in the proof of [2, Proposition 6.5] to see that the group

$$\mathrm{Iso}(c^{-1}(\mathrm{id}_\Gamma))^\circ_u \cong \mathcal{U}\big(\big(C^*_r(\mathrm{Iso}(c^{-1}(\mathrm{id}_\Gamma))^\circ)\big)_u\big)/\mathcal{U}_0\big(\big(C^*_r(\mathrm{Iso}(c^{-1}(\mathrm{id}_\Gamma))^\circ)\big)_u\big),$$

can be recovered from $\big(C_0(G^{(0)})'_{C^*_{c^{-1}(\mathrm{id}_\Gamma)}}\big)_u$, and hence from the triple $(C^*_r(G), C_0(G^{(0)}), \delta_c)$.

Related results. If G is amenable, then [1, Theorem 4.2] implies [2, Corollary 5.3]. If G contains a clopen amenable subgroupoid that contains $\overline{\mathrm{Iso}(G)^\circ}$, we can use the argument of [1, Theorem 4.2] in the same way. If $\mathrm{Iso}(G)^\circ$ is closed, then the argument of [3, Proposition 3.12] shows that $A = C^*(\mathrm{Iso}(G)^\circ)$. However, imposing any of these hypotheses would substantially weaken the main results of [2].

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The correction. The first three paragraphs of the proof of [2, Lemma 5.2] are correct. In the following paragraph, the proof that $J_u \subseteq \{a \in A : f_a|_{\mathrm{Iso}(G)^\circ_u}\}$ is correct. The following four sentences "For the reverse containment...proves the claim" are invalid as they depend upon the erroneous assertion (0.2) mentioned above. The remainder of the proof is correct *provided* that it is indeed true that $J_u \supseteq \{a \in A : f_a|_{\mathrm{Iso}(G)^\circ_u}\}$. Hence Theorem 1.3 below corrects the proof of [2, Lemma 5.2], and then [2, Corollary 5.3] and the subsequent results that depend upon it follow as argued there.

1. The correction for the proof of [2, Lemma 5.2]

Notation 1.1. Throughout this note, G is an étale groupoid. For $x \in G^{(0)}$, we write $Gx := s^{-1}(x)$ and $xG := r^{-1}(x)$ (note that this differs from Renault's notation $G_x = s^{-1}(x)$ and $G^x = r^{-1}(x)$), and we denote by $\mathrm{Iso}(G)$ the isotropy subgroupoid of G; that is, $\mathrm{Iso}(G) := \bigcup_{x \in G^{(0)}}(xG \cap Gx)$. We will assume throughout that the interior $\mathrm{Iso}(G)^\circ$ of the isotropy, is abelian. We make frequent use of the set $\mathrm{Iso}(G)^\circ$, its closure $\overline{\mathrm{Iso}(G)^\circ}$, and their fibres $\mathrm{Iso}(G)^\circ \cap Gx$ and $\overline{\mathrm{Iso}(G)^\circ} \cap Gx$. Because of the utter failure of commutativity amongst the operations of adding an overline, a superscripted $\circ$ or a subscripted x to the expression $\mathrm{Iso}(G)$, we use the following non-standard notation:

$$\mathcal{I} := \mathrm{Iso}(G)^\circ, \qquad \overline{\mathcal{I}} := \overline{\mathrm{Iso}(G)^\circ},$$
$$\mathcal{I}x := \mathrm{Iso}(G)^\circ \cap Gx, \text{ and} \qquad \overline{\mathcal{I}}x := \overline{\mathrm{Iso}(G)^\circ} \cap Gx.$$

We recall that $j\colon C^*_r(G) \to C_0(G)$ is Renault's map extending the identity map on $C_c(G)$ [5, Proposition II.4.2] (in [2] this map is denoted $a \mapsto f_a$); if λ_x denotes the regular representation of $C^*_r(G)$ on $\ell^2(Gx)$ for $x \in G^{(0)}$, then j is defined by $j(a)(\gamma) = \big(\lambda_{s(\gamma)}(a)e_{s(\gamma)} \mid e_\gamma\big)$, and satisfies $(\lambda_x(a)e_\gamma \mid e_\delta) = j(a)(\gamma\delta^{-1})$ for all $x \in G^{(0)}$ and $\gamma, \delta \in Gx$.

For $g \in C_0(G)$, we write $\mathrm{supp}^\circ(g) = \{\gamma \in G : g(\gamma) \neq 0\}$ for the open support of g and $\mathrm{supp}(g) = \overline{\mathrm{supp}^\circ(g)}$ for its support in the usual sense.

Remark 1.2. We frequently write expressions of the form $\prod_{i \in S} \beta_i$ where the β_i are elements of G and S is an un-ordered indexing set. We do this only when the β_i are pairwise commuting isotropy elements, so that the product does not depend on the order in which S is listed. Similarly, we frequently write $\prod_{i \in S}(C_i \cap \mathcal{I})$, where the C_i are open bisections in G, exploiting that $\mathcal{I}$ is abelian so that the product is well defined. If the terms in a product do not necessarily commute, we write it as $\prod_{i=1}^N F_i$, which always means $F_1F_2\cdots F_N$.

We take the convention that if $\alpha \in G$, then $\alpha^0 = s(\alpha)$ and if $C \subseteq G$ is an open bisection, then $C^0 = s(C)$.

As detailed above, to correct the proof of [2, Lemma 5.2], it suffices to establish the following assertion.

Theorem 1.3. *Let $A = \{a \in C^*_r(G) : j(a)|_{G\setminus\mathcal{I}} = 0\} \subseteq C^*_r(G)$. Then for $x \in G^{(0)}$,*

$$\{a \in A : j(a)|_{\mathcal{I}x} = 0\} \subseteq J_x := \overline{\{ad : a \in A, d \in C_0(G^{(0)}) \text{ and } d(x) = 0\}}.$$

For the following lemma observe that since $r, s\colon G \to G^{(0)}$ are continuous, the isotropy $\mathrm{Iso}(G)$ is closed, and in particular each $\overline{\mathcal{I}}x \subseteq \mathrm{Iso}(G)$.

Lemma 1.4. *Suppose that $\mathcal{C} = \{C_i : i \in S\}$ is a finite family of open bisections and that $x \in \bigcap_{i \in S} s(C_i)$. For each i, let γ_i denote the unique element of C_ix, and suppose that $\gamma_i \in \mathrm{Iso}(G)$ for all $i \in S$ and that the γ_i pairwise commute. Let H_S be the subgroup of $\mathrm{Iso}(G)x$ generated by $\{\gamma_i : i \in S\}$. Then there is a neighbourhood U_S of x such that*

(1) *for any integers* $(m_i)_{i\in S}$ *and any* $j \in S$ *such that* $\prod_{i\in S}\gamma_i^{m_i} = \gamma_j$*, we have* $\prod_{i\in S}(C_i \cap \mathcal{I})^{m_i} U_S \subseteq C_j$*; and*
(2) *for each* $y \in U_S$ *such that* $C_i \cap \mathcal{I}y \neq \emptyset$ *for all* $i \in S$*, there is a homomorphism* $q_y\colon H_S \to \mathcal{I}y$ *such that* $q_y(\gamma_i) \in C_i \cap \mathcal{I}y$ *for all* $i \in S$.

Proof. Since S is finite and the γ_i commute, H_S is a finitely generated abelian group. So by the fundamental theorem of finitely generated abelian groups, there are integers $f, t \geq 0$, elements $\eta_1, \dots, \eta_{t+f}$ of H_S, and strictly positive integers $O_1, \dots, O_t$ such that $(a_1, \dots, a_{t+f}) \mapsto \prod_{k=1}^{t+f}\eta_k^{a_k}$ is an isomorphism $\big(\bigoplus_{k\leq t}\mathbb{Z}/O_k\mathbb{Z}\big) \oplus \mathbb{Z}^f \to H_S$. So H_S is generated by $\{\eta_1, \dots, \eta_{t+f}\}$, and if $\{\zeta_1, \dots, \zeta_{t+f}\}$ are elements of a group H' that pairwise commute and satisfy $\zeta_k^{O_k} = e_{H'}$ for all $1 \leq k \leq t$, then $\eta_k \mapsto \zeta_k$ extends to a homomorphism $H_S \to H'$.

For every $k \leq t+f$, since $\eta_k \in H_S$, there is a function $m(k,\cdot)\colon S \to \mathbb{Z}$ such that

$$\eta_k = \prod_{i\in S}\gamma_i^{m(k,i)},$$

using Remark 1.2. Let

$$V_k := \bigcap_{S=\{i_1,\dots,i_{|S|}\}}\prod_{j=1}^{|S|} C_{i_j}^{m(k,i_j)}, \tag{1.1}$$

the intersection over all enumerations of S. Since it is an intersection of products of open bisections, V_k is itself an open bisection neighbourhood of η_k.

For each $k \leq t$, since the unit space of G is open and since multiplication is continuous, we can shrink V_k to a smaller neighbourhood of η_k such that $V_k^{O_k} \subseteq G^{(0)}$. Shrinking V_k further by replacing it with the open subset $V_k \cap s^{-1}(V_k^{O_k})$, we can assume that $V_k^{O_k} = s(V_k)$. Hence $V_k V^{O_k - 1} = s(V_k)$. Since the open bisections of G form an inverse semigroup $S(G)$ [4, Proposition 2.2.4], uniqueness of inverses in inverse semigroups [4, page 21] implies that $V_k^{O_k-1}$ is the inverse $V_k^* = \{\alpha^{-1} : \alpha \in V_k\}$ of V_k in $S(G)$.

We have chosen $V_k \ni \eta_k$ for $k \leq t+f$ such that for each k and each possible enumeration $S = \{i_1, \dots, i_{|S|}\}$, we have $V_k \subseteq \prod_{j=1}^{|S|} C_{i_j}^{m(k,i_j)}$, and such that, in addition,

$$r(V_k) = V_k V_k^* = V_k V_k^{O_k-1} = V_k^{O_k} = s(V_k) \quad \text{for } k \leq t. \tag{1.2}$$

For each $i \in S$, since $\gamma_i \in H_S$, there is a function $b(i,\cdot) : \{1, \dots, t+f\} \to \mathbb{Z}$ such that

$$\prod_{k=1}^{t+f}\eta_k^{b(i,k)} = \gamma_i \quad \text{for each } i \in S; \tag{1.3}$$

if $i \in S$ satisfies $\gamma_i = x$, then we take $b(i,k) = 0$ for all k. For any $i \in S$, both C_i and $\prod_{k=1}^{t+f} V_k^{b(i,k)}$ are open bisection neighbourhoods of γ_i; so we can further shrink each V_k to a smaller neighbourhood of η_k to ensure that

$$\prod_{k=1}^{t+f} V_k^{b(i,k)} \subseteq C_i \quad \text{for each } i \in S. \tag{1.4}$$

Now fix $y \in \bigcap_{i\in S} s(C_i)$, so there is a unique element in $C_i y$ for each $i \in S$, and fix $k \leq t+f$. If y satisfies $C_i y \subseteq \mathcal{I}$ for each $i \in S$ satisfying $m(k,i) \neq 0$, then the unique elements of $(C_i y)^{m(k,i)}$ for $i \in S$ pairwise commute because $\mathcal{I}$ is abelian; so $\prod_{i\in S}(C_i y)^{m(k,i)}$ is well-defined, and is a singleton. For any enumeration $S = \{i_1, \dots, i_{|S|}\}$, the set $\prod_{j=1}^{|S|} C_{i_j}^{m(k,i_j)}$ contains $\prod_{i\in S}(C_i y)^{m(k,i)}$.

Since this set is a bisection, we deduce that

$$\prod_{i\in S}(C_iy)^{m(k,i)} = \Big(\bigcap_{S=\{i_1,\dots,i_{|S|}\}}\prod_{j=1,\dots,|S|} C_{i_j}^{m(k,i_j)}\Big)y \overset{(1.1)}{=} V_ky. \tag{1.5}$$

Since each $\eta_k \in V_k \cap H_S$, we have $x \in \bigcap_{k=1}^{t+f} s(V_k)$, so (1.4) implies that there is a neighbourhood $U_S \subseteq \big(\bigcap_{k=1}^{t+f} s(V_k)\big) \cap \big(\bigcap_{i\in S} s(C_i)\big)$ of x such that

$$\Big(\prod_{k=1}^{t+f} V_k^{b(i,k)}\Big)U_S = C_iU_S \quad \text{for each } i \in S. \tag{1.6}$$

For each $j \in S$ satisfying $\gamma_j = x$, we took $b(j,k) = 0$ for all k; hence for such j, Equation (1.6) becomes $U_S = \Big(\prod_{k=1}^{t+f} s(V_k)\Big)U_S = C_jU_S$. In particular, $C_jU_S \subseteq G^{(0)}$.

We show that U_S satisfies (1) and (2).

For (1), suppose that $\prod_{i\in S}\gamma_i^{m_i} = \gamma_j$. Fix $y \in U_S$, and suppose that $\prod_{i\in S}(C_i \cap \mathcal{I})^{m_i}y \neq \emptyset$; we must show that this set is contained in C_j. As $\prod_{i\in S}(C_i \cap \mathcal{I})^{m_i}y \neq \emptyset$, each $(C_i \cap \mathcal{I})^{m_i}y$ is non-empty. Since $C_i \cap \mathcal{I} \subseteq \operatorname{Iso}(G)$, we have $(C_i \cap \mathcal{I})^{m_i}y = (C_i \cap \mathcal{I}y)^{m_i}$, and so for each $i \in S$, the set $C_i \cap \mathcal{I}y$ is non-empty; we let β_i denote its unique element so that

$$\prod_{i\in S}(C_i \cap \mathcal{I})^{m_i}y = \Big\{\prod_{i\in S}\beta_i^{m_i}\Big\}.$$

Since each η_k is an element of the abelian group H_S, Equation (1.3) implies that $\gamma_i^{m_i} = \prod_{k=1}^{t+f}\eta_k^{m_ib(i,k)}$ for all i. Since, by assumption, $\prod_i \gamma_i^{m_i} = \gamma_j$,

$$\prod_{k=1}^{t+f}\eta_k^{\sum_{i\in S} m_ib(i,k)} = \prod_{i\in S}\Big(\prod_{k=1}^{t+f}\eta_k^{m_ib(i,k)}\Big) = \prod_{i\in S}\gamma_i^{m_i} = \gamma_j = \prod_{k=1}^{t+f}\eta_k^{b(j,k)}.$$

Since $(a_1,\dots,a_{t+f}) \mapsto \prod_{k=1}^{t+f}\eta_k^{a_k}$ is an isomorphism $\big(\bigoplus_{k\le t}\mathbb{Z}/O_k\mathbb{Z}\big) \oplus \mathbb{Z}^f \to H_S$, there are integers $x_k, k \le t$ such that

$$b(j,k) = x_kO_k + \sum_{i\in S} m_ib(i,k) \qquad \text{for } k \le t, \text{ and} \tag{1.7}$$

$$b(j,k) = \sum_{i\in S} m_ib(i,k) \qquad \text{for } k > t. \tag{1.8}$$

We have

$$V_k^{x_kO_k} = (V_k^{O_k})^{x_k} = s(V_k)^{x_k} = s(V_k) \supseteq U_S \quad \text{for } k \le t. \tag{1.9}$$

Since $C_iy \subseteq \mathcal{I}$ for all $i \in S$ and since $\mathcal{I}$ is abelian, for any enumeration $S = \{i_1,\dots,i_{|S|}\}$ of S and any integers $n_1,\dots,n_{|S|}$, we have

$$\Big(\prod_{j=1}^{|S|} C_{i_j}^{n_{i_j}}\Big)y = \prod_{i\in S}(C_iy)^{n_i} = \Big\{\prod_{i\in S}\beta_i^{n_i}\Big\}.$$

So (1.1) and that $\mathcal{I}$ is closed under multiplication imply that for each $k \le t+f$, defining $\zeta_k := \prod_{i\in S}\beta_i^{m(k,i)} \in \mathcal{I}y$, we have

$$V_ky = \{\zeta_k\}. \tag{1.10}$$

Since $\mathcal{I}$ is abelian, the ζ_k pairwise commute. Since $y \in U_S$, for each $i \in S$,

$$\{\beta_i\} = C_i y \overset{(1.6)}{=} \Big(\prod_{k=1}^{t+f} V_k^{b(i,k)}\Big) y = \prod_{k=1}^{t+f} (V_k y)^{b(i,k)} \overset{(1.10)}{=} \Big\{\prod_{k=1}^{t+f} \zeta_k^{b(i,k)}\Big\}. \tag{1.11}$$

Using again that $\mathcal{I}$ is abelian, we obtain

$$\prod_{i\in S} \beta_i^{m_i} = \prod_{i \in S} \Big(\prod_{k=1}^{t+f} \zeta_k^{b(i,k)}\Big)^{m_i} = \prod_{k=1}^{t+f} \zeta_k^{\sum_{i\in S} m_i b(i,k)}. \tag{1.12}$$

Consequently,

$$\begin{aligned}\prod_{i\in S} (C_i \cap \mathcal{I})^{m_i} y &= \Big\{\prod_{k=1}^{t+f} \zeta_k^{\sum_{i\in S} m_i b(i,k)}\Big\} \\ &\subseteq \Big(\prod_{k=1}^{t+f} V_k^{\sum_{i\in S} m_i b(i,k)}\Big) U_S = \Big(\prod_{k=1}^{t} V_k^{\sum_{i\in S} m_i b(i,k)}\Big)\Big(\prod_{k=t+1}^{t+f} V_k^{\sum_{i\in S} m_i b(i,k)}\Big) U_S.\end{aligned}$$

By (1.9), for each $k \le t$ we have

$$V_k^{\sum_{i\in S} m_i b(i,k)} = V_k^{\sum_{i\in S} m_i b(i,k)} V_k^{x_k O_k} = V_k^{x_k O_k + \sum_{i\in S} m_i b(i,k)}.$$

Hence

$$\prod_{i\in S} (C_i \cap \mathcal{I})^{m_i} y = \Big(\prod_{k=1}^{t} V_k^{x_k O_k + \sum_{i\in S} m_i b(i,k)}\Big)\Big(\prod_{k=t+1}^{t+f} V_k^{\sum_{i\in S} m_i b(i,k)}\Big) U_S.$$

Now applying (1.7) to the first product and (1.8) to the second, we obtain

$$\prod_{i\in S} (C_i \cap \mathcal{I})^{m_i} y = \Big(\prod_{k=1}^{t} V_k^{b(j,k)}\Big)\Big(\prod_{k=t+1}^{t+f} V_k^{b(j,k)}\Big) U_S = \prod_{k=1}^{t+f} V_k^{b(j,k)}.$$

Hence (1.6) gives

$$\prod_{i\in S} (C_i \cap \mathcal{I})^{m_i} y = C_j U_S.$$

For (2), suppose that $y \in U_S$ and that $C_i \cap \mathcal{I} y \neq \emptyset$ for each $i \in S$, and denote its unique element by β_i. We must show that there is a homomorphism $q_y \colon H_S \to \mathcal{I} y$ such that $q_y(\gamma_i) = \beta_i$ for all $i \in S$. The elements $\{\zeta_k\} = V_k y$ defined at (1.10) belong to the group $\mathcal{I} y$ and pairwise commute because $\mathcal{I}$ is abelian. For $k \le t$, we have $\zeta_k^{O_k} \in V_k^{O_k}$ by (1.10), and $V_k^{O_k} \subseteq G^{(0)}$ by definition of O_k, so $\zeta_k^{O_k} = y$ is the identity element of $\mathcal{I} y$. So the universal property of H_S yields a homomorphism $q_y \colon H_S \to \mathcal{I} y$ such that $q_y(\eta_k) = \zeta_k$ for all $k \le t + f$. Using (1.3) at the first equality and (1.11) at the third, we see that, for each $i \in S$,

$$q_y(\gamma_i) = \prod_{k=1}^{t+f} q_y(\eta_k)^{b(i,k)} = \prod_{k=1}^{t+f} \zeta_k^{b(i,k)} = \beta_i. \qquad \square$$

Corollary 1.5. *Suppose that $\{C_i : i \in I_0\}$ is a finite family of open bisections and that $x \in \bigcap_{i \in I_0} s(C_i)$. For each i, let γ_i denote the unique element of $C_i x$, and suppose that $\gamma_i \in \operatorname{Iso}(G)$ for all $i \in I_0$. Then there exists a neighbourhood U_0 of x such that*

(i) *for each $y \in U_0$, the set $\{\gamma_i : C_i \cap \mathcal{I} y \neq \emptyset\}$ is a commutative subset of G and the subgroup K_y of $\operatorname{Iso}(G)x$ that it generates is contained in $\overline{\mathcal{I}}x$;*

(ii) *for any $y \in U_0$, any $j \in I_0$, any nonemtpy $S \subseteq \{i \in I_0 : C_i \cap \mathcal{I} y \neq \emptyset\}$, and any integers $(m_i)_{i\in S}$ such that $\prod_{i\in S} \gamma_i^{m_i} = \gamma_j$, we have $\prod_{i \in S} (C_i \cap \mathcal{I})^{m_i} U_0 \subseteq C_j$; and*

(iii) *for each $y \in U_0$, there is a homomorphism q_y from the subgroup K_y of* $\mathrm{Iso}(G)x$ *generated by $\{\gamma_i : C_i \cap \mathcal{I}y \neq \emptyset\}$ to $\mathcal{I}y$ such that for each $i \in I_0$ satisfying $C_i \cap \mathcal{I}y \neq \emptyset$, we have $q_y(\gamma_i) \in C_i y$.*

Proof. Fix a subset S of I_0. If there exists a neighbourhood W of x such that for every $y \in W$, there exist $i \in S$ such that $C_i \cap \mathcal{I}y = \emptyset$, let V_S be such a neighbourhood; otherwise put $V_S = G^{(0)}$. Let $V := \bigcap_{S \subseteq I_0} V_S$. Then V is an open neighbourhood of x. For $y \in V$, let

$$S_y := \{i \in I_0 : C_i \cap \mathcal{I}y \neq \emptyset\}.$$

By definition of the sets V_S,

(1.13) if $y \in V$ and $S \subseteq S_y$, then every neighbourhood of x contains a point z such that $C_i \cap \mathcal{I}z \neq \emptyset$ for all $i \in S$.

This implies that for each $y \in V$ with $S_y \neq \emptyset$, there is a net $(z_\lambda)_{\lambda \in \Lambda_y}$ converging to x such that $C_i \cap \mathcal{I}z_\lambda \neq \emptyset$ for all $i \in S_y$ and all $\lambda \in \Lambda_y$; for each $i \in S_y$ and $\lambda \in \Lambda_y$ we define $\beta_{i,\lambda}$ be the unique element of $C_i \cap \mathcal{I}z_\lambda$. Since each C_i is a bisection and hence each $s|_{C_i}$ is a homeomorphism, for each $i \in S_y$,

$$C_i \cap \mathcal{I}z_\lambda \ni \beta_{i,\lambda} \to \gamma_i. \tag{1.14}$$

We claim that V satisfies (i); it then follows that any open subset U_0 of V also satisfies (i). Fix $y \in V$; we claim that $\{\gamma_i : i \in S_y\}$ is a set of pairwise commuting elements of G. If S_y is empty, the claim is vacuous, so suppose that $S_y \neq \emptyset$ and fix $i, j \in S_y$. Then for each $\lambda \in \Lambda_y$ we have $\beta_{i,\lambda}, \beta_{j,\lambda} \in \mathcal{I}z_\lambda$, and in particular $\beta_{i,\lambda}\beta_{j,\lambda} = \beta_{j,\lambda}\beta_{i,\lambda}$ because $\mathcal{I}$ is abelian. So using (1.14) twice, we see that

$$\gamma_i\gamma_j = \lim_{\lambda \in \Lambda_y} \beta_{i,\lambda}\beta_{j,\lambda} = \lim_{\lambda \in \Lambda_y} \beta_{j,\lambda}\beta_{i,\lambda} = \gamma_j\gamma_i.$$

Hence $\{\gamma_i : i \in S_y\}$ is a set of pairwise commuting elements. For the remaining assertions in (i), suppose that γ belongs to K_y. Then γ has the form $\prod_{i \in S_y} \gamma_i^{m_i}$, and so $\lim_\lambda \prod_{i \in S_y} \beta_{i,\lambda}^{m_i} = \gamma$. Since $\mathcal{I}$ is closed under multiplication, we have $\prod_{i \in S_y} \beta_{i,\lambda}^{m_i} \in \mathcal{I}$ for all λ, and so $\gamma \in \overline{\mathcal{I}}x$. Hence V satisfies (i) as claimed.

Let

$$\mathcal{S} := \{S \subseteq I_0 : S \neq \emptyset \text{ and there exists } y \in V \text{ such that } S \subseteq S_y\}.$$

First suppose that $\mathcal{S} = \emptyset$. Then $S_y = \emptyset$ for all $y \in V$. Hence Condition (ii) is vacuously true for any $U_0 \subseteq V$. Thus Condition (iii) reduces to asserting the existence of the trivial homomorphism from $\{x\}$ to $\mathcal{I}y$, and so we are done. Now suppose that $\mathcal{S}$ is nonempty. For each $S \in \mathcal{S}$, since V satisfies Condition (i), we can apply Lemma 1.4 to the bisections $\{C_i V : i \in S\}$ to obtain a neighbourhood $U_S \subseteq V$ of x satisfying Conditions (1) and (2) of that result. Let $U_0 := \bigcap_{S \in \mathcal{S}} U_S$, which is an open neighbourhood of x.

Since U_0 is an open subset of V, we saw above that it satisfies Condition (i). To see that it satisfies Condition (ii), fix a point $y \in U_0$, an element $j \in I_0$, and a nonempty set $S \subseteq \{i \in I_0 : C_i \cap \mathcal{I}y \neq \emptyset\}$, and suppose that $(m_i)_{i \in S}$ are integers satisfying $\prod_{i \in S} \gamma_i^{m_i} = \gamma_j$. We must show that $\prod_{i \in S} (C_i \cap \mathcal{I})^{m_i} U_0 \subseteq C_j$.

Let $(z_\lambda)_{\lambda \in \Lambda_y}$ and $\beta_{i,\lambda} \in C_i \cap \mathcal{I}z_\lambda$ for $i \in S_y$ be as above. By (1.14), we have $\lim_\lambda \prod_{i \in S} \beta_{i,\lambda}^{m_i} = \prod_{i \in S} \gamma_i^{m_i} = \gamma_j$. Since C_j is a neighbourhood of γ_j, we have $\prod_{i \in S} \beta_{i,\lambda}^{m_i} \in C_j \cap \mathcal{I}z_\lambda$ for large λ. In particular, for large λ, the point $z_\lambda \in V$ satisfies $C_i \cap \mathcal{I}z_\lambda \neq \emptyset$ not only for $i \in S$ but also for $i = j$. Hence $S \cup \{j\} \in \mathcal{S}$. Since $U_0 \subseteq U_{S \cup \{j\}}$, Condition (1) of Lemma 1.4 for $S \cup \{j\}$ implies that $\prod_{i \in S} (C_i \cap \mathcal{I})^{m_i} U_0 \subseteq C_j$. Hence U_0 satisfies Condition (ii).

Finally, to see that U_0 satisfies Condition (iii), fix $y \in U_0 \subseteq U_{S_y}$. Condition (2) of Lemma 1.4 for S_y gives a homomorphism from $H_{S_y} = K_y \leq \overline{\mathcal{I}}x$ to $\mathcal{I}y$ with the desired property. $\square$

We will need a simple technical result regarding expressions for elements of $C_c(G)$ as sums of functions supported on bisections.

Lemma 1.6. *Let G be an étale groupoid (for this lemma, $\mathcal{I}$ need not be abelian) and fix $g \in C_c(G)$ and $x \in G^{(0)}$. There is a finite set $\mathcal{B}$ of open bisections of G and for each $B \in \mathcal{B}$ a function $g_B \in C_c(B) \subseteq C_c(G)$ such that $g = \sum_{B\in\mathcal{B}} g_B$ and such that if $B, B' \in \mathcal{B}$ satisfy $B \cap B' \cap Gx \neq \emptyset$, then $B = B'$.*

Proof. The standard partition-of-unity argument shows that there is a finite set $\mathcal{A}$ of open bisections and for each $A \in \mathcal{A}$ a function $f_A \in C_c(A)$ such that $g = \sum_{A\in\mathcal{A}} f_A$.

For each of the finitely many $\gamma \in \big(\bigcup \mathcal{A}\big)x$, let $\mathcal{A}_\gamma := \{A \in \mathcal{A} : \gamma \in A\}$. Then the set $B(\gamma) := \bigcap \mathcal{A}_\gamma$ is an open bisection containing γ. Fix $h_\gamma \in C_c(B(\gamma), [0, 1])$ such that h_γ is identically 1 on a neighbourhood of γ. Using $\cdot$ to denote pointwise multiplication in $C_c(G)$, define

$$g_{B(\gamma)} := \sum_{A\in\mathcal{A}_\gamma} h_\gamma \cdot f_A \in C_c(G).$$

For each $A \in \mathcal{A}_\gamma$, the set $B(A) := A \setminus (A \cap Gx) = A \setminus \{\gamma\}$ is an open bisection that does not meet Gx; we put $g_{B(A)} = (1 - h_\gamma) \cdot f_A$. Since h_γ is identically 1 on a neighbourhood of γ, each $g_{B(A)}$ vanishes on a neighbourhood of γ and so by choice of f_A, we have $g_{B(A)} \in C_c(B(A))$. By construction, $\sum_{A\in\mathcal{A}_\gamma} f_A = g_{B(\gamma)} + \sum_{A\in\mathcal{A}_\gamma} g_{B(A)}$. Since each $A \in \mathcal{A}$ is a bisection, the $\mathcal{A}_\gamma$ are mutually disjoint, so

$$\sum_{A\in\mathcal{A}, Ax\neq\emptyset} f_A = \sum_{\gamma\in(\bigcup\mathcal{A})x} \Big(\sum_{A\in\mathcal{A}_\gamma} f_A\Big) = \sum_{\gamma\in(\bigcup\mathcal{A})x} \Big(g_{B(\gamma)} + \sum_{A\in\mathcal{A}_\gamma} g_{B(A)}\Big).$$

For $A \in \mathcal{A}$ such that $Ax = \emptyset$, let $B(A) = A$ and $g_A = f_A$. Let $\mathcal{B} := \{B(\gamma) : \gamma \in \big(\bigcup \mathcal{A}\big)x\} \cup \{B(A) : A \in \mathcal{A}\}$. Since $\mathcal{A}$ is finite, so is $\mathcal{B}$. We have

$$g = \sum_{A\in\mathcal{A}} f_A = \Big(\sum_{A\in\mathcal{A}, Ax=\emptyset} f_A\Big) + \sum_{\gamma\in(\bigcup\mathcal{A})x} \Big(g_{B(\gamma)} + \sum_{A\in\mathcal{A}_\gamma} g_{B(A)}\Big) = \sum_{B\in\mathcal{B}} g_B.$$

Suppose that $B, B' \in \mathcal{B}$ satisfy $B \cap B' \cap Gx \neq \emptyset$. Then in particular $B \cap Gx$ and $B' \cap Gx$ are nonempty. By construction, $B(A) \cap Gx = \emptyset$ for all $A \in \mathcal{A}$, so there exist $\gamma, \gamma' \in \big(\bigcup \mathcal{A}\big)x$ such that $B = B(\gamma)$ and $B' = B(\gamma')$. Hence $B \cap Gx = \{\gamma\}$ and $B' \cap Gx = \{\gamma'\}$, so $B \cap B' \cap Gx \neq \emptyset$ forces $\gamma = \gamma'$ and thus $B = B(\gamma) = B(\gamma') = B'$. □

Lemma 1.7. *Fix $x \in G^{(0)}$, $g \in C_c(G)$ and $\varepsilon > 0$. There is a neighbourhood U of x such that for each $y \in U$ there are an abelian subgroup $K_y \leq \operatorname{Iso}(G)x$ with $K_y \subseteq \overline{\mathcal{I}}x$ and a homomorphism $q_y \colon K_y \to \mathcal{I}y$ such that*

(a) *$\mathcal{I}y \cap \operatorname{supp}^\circ(g) \subseteq q_y(K_y \cap \operatorname{supp}(g))$;*
(b) *q_y is injective on $K_y \cap \operatorname{supp}(g)$; and*
(c) *$|g(q_y(\gamma)) - g(\gamma)| < \varepsilon$ for all $\gamma \in K_y \cap \operatorname{supp}(g)$.*

Proof. Since $g \in C_c(G)$, Lemma 1.6 implies that we can write $g = \sum_{i=1}^n g_i$ so that for each i there is an open bisection C_i such that $g_i \in C_c(C_i)$, and so that if $C_i \cap C_j \cap Gx \neq \emptyset$, then $i = j$.

Let

$$I_0 := \{i \leq n : \operatorname{supp}(g_i) \cap \overline{\mathcal{I}}x \neq \emptyset\}.$$

Then for $i \in I_0$, since $\operatorname{supp}(g_i) \subseteq C_i$, it follows that $C_i \cap \overline{\mathcal{I}}x \neq \emptyset$. Since C_i is a bisection, $C_i x$ is a singleton; we write γ_i for the unique element of $\operatorname{supp}(g_i)x \subseteq C_i x$. By choice of the C_i and g_i in the preceding paragraph, the map $i \mapsto \gamma_i$ is an injection from I_0 to $\overline{\mathcal{I}}x$.

Since G is Hausdorff, the γ_i can be separated by mutually disjoint open neighbourhoods. For each i, sets of the form $C_i V$, where V is an open neighbourhood of $s(\gamma_i) = x$ are a neighbourhood base at γ_i. Since $i \mapsto \gamma_i$ is injective, it follows that there is a neighbourhood W of x such that the bisections $C_iW, i \in I_0$ are mutually disjoint and such that whenever $i \in I_0$ satisfies $g_i(\gamma_i) \neq 0$ we have $0 \notin g_i(C_iW)$. Since $g = \sum_{i=1}^n g_i$ and $\{\gamma_i\} = \operatorname{supp}(g_i)x \subseteq C_i$, this implies in particular that

$$\gamma_i \in \operatorname{supp}(g) \text{ for all } i \in I_0. \tag{1.15}$$

Corollary 1.5 applied to these bisections gives a neighbourhood $U_0 \subseteq W$ of x satisfying Conditions (i), (ii), (iii) of that corollary.

For each $i \in I_0$, since g_i is continuous and since $s\colon C_i \to s(C_i) \subseteq G^{(0)}$ is a homeomorphism, there is a neighbourhood U_i of x such that $\sup_{\eta\in C_iU_i} |g_i(\eta) - g_i(\gamma_i)| < \varepsilon$. Let

$$U_{I_0} := \bigcap_{i\in I_0} U_i.$$

Then U_{I_0} is an open neighbourhood of x satisfying

$$\sup_{\eta\in C_iU_{I_0}} |g_i(\eta) - g_i(\gamma_i)| < \varepsilon \text{ for every } i \in I_0. \tag{1.16}$$

Fix $i \in \{1, \ldots, n\}\setminus I_0$. We claim that there is a neighbourhood U_i of x such that $\operatorname{supp}^\circ(g_i)U_i \cap \mathcal{I} = \emptyset$. To see this, consider two cases: either $x \in s(\operatorname{supp}(g_i))$ or not. If $x \in s(\operatorname{supp}(g_i))$, let γ be the unique element of $\operatorname{supp}(g_i) \subseteq C_i$ such that $s(\gamma) = x$. Since $\operatorname{supp}(g_i) \cap \overline{\mathcal{I}}x = \emptyset$, we have $\gamma \notin \overline{\mathcal{I}}x$. Hence there is an open set $Z \subseteq G$ containing γ such that $Z \cap \overline{\mathcal{I}} = \emptyset$. Let $U_i := s(Z\cap C_i)$. This is an open set because s is open, and we have $\operatorname{supp}^\circ(g_i)U_i \subseteq C_iU_i = Z\cap C_i$ because C_i is a bisection. Since $Z\cap\mathcal{I} \subseteq Z\cap\overline{\mathcal{I}} = \emptyset$, this set U_i does the job. Now suppose that $x \notin s(\operatorname{supp}(g_i))$. Since $s|_{C_i}$ is a homeomorphism and $\operatorname{supp}(g_i) \subseteq C_i$, $s(\operatorname{supp}(g_i))$ is closed. So there is a neighbourhood U_i of x such that $U_i \cap s(\operatorname{supp}(g_i)) = \emptyset$. So $\operatorname{supp}^\circ(g_i)U_i$ is empty, and in particular does not intersect $\mathcal{I}$. This proves the claim.

Let

$$U_{I_0^c} := \bigcap_{i\in\{1,\ldots,n\}\setminus I_0} U_i.$$

Then $U_{I_0^c}$ is an open neighbourhood of x (we use the convention that $U_{I_0^c} = G^{(0)}$ if $\{1, \ldots, n\} \setminus I_0 = \emptyset$). So for all $i \in \{1, \ldots, n\} \setminus I_0$, we have $\operatorname{supp}^\circ(g_i)U_{I_0^c} \cap \mathcal{I} = \emptyset$.

Now let

$$U := U_0 \cap U_{I_0} \cap U_{I_0^c}.$$

We show that this U has the desired properties. Fix $y \in U$. Let

$$S := \{i \in I_0 : C_i \cap \mathcal{I}y \neq \emptyset\},$$

and let K_y be the subgroup of $\operatorname{Iso}(G)x$ generated by $\{\gamma_i : i \in S\}$. Since $U \subseteq U_0$, Condition (i) of Corollary 1.5 guarantees that K_y is an abelian subgroup of $\operatorname{Iso}(G)x$ and is contained in $\overline{\mathcal{I}}x$, and Condition (iii) of Corollary 1.5 gives a homomorphism $q_y\colon K_y \to \mathcal{I}y$ such that $q_y(\gamma_i) \in C_iy$ for all $i \in S$.

For Property (a), suppose that $\beta \in \mathcal{I}y$ satisfies $g(\beta) \neq 0$. Then $g_i(\beta) \neq 0$ for some $i \in \{1, \ldots, n\}$. As $y \in U \subseteq U_{I_0^c} \subseteq U_i$, this means $\{\beta\} = \operatorname{supp}^\circ(g_i)U_i \cap \mathcal{I}y \subseteq C_i \cap \mathcal{I}y$. By our choice of the sets U_j for $j \in \{1, \ldots, n\} \setminus I_0$, this implies that $i \in I_0$ and hence $i \in S$. Thus, γ_i is one of the generators of K_y. Since $q_y(\gamma_i)$ is the unique element of C_iy, it must coincide with β, and we deduce with (1.15) that $q_y(\gamma_i) \in q_y(K_y \cap \operatorname{supp}(g))$ as required.

For (b), note first that this is trivial if $S = \emptyset$, since then the set $K_y \cap \operatorname{supp}(g)$ is at most a singleton. We may therefore assume that $S \neq \emptyset$.

We claim that, if $i \in I_0$ satisfies $\gamma_i \in K_y$, then $q_y(\gamma_i) \in C_i y$ (so in particular $i \in S$). To see this, write γ_i as a product of the generators of K_y; that is, fix integers $m_l, l \in S$ such that $\gamma_i = \prod_{l \in S} \gamma_l^{m_l}$. By construction of S, we may apply Condition (ii) of Corollary 1.5 to it:

$$\begin{aligned}
\mathcal{I}y \ni q_y(\gamma_i) &= \prod_{l \in S} q_y(\gamma_l)^{m_l} && \text{by choice of } (m_l)_l \\
&\in \prod_{l \in S} (C_l \cap \mathcal{I}y)^{m_l} && \text{by Corollary 1.5(iii) and choice of } S \\
&\subseteq \prod_{l \in S} (C_l \cap \mathcal{I})^{m_l} U_0 && \text{as } y \in U \subseteq U_0 \\
&\subseteq C_i && \text{by Corollary 1.5(ii) and choice of } (m_l)_l.
\end{aligned}$$

This establishes the claim.

Now, suppose that γ, γ' are distinct elements of $K_y \cap \operatorname{supp}(g)$. Since $K_y \subseteq \overline{\mathcal{I}}x$ and $\operatorname{supp}(g) \subseteq \bigcup_{i \in \{1,\dots,n\}} C_i$, it follows from the opening paragraph of the proof that $\gamma = \gamma_i$ and $\gamma' = \gamma_j$ for distinct $i, j \in I_0$. By the above argument, we have $q_y(\gamma) \in C_i y$ and $q_y(\gamma') \in C_j y$. Since we chose the neighborhood W of x so that the bisections $C_i W$, $i \in I_0$, are mutually disjoint (page 8) and since $y \in U \subset U_0 \subseteq W$, we have $C_i y \cap C_j y = \emptyset$. In other words, $q_y(\gamma) \neq q_y(\gamma')$, as claimed.

It remains to establish (c). Suppose that $\gamma \in K_y \cap \operatorname{supp}(g)$. As in the argument for (b) above, $\gamma = \gamma_j \in C_j$ for some $j \in I_0$, and the claim above shows that $q_y(\gamma) \in C_j y$. By our choice of the sets U_i for $i \notin I_0$ and since $y \in U \subseteq U_i$, we have $g(q_y(\gamma)) = \sum_{i=1}^n g_i(q_y(\gamma)) = \sum_{i \in I_0} g_i(q_y(\gamma))$. As in the preceding paragraph, $C_i U_0 \cap C_j U_0 = \emptyset$ for $i \in I_0 \setminus \{j\}$, so $q_y(\gamma) \notin C_i$ for $i \in I_0 \setminus \{j\}$, and hence $g_i(q_y(\gamma)) = 0$ for $i \in I_0 \setminus \{j\}$. Hence $g(q_y(\gamma)) = g_j(q_y(\gamma))$. In the case $y = x$, so that $q_y = \mathrm{id}$, this proves $g(\gamma) = g_j(\gamma)$. Since $s(q_y(\gamma)) = y \in U \subseteq U_{I_0}$, we deduce from (1.16) that $|g(q_y(\gamma)) - g(\gamma)| = |g_j(q_y(\gamma_j)) - g_j(\gamma_j)| < \varepsilon$. □

Lemma 1.8. *For $y \in G^{(0)}$, let $\lambda_y \colon C_r^*(G) \to \mathcal{B}(\ell^2(Gy))$ be the left regular representation, let $P_{\mathcal{I}y}$ be the orthogonal projection of $\ell^2(Gy)$ onto $\ell^2(\mathcal{I}y)$, and let $\Phi_y \colon C_r^*(G) \to \mathcal{B}(\ell^2(\mathcal{I}y))$ be the map*

$$\Phi_y(a) = P_{\mathcal{I}y} \lambda_y(a) P_{\mathcal{I}y}. \tag{1.17}$$

Fix $x \in G^{(0)}$ and let Q be the orthogonal projection of $\ell^2(Gx)$ onto $\ell^2(\overline{\mathcal{I}}x)$. Then for each $g \in C_c(G)$ and each $\varepsilon > 0$, there is a neighbourhood U of x such that $\|\Phi_y(g)\| \leq \|Q\lambda_x(g)Q\| + \varepsilon$ for all $y \in U$.

Proof. Fix $g \in C_c(G)$ and $\varepsilon > 0$. Let $N := |\overline{\mathcal{I}}x \cap \operatorname{supp}(g)|$. By Lemma 1.7, there exists a neighbourhood U of x such that for each $y \in U$ there are a subgroup K_y of $\operatorname{Iso}(G)x$ contained in $\overline{\mathcal{I}}x$ and a homomorphism $q_y \colon K_y \to \mathcal{I}y$ such that

(a) $\mathcal{I}y \cap \operatorname{supp}^\circ(g) \subseteq q_y(K_y \cap \operatorname{supp}(g))$;
(b) q_y is injective on $K_y \cap \operatorname{supp}(g)$; and
(c) $|g(q_y(\gamma)) - g(\gamma)| < \varepsilon/(N+1)$ for all $\gamma \in K_y \cap \operatorname{supp}(g)$.

Fix $y \in U$. We must show that $\|\Phi_y(g)\| \leq \|Q\lambda_x(g)Q\| + \varepsilon$. Writing $\lambda_\beta^{\mathcal{I}y}$ for the unitaries in the regular representation of $\mathcal{I}y$, we can write Φ_y as

$$\Phi_y(g) = \sum_{\beta \in \mathcal{I}y} g(\beta) \lambda_\beta^{\mathcal{I}y}.$$

In this proof, we denote the universal unitary representations of the groups K_y, $\mathcal{I}y$ and $q_y(K_y) \subseteq \mathcal{I}_y$ respectively by

$$K_y \ni \gamma \mapsto W_\gamma^{K_y} \in C^*(K_y), \qquad \mathcal{I}_y \ni \beta \mapsto W_\beta^{\mathcal{I}y} \in C^*(\mathcal{I}_y), \qquad \text{and}$$
$$q_y(K_y) \ni \beta \mapsto W_\beta^{q_y(K_y)} \in C^*(q_y(K_y)),$$

and we denote the regular representation of K_y by

$$K_y \ni \gamma \mapsto \lambda^{K_y} \in \mathcal{B}(\ell^2(K_y)).$$

Since $\mathcal{I}y$ is abelian and hence amenable,

$$\Big\| \sum_{\beta\in\mathcal{I}y} g(\beta)\lambda_\beta^{\mathcal{I}y} \Big\| = \Big\| \sum_{\beta\in\mathcal{I}y} g(\beta) W_\beta^{\mathcal{I}y} \Big\|. \tag{1.18}$$

Using that $g(\beta) = 0$ for $\beta \in \mathcal{I}y \setminus q_y(K_y \cap \operatorname{supp}(g))$ by Property (a), we see that

$$\Big\| \sum_{\beta\in\mathcal{I}y} g(\beta) W_\beta^{\mathcal{I}y} \Big\| = \Big\| \sum_{\beta\in q_y(K_y\cap\operatorname{supp}(g))} g(\beta) W_\beta^{\mathcal{I}y} \Big\|. \tag{1.19}$$

Since the inclusion $q_y(K_y) \hookrightarrow \mathcal{I}y$ of groups induces an injective, and hence isometric, homomorphism of full C^*-algebras $C^*(q_y(K_y)) \hookrightarrow C^*(\mathcal{I}y)$, we can rewrite the right-hand side,

$$\Big\| \sum_{\beta\in q_y(K_y\cap\operatorname{supp}(g))} g(\beta) W_\beta^{\mathcal{I}y} \Big\| = \Big\| \sum_{\beta\in q_y(K_y\cap\operatorname{supp}(g))} g(\beta) W_\beta^{q_y(K_y)} \Big\|. \tag{1.20}$$

Using (1.18), (1.19) and (1.20) at the first step, and then Property (b) of the homomorphism q_y at the second, we see that

$$\|\Phi_y(g)\| = \Big\| \sum_{\beta\in q_y(K_y\cap\operatorname{supp}(g))} g(\beta) W_\beta^{q_y(K_y)} \Big\| = \Big\| \sum_{\gamma\in K_y\cap\operatorname{supp}(g)} g(q_y(\gamma)) W_{q_y(\gamma)}^{q_y(K_y)} \Big\|.$$

Writing each $g(q_y(\gamma))W_{q_y(\gamma)}^{q_y(K_y)} = g(\gamma)W_{q_y(\gamma)}^{q_y(K_y)} + \big(-g(\gamma) + g(q_y(\gamma))\big)W_{q_y(\gamma)}^{q_y(K_y)}$ and then applying the triangle inequality, we obtain

$$\|\Phi_y(g)\| \le \Big\| \sum_{\gamma\in K_y\cap\operatorname{supp}(g)} g(\gamma) W_{q_y(\gamma)}^{q_y(K_y)} \Big\| + \sum_{\gamma\in K_y\cap\operatorname{supp}(g)} |g(\gamma) - g(q_y(\gamma))|.$$

The homomorphism $q_y \colon K_y \to q_y(K_y)$ induces a homomorphism $\pi_y \colon C^*(K_y) \to C^*(q_y(K_y))$ such that $\pi_y(W_\gamma^{K_y}) = W_{q_y(\gamma)}^{q_y(K_y)}$ for all $\gamma \in K_y$. This combined with Property (c) yields

$$\|\Phi_y(g)\| \le \Big\| \pi_y\Big(\sum_{\gamma\in K_y\cap\operatorname{supp}(g)} g(\gamma) W_\gamma^{K_y} \Big) \Big\| + \sum_{\gamma\in K_y\cap\operatorname{supp}(g)} \varepsilon/(N+1).$$

Since π_y is a C^*-homomorphism, it is norm-decreasing, and so, since $N = |\overline{\mathcal{I}}x \cap \operatorname{supp}(g)| \ge |K_y \cap \operatorname{supp}(g)|$, we obtain

$$\|\Phi_y(g)\| < \Big\| \sum_{\gamma\in K_y\cap\operatorname{supp}(g)} g(\gamma) W_\gamma^{K_y} \Big\| + \varepsilon = \Big\| \sum_{\gamma\in K_y} g(\gamma) W_\gamma^{K_y} \Big\| + \varepsilon. \tag{1.21}$$

Since K_y is abelian and hence amenable,

$$\Big\| \sum_{\gamma\in K_y} g(\gamma) W_\gamma^{K_y} \Big\| = \Big\| \sum_{\gamma\in K_y} g(\gamma) \lambda_\gamma^{K_y} \Big\|. \tag{1.22}$$

Let Q_{K_y} be the orthogonal projection of $\ell^2(Gx)$ onto $\ell^2(K_y)$. Recall that λ_x is the regular representation of $C^*(G)$ on $\ell^2(Gx)$. We claim that

$$\sum_{\gamma\in K_y} g(\gamma)\lambda^{K_y}_\gamma = Q_{K_y}\lambda_x(g)Q_{K_y}, \tag{1.23}$$

regarded as operators on $\ell^2(K_y)$. To see this, we fix $\eta \in K_y$ (so that e_η is a typical basis element for $\ell^2(K_y)$), and show that $\sum_{\gamma\in K_y} g(\gamma)\lambda^{K_y}_\gamma e_\eta = Q_{K_y}\lambda_x(g)Q_{K_y}e_\eta$. We calculate

$$\begin{aligned}
\big(Q_{K_y}\lambda_x(g)Q_{K_y}\big)e_\eta &= Q_{K_y}\lambda_x(g)e_\eta && \text{as } Q_{K_y}e_\eta = e_\eta\\
&= \sum_{\gamma\in G_{r(\eta)}} g(\gamma)Q_{K_y}e_{\gamma\eta} && \text{by definition of } \lambda_x\\
&= \sum_{\gamma\in K_y} g(\gamma)e_{\gamma\eta} && \text{as } K_y \le G\text{, and by defnition of } Q_{K_y}\\
&= \sum_{\gamma\in K_y} g(\gamma)\lambda^{K_y}_\gamma e_\eta && \text{by definition of } \lambda^{K_y},
\end{aligned}$$

estabishing (1.23). Recall that $Q\colon \ell^2(Gx) \to \ell^2(\overline{\mathcal{I}}x)$ is the orthogonal projection. Since $\ell^2(K_y) \subseteq \ell^2(\overline{\mathcal{I}}x)$, we have $Q_{K_y} \le Q$, and so (1.22) gives

$$\begin{aligned}
\Big\|\sum_{\gamma\in K_y} g(\gamma)W^{K_y}_\gamma\Big\| &\overset{(1.22)}{=} \Big\|\sum_{\gamma\in K_y} g(\gamma)\lambda^{K_y}_\gamma\Big\|\\
&\overset{(1.23)}{=} \Big\|Q_{K_y}\lambda_x(g)Q_{K_y}\Big\| = \Big\|Q_{K_y}Q\lambda_x(g)QQ_{K_y}\Big\| \le \Big\|Q\lambda_x(g)Q\Big\|.
\end{aligned}$$

Combining this with (1.21) yields the desired estimate $\|\Phi_y(g)\| \le \|Q\lambda_x(g)Q\| + \varepsilon$. □

Proof of Theorem 1.3. Fix $a \in C^*_r(G)$ such that the open support of $j(a)$ is contained in $\mathcal{I}$ and $j(a)|_{\mathcal{I}x} = 0$. We prove that for every $\varepsilon > 0$ there exists $h \in C_0(G^{(0)})$ such that $h(x) = 0$ and $\|a - ah\| \le \varepsilon$; since each $ah \in J_x$ this proves that $a \in J_x$.

So fix $\varepsilon > 0$. Since $C_c(G)$ is dense in $C^*_r(G)$, there exists $g \in C_c(G)$ such that

$$\|g - a\| < \varepsilon/4. \tag{1.24}$$

Let $Q\colon \ell^2(Gx) \to \ell^2(\overline{\mathcal{I}}x)$ be the orthogonal projection, and let Φ_y be as defined in Equation (1.17). By Lemma 1.8 there is a neighbourhood U of x such that

$$\sup_{y\in U} \|\Phi_y(g)\| < \|Q\lambda_x(g)Q\| + \varepsilon/4. \tag{1.25}$$

Since $j(a)$ is identically zero on $\mathcal{I}x \cup G \setminus \mathcal{I}$ by assumption, we have $j(a)(\gamma) = 0$ for all $\gamma \in Gx$. Hence, for $\gamma, \eta \in \overline{\mathcal{I}}x$, since $\gamma\eta^{-1} \in Gx$, we have $(\lambda_x(a)e_\gamma \mid e_\eta) = j(a)(\gamma\eta^{-1}) = 0$. Hence

$$(Q\lambda_x(a)Qe_\gamma \mid e_\eta)_{\ell^2(\overline{\mathcal{I}}x)} = (\lambda_x(a)Qe_\gamma \mid Qe_\eta)_{\ell^2(Gx)} = 0.$$

Hence $Q\lambda_x(a)Qe_\gamma = 0$ for each basis element e_γ, so $Q\lambda_x(a)Q = 0$. It follows from (1.24) that $\varepsilon/4 > \|Q\lambda_x(g)Q - Q\lambda_x(a)Q\| = \|Q\lambda_x(g)Q\|$, and thus by (1.25),

$$\sup_{y\in U} \|\Phi_y(g)\| < \varepsilon/2. \tag{1.26}$$

If $a \in A$ is nonzero, then $j(a)(\gamma) \ne 0$ for some $\gamma \in \mathcal{I}$, and hence $(\Phi_{s(\gamma)}(a)e_{s(\gamma)} \mid e_\gamma) = j(a)(\gamma)$ is nonzero. So $\bigoplus_{y\in G^{(0)}} \Phi_y$ is injective on A and hence

$$\Big\|\bigoplus_{y\in G^{(0)}} \Phi_y(b)\Big\| = \|b\| \qquad \text{for all } b \in A. \tag{1.27}$$

Fix $h \in C_0(G^{(0)}, [0,1])$ such that $h(x) = 0$ and $h(y) = 1$ for all $y \in s(\mathrm{supp}(g)) \setminus U$. for $y \in G^{(0)}$ and $\eta \in \mathcal{I}_y$, we have

$$\begin{aligned}\Phi_y(g(1-h))e_\eta &= \sum_{\gamma \in Gy} g(\gamma)(1-h(s(\gamma)))P_{\mathcal{I}y}e_{\gamma\eta} \\ &= (1-h)(y)\sum_{\gamma \in Gy} g(\gamma)P_{\mathcal{I}y}e_{\gamma\eta} = (1-h)(y)\Phi_y(g)e_\eta.\end{aligned}$$

Consequently,

$$\Phi_y(g(1-h)) = (1-h)(y)\Phi_y(g) \quad \text{for all } y \in G^{(0)}. \tag{1.28}$$

Using (1.27), we have

$$\|a - ah\| = \Big\|\Big(\bigoplus_{y \in G^{(0)}} \Phi_y\Big)(a - ah)\Big\| = \sup_{y \in G^{(0)}} \|\Phi_y(a - g + g - gh + gh - ah)\|.$$

Applying the triangle inequality and then (1.24) and that $\|h\| \leq 1$, we obtain

$$\begin{aligned}\|a - ah\| &\leq \sup_{y \in G^{(0)}} \|\Phi_y(a-g)\| + \|\Phi_y(g - gh)\| + \|\Phi_y(a-g)\|\|\Phi_y(h)\| \\ &\leq \varepsilon/4 + \sup_{y \in G^{(0)}} \|\Phi_y(g(1-h))\| + \varepsilon/4.\end{aligned}$$

Using that $\Phi_y(g) = 0$ for $y \notin s(\mathrm{supp}(g))$ together with (1.28) and that $1-h$ vanishes off U, we obtain

$$\|a - ah\| \leq \sup_{y \in U} \|\Phi_y(g)\| + \varepsilon/2,$$

and so (1.26) gives $\|a - ah\| < \varepsilon$. □

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