

CARTAN SUBALGEBRAS IN C^* -ALGEBRAS OF HAUSDORFF ÉTALE GROUPOIDS

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ABSTRACT. The reduced C^* -algebra of the interior of the isotropy in any Hausdorff étale groupoid G embeds as a C^* -subalgebra M of the reduced C^* -algebra of G . We prove that the set of pure states of M with unique extension is dense, and deduce that any representation of the reduced C^* -algebra of G that is injective on M is faithful. We prove that there is a conditional expectation from the reduced C^* -algebra of G onto M if and only if the interior of the isotropy in G is closed. Using this, we prove that when the interior of the isotropy is abelian and closed, M is a Cartan subalgebra. We prove that for a large class of groupoids G with abelian isotropy—including all Deaconu–Renault groupoids associated to discrete abelian groups— M is a maximal abelian subalgebra. In the specific case of k -graph groupoids, we deduce that M is always maximal abelian, but show by example that it is not always Cartan.

1. INTRODUCTION

A key tool in the study of graph C^* -algebras and their analogues is the Cuntz–Krieger uniqueness theorem [7, 13]. This result says that if all the cycles in a graph E have an entrance, then any representation of the associated C^* -algebra which is nonzero on all of the generating projections associated to vertices of the graph is faithful. There are numerous ways to prove this theorem. But the key to the argument in each case is showing that any element of the graph C^* -algebra can be compressed to an element close to its canonical abelian subalgebra, and this process is faithful on positive elements. This compression property is reminiscent of Anderson’s study [1] of the state-extension property for inclusions of C^* -algebras.

In the case of graph algebras, the condition on a graph that every cycle has an entrance is equivalent to the condition that the associated groupoid is topologically

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principal. (It is worth pointing out that this is in fact how the Cuntz–Krieger uniqueness theorem was originally proved [14, Theorem 3.7].) It follows from Renault’s work in [20] that this in turn is equivalent to the condition that there is a dense set of units of the groupoid G for which the associated pure state of $C(G^{(0)})$ has unique extension to $C_r^*(G)$. It also follows from Renault’s work that when G is topologically principal $C_0(G^{(0)})$ is a maximal abelian subalgebra—indeed, a Cartan subalgebra—of $C^*(G)$.

This analysis fails if E contains cycles with no entrance. Szymański showed in [24] that to verify faithfulness of a representation π of $C^*(E)$, in addition to checking that each $\pi(p_v)$ is nonzero, one must check that $\pi(s_\mu)$ has full spectrum for every cycle μ with no entrance. The second and third authors systematised and generalised Szymański’s analysis in [16, 17]. They introduced the notion of a pseudo-diagonal M of a C^* -algebra A and showed that representations of A that are faithful on M are automatically faithful on A . They then showed that the subalgebra of a graph C^* -algebra generated by the usual abelian subalgebra and the elements $\{s_\mu : \mu \text{ is a cycle with no entrance}\}$ is a pseudo-diagonal, recovering Szymański’s result.

The first three authors considered the extension of this analysis to C^* -algebras of higher-rank graphs in [5]. They considered an abelian subalgebra M of the k -graph algebra $C^*(\Lambda)$ spanned by partial unitaries of the form $s_\mu s_\nu^*$, and identified it as the completion in the associated groupoid C^* -algebra of the functions supported on the interior of its isotropy. By careful analysis of the set of states of M with unique extension to $C^*(\Lambda)$, they proved that every representation of $C^*(\Lambda)$ that is injective on M is faithful, without proving that M was either maximal abelian or the range of a faithful conditional expectation from $C^*(\Lambda)$. They left open the natural question as to whether M is in fact a pseudo-diagonal in the sense of [17].

Here we answer a more general question about a canonical subalgebra of the reduced C^* -algebra of a Hausdorff étale groupoid G . The reduced C^* -algebra of the interior $\text{Iso}(G)^\circ$ of the isotropy in G embeds as a subalgebra M_r of $C_r^*(G)$. We show that the set of pure states of M_r with unique extension to $C_r^*(G)$ is dense in the set of all pure states of M_r . We conclude from this that any representation of $C_r^*(G)$ that is injective on M_r is faithful. This generalises the result from [5] discussed in the preceding paragraph. Along the way we show that commutativity of the subalgebra can be dropped from the hypotheses of the abstract uniqueness theorem of [5].

We then turn our attention to deciding when M_r is a Cartan subalgebra in the sense of [20], and when it is a pseudodiagonal in the sense of [17]. It turns out that the two are equivalent and hold precisely when there is a conditional expectation $\Psi : C_r^*(G) \rightarrow M_r$, and M_r is maximal abelian. We prove that if $\text{Iso}(G)^\circ$ is closed in G , then the map $f \mapsto f|_{\text{Iso}(G)^\circ}$ from $C_c(G)$ to $C_c(\text{Iso}(G)^\circ)$ extends to a faithful conditional expectation $\Psi : C_r^*(G) \rightarrow M_r$; and conversely, if $\text{Iso}(G)^\circ$ is not closed in G then there does not exist a faithful conditional expectation on $C_r^*(G)$ with range M_r .

To address the question of when M_r is maximal abelian, we restrict our attention to groupoids G such that $\text{Iso}(G)^\circ$ is abelian (that is, a bundle of abelian groups). Renault shows in [19, Proposition II.4.2(i)] that the identity map on $C_c(G)$ extends to a continuous injection $j : C_r^*(G) \rightarrow C_0(G)$. We show that M_r is maximal abelian if and only if it is equal to the set of elements $a \in C_r^*(G)$ such that $j(a) \in C_0(\text{Iso}(G)^\circ)$. We then establish two sufficient conditions under which M_r is maximal abelian: (a) that the interior of the isotropy is closed; or (b) that there is a continuous 1-cocycle from G into an abelian group H that is injective on each fibre of the isotropy of G . It follows from this and our previous result that whenever $\text{Iso}(G)^\circ$ is abelian and closed, M_r is a Cartan subalgebra and a pseudodiagonal. It also follows that if G is a Deaconu–Renault groupoid associated to an action of a discrete abelian group, and if the interior of the isotropy is not closed in G , then $M_r \subseteq C_r^*(G)$ is a maximal abelian subalgebra of $C_r^*(G)$ but is not Cartan or a pseudodiagonal because it is not the range of a conditional expectation.

Specialising to k -graphs we are able to answer the questions left open in [17]: given a k -graph Λ , the subalgebra M of $C^*(\Lambda)$ described in [17] is always a maximal abelian subalgebra, but is a Cartan subalgebra only under the additional hypothesis that the interior of the isotropy in the k -graph groupoid is closed. We prove by example that the latter is not automatic. We also tie up a loose end by proving that if D is a Cartan subalgebra of any C^* -algebra A , then D is a pseudodiagonal as well.

The paper is organised as follows. After a short preliminaries section to establish notation, we break our analysis up into two sections. In Section 3 we prove our main uniqueness result about the reduced C^* -algebra of a Hausdorff étale groupoid in terms of the subalgebra M_r corresponding to the interior of its isotropy. The results in this section do not require $\text{Iso}(G)^\circ$ to be abelian or closed. We have tried to be explicit about which parts of our results apply to full C^* -algebras, and in particular what additional consequences follow from amenability of G or of $\text{Iso}(G)^\circ$.

Section 4 then deals with the questions of when there is a conditional expectation of $C_r^*(G)$ onto M_r , and when M_r is maximal abelian. We prove that $C_r^*(G)$ admits an expectation onto M_r if and only if $\text{Iso}(G)^\circ$ is closed in Proposition 4.1. We then restrict to the special case where $\text{Iso}(G)^\circ$ is abelian and hence also amenable by results of [21]. We establish our sufficient conditions for M_r to be maximal abelian in Theorem 4.3. We also discuss the consequences of our results for higher-rank-graph C^* -algebras, and provide an example of a 2-graph for which the interior of the isotropy in the associated groupoid is not closed. We finish the section by proving that every Cartan subalgebra is a pseudo-diagonal.

2. PRELIMINARIES

Throughout this paper, G will denote a locally compact second-countable Hausdorff groupoid which is étale in the sense that $r, s : G \rightarrow G^{(0)}$ are local homeomorphisms.

For subsets $A, B \subset G$, we write

$$AB := \{\alpha\beta \in G : (\alpha, \beta) \in (A \times B) \cap G^{(2)}\}.$$

We use the standard groupoid conventions that $G^u = r^{-1}(u)$, $G_u = s^{-1}(u)$, and $G_u^u = G^u \cap G_u$ for $u \in G^{(0)}$. For $K \subset G^{(0)}$, the restriction of G to K is the subgroupoid $G|_K = \{\gamma \in G : r(\gamma), s(\gamma) \in K\}$. We will be particularly interested in the *isotropy subgroupoid*

$$\text{Iso}(G) = \{\gamma \in G : r(\gamma) = s(\gamma)\} = \bigcup_{u \in G^{(0)}} G_u^u.$$

Note that $\text{Iso}(G)$ is closed in G as well as a group bundle over $G^{(0)}$.

The I -norm on $C_c(G)$ is defined by

$$\|f\|_I = \sup_{u \in G^{(0)}} \max \left\{ \sum_{\gamma \in G_u} |f(\gamma)|, \sum_{\gamma \in G^u} |f(\gamma)| \right\}.$$

The groupoid C^* -algebra $C^*(G)$ is the completion of $C_c(G)$ in the norm $\|a\| = \sup\{\pi(a) : \pi \text{ is an } I\text{-norm bounded } *\text{-representation}\}$. For $u \in G^{(0)}$ there is a representation $L^u : C^*(G) \rightarrow B(\ell^2(G_u))$ given by $L^u(f)\delta_\gamma = \sum_{s(\alpha)=r(\gamma)} f(\alpha)\delta_{\alpha\gamma}$. This is called the (left-)regular representation associated to u . The reduced groupoid C^* -algebra $C_r^*(G)$ is the image of $C^*(G)$ under $\bigoplus_{u \in G^{(0)}} L^u$.

A *bisection* in G , also known as a G -set, is a set $U \subset G$ such that r, s restrict to homeomorphisms on U . An important feature of étale groupoids is that they have plenty of open bisections: Proposition 3.5 of [9] together with local compactness implies that the topology on an étale groupoid has a basis consisting of precompact open bisections.

Because G is étale, there is a homomorphism $C_0(G^{(0)}) \hookrightarrow C^*(G)$ implemented on $C_c(G)$ by extension of functions by 0. We regard $C_0(G^{(0)})$ as a $*$ -subalgebra of $C^*(G)$. Since $G^{(0)}$ is closed, $f|_{G^{(0)}} \in C_c(G^{(0)})$ for all $f \in C_c(G)$. This map extends to a faithful conditional expectation $\Phi_r : C_r^*(G) \rightarrow C_0(G^{(0)})$ and a (not necessarily faithful) conditional expectation $\Phi : C^*(G) \rightarrow C_0(G^{(0)})$ (this is proved for principal groupoids in the final sentence of the proof of [19, Proposition II.4.8], and the same proof applies for non-principal groupoids).

We write $\text{Iso}(G)^\circ$ for the interior of $\text{Iso}(G)$ in G . Since G is étale, $G^{(0)} \subset \text{Iso}(G)^\circ$ and $\text{Iso}(G)^\circ$ is an open étale subgroupoid of G . We will need the following consequence of [23, Proposition 2.5].

Lemma 2.1 ([23, Proposition 2.5(b) and (c)]). *Suppose that G is a second-countable locally compact Hausdorff étale groupoid. For each $\gamma \in G$, the map $\alpha \mapsto \gamma\alpha\gamma^{-1}$ is a bijection from $\text{Iso}(G)_{s(\gamma)}^\circ$ onto $\text{Iso}(G)_{r(\gamma)}^\circ$. Each $\text{Iso}(G)_u^\circ$ is a normal subgroup of G_u^u .*

3. A UNIQUENESS THEOREM

The paper [5] presents a uniqueness theorem for the C^* -algebras $C^*(\Lambda)$ of k -graphs Λ that characterises injectivity of homomorphisms induced by the universal property. The hypotheses of this theorem are in terms of the abelian subalgebra M generated by elements $s_\mu s_\nu^*$ of $C^*(\Lambda)$ such that $\mu x = \nu x$ for every infinite path x of Λ . As discussed in Remark 4.11 of [5], M is the completion of $C_c(\text{Iso}(G_\Lambda)^\circ) \subseteq C_c(G_\Lambda)$, where G_Λ is the groupoid associated to Λ as in [12]. Here we use different methods to generalize the uniqueness theorem of [5] to the reduced C^* -algebras of Hausdorff étale groupoids. Our result characterises injectivity of homomorphisms of $C_r^*(G)$ in terms of injectivity of their restrictions to the canonical copy of $C_r^*(\text{Iso}(G)^\circ)$ in $C_r^*(G)$.

To see that such a copy exists, note that $\text{Iso}(G)^\circ$ is open in G so there is an injective $*$ -homomorphism $\iota : C_c(\text{Iso}(G)^\circ) \hookrightarrow C_c(G)$ given by extension by zero. Since ι is isometric for the respective I -norms, and by [18, Proposition 1.9], this map ι extends to inclusions

$$(3.1) \quad \iota : C^*(\text{Iso}(G)^\circ) \rightarrow C^*(G) \quad \text{and} \quad \iota_r : C_r^*(\text{Iso}(G)^\circ) \rightarrow C_r^*(G).$$

Theorem 3.1. *Let G be a locally compact Hausdorff étale groupoid. Let $M := \iota(C^*(\text{Iso}(G)^\circ)) \subseteq C^*(G)$ and $M_r := \iota_r(C_r^*(\text{Iso}(G)^\circ)) \subseteq C_r^*(G)$.*

- (a) *Suppose that $u \in G^{(0)}$ satisfies $G_u^u = \text{Iso}(G)_u^\circ$. If φ_r is a state of M_r that factors through $C_r^*(G_u^u)$, then φ_r extends uniquely to $C_r^*(G)$. If φ is a state of M that factors through $C^*(G_u^u)$, then φ extends uniquely to $C^*(G)$.*
- (b) *If $\pi : C_r^*(G) \rightarrow D$ is a C^* -homomorphism, then π is injective if and only if $\pi \circ \iota_r$ is an injective homomorphism of $C_r^*(\text{Iso}(G)^\circ)$.*

To prove the theorem, we need a few preliminary results. The first is a slight improvement of the uniqueness theorem of [5] in that we do not require that the subalgebra M be abelian.

Theorem 3.2. *Let A be a C^* -algebra and M a C^* -subalgebra of A . Suppose that S is a collection of states of M such that*

- (a) *every $\varphi \in S$ has a unique extension to a state $\tilde{\varphi}$ of A ; and*
- (b) *the direct sum $\bigoplus_{\varphi \in S} \pi_{\tilde{\varphi}}$ of the GNS representations associated to extensions of elements of S to A is faithful on A .*

Let $\rho : A \rightarrow B$ be a C^ -homomorphism. Then ρ is injective if and only if it is injective on M .*

Proof. The “only if” statement is trivial. So suppose that ρ is injective on M . Let $J = \ker \rho$; we must show that $J = \{0\}$. By hypothesis, we have $J \cap M = \{0\}$. Let $A_0 := J + M$; then A_0 is a C^* -subalgebra of A by, for example, [8, Corollary 1.8.4]. Let $\gamma : A_0 \rightarrow M$ denote the quotient map. Since any state of A_0 extends to a state of A , hypothesis (a) implies that each $\varphi \in S$ has a unique state extension to A_0 . Since $\varphi \circ \gamma$ is an extension of φ to A_0 , we deduce that $\varphi \circ \gamma$ is the only extension of φ to a

state of A_0 for each $\varphi \in S$. Since $\tilde{\varphi}|_{A_0}$ is also an extension of φ to A_0 , we obtain

$$(3.2) \quad \tilde{\varphi}(a) = \varphi(\gamma(a)) \quad \text{for all } a \in A_0.$$

Now fix $x \in J$. We have $a^*x^*xa \in J$ for all $a \in A$. Take $\varphi \in S$. Since $J \subseteq A_0$, it follows from (3.2) that $\tilde{\varphi}(a^*x^*xa) = 0$ for all $a \in A$. Hence $(\pi_{\tilde{\varphi}}(x)h \mid \pi_{\tilde{\varphi}}(x)h) = 0$ for all $h \in \mathcal{H}_{\tilde{\varphi}}$, giving $\tilde{\varphi}(x) = 0$. Since $\varphi \in S$ was arbitrary, we deduce that $\bigoplus_{\varphi \in S} \pi_{\tilde{\varphi}}(x) = 0$, and so $x = 0$ by (b). \square

Next we need a technical lemma.

Lemma 3.3. *Let G be a locally compact Hausdorff étale groupoid.*

- (a) *The set $X := \{u \in G^{(0)} : G_u^u = \text{Iso}(G)_u^\circ\}$ is dense in $G^{(0)}$.*
- (b) *Suppose that $u \in G^{(0)}$ satisfies $G_u^u = \text{Iso}(G)_u^\circ$, and take $f \in C_c(G)$. Then there exists $b \in C_c(G^{(0)})^+$ such that $\|b\| = b(u) = 1$ and $bfb \in C_c(\text{Iso}(G)^\circ)$.*

Proof. (a) We say that B is an open *nested* bisection of G if there is a precompact open bisection D of G such that $\overline{B} \subset D$. This forces $\overline{r(B)} \subset r(D)$ because r is a homeomorphism on D . Note that G has a countable basis of open nested bisections.

Fix an open nested bisection B of G with $\overline{B} \subset D$ as above. Let $B' := B \cap \text{Iso}(G) \setminus \text{Iso}(G)^\circ$. We claim that $r(B')$ is nowhere dense in $G^{(0)}$. To see this, suppose that $V \subset \overline{r(B')}$ is open. We show that V is empty. Since D is an open bisection, $r|_D$ is a homeomorphism onto $r(D)$. Since $\overline{B'} \subset D$ we have

$$r(VD) = V \subset \overline{r(B')} = r(\overline{B'}).$$

Thus $VD = r^{-1}(V) \cap D$ is an open subset of $\overline{B'} \subset \text{Iso}(G) - \text{Iso}(G)^\circ$, which has empty interior. Therefore $V = \emptyset$.

Since G is étale, we have

$$\begin{aligned} & \{u \in G^{(0)} : G_u^u \neq \text{Iso}(G)_u^\circ\} \\ &= \{r(B \cap \text{Iso}(G) \setminus \text{Iso}(G)^\circ) : B \text{ is an open nested bisection}\}. \end{aligned}$$

Since G is second countable, it follows from the preceding paragraph that $\{u \in G^{(0)} : G_u^u \neq \text{Iso}(G)_u^\circ\}$ is a countable union of nowhere-dense sets, and hence nowhere dense by the Baire Category Theorem as stated in, for example, [11, Theorem 6.34]. Hence $\{u \in G^{(0)} : G_u^u = \text{Iso}(G)_u^\circ\}$ is dense in $G^{(0)}$.

(b) Fix $f \in C_c(G)$. Express $f = \sum_{D \in F} f_D$ where F is a finite collection of precompact open bisections of G and each $f_D \in C_c(D)$. Choose open neighbourhoods $\{V_D \subseteq G^{(0)} : D \in F\}$ of u as follows:

- if $u = r(\alpha) = s(\alpha)$ for some $\alpha \in D$, take $V_D = r(D \cap \text{Iso}(G)^\circ) = s(D \cap \text{Iso}(G)^\circ)$ so that $V_D D V_D \subseteq D \cap \text{Iso}(G)^\circ$ (this V_D is nonempty because $\alpha \in D \cap G_u^u \subset D \cap \text{Iso}(G)^\circ$ by choice of u);
- if there exists $\alpha \in D$ such that $r(\alpha) = u$ and $s(\alpha) \neq u$ or $s(\alpha) = u$ and $r(\alpha) \neq u$, choose an open subset $D' \subset D$ containing α such that $r(D') \cap s(D') = \emptyset$, and take $V_D = r(D')$, so that $V_D D V_D = \emptyset$; and

- if $u \notin r(D)$ and $u \notin s(D)$, use that $f_D \in C_c(D)$ to choose a neighbourhood V_D of u such that $f_D|_{V_D D V_D} = 0$.

Let $V := \bigcap_{D \in F} V_D$. Then V is open and contains u . Choose $b \in C_c(V)^+$ such that $b(u) = \|b\| = 1$. By construction, $\text{supp}(b f_D b)$ is a compact subset of $V_D D V_D \subset \text{Iso}(G)^\circ$, so $b f_D b \in C_c(\text{Iso}(G)^\circ)$. Thus $b f b = \sum_{D \in F} b f_D b$ is also in $C_c(\text{Iso}(G)^\circ)$. \square

Remark 3.4. Since $G^{(0)} \subset \text{Iso}(G)^\circ$, if $G_u^u = \{u\}$ then $u \in X = \{u : G_u^u = \text{Iso}(G)_u^\circ\}$. Now if G is topologically principal then $\text{Iso}(G)^\circ = G^{(0)}$ (see [4]) so in this case $X = \{u : G_u^u = \{u\}\}$.

Lemma 3.5. *Let G be a locally compact Hausdorff étale groupoid and $u \in G^{(0)}$ such that $G_u^u \subset \text{Iso}(G)^\circ$. Let $\epsilon > 0$ be given.*

- Let $a \in C_r^*(G)$. Then there exist $b, c \in C_r^*(\text{Iso}(G)^\circ)$ such that b positive of norm 1, such that $\varphi(b) = 1$ for all states φ that factor through $C_r^*(G_u^u)$, and such that $\|bab - c\|_r < \epsilon$.
- Let $a \in C^*(G)$. Then there exist $b, c \in C^*(\text{Iso}(G)^\circ)$ such that b positive of norm 1, such that $\varphi(b) = 1$ for all states φ that factor through $C^*(G_u^u)$, and such that $\|bab - c\| < \epsilon$.

Proof. We prove (a); the proof of (b) is exactly the same. By continuity it suffices to show that for $a \in C_c(G)$ we can find $b \in C_c(\text{Iso}(G)^\circ)$ such that $bab \in C_c(\text{Iso}(G)^\circ)$ and $\varphi(b) = \|b\|_r = 1$ for all states φ that factor through $C_r^*(G_u^u)$.

Fix $f \in C_c(G)$. By Lemma 3.3(b), there exists $b \in C_c(G^{(0)})^+ \subseteq C_c(\text{Iso}(G)^\circ)^+$ such that $b(u) = \|b\| = 1$ and $b f b \in C_r^*(\text{Iso}(G)^\circ)$. Since the quotient map from $C_r^*(\text{Iso}(G)^\circ)$ onto $C_r^*(G_u^u)$ carries b to $1_{C_r^*(G_u^u)}$, we have $\varphi(b) = \|b\| = 1$ for all states φ that factor through $C_r^*(G_u^u)$. \square

We now have the wherewithal to prove Theorem 3.1.

Proof of Theorem 3.1. For (a), we just prove the assertion about reduced C^* -algebras; the assertion about full C^* -algebras follows from exactly the same argument using part (b) of Lemma 3.5 instead of part (a). Fix $u \in G^{(0)}$ such that $G_u^u \subseteq \text{Iso}(G)^\circ$, and a state φ of $C_r^*(\text{Iso}(G)^\circ)$ that factors through $C_r^*(G_u^u)$. By the argument preceding [1, Theorem 3.2] ([1] is about unital C^* -algebras, but the argument also works in the non-unital setting) it will suffice to show that for each $a \in C_r^*(G)$ and $\epsilon > 0$ there exists a positive element $b \in C_r^*(\text{Iso}(G)^\circ)$ such that $\varphi(b) = \|b\| = 1$ and an element $c \in C_r^*(\text{Iso}(G)^\circ)$ such that $\|bab - c\| < \epsilon$. But this is just Lemma 3.5(a).

For (b), since ι_r is injective, the “only if” is clear. Suppose that $\pi \circ \iota_r$ is injective, so π is injective on $C_r^*(\text{Iso}(G)^\circ)$. Let $X = \{u \in G^{(0)} : G_u^u = \text{Iso}(G)_u^\circ\}$. For each $u \in X$, let S_u be the collection of pure states of $C_r^*(\text{Iso}(G)^\circ)$ that factor through $C_r^*(G_u^u)$. Let $S = \bigcup_{u \in X} S_u$. By part (a) above, each φ in S has unique extension $\tilde{\varphi}$ to $C^*(G)$. For each $\varphi \in S$, write $\pi_{\tilde{\varphi}}$ for the GNS representation of $C_r^*(G)$ associated to $\tilde{\varphi}$ and for $A \subset S$, let $\pi_A = \bigoplus_{\varphi \in A} \pi_{\tilde{\varphi}}$. By Theorem 3.2, it suffices to show that $\pi_S := \bigoplus_{\varphi \in S} \pi_{\tilde{\varphi}}$ is faithful on $C_r^*(G)$.

Let $\Phi_I^r : C_r^*(\text{Iso}(G)^\circ) \rightarrow C_0(G^{(0)})$ be the conditional expectation that extends restriction of functions and ev_u be the evaluation map at u for each $u \in X$. We claim that for each $u \in X$, $\text{ev}_u \circ \Phi_I^r$ factors through $C^*(G_u^u)$. Let $q_u : C_r^*(\text{Iso}(G)^\circ) \rightarrow C_r^*(G_u^u)$ be the quotient map and let \mathcal{K} be an increasing net of compact subsets of $G^{(0)} \setminus \{u\}$ such that $\bigcup \mathcal{K} = G^{(0)} \setminus \{u\}$. For each $K \in \mathcal{K}$, choose $f_K \in C_c(G^{(0)} \setminus \{u\})$ such that $f_K|_K \equiv 1$ and $0 \leq f_K \leq 1$. Then $\{f_K\}_{K \in \mathcal{K}}$ is an approximate unit for $\ker(q_u)$. So for $a \in \ker(q_u)$, we have $a = \lim_K f_K a$, and hence

$$\text{ev}_u \circ \Phi_I^r(a) = \lim_K \text{ev}_u \circ \Phi_I^r(f_K a) = \lim_K \text{ev}_u(f_K) \text{ev}_u \circ \Phi_I^r(a) = 0.$$

This proves the claim.

Suppose $a \in C_r^*(G)$ with $\pi_S(a) = 0$. We want to show that $a = 0$. Let $\Phi_r : C_r^*(G) \rightarrow C_0(G^{(0)})$ be the faithful conditional expectation extending restriction of functions. Since Φ_r is faithful, it is enough to show that $\Phi_r(a^*a) = 0$. By way of contradiction assume that $\Phi_r(a^*a) \neq 0$. By Lemma 3.3, X is dense, so there exists $u \in X$ with $\Phi_r(a^*a)(u) > 0$. Pick ϵ such that

$$(3.3) \quad \Phi_r(a^*a)(u) > \epsilon > 0.$$

By Lemma 3.5, there exists $b \in C_r^*(G)$ and $c \in C_r^*(\text{Iso}(G)^\circ)$ such that $\varphi(b) = 1$ for all states of $C_r^*(\text{Iso}(G)^\circ)$ that factor through $C^*(G_u^u)$ and such that

$$(3.4) \quad \|ba^*ab - c\| < \epsilon/2.$$

We have $\pi_{S_u}(ba^*ab) = \pi_{S_u}(a^*a) = 0$ by assumption. Thus

$$\tilde{\varphi}(ba^*ab) = \lim_\lambda \langle \pi_{\tilde{\varphi}}(ba^*ab)(e_\lambda + N_{\tilde{\varphi}}), e_\lambda + N_{\tilde{\varphi}} \rangle = 0$$

for all $\varphi \in S_u$ where $\{e_\lambda\}$ is an approximate unit for $C_r^*(G)$. Now from Equation (3.4) we get $|\varphi(c)| < \epsilon/2$ for all $\varphi \in S_u$. Thus $\|q_u(c)\| \leq \epsilon/2$. Since $\text{ev}_u \circ \Phi_I^r$ factors through $C^*(G_u^u)$ we deduce that $|\text{ev}_u \circ \Phi_r(c)| \leq \epsilon/2$.

On the other hand, if $\tilde{\psi}$ is the unique extension of a state ψ on $C_r^*(\text{Iso}(G)^\circ)$ that factors through $C^*(G_u^u)$ then, since $\psi(b) = 1$, we have $\tilde{\psi}(bdb) = \tilde{\psi}(d)$ for all $d \in C_r^*(G)$. In particular, $\text{ev}_u \circ \Phi_r(ba^*ab) = \text{ev}_u \circ \Phi_r(a^*a)$. Thus using Equation (3.4) again, we have

$$|\text{ev}_u \circ \Phi_r(a^*a) - \text{ev}_u \circ \Phi_r(c)| = |\text{ev}_u \circ \Phi_r(ba^*ab) - \text{ev}_u \circ \Phi_r(c)| < \epsilon/2.$$

Hence $|\text{ev}_u \circ \Phi_r(a^*a)| < \epsilon/2 + \epsilon/2 = \epsilon$. This contradicts Equation (3.3). \square

4. MAXIMAL ABELIAN SUBALGEBRAS, CARTAN SUBALGEBRAS AND PSEUDO-DIAGONALS

In [5, Remark 4.11], the authors conjecture that if Λ is a k -graph, and M is the C^* -subalgebra of $C^*(\Lambda)$ spanned by the elements $s_\mu s_\nu^*$ such that $\mu x = \nu x$ for every infinite path x , then M is a maximal abelian subalgebra of $C^*(\Lambda)$. Yang established this in [26] for cofinal k -graphs, which are those whose groupoids are minimal. She has recently communicated to us the preprint [27] in which she proves the same result

for those k -graphs with the property that the interior of the isotropy in the associated groupoid is closed.

We will show more generally that if G is an étale groupoid G in which the interior of the isotropy is abelian, and if either (a) the interior of the isotropy is closed in G , or (b) G carries a continuous cocycle into a discrete abelian group H that is injective on the isotropy over every unit of G , then the subalgebra M_r of Theorem 3.1 is a maximal abelian subalgebra of $C_r^*(G)$.

We also investigate when M_r is a Cartan subalgebra in the sense of Renault and a pseudo-diagonal in the sense of [17]. Both conditions require the existence of a conditional expectation $\Psi : C_r^*(G) \rightarrow M_r$. We prove that such an expectation exists if and only if $\text{Iso}(G)^\circ$ is closed in G . We show by example that there exist k -graphs Λ for which the interior of the isotropy in the associated infinite-path groupoid is not closed, and therefore for which M_r is not a Cartan subalgebra of $C^*(\Lambda)$. We tie up a loose thread from [16] by proving that every Cartan subalgebra is a pseudo-diagonal.

Throughout this section we will make frequent use of the following fact: if G is a second-countable étale Hausdorff groupoid, then [19, Proposition II.4.2(i)] implies that the injection $j : C_c(G) \rightarrow C_0(G)$ extends to an injective norm-decreasing linear map $j : C_r^*(G) \rightarrow C_0(G)$.

Proposition 4.1. *Let G be a locally compact Hausdorff étale groupoid. Let M and M_r be as in Theorem 3.1.*

- (a) *There exists a conditional expectation from $C_r^*(G)$ to M_r if and only if $\text{Iso}(G)^\circ$ is closed in G .*
- (b) *If $\text{Iso}(G)^\circ$ is closed, then there is a faithful conditional expectation*

$$\Psi_r : C_r^*(G) \rightarrow M_r$$

such that $\Psi_r(f) = \iota_r(f|_{\text{Iso}(G)^\circ})$ for all $f \in C_c(G)$.

- (c) *If $\text{Iso}(G)^\circ$ is closed and amenable, then there is also a conditional expectation (not necessarily faithful) $\Psi : C^*(G) \rightarrow M$ satisfying $\Psi(f) = \iota(f|_{\text{Iso}(G)^\circ})$ for all $f \in C_c(G)$.*

To prove this proposition we need a lemma.

Lemma 4.2. *If $\text{Iso}(G)^\circ$ is amenable then $\iota : C^*(\text{Iso}(G)^\circ) \rightarrow C^*(G)$ is injective.*

Proof. Observe that the conditional expectations $\Phi_I : C^*(\text{Iso}(G)^\circ) \rightarrow C_0(G^{(0)})$ and $\Phi : C^*(G) \rightarrow C_0(G^{(0)})$ determined by restriction of functions satisfy $\Phi \circ \iota = \iota \circ \Phi_I$. Restriction of functions also determines a faithful conditional expectation $\Phi_I^r : C_r^*(\text{Iso}(G)^\circ) \rightarrow C_0(G^{(0)})$. Since $\text{Iso}(G)^\circ$ is amenable, $C^*(\text{Iso}(G)^\circ) = C_r^*(\text{Iso}(G)^\circ)$. So $\Phi_I = \Phi_I^r$ and the latter is faithful. So a standard argument (see, for example, [22, Lemma 3.13]) shows that ι is injective. \square

Proof of Proposition 4.1. We first show that if $\text{Iso}(G)^\circ$ is not closed, then there does not exist a conditional expectation from $C_r^*(G)$ to M_r . We argue by contradiction:

that is, we suppose that $\text{Iso}(G)^\circ$ is not closed and that there is a conditional expectation $\Phi : C_r^*(G) \rightarrow M_r$, and we derive a contradiction. Fix $\gamma \in \overline{\text{Iso}(G)^\circ} \setminus \text{Iso}(G)^\circ$, and choose a sequence γ_n in $\text{Iso}(G)^\circ$ converging to γ . Choose a precompact open bisection U containing γ and a function $f \in C_c(G)$ such that $\text{supp}(f) \subset U$ and $f(\gamma) = 1$. Without loss of generality, the γ_n are all in the support of f . Since each γ_n is interior to the isotropy, we can choose open sets $V_n \subseteq U \cap \text{Iso}(G)^\circ$ such that $\gamma_n \in V_n$. The sets $r(V_n) \subseteq G^{(0)}$ are then open sets of units, and we can choose functions $g_n \in C_c(r(V_n))$ such that $g_n(r(\gamma_n)) = 1$.

Now each $g_n \in M_r$, and so $\Phi(g_n * f) = g_n * \Phi(f)$ for each n . Since f is supported on a bisection and g_n is supported in $G^{(0)}$, the element $g_n * f$ is supported on $\text{supp}(f) \cap r^{-1}(\text{supp}(g_n)) \subseteq \text{supp}(f) \cap r^{-1}(r(V_n)) \subseteq \text{Iso}(G)^\circ$. Since g_n and f are compactly supported, we obtain $g_n * f \in C_c(\text{Iso}(G)^\circ) \subseteq M_r$. Hence $\Phi(g_n * f) = g_n * f$ for each n . We have $j(g_n * h)(\gamma_n) = j(h)(\gamma_n)$ for all $h \in C_r^*(G)$, and so

$$j(\Phi(f))(\gamma_n) = j(g_n * \Phi(f))(\gamma_n) = j(\Phi(g_n * f))(\gamma_n) = j(g_n * f)(\gamma_n) = f(\gamma_n).$$

Since f is continuous and $f(\gamma) = 1$, we have $j(\Phi(f))(\gamma_n) \rightarrow 1$.

We have $\Phi(f) \in M_r$, say $\Phi(f) = \iota_r(a)$ where $a \in C_r^*(\text{Iso}(G)^\circ)$. We can choose elements $h_n \in C_c(\text{Iso}(G)^\circ)$ such that $h_n \rightarrow a$ in $C_r^*(\text{Iso}(G)^\circ)$. We therefore have $\iota_r(h_n) \rightarrow \Phi(f)$ in $C_r^*(G)$. Since $j : C_r^*(G) \rightarrow C_0(G)$ is continuous, it follows that $j(\iota_r(h_n)) \rightarrow j(\Phi(f))$ in $C_0(G)$, and in particular $j(\iota_r(h_n))(\gamma) \rightarrow j(\Phi(f))(\gamma)$. Since each h_n belongs to $C_c(\text{Iso}(G)^\circ)$ and since $\gamma \notin \text{Iso}(G)^\circ$, we have $j(\iota_r(h_n))(\gamma) = 0$ for all n and it follows that $j(\Phi(f))(\gamma) = 0$.

Putting all this together, we have $\gamma_n \rightarrow \gamma$, $j(\Phi(f))(\gamma_n) \rightarrow 1$ and $j(\Phi(f))(\gamma) = 0$, which contradicts continuity of $j(\Phi(f))$. This proves the ‘‘only if’’ part of (a).

Now suppose that $\text{Iso}(G)^\circ$ is closed. Then $f|_{\text{Iso}(G)^\circ} \in C_c(\text{Iso}(G)^\circ)$ for all $f \in C_c(G)$. Define $\Psi_0 : C_c(G) \rightarrow C_c(\text{Iso}(G)^\circ)$ by $\Psi_0(f) = \iota(f|_{\text{Iso}(G)^\circ})$. The image of Ψ_0 is $C_c(\text{Iso}(G)^\circ)$ since $\text{Iso}(G)^\circ$ is closed. Also, Ψ_0 is a linear idempotent. We claim that

$$\|\Psi_0(f)\|_{C_r^*(\text{Iso}(G)^\circ)} \leq \|f\|_{C_r^*(G)} \quad \text{for all } f \in C_c(G).$$

Fix $f \in C_c(G)$ and $\varepsilon > 0$. We will show that $\|\Psi_0(f)\|_r \leq \|f\|_r + \varepsilon$. Let $f_0 := f|_{\text{Iso}(G)^\circ}$. There is a unit $u \in G^{(0)}$ such that the associated regular representation $L^u : C^*(\text{Iso}(G)^\circ) \rightarrow \mathcal{B}(\ell^2(\text{Iso}(G)_u^\circ))$ satisfies $\|L^u(f_0)\| \geq \|f_0\|_r - \varepsilon$. Let π_u be the regular representation of $C^*(G)$ on $\ell^2(G_u)$. Let $P \in \mathcal{B}(\ell^2(G_u))$ be the orthogonal projection into $\overline{\text{span}}\{\delta_\gamma : \gamma \in \text{Iso}(G)_u^\circ\} \subset \ell^2(G_u)$.

For $\alpha, \beta \in G_u$, we have

$$\begin{aligned} (P\pi_u(f)P\delta_\alpha \mid \delta_\beta) &= \begin{cases} (\pi_u(f)\delta_\alpha \mid \delta_\beta) & \text{if } \alpha, \beta \in \text{Iso}(G)^\circ \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} f(\beta\alpha^{-1}) & \text{if } \alpha, \beta \in \text{Iso}(G)^\circ \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

$$= \begin{cases} (L^u(f_0)\delta_\alpha \mid \delta_\beta) & \text{if } \alpha, \beta \in \text{Iso}(G)^\circ \\ 0 & \text{otherwise.} \end{cases}$$

So the canonical unitary isomorphism $\ell^2(\text{Iso}(G)_u^\circ) \cong P\ell^2(G_u)$ intertwines $P\pi_u(f)P$ and $L^u(f_0)$, giving $\|L^u(f_0)\| = \|P\pi_u(f)P\|$. Hence

$$\|\Psi_0(f)\|_r = \|\iota_r(f_0)\|_r \leq \|L^u(f_0)\| + \varepsilon = \|P\pi_u(f)P\| + \varepsilon \leq \|f\|_r + \varepsilon.$$

Hence Ψ_0 extends to a linear idempotent $\Psi_r : C_r^*(G) \rightarrow M_r$. Theorem II.6.10.2 of [3] shows that Ψ_r is a conditional expectation. Since $G^{(0)} \subset \text{Iso}(G)^\circ$, the canonical expectation $\Phi_r : C_r^*(G) \rightarrow C_0(G^{(0)})$ satisfies $\Phi_r = \Phi_r \circ \Psi_r$. Since Φ_r is faithful, it follows that Ψ_r is too.

This gives us the remaining implication for part (a) as well as part (b).

To establish (c), suppose that $\text{Iso}(G)^\circ$ is amenable. Then Lemma 4.2 shows ι is injective and hence isometric, and so we have $\|\iota(f)\| = \|f\|_{C^*(\text{Iso}(G)^\circ)} = \|f\|_{C_r^*(\text{Iso}(G)^\circ)}$ for all $f \in C_c(\text{Iso}(G)^\circ)$. In particular we saw above that for $f \in C_c(G)$ we have $\|\iota(f|_{\text{Iso}(G)^\circ})\| \leq \|f\|_{C_r^*(G)} \leq \|f\|_{C^*(G)}$. Hence Ψ_0 extends to a linear idempotent Ψ of norm 1 from $C^*(G)$ to M . Once again, [3, Theorem II.6.10.2] shows that Ψ is a conditional expectation. \square

We now consider when M_r is a maximal abelian subalgebra of $C_r^*(G)$.

Theorem 4.3. *Let G be a locally compact Hausdorff étale groupoid, and suppose that $\text{Iso}(G)^\circ$ is abelian. Then $\text{Iso}(G)^\circ$ is amenable, $M := \iota(C^*(\text{Iso}(G)^\circ))$ is an abelian subalgebra of $C^*(G)$, and $M_r = \iota_r(C^*(\text{Iso}(G)^\circ))$ is an abelian subalgebra of $C_r^*(G)$. Suppose that either*

- (a) $\text{Iso}(G)^\circ$ is closed in G , or
- (b) there exist a countable discrete abelian group H and a continuous 1-cocycle $c : G \rightarrow H$ such that $c|_{G_x^\pm}$ is injective for each $x \in G^{(0)}$.

Then M_r is maximal abelian in $C_r^*(G)$.

Before proving Theorem 4.3, we establish a technical result that may be useful in future.

Lemma 4.4. *Let G be a locally compact Hausdorff étale groupoid, and suppose that $\text{Iso}(G)^\circ$ is abelian. Then M_r is maximal abelian in $C_r^*(G)$ if and only if*

$$\{a \in C_r^*(G) : j(a) \in C_0(\text{Iso}(G)^\circ)\} \subseteq M_r.$$

Proof. Let $A := \{a \in C_r^*(G) : j(a) \in C_0(\text{Iso}(G)^\circ)\}$. Since j is continuous, A is closed, and hence a C^* -subalgebra of $C_r^*(G)$. Renault proves that j is multiplicative for the usual convolution product, and so A is an abelian C^* -subalgebra of $C_r^*(G)$ containing M_r . So if M_r is maximal abelian, then it is equal to A .

Conversely suppose that $M_r = A$. Suppose that $a \in C_r^*(G)$ commutes with every element of M_r . We must show that $a \in M_r$. Fix $\alpha \in G \setminus \text{Iso}(G)$. Since $r(\alpha) \neq s(\alpha)$

there exists $b \in C_0(G^{(0)})$ such that $b(s(\alpha)) = 1$, and $b(r(\alpha)) = 0$. Since $ab = ba$, it follows from [19, Proposition II.4.2(iii)] that

$$0 = |j(ab - ba)(\alpha)| = |j(a)(\alpha)b(s(\alpha)) - b(r(\alpha))j(a)(\alpha)| = |j(a)(\alpha)|.$$

So $j(a)$ vanishes on $G \setminus \text{Iso}(G)$. Now for $\alpha \in G \setminus \text{Iso}(G)^\circ$, there is a sequence $\alpha_n \in G \setminus \text{Iso}(G)$ with $\alpha_n \rightarrow \alpha$. Since $j(a)$ is continuous, we deduce that $j(a)(\alpha) = \lim j(a)(\alpha_n) = 0$. \square

Proof of Theorem 4.3. Since $\text{Iso}(G)^\circ$ is an abelian-group bundle, [21, Theorem 3.5] shows that it is amenable. Since $\text{Iso}(G)^\circ$ is abelian, M and M_r are abelian.

Now fix $a \in C_r^*(G)$ such that $j(a) \in C_0(\text{Iso}(G)^\circ)$. By Lemma 4.4, to complete the proof it suffices to show that if either (a) or (b) holds, then $a \in M_r$.

First suppose that (a) holds. Then Proposition 4.1 shows that restriction of compactly supported functions extends to a conditional expectation $\Psi_r : C_r^*(G) \rightarrow M_r$. Take $a_n \in C_c(G)$ with $a_n \rightarrow a$ in $C_r^*(G)$. We have $a_n - \Psi_r(a_n) \rightarrow a - \Psi_r(a)$ in $C_r^*(G)$. Therefore $j(a_n - \Psi_r(a_n)) \rightarrow j(a - \Psi_r(a))$. But j is the identity on $C_c(G)$ and each $a_n - \Psi_r(a_n)$ vanishes on $\text{Iso}(G)^\circ$. Hence $j(a - \Psi_r(a))$ vanishes on $\text{Iso}(G)^\circ$. But then $j(a - \Psi_r(a)) = 0$. Since j is injective, we obtain $a = \Psi_r(a)$. Since $\Psi_r(C_r^*(G)) = M_r$, we deduce that $a \in M_r$ as required.

Now suppose that (b) holds. Let $A := \{b \in C_r^*(G) : j(b) \in C_0(\text{Iso}(G)^\circ)\}$. Let $\gamma : \widehat{H} \rightarrow \text{Aut}(C_r^*(G))$ be the action such that $\gamma_\chi(f)(\alpha) = \chi(c(\alpha))f(\alpha)$ for $f \in C_c(G)$. By continuity, for $b \in C_r^*(G)$, $\chi \in \widehat{H}$ and $\alpha \in G$, we have

$$j(\gamma_\chi(b))(\alpha) = \chi(c(\alpha))j(b)(\alpha).$$

Hence A is invariant under γ . In particular, the Fourier coefficients

$$\Phi_h(a) := \int_{\widehat{H}} \overline{\chi(h)} \gamma_\chi(a) d\mu(\chi)$$

of a belong to A . Since H is amenable, a can be expressed as a norm-convergent sum of the $\Phi_h(a)$ (see, for example, the argument of [2, Theorem 5.6]), so we just need to show that each $\Phi_h(a) \in M_r$.

Fix $h \in H$ and consider $a_h := \Phi_h(a)$. Fix $\varepsilon > 0$. It suffices to show that there exists $b \in C_c(\text{Iso}(G)^\circ)$ such that $\|a_h - b\|_r \leq \varepsilon$. For this, observe that $\text{supp}(j(a_h)) = \text{supp}(a) \cap c^{-1}(h) \subseteq \overline{\text{Iso}(G)^\circ} \cap c^{-1}(h)$. Since c is injective on each G_x^x , this set is a bisection. The I -norm is finite and agrees with the supremum norm on $C_0(V)$ for any bisection V . Since the I -norm dominates the full norm, and hence the reduced norm, on $C_c(G)$, it follows that for $b \in j^{-1}(C_0(\{\alpha : j(a_h)(\alpha) \neq 0\}))$, we have $\|j(b)\|_\infty = \|j(b)\|_I \geq \|b\|_r \geq \|j(b)\|_\infty$, giving equality throughout. Thus while it is not true for arbitrary $a \in C_c(G)$ that $\|j(a)\|_\infty = \|a\|_r$, we do have

$$(4.1) \quad \|j(b)\|_\infty = \|b\|_r \quad \text{for all } b \in j^{-1}(C_0(\{\alpha : j(a_h)(\alpha) \neq 0\})).$$

Fix $\varepsilon > 0$. The sets $U_\varepsilon := \{\alpha \in G : |j(a_h)(\alpha)| \geq \varepsilon\}$ and $U_{\varepsilon/2} := \{\alpha \in G : |j(a_h)(\alpha)| \geq \varepsilon/2\}$ are compact subsets of $\{\alpha : j(a_h)(\alpha) \neq 0\} \subseteq \text{Iso}(G)^\circ$. Since

$\overline{\text{Iso}(G)^\circ} \cap c^{-1}(h)$ is a bisection, there exists $g \in C_c(G^{(0)}, [0, 1])$ such that $g(r(\alpha)) = 1$ for all $\alpha \in U_\varepsilon$, and $\text{supp}(g) \subseteq r(U_{\varepsilon/2})$. We have $g * a_h = g * j(a_h) \in C_c(\text{Iso}(G)^\circ)$. Furthermore, $a_h - g * a_h \in C_0(\{\alpha : j(a_h)(\alpha) \neq 0\})$. By the preceding paragraph, we therefore have

$$\|a_h - g * a_h\|_r = \|j(a_h - g * a_h)\|_\infty = \|j(a_h) - g * a_h\|_\infty.$$

By construction of g , we have $(g * a_h)(\alpha) = j(a_h)(\alpha)$ whenever $|a_h(\alpha)| \geq \varepsilon$, and $|j(a_h - g * a_h)(\alpha)| = (1 - g(r(\alpha)))|a_h(\alpha)| < |a_h(\alpha)|$ for all α . Thus $\|j(a_h) - g * a_h\|_\infty \leq \varepsilon$ as required. \square

The previous theorem resolves some issues left unanswered in [20] and [17]. For the next result, observe that when $\text{Iso}(G)^\circ$ is abelian, the homomorphisms ι and ι_r of (3.1) are injective by Lemma 4.2 and [18, Proposition 1.9] respectively.

Corollary 4.5. *Let G be a locally compact Hausdorff étale groupoid and suppose that $\text{Iso}(G)^\circ$ is abelian. The following are equivalent:*

- (a) $\iota_r(C^*(\text{Iso}(G)^\circ)) \subset C_r^*(G)$ is a pseudo-diagonal in the sense of [17, page 268];
- (b) $\iota_r(C^*(\text{Iso}(G)^\circ)) \subset C_r^*(G)$ is a Cartan subalgebra in the sense of [20, Definition 4.5]; and
- (c) $\text{Iso}(G)^\circ$ is closed in G .

In particular, if G is amenable and $\text{Iso}(G)^\circ$ is closed, then $\iota(C^(\text{Iso}(G)^\circ)) \subseteq C^*(G)$ is both a pseudo-diagonal and a Cartan subalgebra.*

Proof. Both (a) and (b) imply by definition that there is a conditional expectation from $C_r^*(G)$ to $\iota_r(C^*(\text{Iso}(G)^\circ))$, and then the ‘‘only if’’ implication in the first statement of Theorem 4.1 gives (c). So it suffices to prove that (c) implies (a) and (b).

Suppose, then, that $\text{Iso}(G)^\circ$ is closed. Let $M := \iota_r(C^*(\text{Iso}(G)^\circ))$. Theorem 4.3 implies that M is a maximal abelian subalgebra of $C_r^*(G)$. Let $S_{\text{Iso}(G)^\circ}$ be the set of pure states of M that factor through $C_r^*(G_u^u)$ for some unit u with $G_u^u = \text{Iso}(G)^\circ_u$. We claim that $S_{\text{Iso}(G)^\circ}$ is dense in the set of all pure states of M ; that is, the corresponding set $S_{\text{Iso}(G)^\circ}^\wedge$ is dense in the Gelfand dual \widehat{M} of M . By [15, Corollary 3.4] and the subsequent remarks, and by [15, Proposition 3.6], the map $p : \widehat{M} \rightarrow G^{(0)}$ is an open map making $\text{Iso}(G)^\circ$ into an abelian-group bundle over $G^{(0)}$. By Lemma 3.3 part (a), it suffices to show that if $D \subset G^{(0)}$ is dense then $p^{-1}(D)$ is dense in \widehat{M} . To see this, fix $\sigma \in \widehat{M}$. There exist $u_n \in D$ such that $u_n \rightarrow p(\sigma)$. Since p is open, we can invoke [25, Proposition 1.15], pass to a subsequence and relabel so that there exist $\sigma_n \in \widehat{M}$ such that $\sigma_n \rightarrow \sigma$ and $p(\sigma_n) = u_n$. This suffices and the claim is established.

Now Theorem 3.1(a) implies that $S_{\text{Iso}(G)^\circ}$ is a weak- $*$ dense set of pure states φ of M for which $\varphi \circ \Psi$ is the unique extension of φ to a state of $C^*(G)$. In the terminology of [17, page 266] (the definition just below Remark 2.4), we have just established that $M \subset C^*(G)$ has the canonical almost extension property with associated expectation

Ψ . Proposition 4.1 shows that Ψ is a faithful conditional expectation, and so M is a pseudo-diagonal as defined in [17, p. 268].

To see that M is also a Cartan subalgebra, we have to check that it is a regular maximal abelian subalgebra containing an approximate identity for $C^*(G)$ and admitting a faithful conditional expectation. It contains an approximate identity because $C_0(G^{(0)}) \subset M$ does. We have already checked that it is a maximal abelian subalgebra and admits a faithful conditional expectation. For regularity, we must show that $\{n \in C^*(G) : n^*Mn \cup nMn^* \subset M\}$ generates $C^*(G)$ as a C^* -algebra. For this, observe that if $a \in C_c(\text{Iso}(G)^\circ)$ is supported in an open bisection $U \subset \text{Iso}(G)^\circ$ and $n \in C_c(G)$ is supported in an open bisection B in G , then n^*an is supported in $B^{-1}UB$ and nan^* is supported in BUB^{-1} . Since $\text{Iso}(G)^\circ$ is invariant under conjugation in G , it follows that both n^*an and nan^* belong to $C_c(\text{Iso}(G)^\circ)$. Now continuity and linearity shows that if $a \in M$ and $n \in C_c(G)$ is supported on an open bisection, then $n^*an, nan^* \in M$. Since $\{n \in C_c(G) : n \text{ is supported on an open bisection}\}$ generates $C^*(G)$, we deduce that M is regular, and hence Cartan. \square

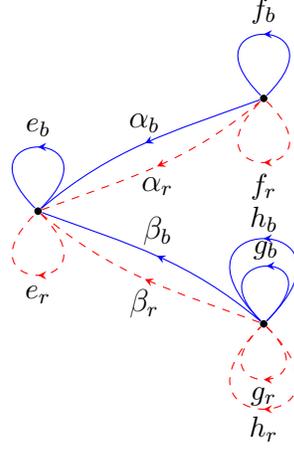
In particular, we obtain from the above a complete answer to the question asked in [5, Remark 4.11]. For background and notation for k -graphs and their infinite-path spaces, see [12]. For our purposes it suffices to recall that each k -graph Λ (with degree map $d : \Lambda \rightarrow \mathbf{N}^k$) has an infinite-path space Λ^∞ , that if $x \in \Lambda^\infty$ and $\lambda \in \Lambda r(x)$, then we can form the infinite path λx , and that the k -graph groupoid $G = G_\Lambda$ consists of triples of the form $(\mu x, d(\mu) - d(\nu), \nu x)$ where $x \in \Lambda^\infty$ and $\mu, \nu \in \Lambda r(x)$.

Corollary 4.6 (Yang [27]). *Let Λ be a row-finite k -graph with no sources, and let G be the groupoid associated to Λ in [12]. Then $M := \overline{\text{span}}\{s_\mu s_\nu^* : \mu x = \nu x \text{ for every } x \in \Lambda^\infty\}$ is a maximal abelian subalgebra of $C^*(\Lambda)$. The following are equivalent:*

- (a) M is a pseudo-diagonal;
- (b) M is a Cartan subalgebra;
- (c) the set $\{(\mu x, d(\mu) - d(\nu), \nu x) \in G : \mu y = \nu y \text{ for all } y \in r(x)\Lambda^\infty\}$ is closed.

Proof. Let G be the groupoid associated to Λ in [12]. As observed in Remark 4.11 of [5], the isomorphism $C^*(\Lambda) \cong C^*(G)$ of [12, Corollary 3.5] carries $\overline{\text{span}}\{s_\mu s_\nu^* : \mu y = \nu y \text{ for every } y \in s(\mu)\Lambda^\infty\}$ to $\iota(C^*(\text{Iso}(G)^\circ)) \subset C^*(G)$. The canonical cocycle $(x, m, y) \mapsto m$ from G to \mathbf{Z}^k is injective on each G_x^x , so the result follows from Theorem 4.3 and Corollary 4.5. \square

The preceding result begs the question: is the interior of the isotropy always closed in the infinite-path groupoid of a k -graph? The answer when $k = 1$ is “yes”, as can be deduced from Proposition 4.1 and [16, Theorem 3.6]. However, this happy situation does not persist for $k \geq 2$, as the next example shows.


 FIGURE 1. A 2-graph such that $\text{Iso}(G)^\circ$ is not closed.

Example 4.7. Consider the 2-coloured graph in Figure 1 where the factorisation rules are given by

$$\begin{aligned} e_b \alpha_r &= e_r \alpha_b, \quad e_b \beta_r = e_r \beta_b, \quad \alpha_b f_r = \alpha_r f_b, \quad f_b f_r = f_r f_b, \quad \beta_b g_r = \beta_r g_b, \quad \beta_b h_r = \beta_r h_b, \\ g_b g_r &= g_r g_b, \quad g_b h_r = h_r g_b, \quad h_b g_r = g_r h_b, \quad \text{and } h_b h_r = h_r h_b. \end{aligned}$$

Let Λ be the resulting 2-graph and $w = r(g_r)$. By construction, $w\Lambda$ is isomorphic to the 2-graph $B_2 \times B_2$ where B_2 is the bouquet of two loops. In particular, $w\Lambda$ is aperiodic. Fix an infinite path $y_0 \in w\Lambda^\infty$ such that $\sigma^m(y_0) = \sigma^n(y_0)$ only when $m = n \in \mathbf{N}^2$. Put

$$y := \beta_b y_0.$$

So $\sigma^m(y) = \sigma^n(y)$ implies $m = n$, and $r(y) = r(\beta_b) =: v$.

Note that $Z(\alpha_b) = Z(\alpha_r) = \{\alpha_b(f_b f_r)^\infty\} = \{\alpha_r(f_b f_r)^\infty\}$. Also

$$\sigma^{(1,0)}(\alpha_b(f_b f_r)^\infty) = (f_b f_r)^\infty = \sigma^{(0,1)}(\alpha_b(f_b f_r)^\infty),$$

and we deduce that $Z(\alpha_b, \alpha_r)$ is contained in $\text{Iso}(G_\Lambda)^\circ$. Let

$$x := \alpha_b(f_b f_r)^\infty.$$

Also note that

$$z := (e_b e_r)^\infty$$

satisfies $\sigma^{(1,0)}(z) = \sigma^{(0,1)}(z)$.

For each n , let

$$\gamma_n := ((e_b e_r)^n e_b x, (1, -1), (e_b e_r)^n e_r x) \in G_\Lambda.$$

Using that $Z(\alpha_b, \alpha_r) = \{(x, (1, -1), x)\}$, we see that

$$Z((e_b e_r)^n e_b, v) Z(\alpha_b, \alpha_r) Z(v, (e_b e_r)^n e_b) = \{\gamma_n\},$$

and so the γ_n all belong to $\text{Iso}(G_\Lambda)^\circ$.

The sets $Z((e_b e_r)^n e_b, (e_b e_r)^n e_r)$ form a decreasing base of neighbourhoods of $\gamma := (z, (1, -1), z)$, and each $\gamma_n \in Z((e_b e_r)^n e_b, (e_b e_r)^n e_r)$. Hence $\gamma_n \rightarrow \gamma$, giving $\gamma \in \overline{\text{Iso}(G_\Lambda)^\circ}$.

The elements

$$\gamma'_n := ((e_b e_r)^n e_b y, (1, -1), (e_b e_r)^n e_r y) \in G_\Lambda$$

satisfy $\gamma'_n \in Z((e_b e_r)^n e_b, (e_b e_r)^n e_r)$ for each n , and so $\gamma'_n \rightarrow \gamma$. We claim that $\gamma'_n \notin \text{Iso}(G_\Lambda)$ for each n . To see this, we calculate:

$$\sigma^{((n+1), (n+1))}(r(\gamma'_n)) = \sigma^{(0,1)}(y) \neq \sigma^{(1,0)}(y) = \sigma^{((n+1), (n+1))}(s(\gamma'_n)).$$

so $r(\gamma'_n) \neq s(\gamma'_n)$. Hence $\gamma = \lim \gamma'_n \notin \text{Iso}(G_\Lambda)^\circ$.

Remark 4.8. The preceding example, combined with Corollary 4.6, shows that there exist k -graphs Λ such that the subalgebra M of $C^*(\Lambda)$ described above is maximal abelian and has the property that every representation of $C^*(\Lambda)$ that is faithful on M is faithful, but is nevertheless not the range of a conditional expectation of $C^*(\Lambda)$.

To finish, we clarify the relationship between Cartan subalgebras and pseudo-diagonals. On page 890 of [16], the authors comment that the maximal abelian subalgebra that they construct in each a graph algebra $C^*(E)$ is in fact a Cartan subalgebra. In their subsequent paper [17], they show that it is a pseudo-diagonal. The relationship in general between these two conditions is not addressed. We show that every Cartan subalgebra B of a C^* -algebra A is a pseudo-diagonal in A . This provides an alternative proof of the assertion that $C^*(\text{Iso}(G)^\circ)$ is a pseudo-diagonal in Corollary 4.5, although the proof via Cartan subalgebras provides less-direct information about which pure states of $C^*(\text{Iso}(G)^\circ)$ have unique extension.

Lemma 4.9. *Let B be a Cartan subalgebra of a C^* -algebra A . Then B is a pseudo-diagonal in A .*

For the proof of Lemma 4.9, we need to recall some ideas from [20]. A *twist* over a Hausdorff étale groupoid G is a Hausdorff groupoid Σ equipped with an injective groupoid homomorphism $i : \mathbf{T} \times G^{(0)} \rightarrow \Sigma$ and a surjective groupoid homomorphism $q : \Sigma \rightarrow G$ such that the kernel $\{\gamma \in \Sigma : q(\gamma) \in G^{(0)}\}$ of q is the image of i . We write $C_c(\Sigma, G)$ for the convolution algebra

$$\{f \in C_c(\Sigma) : f(i(z, r(\gamma))\gamma) = z f(\gamma) \text{ for all } \gamma \in \Sigma \text{ and } z \in \mathbf{T}\}.$$

There is an inclusion $\iota : C_c(G^{(0)}) \hookrightarrow C_c(G, \Sigma)$ such that each $\iota(f)$ is supported on $i(\mathbf{T} \times G^{(0)})$ and satisfies $\iota(f)(i(z, x)) = z f(x)$ for $z \in \mathbf{T}$ and $x \in G^{(0)}$. We identify $C_c(G^{(0)})$ with its image under ι .

Also recall that a groupoid G is *topologically principal* if the set of units in $G^{(0)}$ with trivial isotropy is dense in $G^{(0)}$. That is, $\overline{\{x \in G^{(0)} : G_x^x = \{x\}\}} = G^{(0)}$. It is worth pointing out that the condition we are here calling topologically principal has

gone under a variety of names in the literature and that those names have not been used consistently (see [4, Remark 2.3]).

Proof of Lemma 4.9. We must show that B is maximal abelian in A , that there is a faithful conditional expectation from A onto B , and that the set of pure states of B with unique extension to A is weak*-dense in the set of pure states of B .

By [20, Theorem 5.9(i)], there exist a topologically principal étale groupoid G and a twist Σ over G for which there exists an isomorphism $\pi : A \rightarrow C_r^*(G, \Sigma)$ that carries B to the canonical copy of $C_0(G^{(0)})$. So it suffices to show that there is a dense set of points x in $G^{(0)}$ for which the state $\widehat{x}(f) = f(x)$ on $C_0(G^{(0)})$ has unique extension to $C_r^*(G, \Sigma)$. Since G is topologically principal, the set $\{x \in G^{(0)} : G_x^x = \{x\}\}$ is dense, so it suffices to show that if $G_x^x = \{x\}$, then \widehat{x} has unique extension.

The argument is very similar to that of Theorem 3.1(a), so we just give a quick sketch. By the argument preceding [1, Theorem 3.2] we just have to show that for $a \in C_c(G, \Sigma)$ there exists a positive element $b \in C_0(G^{(0)})$ such that $\widehat{x}(b) = \|b\| = 1$ and $bab \in C_0(G^{(0)})$. Write $q : \Sigma \rightarrow G$ for the quotient map. Fix $a \in C_c(G, \Sigma)$. Use a partition of unity to express $a = \sum_{B \in F} a_B$ where F is a finite collection of precompact open bisections of G and each $a_B \in C_c(q^{-1}(B))$. For each B such that $x \notin B$, since $G_x^x = \{x\}$, there is a neighbourhood V_B of x such that $V_B B V_B = \emptyset$. And for B such that $x \in B$, there is a neighbourhood V_B of x such that $V_B B V_B = V_B \subset G^{(0)}$. Let $V := \bigcap_B V_B$ and choose $b \in C_c(V)$ such that $b(x) = 1$; that is $\widehat{x}(b) = 1$. Then $bab = \sum_{x \in B} ba_B b \in C_0(G^{(0)})$ by choice of the V_B . \square

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