arXiv:2302.03989v1 [math.KT] 8 Feb 2023

# $KK$-DUALITY FOR SELF-SIMILAR GROUPOID ACTIONS ON GRAPHS

NATHAN BROWNLOWE, ALCIDES BUSS, DANIEL GONÇALVES, JEREMY B. HUME, AIDAN SIMS, AND MICHAEL F. WHITTAKER

Abstract. We extend Nekrashevych's $KK$-duality for $C^*$-algebras of regular, recurrent, contracting self-similar group actions to regular, contracting self-similar groupoid actions on a graph, removing the recurrence condition entirely and generalising from a finite alphabet to a finite graph.

More precisely, given a regular and contracting self-similar groupoid $(G, E)$ acting faithfully on a finite directed graph $E$, we associate two $C^*$-algebras, $\mathcal{O}(G, E)$ and $\widehat{\mathcal{O}}(G, E)$, to it and prove that they are strongly Morita equivalent to the stable and unstable Ruelle C*-algebras of a Smale space arising from a Wieler solenoid of the self-similar limit space. That these algebras are Spanier-Whitehead dual in $KK$-theory follows from the general result for Ruelle algebras of irreducible Smale spaces proved by Kaminker, Putnam, and the last author.

## 1. Introduction

In the last 40 years self-similar groups have been fundamental in answering a wide range of outstanding conjectures; for example, the Grigorchuk group [10] was the first group shown to have intermediate growth, and also the first known example of a group that is amenable but not elementary amenable. Since self-similar groups are defined by their actions on trees, and hence induce actions on their boundaries, they lend themselves to study via noncommutative analysis. Segal [30], building on the work of Maharam [18] showed that commutative von Neumann algebras are precisely those that arise as the $L^\infty$-algebras of localisable measure spaces, and the Gelfand–Naimark Theorem [9] characterises commutative $C^*$-algebras as $C_0$-algebras of locally compact Hausdorff spaces. Building on this foundation, much of modern operator-algebra theory, including Connes' noncommutative geometry, investigates noncommutative $C^*$-algebras by analogy with a kind of noncommutative measure space or topological space. An excellent example is Connes' notion of noncommutative Poincaré duality [4], which we refer to as *$KK$-duality* between a pair of noncommutative $C^*$-algebras (it's actually a version of Spanier–Whitehead duality, see [13]). A very general example of this is the $KK$-duality between the stable and unstable Ruelle algebras of irreducible Smale spaces [12]. Recently, pioneering work of Nekrashevych [22] proved that contracting, recurrent, and regular self-similar groups each give rise to a pair of $C^*$-algebras that can be realised as the Ruelle algebras of an underlying Smale space and used the duality result of [12] to see that these algebras are $KK$-dual.

2020 *Mathematics Subject Classification.* 47L05, 19K35 (Primary); 37B05 (Secondary).

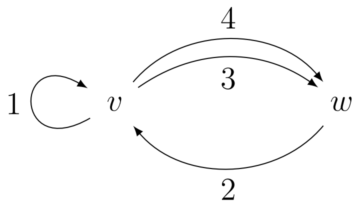
*Key words and phrases.* $C^*$-algebra; self-similar group; limit space; Poincaré duality; Spanier-Whitehead duality; KK-duality; Smale space.

Sims was supported by Australian Research Council grant DP220101631 and by CAPES grant 88887.370640. The second and third authors were partially supported by CNPq and CAPES - Brazil. Hume was supported by the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No. 817597). We thank Isnie Yusnitha for careful reading and helpful comments and Volodia Nekrashevych for insightful conversations.

In this paper we extend this duality result to the self-similar groupoids defined in [17], and simultaneously extend Nekrashevych's Smale space result [22] to self-similar actions that are not necessarily recurrent. We also note that our proof is completely different; in particular, we show that the underlying Smale space is a Wieler solenoid [31]. Our main theorem, Theorem 8.1, states that the $C^*$-algebra $\mathcal{O}(G,E)$ of a contracting self-similar groupoid action as defined in [17], and the $C^*$-algebra, suggestively denoted $\widehat{\mathcal{O}}(G,E)$, of the Deaconu–Renault groupoid of the canonical local homeomorphism on an associated limit space are $KK$-dual. In conjunction with consequences of classification theory for $C^*$-algebras (see [12, Theorem 1.1 and Section 4.4] and [24, Theorem 5.11]), our result implies that $\mathcal{O}(G,E) \cong \widehat{\mathcal{O}}(G,E)$, so our $KK$-duality mimics Poincaré duality in topology.

To prove our main theorem, we generalise the results of Nekrashevych [22] to the self-similar groupoid setting of [17]. We extend Nekrashevych's construction of the limit space $\mathcal{J}$ of a self-similar group to the setting of self-similar groupoids, and show that the shift map on $\mathcal{J}$ is open and expansive. We employ Wieler's classification of Smale spaces with totally disconnected stable sets to see that the projective limit of $\mathcal{J}$ with respect to the shift map, which we identify with a natural limit solenoid $\mathcal{S}$, is a Smale space with respect to a homeomorphism $\tilde{\tau}$ induced by the shift map on $\mathcal{J}$. Nekrashevych's construction of Smale spaces from self-similar groups is subtle, so we are careful to include all the details in extending to the situation of self-similar groupoids. In doing so, we are able to weaken the existing hypotheses, even for self-similar groups. The remainder of our work goes into proving that the Cuntz–Pimsner algebra $\mathcal{O}(G,E)$ is Morita equivalent (i.e. stably isomorphic) to the unstable Ruelle algebra of $(\mathcal{S},\tilde{\tau})$ and the Deaconu–Renault groupoid $C^*$-algebra $\widehat{\mathcal{O}}(G,E)$ is Morita equivalent to the stable Ruelle algebra of $(\mathcal{S},\tilde{\tau})$.

We illustrate our results via several examples. In particular, the main example from [17] is defined as the self-similar groupoid $(G,E)$ arising from the following graph and automaton:




$$a\cdot 1 = 4,\ a|_1 = v;$$
$$a\cdot 2 = 3,\ a|_2 = b;$$
$$b\cdot 3 = 1,\ b|_3 = v;$$
$$b\cdot 4 = 2,\ b|_4 = a.$$

The limit space of this self-similar action is homeomorphic to the complex unit circle with the map $z \mapsto z^2$. The associated limit solenoid is the classical dyadic solenoid Smale space. Thus, using [32] and Theorem 8.1 we deduce that

$$K_0(\mathcal{O}_{(G,E)}) \cong \mathbb{Z}\oplus\mathbb{Z}, \quad K_1(\mathcal{O}_{(G,E)}) \cong \mathbb{Z}, \quad K^0(\mathcal{O}_{(G,E)}) \cong \mathbb{Z}, \quad \text{and } K^1(\mathcal{O}_{(G,E)}) \cong \mathbb{Z}\oplus\mathbb{Z}.$$

The Kirchberg-Phillips Theorem then implies that $\mathcal{O}_{(G,E)}$ is isomorphic to the Cuntz—Pimsner algebra of the odometer.

Another source of interesting examples are the Katsura algebras [14]. These were recognised as self-similar actions on graphs by Exel and Pardo [7]. To see how they fit into our framework, see [17, Example 7.7] and [17, Appendix A] for the general translation from the Exel-Pardo situation to self-similar groupoid actions. In [7, Section 18], Exel and Pardo show that all unital Kirchberg algebras in the UCT class have representations as self-similar groupoids. They also prove that the $K$-theory of the Cuntz–Pimsner algebra of a self-similar groupoid of this sort is directly computable from the graph adjacency matrix and restriction matrix for the self-similar action. An analysis using Schreier graphs shows that the limit space of such a self-similar groupoid is the total space of a bundle over the circle of copies of the Cantor set, each with an odometer action specified by the restriction matrix. Once again, a combination of $KK$-duality with classification theory shows that the Cuntz–Pimsner algebra of such a system is isomorphic to the Deaconu–Renault groupoid of the limit space. This allows us to compute the $K$-theory of these interesting Deaconu–Renault $C^*$-algebras.

Example 3.22 introduces a new example whose limit dynamical system is conjugate to that of the basilica group. By definition, the basilica group is the iterated monodromy group (see [21, Chapter 5]) of the function $f(z) = z^2 - 1$ as a complex map from $\mathbb{C} \setminus \{-1, 0, 1\} \to \mathbb{C} \setminus \{0, 1\}$. The $K$-groups of its Cuntz–Pimsner algebra, and those of its dual algebra, are computed in [22, Theorem 4.8] and [22, Theorem 6.6] (see also [11]). Our main theorem therefore allows us also to compute the $K$-homology of both algebras.

The paper is organised as follows. In Section 2 we give the necessary background for the paper. We begin with directed graphs and their $C^*$-algebras. This leads to a section on self-similar groupoid actions on graphs and we recall relevant information from [17]. We conclude the section with Smale spaces and their $C^*$-algebras.

In Section 3, we generalise Nekrashevych's notion of a limit space to self-similar groupoid actions. Nekrashevych's construction is clever and subtle, so we provide substantial details regarding the metric topology on the limit space that are omitted in Nekrashevych's work. We complete this section by defining the level Schreier graphs of a self-similar groupoid action and how these relate to the limit space. In the following two sections we seek to understand the dynamics on the limit space. In particular, we show that the shift map on the limit space is a Wieler solenoid, and hence the natural extension is a Smale space with totally disconnected stable sets.

Sections 6 and 7 define two natural $C^*$-algebras associated to a self-similar groupoid action. One is the Cuntz–Pimsner algebra of [17]; our main goal is to provide a groupoid model for this algebra, extending that given by Nekrashevych in [22, Section 5]. The other is the $C^*$-algebra of a generalisation to self-similar groupoids of Nekrashevych's Deaconu–Renault groupoid of the limit space of a self-similar group. This becomes the dual algebra for the Cuntz–Pimsner algebra. Corollary 6.7 establishes the exact condition required for the groupoid of germs to be Hausdorff.

Our main result appears in Section 8. We prove that the Cuntz–Pimsner algebra is strongly Morita equivalent to the unstable Ruelle algebra of a Smale space and that the Deaconu–Renault groupoid algebra is strongly Morita equivalent to its stable Ruelle algebra. We then deduce our main $KK$-duality result from [12].

## 2. Background

2.1. **Graphs and $C^*$-algebras.** In this paper, we use the notation and conventions of [28] for graphs and their $C^*$-algebras. A *directed graph* $E$ is a quadruple $E = (E^0, E^1, r, s)$ consisting of sets $E^0$ and $E^1$ and maps $r, s : E^1 \to E^0$. The elements of $E^0$ are called *vertices* and we think of them as dots, and elements of $E^1$ are called edges, and we think of them as arrows pointing from one vertex to another: $e \in E^1$ points from $s(e)$ to $r(e)$.

A *path* in $E$ is either a vertex, or a string $\mu = e_1 \dots e_n$ such that each $e_i \in E^1$ and $s(e_i) = r(e_{i+1})$. For $n \geq 2$ we define $E^n = \{e_1 \dots e_n \mid e_i \in E^1, s(e_i) = r(e_{i+1})\}$ for the set of paths of length $n$ in $E$. The *length* $|\mu|$ of the path $\mu$ is given by $|\mu| = n$ if and only if $\mu \in E^n$. The collection $E^* := \bigcup_{n=0}^{\infty} E^n$ of all paths in $E$ is a small category with identity morphisms $E^0$, composition given by concatenation of paths of nonzero length together with the identity rules $r(\mu)\mu = \mu = \mu s(\mu)$, and domain and codomain maps $r, s$.

For $\mu \in E^*$ and $X \subseteq E^*$, we write $\mu X = \{\mu\nu \mid \nu \in X, s(\mu) = r(\nu\}$ and $X\mu = \{\nu\mu \mid \nu \in X, r(\mu) = s(\nu)\}$. We write $\mu X \nu$ for $\mu X \cap X\nu$.

We say that a graph $E$ is *row finite* if $vE^1$ is finite for each $v \in E^0$ and that it has *no sources* if each $vE^1$ is nonempty. We say that it is *finite* if both $E^0$ and $E^1$ are finite. We say that $E$ is *strongly connected* if for all $v, w \in E^0$, the set $vE^*w$ is nonempty, and $E$ is not the graph with one vertex and no edges. If $E$ is strongly connected, then $vE^1$ and $E^1v$ are nonempty for all $v$ in $E^0$.

In this paper, we will need to work with left-infinite, right-infinite and bi-infinite paths in a directed graph $E$. We will use the following notation:

$$\begin{aligned} E^\infty &= \{e_1e_2e_3\cdots \mid e_i \in E^1, s(e_i) = r(e_{i+1}) \text{ for all } i\},\\ E^{-\infty} &= \{\dots e_{-3}e_{-2}e_{-1} \mid e_i \in E^1, s(e_i) = r(e_{i+1}) \text{ for all } i\}, \quad \text{and}\\ E^{\mathbb{Z}} &= \{\dots e_{-2}e_{-1}e_0e_1e_2\cdots \mid e_i \in E^1, s(e_i) = r(e_{i+1}) \text{ for all } i\}. \end{aligned}$$

For $x = x_1x_2\cdots \in E^\infty$ we write $r(x) = r(x_1)$ and for $x = \dots x_{-2}x_{-1} \in E^{-\infty}$, we write $s(x) = s(x_{-1})$.

We endow these spaces with the topologies determined by cylinder sets. These cylinder sets are indexed by finite paths in each of the three spaces involved, so we will distinguish them with the following slightly non-standard notation: for $\mu \in E^n$, we define

$$\begin{aligned} Z[\mu) &:= \{x \in E^\infty \mid x_1 \dots x_n = \mu\}, \quad \text{and}\\ Z(\mu] &:= \{x \in E^{-\infty} \mid x_{-n} \dots x_{-1} = \mu\}. \end{aligned}$$

For $n \geq 0$ and $\mu \in E^{2n+1}$, we write

$$Z(\mu) := \{x \in E^{\mathbb{Z}} \mid x_{-n} \dots x_n = \mu\}.$$

In this paper, all graphs will be finite. The spaces $E^\infty$, $E^{-\infty}$ and $E^{\mathbb{Z}}$ are then totally disconnected compact Hausdorff spaces and the collections of cylinder sets are bases for the topologies that are closed under intersections.

There are standard metrics realising these topologies. The metric on $E^{-\infty}$ is given by

$$d(x, y) = \begin{cases} \inf\{2^{-n} \mid x, y \in Z(\mu] \text{ for some } \mu \in E^n\} & \text{if } s(x) = s(y)\\ 2 & \text{if } s(x) \neq s(y), \end{cases} \tag{2.1}$$

and the other two are defined analogously: for $E^\infty$, we replace $Z(\mu]$ with $Z[\mu)$ and the source map with the range map; for $E^{\mathbb{Z}}$ we replace "$Z(\mu]$ for some $\mu \in E^n$" with "$Z(\mu)$ for some $\mu \in E^{2n+1}$," and the conditions "$s(x) = s(y)$" and "$s(x) \neq s(y)$" with "$x_0 = y_0$" and "$x_0 \neq y_0$."

Given a finite directed graph with no sources, a *Cuntz–Krieger $E$-family* in a $C^*$-algebra $A$ is a pair $(p, s)$ of functions $p : v \mapsto p_v$ from $E^0$ to $A$ and $s : e \mapsto s_e$ from $E^1$ to $A$ such that the $p_v$ are mutually orthogonal projections, each $s_e^*s_e = p_{s(e)}$, and $p_v = \sum_{e \in vE^1} s_e s_e^*$ for all $v \in E^0$. The *graph $C^*$-algebra*, denoted $C^*(E)$, is the universal $C^*$-algebra generated by a Cuntz–Krieger $E$-family, see [28].

2.2. **Self similar actions of groupoids on graphs.** Recall that a *groupoid* is a small category $\mathcal{G}$ with inverses. The identity morphisms are called *units* and the collection of all identity morphisms is called the *unit space* and denoted $\mathcal{G}^{(0)}$. The set of composable pairs of elements in $\mathcal{G}$ is denoted $\mathcal{G}^{(2)}$.

Self-similar actions of groupoids on graphs were introduced in [17], inspired by Exel and Pardo's work in [7]. The precise relationship between the two constructions is detailed in the appendix of [17].

Given a directed graph $E$ with no sources, and given $v, w \in E^0$, a *partial isomorphism* of $E^*$ is a bijection $g : vE^* \to wE^*$ that preserves length and preserves concatenation in the sense that $g(\mu e) \in g(\mu)E^1$ for all $\mu \in E^*$ and $e \in E^1$. The expected formula $g(\mu e) = g(\mu)g(e)$ does not even make sense since $g$ is not typically defined on $s(\mu)E^*$. For each $v \in E^*$, the identity map $\mathrm{id}_v : vE^* \to vE^*$ is a partial isomorphism.

The set $\mathrm{PIso}(E^*)$ of all partial isomorphisms of $E^*$ is a groupoid with units $\mathrm{id}_v$ indexed by the vertices of $E$ and multiplication given by composition of maps. We will identify the unit space of $\mathrm{PIso}(E^*)$ with $E^0$ in the canonical way; this is consistent with our notation for graphs since the map $\mu \mapsto v\mu$ coincides with $\mathrm{id}_v : vE^* \to vE^*$.

We will write $c, d : \mathrm{PIso}(E^*) \to E^0$ for the codomain and domain maps on the groupoid $\mathrm{PIso}(E^*)$, because the symbols $s$ and $r$ are already fairly overloaded. So if $g : vE^* \to wE^*$ is a partial isomorphism, then $c(g) = w$ and $d(g) = v$.

A *faithful action* of a groupoid $G$ with unit space $E^0$ on the graph $E$ is an injective groupoid homomorphism $\phi : G \to \mathrm{PIso}(E^*)$ that restricts to the identity map on $E^0$. We will generally write $g \cdot \mu$ in place of $\phi(g)(\mu)$.

If $E = (E^0, E^1, r, s)$ is a directed graph, and $G$ is a groupoid with unit space $E^0$, that acts faithfully on $E^*$, then we say that $(G, E)$ is a *self similar groupoid action* if for every $g \in G$ and every $e \in d(g)E^1$ there exists $h \in G$ such that $c(h) = s(g \cdot e)$ and

$$g \cdot (e\mu) = (g \cdot e)(h \cdot \mu) \quad \text{for all } \mu \in s(g \cdot e)E^*. \tag{2.2}$$

Since the groupoid action $G \curvearrowright E^*$ is faithful, for each $g \in G$ and $e \in d(g)E^1$ there is a *unique* $h$ satisfying (2.2). We denote this element by $g|_e$, and call it the *restriction* of $g$ to $e$. Restriction extends to finite paths by iteration: for $g \in G$, we define $g|_{d(g)} = g$, and for $e \in E^1$ and $\mu \in s(e)E^*$, we recursively define

$$g|_{e\mu} = (g|_e)|_\mu.$$

So $g|_{e_1 \dots e_n} = (\dots(g|_{e_1})|_{e_2} \dots)|_{e_n}$, and then (2.2) extends to

$$g \cdot (\mu\nu) = (g \cdot \mu)(g|_\mu \cdot \nu)$$

whenever $g \in G$, $\mu \in d(g)E^*$ and $\nu \in E^*s(\mu)$.

We will use the following fundamental formulas without comment throughout the paper.

**Lemma 2.1** ([17, Lemma 3.4 and Proposition 3.6])**.** *Let $(G, E)$ be a self-similar groupoid action on a finite directed graph $E$. For $(g, h) \in G^{(2)}$, $\mu \in d(g)E^*$, $\nu \in E^*s(\mu)$ and $\eta \in c(g)E^*$, we have*

(1) $r(g \cdot \mu) = c(g)$ *and* $s(g \cdot \mu) = g|_\mu \cdot s(\mu)$*;*
(2) $g|_{\mu\nu} = (g|_\mu)|_\nu$*;*
(3) $\mathrm{id}_{r(\mu)}|_\mu = \mathrm{id}_{s(\mu)}$*;*
(4) $(hg)|_\mu = (h|_{g\cdot\mu})(g|_\mu)$*; and*
(5) $g^{-1}|_\eta = (g|_{g^{-1}\cdot\eta})^{-1}$*.*

We now give an example of a self-similar groupoid action by defining an $E$-automaton as described in [17, Definition 3.7, Proposition 3.9 and Theorem 3.9]. The key point

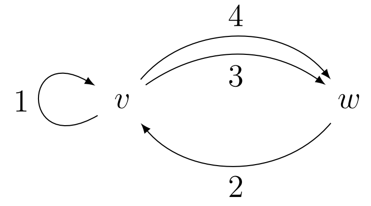



FIGURE 1. Graph $E$ for Example 2.2

of an $E$-automaton is that an action on the edges of the graph and a restriction map satisfying specified range and source conditions ensures that the action extends to a self-similar groupoid action on finite paths of the graph.

*Example* 2.2. The following example is carried through [17]. Consider the graph $E$ in Figure 1, and define

$$\begin{aligned} a\cdot 1 &= 4,\ a|_1 = v; & b\cdot 3 &= 1,\ b|_3 = v; \\ a\cdot 2 &= 3,\ a|_2 = b; & b\cdot 4 &= 2,\ b|_4 = a. \end{aligned} \tag{2.3}$$

See [17, Example 3.10] for a detailed exposition. To see explicitly how the groupoid action on $E^*$ manifests we compute

$$a\cdot 242312 = 3(b\cdot 42312) = 32(a\cdot 2312) = 323(b\cdot 312) = 3231(v\cdot 12) = 323112.$$

2.3. **Smale spaces and $C^*$-algebras.** A Smale space $(X,\varphi)$ consists of a compact metric space $X$ and a homeomorphism $\varphi : X \to X$ along with constants $\varepsilon_X > 0$ and $0 < \lambda < 1$ and a locally defined continuous map

$$[\cdot,\cdot] : \{(x,y) \in X\times X \mid d(x,y) \leq \varepsilon_X\} \to X, \quad (x,y) \mapsto [x,y]$$

satisfying

(B1) $[x,x] = x$,
(B2) $[x,[y,z]] = [x,z]$ if both sides are defined,
(B3) $[[x,y],z] = [x,z]$ if both sides are defined,
(B4) $\varphi[x,y] = [\varphi(x),\varphi(y)]$ if both sides are defined,
(C1) For $x,y \in X$ such that $[x,y] = y$, we have $d(\varphi(x),\varphi(y)) \leq \lambda d(x,y)$, and
(C2) For $x,y \in X$ such that $[x,y] = x$, we have $d(\varphi^{-1}(x),\varphi^{-1}(y)) \leq \lambda d(x,y)$.

The bracket map defines a local product structure on a Smale space as follows, for $x \in X$ and $0 < \varepsilon \leq \varepsilon_X$, define

$$\begin{aligned} X^s(x,\varepsilon) &:= \{y \in X \mid d(x,y) < \varepsilon, [y,x] = x\} \text{ and} \\ X^u(x,\varepsilon) &:= \{y \in X \mid d(x,y) < \varepsilon, [x,y] = x\}. \end{aligned}$$

We call $X^s(x,\varepsilon)$ a *local stable set* of $x$ and $X^u(x,\varepsilon)$ a *local unstable set* of $x$. Figure 2 gives a pictorial representation of the local stable sets and their interactions (provided $d(x,y) < \varepsilon_X/2$).

Suppose $(X,\varphi)$ is a Smale space. Then for $x,y \in X$ the global stable and unstable equivalence relations are given by

$$\begin{aligned} x \sim_s y &\text{ whenever } d(\varphi^n(x),\varphi^n(y)) \to 0 \text{ as } n\to\infty \text{ and} \\ x \sim_u y &\text{ whenever } d(\varphi^{-n}(x),\varphi^{-n}(y)) \to 0 \text{ as } n\to\infty. \end{aligned}$$

The stable equivalence class of $x \in X$ is denoted $X^s(x)$ and we have $X^s(x,\varepsilon) \subset X^s(x)$. Similarly, the unstable equivalence class of $x \in X$ is denoted by $X^u(x)$ and $X^u(x,\varepsilon) \subset X^u(x)$. We consider each of the stable equivalence classes as locally compact and

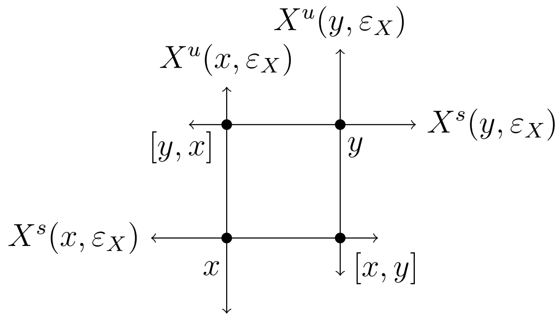



Figure 2. The local stable and unstable sets of $x, y \in X$ and their bracket maps

Hausdorff topological spaces whose topology is generated by $\{X^s(y, \varepsilon) \mid y \in X^s(x), 0 < \varepsilon < \varepsilon_X\}$. A similar topology is defined in the unstable case.

A Smale space $(X, \varphi)$ is said to be *irreducible* if, for all non-empty open sets $U, V \subseteq X$, there exists $N$ such that $\varphi^N(U) \cap V \neq \varnothing$.

It is said to be *mixing* if, for all non-empty open sets $U, V \subseteq X$, there exists $N$ such that $\varphi^n(U) \cap V \neq \varnothing$, for all $n \geq N$.

We now consider various $C^*$-algebras associated with Smale spaces. Ruelle first defined $C^*$-algebras associated to Smale spaces in [29], and these $C^*$-algebras are usually referred to as the stable and unstable algebras of the Smale space. In [25], Putnam then defined the *Ruelle algebras* as crossed product $C^*$-algebras of the stable and unstable algebras. Putnam showed that the Ruelle algebras generalise Cuntz–Krieger algebras. More recently, Putnam and Spielberg [27] considerably simplified the groupoid constructions of the above algebras when the Smale space is mixing, up to Morita equivalence. Putnam discussed in [26, Section 2] how this simplification extends to the *non-wandering* case. We shall only be interested in Smale spaces that are irreducible, which is a stronger condition than non-wandering.

For the remainder of this section we will outline the construction of the stable algebra $S(X, P)$ and the stable Ruelle algebra $S(X, P) \rtimes \mathbb{Z}$. A more detailed version of these constructions is given in [27] and [12, Section 3].

Given an irreducible Smale space $(X, \varphi)$, we fix a non-empty finite $\varphi$-invariant set of periodic points $P$ (in the irreducible case periodic points are dense). Then we define $X^u(P) = \bigcup_{p \in P} X^u(p)$, which is given a locally compact and Hausdorff topology generated by the collection $\{X^u(x, \varepsilon) \mid x \in X^u(P),\ \varepsilon \in (0, \varepsilon_X]\}$.

The groupoid of the stable equivalence relation is

$$G^s(P) := \{(v, w) \in X \times X \mid v \sim_s w \text{ and } v, w \in X^u(P)\}. \tag{2.4}$$

The stable groupoid can be endowed with an étale topology, see [12, Lemma 3.1] for details. With this structure $G^s(P)$ is an amenable locally compact Hausdorff étale groupoid. The stable $C^*$-algebra $S(X, P)$ is defined to be the groupoid $C^*$-algebra associated with $G^s(P)$.

There is a canonical automorphism of the $C^*$-algebra $S(X, P)$ induced by the automorphism of the underlying groupoid $G^s(P)$ defined by $\alpha := \varphi \times \varphi$. This automorphism of $G^s(P)$ gives rise to a semidirect product groupoid $G^s(P) \rtimes_\alpha \mathbb{Z}$, which is again an amenable locally compact Hausdorff étale groupoid. The stable Ruelle algebra is the crossed product $S(X, P) \rtimes_\alpha \mathbb{Z} \cong C^*(G^s(P) \rtimes_\alpha \mathbb{Z})$, where we also write $\alpha$ for the automorphism of $S(X, P)$ induced by $\alpha \in \mathrm{Aut}(G^s(P))$. Putnam explains in [26, Section 2] how $S(X,P) \rtimes_\alpha \mathbb{Z}$ is strongly Morita equivalent to the Ruelle algebra originally defined by Putnam in [25], building from the similar result of Putnam and Spielberg in the mixing case ([27]). The result of [27] that the Ruelle algebra is separable, simple, stable, nuclear, purely infinite, and satisfies the UCT extends readily to the irreducible case.

A similar construction gives the unstable groupoid $G^u(P)$, the unstable algebra $U(X,P)$ and the associated unstable Ruelle algebra $U(S,P) \rtimes \mathbb{Z} \cong C^*(G^u(P) \rtimes \mathbb{Z})$. Alternatively, the stable algebras for the Smale space $(X, \varphi^{-1})$ with the opposite bracket map are isomorphic to the relevant unstable algebras for $(X, \varphi)$.

## 3. The limit space of a self-similar groupoid action

In this section we generalise Nekrashevych's construction of the limit space of a self-similar group [21, Chapter 3] to the situation of self-similar groupoid actions.

**Definition 3.1.** Let $E$ be a finite directed graph. Let $(G,E)$ be a self-similar groupoid action. We say that left-infinite paths $x, y \in E^{-\infty}$ are *asymptotically equivalent*, and write $x \sim_{\text{ae}} y$ if there is a sequence $(g_n)_{n<0}$ in $G$ such that $\{g_n \mid n < 0\}$ is a finite set, and such that

$$g_n \cdot x_n \dots x_{-1} = y_n \dots y_{-1} \qquad \text{for all } n < 0.$$

If the sequence $(g_n)$ implements an asymptotic equivalence $x \sim_{\text{ae}} y$ and the sequence $(h_n)$ an asymptotic equivalence $y \sim_{\text{ae}} z$ then $d(h_n) = c(g_n)$ for all $n$, and $(h_n g_n)$ implements and asymptotic equivalence $x \sim_{\text{ae}} z$. Moreover, the sequence $g_n^{-1}$ implements an asymptotic equivalence $y \sim_{\text{ae}} x$, and the sequence $(r(x_n))_{n<0}$ implements an asymptotic equivalence $x \sim_{\text{ae}} x$. So $\sim_{\text{ae}}$ is an equivalence relation.

**Definition 3.2.** Let $E$ be a finite directed graph. Let $(G,E)$ be a self-similar groupoid action. The *limit space* of $(G,E)$ is defined to be the quotient space $\mathcal{J}_{G,E} := E^{-\infty}/{\sim_{\text{ae}}}$.

The limit space is typically not a Hausdorff space, but, just as in the setting of [21], it is guaranteed to be Hausdorff if the self-similar action is contracting in the following sense, introduced in [17, 21].

**Definition 3.3.** We say that a self-similar groupoid action $(G,E)$ on a finite directed graph $E$ is *contracting* if there is a finite subset $F \subseteq G$ such that for every $g \in G$ there exists $n \geq 0$ such that $\{g|_\mu : \mu \in d(g)E^n\} \subseteq F$. Any such finite set $F$ is called a *contracting core* for $(G,E)$. The *nucleus* of $G$ is the set

$$\mathcal{N}_{G,E} := \bigcap \{F \subseteq G \mid F \text{ is a contracting core for } (G,E)\}.$$

We will frequently just write $\mathcal{N}$ rather than $\mathcal{N}_{G,E}$ when the self-similar groupoid action in question is clear from context.

Just as in the setting of self similar groups, the nucleus is the minimal contracting core for $(G,E)$, and is symmetric, closed under restriction and contains $G^{(0)}$:

**Lemma 3.4.** *Let $E$ be a finite directed graph. Let $(G,E)$ be a contracting self-similar groupoid action. Then $\mathcal{N}$ is a contracting core for $(G,E)$ and is contained in any other contracting core for $(G,E)$. We have*

$$\mathcal{N} = \bigcup_{g \in G} \bigcap_{n=1}^{\infty} \{g|_\mu \mid \mu \in d(g)E^*, |\mu| \geq n\}. \tag{3.1}$$

*We have $\mathcal{N} = \mathcal{N}^{-1}$, $\mathcal{N}$ is closed under restriction, and if $E$ has no sinks, then $G^{(0)} \subseteq \mathcal{N}$.*

*Proof.* For the first statement, we first show that the collection of contracting cores for $(G,E)$ is closed under intersections. If $F,K$ are contracting cores for $G$ and $g \in G$, then there exist $M,N$ such that $g|_\mu \in F$ whenever $|\mu| \geq M$ and $g|_\nu \in K$ whenever $|\nu| \geq N$. In particular, if $|\mu| \geq \max\{M,N\}$ then $g|_\mu \in F \cap K$. So $F \cap K$ is a contracting core.

Now since contracting cores are, by definition, finite, there is a finite collection $\mathcal{F}$ of contracting cores such that $\mathcal{N} = \bigcap \mathcal{F}$. So the preceding paragraph shows that $\mathcal{N}$ is a contracting core. It is then contained in any other contracting core by definition.

Let $\mathcal{M} := \bigcup_{g\in G} \bigcap_{n=1}^\infty \{g|_\mu \mid \mu \in d(g)E^*, |\mu| \geq n\}$. Fix $h \in \mathcal{M}$. Then there exists $g \in G$ and a sequence $(\mu_i)_{i=1}^\infty$ of finite paths such that $|\mu_i| \to \infty$ and $g|_{\mu_i} = h$ for all $i$. By definition of $\mathcal{N}$ there exists $N$ such that $g|_\mu \in \mathcal{N}$ whenever $|\mu| \geq N$. Since $n_i \to \infty$ we have $n_i > N$ for some $i$, and so $h = g|_{\mu_i} \in \mathcal{N}$. So $\mathcal{M} \subseteq \mathcal{N}$. For the reverse containment, observe that we have just seen that $\mathcal{M}$ is finite. Fix $g \in G$. Since $G$ is contracting, the sets $R_n := \{g|_\mu \mid |\mu| \geq n\}$ indexed by $n \in \mathbb{N}$ are all finite, and they are decreasing with respect to set containment. So there exists $N$ such that $R_n = R_N$ for all $n \geq N$. It follows that $g|_\mu \in R_N \subseteq \mathcal{M}$ whenever $|\mu| \geq N$. So $\mathcal{M}$ is a contracting core for $(G,E)$ and therefore $\mathcal{N} \subseteq \mathcal{M}$ by the first assertion of the lemma.

To see that $\mathcal{N} = \mathcal{N}^{-1}$, fix $h \in \mathcal{N}$. Then (3.1) shows that there exists $g \in G$ and a sequence $(\mu_i)_{i=1}^\infty$ in $d(g)E^*$ such that $|\mu_i| \to \infty$ and $g|_{\mu_i} = h$ for all $i$. We then have $h^{-1} = (g|_{\mu_i})^{-1} = g^{-1}|_{g\cdot\mu_i}$ for all $i$, and so (3.1) gives that $h^{-1} \in \mathcal{N}$.

That $\mathcal{N}$ is closed under restriction follows immediately from (3.1).

Finally, if $E$ has no sinks, then for each $v \in E^0$ and $n \geq 0$, $E^n v \neq \emptyset$. Let $w$ be a vertex such that $wE^n v \neq \emptyset$ for infinitely many $n \in \mathbb{N}$. Then, $v \in \{w|_\mu \ : \ \mu \in wE^n, |\mu| \geq n\}$ for all $n \in \mathbb{N}$, so that $v \in \mathcal{N}$. □

**Notation 3.5.** Let $E$ be a finite directed graph with no sources. Let $(G,E)$ be a contracting self-similar groupoid action with nucleus $\mathcal{N}$. Since $\mathcal{N}$ is finite, so are the sets

$$\mathcal{N}^k := \Big\{ \prod_{i=1}^k g_i \mid g_1,\dots,g_k \in \mathcal{N} \text{ and } d(g_i) = c(g_{i+1}) \text{ for all } i \Big\}.$$

We write $R_k$ for the integer

$$R_k := \min\{j \in \mathbb{N} \mid h|_\mu \in \mathcal{N} \text{ for all } h \in \mathcal{N}^k \text{ and } \mu \in E^j\}.$$

So $R_1 = 0$, and $R_i \leq R_{i+1}$ for all $i$.

We now show that if $x,y \in E^{-\infty}$ are asymptotically equivalent, then the sequence $g_i$ implementing the asymptotic equivalence can be taken to belong to $\mathcal{N}$ and to be consistent with respect to restriction.

**Lemma 3.6.** *Let $E$ be a finite directed graph. Let $(G,E)$ be a contracting self-similar groupoid action with nucleus $\mathcal{N}$. Then $x,y \in E^{-\infty}$ are asymptotically equivalent if and only if there exists a sequence $(h_n)_{n<0}$ of elements of $\mathcal{N}$ such that $h_n \cdot x_n = y_n$ and $h_n|_{x_n} = h_{n+1}$ for all $n$.*

*Proof.* If there is such a sequence $(h_n)$ of elements of $\mathcal{N}$, then for each $n$ we have

$$h_n \cdot (x_n \dots x_{-1}) = y_n(h_n|_{x_n} \cdot x_{n+1} \dots x_{-1}) = y_n(h_{n+1} \cdot x_{n+1} \dots x_{-1}) = \cdots = y_n \dots y_{-1}.$$

So $x \sim_{\text{ae}} y$.

Conversely suppose that $x \sim_{\text{ae}} y$, and fix a sequence $(g_n)_{n<0}$ in $G$ with just finitely many distinct terms and satisfying $g_n \cdot x_n \dots x_{-1} = y_n \dots y_{-1}$ for all $n$. Let $S = \{g_n \mid n < 0\}$ be the finite set of elements appearing in the sequence $(g_n)$. Since $(G,E)$ is contracting, for each $g \in S$ there exists $k_g$ such that $g|_\mu \in \mathcal{N}$ whenever $|\mu| \geq k_g$. Let $k := \max_{g\in S} k_g$. We construct a sequence $(h_n)_{n<\infty}$ in $\mathcal{N}$ iteratively as follows. Consider the sequence $(g_n|_{x_n \dots x_{-2}})_{n<-k-1}$. By the choice of $k$, every term of this sequence belongs to $\mathcal{N}$. Since $\mathcal{N}$ is finite, there exists $h_{-1} \in \mathcal{N}$ and a strictly decreasing infinite sequence $(n_i^1)_{i=1}^\infty$ of integers $n_i^1 < -2-k$ such that $g_{n_i^1}|_{x_{n_i^1} \dots x_{-2}} = h_{-1}$ for all $i$. Since each $g_{n_i^1} \cdot (x_{n_i^1} \dots x_{-1}) = y_{n_i^1} \dots y_{-1}$, we have $h_{-1} \cdot x_{-1} = g_{n_i^1}|_{x_{n_i^1} \dots x_{-2}} \cdot x_{-1} = (g_{n_i^1} \cdot x_{n_i^1} \dots x_{-1})_{-1} = y_{-1}$.

Now suppose that we have chosen $h_m, \dots, h_{-1} \in \mathcal{N}$ such that each $h_j \cdot x_j = y_j$ and $h_j|_{x_j} = h_{j+1}$, and a strictly decreasing sequence $(n_i^m)_{i=1}^\infty$ of integers $n_i^m < m-k-1$ such that $g_{n_i^m}|_{x_{n_i^m} \dots x_{m-1}} = h_m$ for all $i$. Then the sequence $(g_{n_i^m}|_{x_{n_i^m} \dots x_{m-2}})_i$ is contained in $\mathcal{N}$, so there exists $h_{m-1} \in \mathcal{N}$ and a subsequence $n_i^{m-1}$ of the sequence $n_i^m$ with the property that $n_i^{m-1}|_{x_{n_i^m} \dots x_{m-2}} = h_{m-1}$ for all $i$. We have

$$h_{m-1}|_{m-1} = (g_{n_1^{m-1}}|_{x_{n_i^m} \dots x_{m-2}})|_{x_{m-1}} = g_{n_1^{m-1}}|_{x_{n_i^m} \dots x_{m-1}} = h_m,$$

and $h_{m-1} \cdot x_{m-1} = y_{m-1}$ by a calculation just like the one we used to see that $h_{-1} \cdot x_{-1} = y_{-1}$.

The above procedure produces a sequence $(h_n)_{n<1}$ in $\mathcal{N}$ with the desired properties. □

**Corollary 3.7.** *Let $E$ be a finite directed graph. Let $(G, E)$ be a contracting self-similar groupoid action with limit space $\mathcal{J} = \mathcal{J}_{G,E}$. Let $q : E^{-\infty} \to \mathcal{J}$ be the quotient map. For each $x \in E^{-\infty}$, the equivalence class $[x] := q^{-1}(q(x))$ satisfies $|[x]| \leq |\mathcal{N}|$.*

*Proof.* Fix $x \in E^{-\infty}$. Let $y^1, \dots, y^l$ be distinct elements of $[x]$. We must show that $l \leq |\mathcal{N}|$. Fix $m < 0$ such that the finite paths $\mu^i := y^i_m \dots y^i_{-1}$ for $i \leq l$ are all distinct. Lemma 3.6 implies that there are elements $n^1, \dots, n^l \in \mathcal{N}$ such that $\mu^i = n^i \cdot x_m \dots x_{-1}$ for all $i$. Since the $\mu^i$ are distinct, the $n^i$ are distinct, forcing $l \leq |\mathcal{N}|$. □

To construct a Smale space from the limit space $\mathcal{J}$ we will show that the shift map on $E^{-\infty}$ descends to a self-mapping of $\mathcal{J}$, and that under a regularity hypothesis similar to that used by Nekrashevych [21], this self-mapping is locally expanding and hence a local homeomorphism.

To do this, we need to describe a basis for the topology on $\mathcal{J}$ and then a metric that induces that topology.

We start with a preliminary lemma about quotient topologies.

**Lemma 3.8.** *Let $X$ be a compact and metrisable Hausdorff space and let $\sim$ be an equivalence relation on $X$. Let $Y := X/\sim$ be the quotient space, and $q : X \to Y$ the quotient map. For each $A \subseteq X$, let $U_A := \{y \in Y \mid q^{-1}(y) \subseteq A\}$. If $A$ is open in $X$, then $U_A$ is open in $Y$. If $|q^{-1}(y)| < \infty$ for each $y \in Y$, then for any basis $\mathcal{B}$ for the topology on $X$, the set*

$$\mathcal{U}_\mathcal{B} := \{U_B \mid B \text{ is a finite union of elements of } \mathcal{B}\}$$

*is a basis for the quotient topology on $Y$. If $q : X \to Y$ is a closed map, then $Y$ is metrisable.*

*Proof.* By definition of the quotient topology, $U_A$ is open in $Y$ if and only if $q^{-1}(U_A)$ is open in $X$. By definition of $U_A$, we have $[x] \in U_A$ if and only if $[x] \subseteq A$, and so

$$q^{-1}(U_A) = \{x \in X \mid [x] \subseteq A\} = X \setminus \{x \in X \mid [x] \setminus A \neq \varnothing\}.$$

So it suffices to show that if a net $(x_i)_{i \in I}$ in $X$ converges to some $x \in X$, and if each $[x_i] \setminus A$ is nonempty, then $[x] \setminus A$ is nonempty. To see this, note that for each $i$, there exists $y_i \in [x_i] \setminus A$. Since $X$ is compact we can pass to a subnet $(y_{i_j})$ that converges in $X$. Since $A$ is open, we have that $y := \lim_j y_{i_j} \notin A$. Since the quotient map is continuous we have $q(y) = \lim_j q(y_{i_j}) = \lim_j q(x_{i_j}) = \lim_i q(x_i) = [x]$, and so $y \in [x] \setminus A$.

Now suppose that $\mathcal{B}$ is a basis for the topology on $X$. Let $V$ be an open subset of $Y$ and fix $y \in V$. Since $q^{-1}(V)$ is open in $X$, for each point $x \in q^{-1}(y)$, we can find $B_x \in \mathcal{B}$ such that $x \in B_x \subseteq q^{-1}(V)$. Let $B := \bigcup_{x \in q^{-1}(y)} B_x$. Since $q^{-1}(y)$ is finite, this is a finite union of elements of $\mathcal{B}$, so it suffices to show that $y \in U_B \subseteq V$. By definition of $B$ we have $q^{-1}(y) \subseteq B$ and so $y \in U_B$. To see that $U_B \subseteq V$, take $y' \in U_B$. Then $q^{-1}(y) \subseteq B$ by definition of $U_B$. Since each $B_x \subseteq q^{-1}(V)$, we have $B \subseteq q^{-1}(V)$ and hence $q(B) \subseteq V$. So $y \in q(B) \subseteq V$, as required.

The last statement follows from [5, Theorem 4.2.13]. □

Our next lemma describes how asymptotic equivalence interacts with the action of the nucleus on cylinder sets.

**Lemma 3.9.** *Let $E$ be a finite directed graph. Let $(G, E)$ be a contracting self-similar groupoid action. If there exists $g \in \mathcal{N}$ and $\mu, \nu$ in $E^*$ such that $g \cdot \mu = \nu$, then $q(Z(\mu]) \cap q(Z(\nu]) \neq \varnothing$.*

*Proof.* Fix $\mu, \nu$, and $g$. Since $\mathcal{N}$ is closed under restriction, there exist $e \in E^1$ and $h \in \mathcal{N}$ such that $h|_e = g$. Let $f := h \cdot e$. We claim that $s(e) = r(\mu)$ and $s(f) = r(\nu)$ so that $h \cdot e\mu = f\nu$. Indeed, by Lemma 2.1(1) we have

$$s(f) = s(h \cdot e) = h|_e \cdot s(e) = g \cdot s(e).$$

Since $d(g) = r(\mu)$ we have that $s(e) = r(\mu)$ and then $s(f) = g \cdot r(\mu) = r(\nu)$, proving the claim. By applying the above procedure recursively, we can construct paths $x \in Z(\mu]$ and $y \in Z(\nu]$ such that $x \sim_{\mathrm{ae}} y$. □

We can now describe a basis for the topology on the limit space of a contracting self-similar groupoid action.

**Corollary 3.10.** *Let $E$ be a finite directed graph with no sources or sinks. Let $(G, E)$ be contracting self-similar groupoid action. The sets*

$$U_\mu := \Big\{ y \in \mathcal{J} \mid q^{-1}(y) \subseteq \bigcup_{g \in \mathcal{N} \cap d^{-1}(r(\mu))} Z(g \cdot \mu] \Big\},$$

*indexed by $\mu \in E^*$, are a basis for the topology on $\mathcal{J}$.*

*Proof.* By Lemma 3.8 we know that $U_\mu$ is open. Now fix an open set $V \subseteq \mathcal{J}$ and $y \in V$.

If $x \in Z(\mu]$ for some $\mu \in E^n$, and $x' \sim_{a.e} x$, then by Lemma 3.6 there exists $g \in \mathcal{N} \cap d^{-1}(r(\mu))$ such that $x' \in Z(g \cdot \mu]$. So, if $q(x) = y$, then $y \in U_{x_{-n} \dots x_{-1}}$ for all $n \in \mathbb{N}$. Let $X_n = \bigcup_{g \in \mathcal{N} \cap d^{-1}(r(x_{-n}))} Z(g \cdot x_{-n} \dots x_{-1}]$. Since $\mathcal{N}$ is closed under restriction, $X_{n+1} \subseteq X_n$ for all $n \in \mathbb{N}$. By Lemma 3.6, $\bigcap_{n \in \mathbb{N}} X_n = q^{-1}(y)$. Hence, the compact sets $Y_n := q(X_n)$ satisfy $Y_{n+1} \subseteq Y_n$ and $\bigcap_{n \in \mathbb{N}} Y_n = \{y\}$. Therefore, there exists $k \in \mathbb{N}$ such that $Y_k \subseteq V$. Since $y \in U_{x_{-k} \dots x_{-1}} \subseteq Y_k$, the result follows. □

Our eventual goal is to show that the projective limit of copies of $\mathcal{J}$ with respect to the endomorphism induced by the shift map on $E^{-\infty}$, is a Smale space. This requires a metric that induces the topology in $\mathcal{J}$. We will build this from the following semi-metric.

**Definition 3.11** ([3, Definition 3.1.2])**.** Suppose $(X,d)$ is a metric space and $R$ is an equivalence relation on $X$. The *quotient semi-metric* $d_R$ is defined by
(3.2)

$$d_R(x,y) = \inf\Big\{ \sum_{i=0}^{k} d(p_i,q_i) \mid p_i,q_i \in X, x = p_0, y = q_k \text{ and } q_i \sim p_{i+1} \text{ for } i < k \Big\}.$$

The following fairly straightforward diagonal argument shows that if $X$ is compact and $R$ is a closed equivalence relation, then $d_R$ is a metric; this is surely known, but we could not find the result in the literature so we give a proof.

**Lemma 3.12.** *Suppose $(X,d)$ is a compact metric space and $R$ is a closed equivalence relation on $X$. Then there is a metric $\tilde{d}_R$ on $X/R$ such that $\tilde{d}_R([x],[y]) = d_R(x,y)$ for all $x,y \in X$.*

*Proof.* To see that the formula for $\tilde{d}_R$ is well defined, suppose that $(x,x')$ and $(y,y')$ belong to $R$. We must show that $d_R(x,y) = d_R(x',y')$. By symmetry it suffices to show that $d_R(x,y) \leq d_R(x',y')$. For this, observe that $p_0 = x$, $q_0 = x'$, $p_1 = y'$ and $q_1 = y$ determines a term in the infimum in (3.2) that defines $d_R(x,y)$ with value $d(x',y')$.

Since $d_R$ is a semi-metric [3, Definition 3.1.2], we now only need to show that if $d_R(x,y) = 0$ then $(x,y) \in R$. Suppose that $d_R(x,y) = 0$. Choose sequences $(p_{i,n})_{i=1}^{k_n}$ and $(q_{i,n})_{i=1}^{k_n}$ such that $p_{0,n} = x$, $q_{0,n} = y$, each $q_{i,n} \sim p_{i+1,n}$ and $\sum_{i=0}^{k_n} d(p_{i,n},q_{i,n}) < \frac{1}{2^n}$. For $n \in \mathbb{N}$ and $i > k_n$, define $p_{i,n} = q_{i,n} = y$, so that $\sum_{i=0}^{\infty} d(p_{i,n},q_{i,n}) < \frac{1}{2^n}$ for each $n$.

The closed subset $R$ of the compact set $X \times X$ is itself compact, so the sequence $\big((q_{0,n},p_{1,n})\big)_{n=1}^{\infty}$ has a subsequence, say $(q_{0,n_{1,j}},p_{1,n_{1,j}})$, that converges to some $(q_0,p_1) \in R$.

Recursively, given $i \geq 1$ and a sequence $(n_{i,j})_{j=1}^{\infty}$ such that $(q_{i-1,n_{i,j}},p_{i,n_{i,j}})$ converges in $R$, choose a subsequence $(n_{i+1,j})_{j=1}^{\infty}$ such that $n_{i+1,1} > n_{i,1}$ and $(q_{i,n_{i+1,j}},p_{i+1,n_{i+1,j}})$ converges to some $(q_i,p_{i+1}) \in R$. The resulting sequence $(q_i,p_{i+1})_{i=0}^{\infty}$ converges to $(y,y)$. We have $p_0 = x$, and for each $i$, since $d(p_{i+1,n_{i+1,j}},q_{i+1,n_{i+1,j}}) < 2^{-n_{i+1,j}}$, we have $p_i = q_i$ for each $i$. That is $x = p_0 \sim p_1 \sim p_2 \cdots \to y$. In particular, each $(x,p_i) \in R$ and we have $(x,p_i) \to (x,y)$. So using once more that $R$ is closed, we see that $(x,y) \in R$. ☐

**Corollary 3.13.** *Let $(G,E)$ be a contracting self-similar groupoid action. Then its limit space $(\mathcal{J},d_{\mathcal{J}})$ is a compact metric space.*

*Proof.* Recall that $(E^{-\infty},d)$ is compact in the product metric $d$ of (2.1). By Lemma 3.8 and Lemma 3.12, it suffices to show that $\sim_{\text{ae}}$ is a closed equivalence relation. For this, suppose that $C$ is a closed subset of $E^{-\infty}$, and fix a net $(x_\alpha) \in C$ and a point $x$ in $E^{-\infty}$ such that $q(x_\alpha)$ converges to $q(x)$. We must show that $q(x) \in q(C)$. Since $E^{-\infty}$ is compact, so is $C$, so we may assume that $x_\alpha$ converges to some $z \in C$. Hence $q(x_\alpha) \to q(z)$. So we must show that $x \sim_{\text{ae}} z$.

Fix an integer $n < 0$. By Corollary 3.10, we have that $q(x_\alpha) \in U_{x_n \dots x_{-1}}$ for large $\alpha$, and since $x_\alpha$ converges to $z$ we have that $x_\alpha \in Z(z_n \dots z_{-1}]$ for large $\alpha$. It follows that there exist $g_n \in \mathcal{N}$ such that $Z(g_n \cdot x_n \dots x_{-1}] \cap Z(z_n \dots z_{-1}] \neq \varnothing$. Since $Z(\mu] \cap Z(\nu] = \varnothing$ for distinct $\mu,\nu \in E^n$, we deduce that $g_n \cdot x_n \dots x_{-1} = z_n \dots z_{-1}$. Since $n < 0$ was arbitrary, it follows that $x \sim_{\text{ae}} z$ as claimed. ☐

3.1. **Schreier graphs and recurrent self-similar actions.** Schreier graphs define useful combinatorial approximations to the limit space of self-similar actions. We begin with the definition that suits our situation. Note that Schreier graphs of groups have a rather general definition that generalises Cayley graphs.

**Definition 3.14.** Let $(G, E)$ be a finitely generated self-similar groupoid, and let $A$ be a generating set for $G$ that is closed under inverses and restriction. The level-$n$ Schreier graph $\Gamma_n := \Gamma_n(G, A)$ is the (undirected) graph with vertex set $\Gamma_n^0 := E^n$ and an edge labelled by $a \in A$ between $\mu$ and $\nu$ if and only if $d(a) = r(\mu)$ and $a \cdot \mu = \nu$.

Note that we could label an edge in $\Gamma_n$ by either $a \in A$ or $a^{-1}$ since $A$ is closed under inverses and if $a \cdot \mu = \nu$, then $a^{-1} \cdot \nu = \mu$. We will make use of the geodesic distance, $d_{\text{geo}}$, on the vertex set of an undirected graph: $d_{\text{geo}}(v, w)$ is the minimum length of a path between $v$ and $w$. The following generalises [21, Proposition 3.6.6]; the proof is virtually identical.

**Proposition 3.15.** *Let $(G, E)$ be a finitely generated, contracting self-similar groupoid action on a finite directed graph $E$. Let $A$ be a finite generating set for $G$ that is closed under inverses and restriction. Let $\Gamma$ be the level-$n$ Schreier graph $\Gamma_n(G, A)$. There is a map $\psi_n : \Gamma_n \to \Gamma_{n-1}$ defined by*

$$\psi_n(e\mu) = \mu \quad \textit{for } e \textit{ in } E^1 \textit{ and } v \textit{ in } s(e)E^{n-1}$$
$$\psi_n(a : e\mu \to f\nu) = a|_e : \mu \to \nu.$$

*For $x, y \in E^{-\infty}$, the sequence $\big(d_{geo}(x_{-n} \dots x_{-1}, y_{-n} \dots y_{-1})\big)_{n=1}^{\infty}$ is bounded if and only if $x$ and $y$ are asymptotically equivalent.*

We now generalise Nekrashevych's notion of a recurrent self-similar group action to groupoid actions. While we do not require recurrence for the main results of this paper, it does illuminate interesting topological properties of the dynamics and the limit space. We note that Nekrashevych synonymously uses *recurrence*, *self-replicating*, and *fractal* for the notion below.

**Definition 3.16.** A self-similar groupoid action $(G, E)$ is said to be *recurrent* if, for any $e, f \in E^1$ and $h \in G$ with $d(h) = s(e)$ and $c(h) = s(f)$, there is $g$ in $Gr(e)$ such that $g \cdot e = f$ and $g|_e = h$.

Recurrence of a self-similar groupoid action is obviously a rather strong condition. For example, if $(G, E)$ is recurrent, then we immediately see that the in-degree of all vertices of the graph must be equal. Another immediate consequence of recurrence is the following.

**Proposition 3.17.** *Suppose $(G, E)$ is a recurrent self-similar groupoid action on a finite directed graph $E$. Then the action of $G$ on $E^*$ is level-transitive.*

*Proof.* For paths of length one, transitivity follows immediately from recurrence. Now suppose that for any paths $\mu$ and $\nu$ of length $n$ and any $h \in G$ with $d(h) = s(\mu)$ there exists $g \in G$ with $d(g) = r(\mu)$ such that $g \cdot \mu = \nu$ and $g|_\mu = h$; that is, $G$ acts transitively on paths of length $n$ with specified restriction as in the definition of recurrence. We now consider paths $\lambda$ and $\rho$ of length $n + 1$ and aim to show that there exists $g \in G$ such that $g \cdot \lambda = \rho$. Recurrence implies that there exists $g_{n+1} \in G$ with $d(g_{n+1}) = r(\lambda_{n+1})$ such that $g_{n+1} \cdot \lambda_{n+1} = \rho_{n+1}$. Now the inductive hypothesis implies that there exists $g \in G$ with $d(g) = r(\lambda)$ such that $g \cdot \lambda_1 \cdots \lambda_n = \rho_1 \cdots \rho_n$ and $g|_{\lambda_1 \cdots \lambda_n} = g_{n+1}$. Thus we have

$$g \cdot \lambda = g \cdot \lambda_1 \cdots \lambda_n \lambda_{n+1} = (\rho_1 \cdots \rho_n) g_{n+1} \cdot \lambda_{n+1} = \rho_1 \cdots \rho_n \rho_{n+1} = \rho,$$

the desired result. □

Following Nekrashevych, we now look to connectedness of the limit space, but first we will generalise [21, Proposition 2.11.3].

**Proposition 3.18.** *Suppose that $(G, E)$ is a contracting, recurrent self-similar groupoid action on a finite directed graph $E$ and that $G$ is finitely generated. Then the nucleus $\mathcal{N}$ of $(G, E)$ is a generating set.*

*Proof.* Let $H$ be a finite generating set for $G$. Then there exists $m \in \mathbb{N}$ such that for every $h \in H$, the set $\{h|_\mu \mid |\mu| \geq m\}$ is contained in $\mathcal{N}$. Given $g \in G$, we have $g = h_1 h_2 \cdots h_n$ with $h_i \in H$ for $1 \leq i \leq n$. Since $(G, E)$ is recurrent, for each $h_i$, there exists $a_i \in G$ and $\mu_i \in s(a_i)E^m$ such that $a_i|_{\mu_i} = h_i$. Since $a_i$ is a product of elements of $H$ it follows that $a_i|_{\mu_i} = h_i$ is a product of elements of $\mathcal{N}$. Thus $G$ is generated by $\mathcal{N}$. □

The following proof follows Nekrashevych's [21, Proposition 3.3.10 and Theorem 3.5.1], which he in turn partially attributes to K. Pilgrim and P. Haissinsky (private communication).

**Theorem 3.19.** *Suppose $(G, E)$ is a contracting self-similar groupoid action on a finite directed graph $E$ and that $G$ is finitely generated. Then the limit space $\mathcal{J}_{(G,E)}$ is connected if and only if $(G, E)$ level-transitive.*

*Proof.* First suppose that $(G, E)$ is level transitive. We suppose that $\mathcal{J} = \mathcal{J}_{(G,E)}$ is not connected, and derive a contradiction. Let $H$ be a finite generating set for $G$. Fix closed, non-empty subsets $A, B \subset \mathcal{J}$ such that $A \cup B = \mathcal{J}$ and $A \cap B = \varnothing$. Let $X_A = q^{-1}(A)$ and $X_B = q^{-1}(B)$. Then $X_A$ and $X_B$ are closed, non-empty subsets of $E^{-\infty}$ such that $X_A \cup X_B = E^{-\infty}$ and $X_A \cap X_B = \varnothing$. Define

$$X_A^n := \{a_{-n} \cdots a_{-1} \mid a = \ldots a_{-n-1} a_{-n} \ldots a_{-1} \in X_A\}.$$

Since $X_A$ is open, we write it as a union $X_A = \bigcup_{\mu \in I} Z(\mu]$ of cylinder sets. Since $X_A$ is also compact, there is a finite $F \subseteq I$ such that $X_A = \bigcup_{\mu \in F} Z(\mu]$. Put $N = \max\{|\mu| : \mu \in F\}$. For each $\mu \in F$, we have $Z(\mu] = \bigcup_{\nu \in E^*\mu \cap E^n} Z(\nu]$. So $Z(\nu] \subset X_A$ whenever for any $\nu \in X_A^n$.

Since $(G, E)$ is level-transitive and contracting, for each $n \geq N$, there exist $a_{-n} \ldots a_{-1}$ in $X_A^n$ and $g_n \in \mathcal{N}$ such that $g_n \cdot a_{-n} \cdots a_{-1} \in X_B$. By Lemma 3.9, there exist $\mu_n \in X_A \cap E^*(a_{-n} \cdots a_{-1})$ and $\nu_n \in X_B \cap E^*(g_n \cdot a_{-n} \cdots a_{-1})$. Since $X_A, X_B$ are compact, there is an increasing sequence $(n_k)_{k=1}^\infty$ of natural numbers such that $(\mu_{n_k})_k$ and $(\nu_{n_k})_k$ both converge, say to $\mu_{n_k} \to \mu \in X_A$ and $\nu_{n_k} \to \nu \in X_B$. Since $\mathcal{N}$ is finite, $\mu \sim_{\text{ae}} \nu$. So $q_{\text{ae}}(\mu) = q_{\text{ae}}(\nu) \in A \cap B$, a contradiction. Thus $\mathcal{J}$ is connected.

Now suppose that $(G, E)$ is not level-transitive. Fix $n \in \mathbb{N}$ and $a \in E^n$ such that $G \cdot a \neq E^n$. Define $A' := \bigcup_{a' \in G \cdot p} Z(a']$ and $B' := \bigcup_{b' \in E^n \setminus G \cdot p} Z(b']$. Then, $A := q(A')$ and $B := q(B')$ are disjoint compact sets in $\mathcal{J}$ such that $A \cup B = \mathcal{J}$. □

**Corollary 3.20.** *Suppose that $(G, E)$ is a contracting and recurrent self-similar groupoid action on a finite directed graph $E$ such that $G$ is finitely generated. Then the limit space $\mathcal{J}_{(G,E)}$ is connected.*

*Example* 3.21. Consider Example 2.2. We claim that the action is contracting with nucleus $\mathcal{N} = \{v, w, a, b, a^{-1}, b^{-1}\}$. Indeed, since all elements of the automaton appear as restrictions, $v, w, a, b, a^{-1}, b^{-1} \in \mathcal{N}$. To see that this is everything we compute

$$(ab)|_3 = a|_{b \cdot 3} b|_3 = v \qquad (ba)|_1 = b|_{a \cdot 1} a|_1 = a$$
$$(ab)|_4 = a|_{b \cdot 4} b|_4 = ba \qquad (ba)|_2 = b|_{a \cdot 2} a|_2 = b$$

and all groupoid elements of length 2 restrict to the nucleus.

The first two Schreier graphs are depicted in Figure 3. More generally, the $n$th Schreier graph is a cycle of length $2^n$ with a loop at each vertex, showing that the

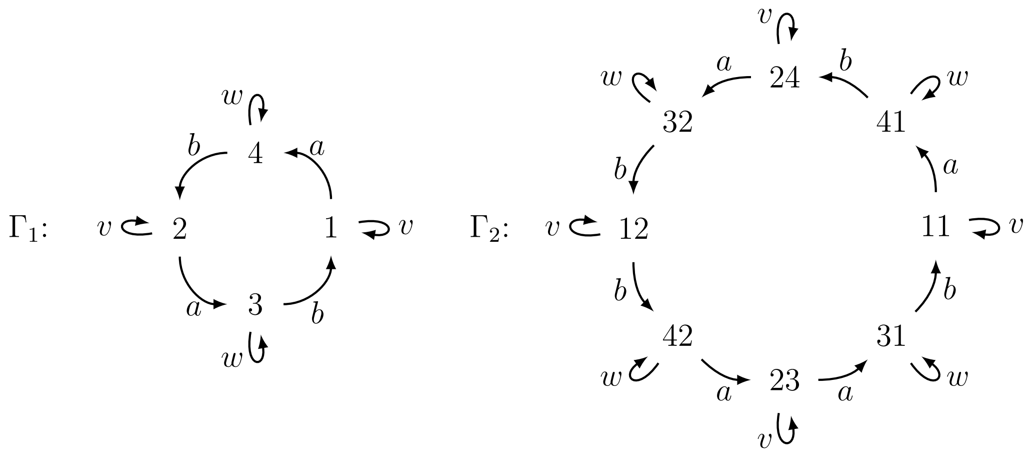



FIGURE 3. The first two Schreier graphs of Example 3.21.

action is level-transitive. This also suggests that the limit space is homeomorphic to a circle. One can prove this, by showing inductively that the vertices of the $n$th Schreier graph can be mapped to the $n$th roots of unity on the complex circle, metrised so that it has circumference 1, in a way that extends the pictures in Figure 3. Specifically, each vertex is connected in the Schreier graph to its two nearest neighbours on the circle, and for any infinite path $\mu \in E^{-\infty}$ the images of its initial segments, regarded as vertices of Schreier graphs, on the unit circle converge. The map that sends $\mu$ to the limit-point is the desired homeomorphism: it is continuous because it is a contraction; it is surjective because its image is both dense and compact; and one checks that it is injective using the final statement of Proposition 3.15.

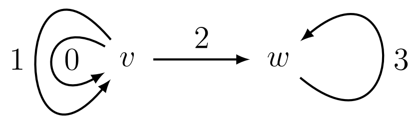



FIGURE 4. Graph $E$ for Example 3.22

*Example* 3.22. Consider the graph $E$ in Figure 3.22, and define a self-similar groupoid through the $E$-automaton

$$\begin{array}{llllll} a\cdot 0 = 2 & a|_0 = v & b\cdot 2 = 0 & b|_2 = v & c\cdot 2 = 1 & c|_2 = v \\ a\cdot 1 = 3 & a|_1 = a & b\cdot 3 = 1 & b|_3 = c & c\cdot 3 = 0 & c|_3 = b. \end{array} \tag{3.3}$$

We claim that this action is contracting with nucleus

$$\mathcal{N} = \{v, w, a, b, c, ba, ca, a^{-1}, b^{-1}, c^{-1}, (ba)^{-1}, (ca)^{-1}\}.$$

To see this we note that all elements of the automaton appear as restrictions, so $v, w, a, b, c$ and their inverses are in the nucleus. That $ba$ and $ca$ are in the nucleus follow from the computations

$$(ba)|_1 = b|_{a\cdot 1}a|_1 = ca \quad \text{and} \quad (ca)|_1 = c|_{a\cdot 1}a|_1 = ba.$$

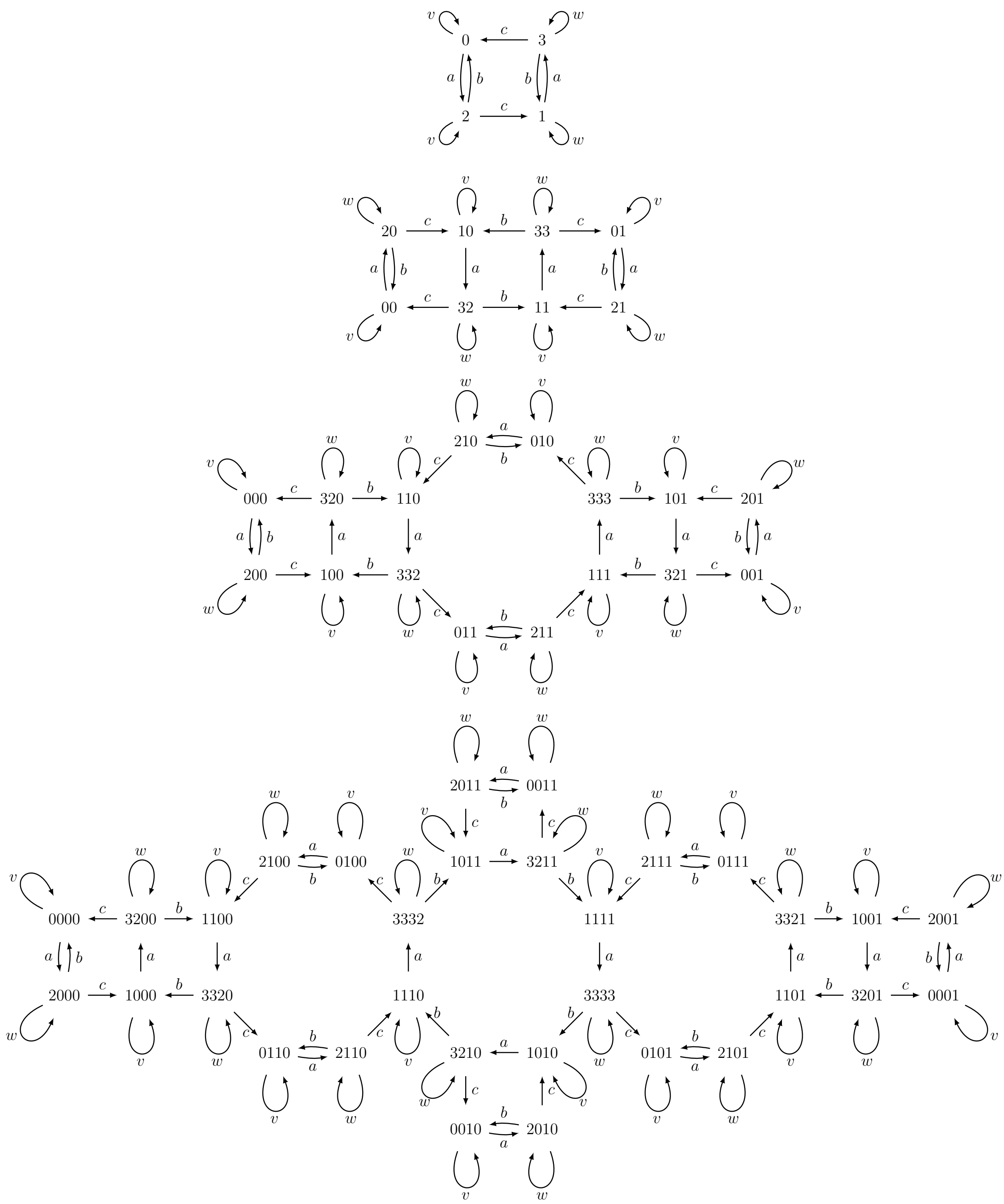



FIGURE 5. The first four Schreier graphs of Example 3.22.

One can now compute that all groupoid elements of length 3 reduce to one of the elements of the nucleus after restriction to length 2 words.

The first four Schreier graphs are presented in Figure 5. They suggest that the action is level-transitive and that the limit-space is homeomorphic to the basilica fractal (the Julia set of $z^2 - 1$); one can prove this via an argument of the sort outlined in Example 3.21.

## 4. Dynamics on the limit space

In this section we describe an action of $\mathbb{N}$ by locally expansive local homeomorphisms of the limit space $\mathcal{J}$ of a contracting, regular self-similar groupoid action. We will use this in the next section to construct a Smale space from the self-similar groupoid action.

Let $E$ be a finite directed graph. The *shift map* $\sigma : E^{-\infty} \to E^{-\infty}$ is defined by $\sigma(\dots x_{-3}x_{-2}x_{-1}) = \dots x_{-3}x_{-2}$; that is, $\sigma$ deletes the right-most edge of a left-infinite path. This $\sigma$ is a local homeomorphism because it restricts to a homeomorphism $\sigma : Z(\mu e] \to Z(\mu]$ for any finite path $\mu$ and any edge $e$ such that $s(\mu) = r(e)$. The main result in this section is about self-similar groupoid actions that are *regular* in the following sense, which is based on the regularity condition used by Nekrashevych in [22].

**Definition 4.1** (cf. [22, Definition 6.1])**.** Let $E$ be a finite directed graph. Let $(G, E)$ be a self-similar groupoid action. We say that $(G, E)$ is *regular* if for every $g \in G$ and every $y \in E^\infty$ such that $g \cdot y = y$, there exists $\mu$ in $E^*$ such that $y \in Z[\mu)$, $g \cdot \mu = \mu$ and $g|_\mu = s(\mu)$.

*Remark* 4.2. Since, by definition, self-similar groupoid actions are faithful, the regularity condition is equivalent to the condition that if $y \in E^\infty$ and $g \cdot y = y$, then there is a clopen neighbourhood of $y$ that is pointwise fixed by $g$.

Our main theorem in this section says that for contracting, regular self-similar groupoid actions, the shift map induces a locally expanding local homeomorphism of $\mathcal{J}$.

**Theorem 4.3.** *Let $E$ be a finite directed graph with no sources. Let $(G, E)$ be a contracting, regular self-similar groupoid action with limit space $\mathcal{J}$ as in Definition 3.2. Let $\sigma : E^{-\infty} \to E^{-\infty}$ be the shift map. Then there is a surjective map $\tilde{\sigma} : \mathcal{J} \to \mathcal{J}$ such that $\tilde{\sigma}([x]) = [\sigma(x)]$ for all $x \in E^{-\infty}$. Furthermore, there exists $\varepsilon > 0$ such that*

1. *whenever $d_{\mathcal{J}}([x], [y]) < \varepsilon$, we have $d_{\mathcal{J}}(\tilde{\sigma}([x]), \tilde{\sigma}([y])) = 2d_{\mathcal{J}}([x], [y])$, and*
2. *whenever $\alpha \le \varepsilon$, we have $\tilde{\sigma}(B([x], \alpha)) = B(\tilde{\sigma}([x]), 2\alpha)$.*

*In particular, $\tilde{\sigma}$ is a locally expanding local homeomorphism.*

Before proving the theorem, we need to establish some preliminary results. To get started, observe that if $x \sim_{\mathrm{ae}} y$, then there is a sequence $(g_n)_{n<0} \in \mathcal{N}$ such that $g_n \cdot x_n \dots x_{-1} = y_n \dots y_{-1}$ for all $n$, and it follows that $g_{n-1} \cdot \sigma(x)_n \dots \sigma(x)_{-1} = g_{n-1} \cdot x_{n-1} \dots x_{-2} = y_{n-1} \dots y_{-2} = \sigma(y)_n \dots \sigma(y)_{-1}$. That is,

$$x \sim_{\mathrm{ae}} y \Longrightarrow \sigma(x) \sim_{\mathrm{ae}} \sigma(y). \tag{4.1}$$

Therefore, there exists a map $\tilde{\sigma} : \mathcal{J} \mapsto \mathcal{J}$ as described in Theorem 4.3.

**Lemma 4.4.** *Let $E$ be a finite directed graph with no sources. Let $(G, E)$ be a regular self-similar groupoid action. For any finite set $F \subseteq G$, there exists $k \in \mathbb{N}$ such that for all $g, h \in F$ such that $d(g) = d(h)$ and all $\mu \in d(g)E^*$ with $|\mu| \ge k$, if $g \cdot \mu = h \cdot \mu$, then $g|_\mu = h|_\mu$.*

*Proof.* Fix $g, h \in F$.

For each $y \in E^\infty$ satisfies $g \cdot y = h \cdot y$, we have $h^{-1}g \cdot y = y$, and so regularity implies that there exists $\lambda_y \in E^*$ such that $y \in Z[\lambda_y)$ and $(h^{-1}g)|_{\lambda_y} = s(\lambda_y)$. For each $x \in E^\infty$ such that $g \cdot x \ne h \cdot x$, we have $h^{-1}g \cdot x \ne x$, and so there exists $\lambda_x \in E^*$ such that $x \in Z[\lambda_x)$ and $(h^{-1}g) \cdot \lambda_x \ne \lambda_x$.

Since $E^\infty = \bigcup_{x \in E^\infty} Z[\lambda_x)$ and since $E^\infty$ is compact, there exists a finite $K \subseteq E^\infty$ such that $E^\infty = \bigcup_{x \in K} Z[\lambda_x)$. Let $k_{g,h} := \max\{|\lambda_x| : x \in K\}$. Suppose that $\mu \in E^*$

with $|\mu| \geq k_{g,h}$ and that $g \cdot \mu = h \cdot \mu$. Since $E$ has no sources we have $Z[\mu) \neq \emptyset$. Since the $Z[\lambda_x)$ cover $E^\infty$ we have $Z[\mu) \cap Z[\lambda_x) \neq \varnothing$ for some $x \in K$. Since $|\mu| \geq |\lambda_x|$, it follows that $\mu = \lambda_x \mu'$ for some $\mu'$ in $E^\infty$. Since $g \cdot \mu = h \cdot \mu$, we have $g \cdot \lambda_x = h \cdot \lambda_x$ and therefore $h^{-1}g \cdot \lambda_x = \lambda_x$. By the choice of $\lambda_x$, we have $h^{-1}g \cdot x = x$ and $(h^{-1}g)|_{\lambda_x} = s(\lambda_x)$. Hence $(h^{-1}g)|_\mu = (h^{-1}g)|_{\lambda_x\mu'} = s(\lambda_x)|_{\mu'} = s(\mu') = s(\mu)$. Hence $g|_\mu = h|_\mu$.

We have now proved that for each $g, h \in F$ with $d(g) = d(h)$, there exists $k_{g,h} \in \mathbb{N}$ such that whenever $\mu \in d(g)E^{k_{g,h}}$ satisfies $g \cdot \mu = h \cdot \mu$, we have $g|_\mu = h|_\mu$. So $k = \max_{g,h \in F} k_{g,h}$ has the required property. □

Our next result is essentially a version of Theorem 4.3 in which the metric balls and $\varepsilon$-approximations are replaced by conditions in terms of the basic open sets from Corollary 3.10. We will bootstrap from this result to prove Theorem 4.3.

**Proposition 4.5.** *Let $E$ be a finite directed graph with no sources. If $(G, E)$ is a contracting and regular self-similar group action, then*

1. *for each $z \in \mathcal{J}$, $\sigma$ maps $q^{-1}(z)$ bijectively onto $q^{-1}(\tilde{\sigma}(z))$;*
2. *there exists $k \in \mathbb{N}$ such that for every $n \geq k+1$ and every $\mu \in E^n$, the map $\tilde{\sigma}$ restricts to a bijection of $U_\mu$ onto $U_{\sigma(\mu)}$; and*
3. *for every $n \geq k$, every $\omega \in E^n$, every $w \in U_\omega$, and every $z \in \tilde{\sigma}^{-1}(w)$, there exists $\mu \in E^{n+1}$ such that $z \in U_\mu$ and $\sigma(\mu) = \omega$.*

*In particular, $\tilde{\sigma}$ is a local homeomorphism.*

*Proof.* Applying Lemma 4.4 to the finite set $F = \mathcal{N}^2 \cup \mathcal{N} \cup E^0$ yields $k \in \mathbb{N}$ such that for all $n_1, n_2 \in F$, if $\mu \in E^*$ with $|\mu| \geq k$ satisfies $n_1 \cdot \mu = n_2 \cdot \mu$, then $n_1|_\mu = n_2|_\mu$. We fix $k$ with this property for the remainder of the proof.

(1) Since $q \circ \sigma = \tilde{\sigma} \circ q$, if $q(x) = z$ then $q(\sigma(x)) = \tilde{\sigma}(q(x)) = \tilde{\sigma}(z)$, and so $\sigma(q^{-1}(z)) \subseteq q^{-1}(\tilde{\sigma}(z))$. So we must prove the reverse inclusion. Suppose that $q(x) = z$ and $y' \sim_{\mathrm{ae}} \sigma(x)$. Let $(g_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{N}$ such that $d(g_n) = r(x_{-n-1})$ and $g_n \cdot x_{-n-1}x_{-n}...x_{-2} = y'_{-n}y'_{-n+1}...y'_{-1}$ for every $n \in \mathbb{N}$. By the choice of $k$, for all $n, n' \geq k$, it follows that $g_n|_{x_{-n-1}x_{-n}...x_{-2}} = g_{n'}|_{x_{-n'-1}x_{-n'}...x_{-2}}$. Let $x'_{-1} = (g_k|_{x_{-k-1}x_{-k}...x_{-2}}) \cdot x_{-1}$ and $x' = y'x'_{-1}$, and let $g'_n = g_k|_{x_{-k-1}x_{-k}...x_{-n-1}}$ for $n < k$, and $g'_n = g_{n-1}$ for $n \geq k$. Then $g'_n \cdot x_{-n}x_{-n+1}...x_{-1} = x'_{-n}x'_{-n+1}...x'_{-1}$ for every $n \in \mathbb{N}$. So, $x \sim_{\mathrm{ae}} x'$ and $\sigma(x') = y'$. Therefore, $\sigma(q^{-1}(z)) = q^{-1}(\tilde{\sigma}(z))$.

Suppose that $x', x'' \in q^{-1}(z)$ satisfy $\sigma(x'') = \sigma(x') = y'$. Since $x' \sim_{\mathrm{ae}} x \sim_{\mathrm{ae}} x''$, there exists $g$ in $\mathcal{N}$ such that $g \cdot x'_{-k-1}x'_{-k}...x'_{-1} = x''_{-k-1}x''_{-k}...x''_{-1}$. By the choice of $k$ and since $x'_{-k-1}x'_{-k}...x'_{-2} = x''_{-k-1}x''_{-k}...x''_{-2}$, we have $g|_{x'_{-k-1}x'_{-k}...x'_{-2}} = s(x'_{-2})$. Hence $x''_{-1} = (g|_{x'_{-k-1}x'_{-k}...x'_{-2}}) \cdot x'_{-1} = x_{-1}$. Therefore, $x' = x''$.

(2) Fix $\mu \in E^*$ with $|\mu| > k$ and $w \in U_{\sigma(\mu)}$. For each $y \in q^{-1}(w)$, there exists $g \in \mathcal{N}$ such that $y \in Z[g \cdot \sigma(\mu))$. By the choice of $k$, the element $g|_{\sigma(\mu)}$ does not depend on the choice of $g$. So there is a unique map $\delta : q^{-1}(U_{\sigma(\mu)}) \to E^{-\infty}$ such that $\delta(y) = y\big((g|_{\sigma(\mu)}) \cdot \mu_{-1}\big)$ for any $g \in \mathcal{N}$ such that $y \in Z[g \cdot \sigma(\mu))$. We claim that $\delta$ descends to a map $\tilde{\delta} : U_{\sigma(\mu)} \to U_\mu$ that is an inverse for $\tilde{\sigma}|_{U_\mu}$.

For this, suppose that $y, y' \in q^{-1}(w)$ are asymptotically equivalent. Fix $g \in \mathcal{N}$ such that $y \in Z(g \cdot \sigma(\mu)]$ and $g' \in \mathcal{N}$ such that $y' \in Z(g' \cdot \sigma(\mu)]$. Then $x := \delta(y)$ satisfies $x = y\big((g|_{\sigma(\mu)}) \cdot \mu_{-1}\big) \in Z(g \cdot \mu]$. By (1) there is a unique element $x' \in E^{-\infty}$ such that $x' \sim x$ and $\sigma(x') = y'$. Since $x \sim_{\mathrm{ae}} x'$, there exists $h \in \mathcal{N}$ such that $h \cdot (x_{-k-1} \cdots x_{-1}) = x'_{-k-1}x'_{-k} \cdots x'_{-1}$. Hence $(hg) \cdot \sigma(\mu) = x'_{-k-1}x'_{-k} \cdots x'_{-2} = g' \cdot \sigma(\mu)$. Since $hg, g' \in F$, by the choice of $k$ we have $(hg)|_{\sigma(\mu)} = g'|_{\sigma(\mu)}$ and we deduce that $x'_{-1} = \big((hg)|_{\sigma(\mu)}\big) \cdot \mu_{-1} = \big(g'|_{\sigma(\mu)}\big) \cdot \mu_{-1}$. By definition, we have $\delta(y') = y'\big(g'|_{\sigma(\mu)}\big) \cdot \mu_{-1} =$

$y'x'_{-1}$, and since $\sigma(x') = y'$ by definition of $x'$, we deduce that $\delta(y') = \sigma(x')x'_{-1} = x'$. So $\delta(y') = x' \sim_{\mathrm{ae}} x = \delta(y)$, and it follows that $\delta$ descends to a map $\tilde{\delta} : U_{\sigma(\mu)} \to \mathcal{J}$.

To see that $\tilde{\delta}(U_{\sigma(\mu)}) \subseteq U_\mu$, fix $y \in q^{-1}(w)$ and let $x = \delta(y)$. We must show that $[x] \subseteq \bigcup_{\nu\in\mathcal{N}\cdot\mu} Z(\nu]$. Fix $x' \in [x]$, and let $y' = \sigma(x)$. Applying the argument of the preceding paragraph we obtain $x' = \delta(y') \in Z((]g' \cdot \mu)$ for some $g' \in \mathcal{N}$ as required.

It remains to show that $\tilde{\delta}$ is an inverse for $\tilde{\sigma}|_{U_\mu}$. By construction, $\sigma\delta(y) = y$ for $y \in q^{-1}(U_{\sigma(\mu)})$, so $\tilde{\sigma}\tilde{\delta}(q(y)) = q(y)$. Therefore, $\tilde{\sigma}\tilde{\delta} = \mathrm{id}_{U_{\sigma(\mu)}}$.

We now show that $\tilde{\delta}\tilde{\sigma} = id_{U_\mu}$. If $q^{-1}(z) \subseteq \bigcup_{g\in\mathcal{N}:d(g)=r(\mu)} Z[g\cdot\mu)$, then $q^{-1}(\tilde{\sigma}(z)) = \sigma(q^{-1}(z)) \subseteq \sigma(\bigcup_{g\in\mathcal{N}:d(g)=r(\mu)} Z[g\cdot\mu)) = \bigcup_{g\in\mathcal{N}:d(g)=r(\sigma(\mu))} Z[g\cdot\sigma(\mu))$. Hence, $\tilde{\sigma}(U_\mu) \subseteq U_{\sigma(\mu)}$, so the composites $\tilde{\delta}\tilde{\sigma}$ and $\delta\sigma$ are well defined on $U_\mu$ and $q^{-1}(U_\mu)$ respectively. We have $\delta(y) = y(g|_{\sigma(\mu)}) \cdot \mu_{-1}$ for $y \in q^{-1}(U_{\sigma(\mu)}) \cap Z[g\cdot\sigma(\mu))$, and so $\delta\sigma(x) = x$ for $x$ in $q^{-1}(U_\mu) \cap Z[g\cdot\mu)$. Hence, $\tilde{\delta}\tilde{\sigma}(q(x)) = q(x)$ for $x \in q^{-1}(U_\mu)$. Therefore, $\tilde{\delta}\tilde{\sigma} = \mathrm{id}_{U_\mu}$, and we have shown $\tilde{\sigma}$ maps $U_\mu$ bijectively onto $U_{\sigma(\mu)}$.

(3) Fix $\omega \in E^*$ with $|\omega| \geq k$ and $w \in U_\omega$, and fix $z \in \tilde{\sigma}^{-1}(w)$. Choose $x, y \in E^{-\infty}$ such that $q(x) = z$, $q(y) = w$, and $\sigma(x) = y$. Then there exists $g \in Gr(\omega)$ such that $y \in Z[g\cdot\omega)$. Since $\sigma(x) = y$, it follows that $x_{-n-1}x_{-n}\dots x_{-1} = (g\cdot\omega)x_{-1}$. We have $d((g|_\omega)^{-1}) = c(g|_\omega) = s(g\cdot\omega) = r(x_{-1})$. Let $f = (g|_\omega)^{-1}\cdot x_{-1}$ and $\mu = \omega f$. Then $x \in Z(g\cdot\mu)$ and $\sigma(\mu) = \omega$, and the map $\delta : q^{-1}(U_{\sigma(\mu)}) \to q^{-1}(U_\mu)$ constructed in the proof of (2) satisfies $\delta(y) = x$. Hence $z = q(x) \in U_\mu$. □

To establish Condition (2) in Theorem 4.3, we will use the following technical lemma, which we will need again in the proof of Lemma 8.6.

**Lemma 4.6.** *Resume the hypotheses of Theorem 4.3. Suppose that $\varepsilon > 0$ satisfies Condition (1) of that theorem. Then there exist $\eta < \varepsilon$ and $n \in \mathbb{N}$ such that*

1. *for every $w \in \mathcal{J}$, and $\alpha \leq \eta$, there exists $\omega \in E^*$ such that $|\omega| \geq n-1$ and $B(w, 2\alpha) \subseteq U_\omega$; and*
2. *for every $\mu \in E^*$ with $|\mu| \geq n$, the map $\tilde{\sigma}$ restricts to a homeomorphism of $U_\mu$ onto $U_{\sigma(\mu)}$; and if $\tilde{\delta}$ denotes its inverse, then for all $z \in U_\mu$ and all $\alpha \leq \eta$ such that $B(\tilde{\sigma}(z), 2\alpha) \subseteq U_{\sigma(\mu)}$, we have $B(z, \alpha) \subseteq U_\mu$, and $\tilde{\delta}$ restricts to a homeomorphism of $B(\tilde{\sigma}(z), 2\alpha)$ onto $B(z, \alpha)$.*

*Proof.* For $\mu, \omega \in E^*$ such that $s(\mu) = r(\omega)$, we have $U_{\mu\omega} \subseteq U_\omega$, and for any infinite path $x \in E^{-\infty}$, we have $\bigcap_{n\in\mathbb{N}} U_{x_{-n}\dots x_{-1}} = [x]$. Hence $\lim_{n\to\infty}\sup_{\mu\in E^n} \mathrm{diam}(U_\mu) = 0$. Let $k$ be as in Proposition 4.5(2), and fix $n > k$ so that $\mathrm{diam}(U_\mu) < \varepsilon$ whenever $|\mu| \geq n$. Using compactness of $\mathcal{J}$, fix $K \subseteq \bigcup_{m\geq n-1} E^m$ such that $\mathcal{J} \subseteq \bigcup_{\omega\in K} U_\omega$. The Lebesgue Number Lemma yields $0 < \eta < \varepsilon$ such that for every $w$ in $\mathcal{J}$, there exist $\omega \in K$ such that $B(w, 2\eta) \subseteq U_\omega$. These values of $n, \eta$ satisfy (1) by construction, so we just have to establish (2).

For this, let $\mu$ be a path such that $|\mu| \geq n$. Since $n \geq k$, by Proposition 4.5(2), $\tilde{\sigma}$ maps $U_\mu$ homeomorphically onto $U_{\sigma(\mu)}$. Suppose that $B(\tilde{\sigma}(z), 2\alpha) \subseteq U_{\sigma(\mu)}$ for $z \in U_\mu$ and $\alpha \leq \eta$. By hypothesis, $\varepsilon$ satisfies Theorem 4.3(1), and so since $\mathrm{diam}(U_\mu) < \varepsilon$, we have

$$2d_{\mathcal{J}}(\tilde{\delta}(\tilde{\sigma}z), \tilde{\delta}(w)) = d_{\mathcal{J}}(\tilde{\sigma}\tilde{\delta}(\tilde{\sigma}z), \tilde{\sigma}\tilde{\delta}(w)) = d_{\mathcal{J}}(\tilde{\sigma}z, w)$$

whenever $d_{\mathcal{J}}(\tilde{\sigma}z, w) < 2\alpha$. Hence, $\tilde{\delta}(B(\tilde{\sigma}z, 2\alpha)) \subseteq B(z, \alpha) \cap U_\mu$, so that $B(\tilde{\sigma}z, 2\alpha) \subseteq \tilde{\sigma}(B(z, \alpha) \cap U_\mu)$. Since $\alpha \leq \eta < \varepsilon$, another application of Theorem 4.3(1) implies that $\tilde{\sigma}$ restricts to a homeomorphism of $B(z, \alpha)$, and that $\tilde{\sigma}(B(z, \alpha)) \subseteq B(\tilde{\sigma}(z), 2\alpha)$. Therefore, $B(z, \alpha) \cap U_\mu = B(z, \alpha)$, and $\tilde{\sigma}(B(z, \alpha)) = B(\tilde{\sigma}(z), 2\alpha)$. Hence, $B(z, \alpha) \subseteq U_\mu$, so that $\tilde{\delta} \circ \tilde{\sigma}|_{B(z,\alpha)}$ is well defined, and $\tilde{\delta}(B(\tilde{\sigma}(z), 2\alpha)) = \tilde{\delta}\tilde{\sigma}(B(z, \alpha)) = B(z, \alpha)$. □

Now we prove the main result of this section. For the following proof, given $x, y \in E^{-\infty}$, a *chain* from $x$ to $y$ is a pair $(P, Q)$ of finite sequences $P = (p_i)_{i=0}^k$ and $Q = (q_i)_{i=0}^k$ in $E^{-\infty}$ such that $p_0 = x$, $q_k = y$, and $q_i \sim_{\text{ae}} p_{i+1}$ for all $1 \le i \le k-1$.

*Proof of Theorem 4.3.* Since $E$ has no sources, $\sigma$, and hence also $\tilde\sigma$ is surjective.

Let $R := \{(x_1, x_2) \in E^{-\infty} \times E^{-\infty} \mid x_1 \sim_{\text{ae}} x_2\}$ so that $\sigma^* R = \{(x_1, x_2) \in E^{-\infty} \times E^{-\infty} \mid \sigma(x_1) \sim_{\text{ae}} \sigma(x_2)\}$. By Proposition 4.5(2), $\tilde\sigma$ is a local injection, so $R$ is an open subspace of $\sigma^* R$. Therefore, $\sigma^* R \setminus R$ is a compact set. It follows that there exists $\varepsilon' > 0$ such that $d_{\mathcal{J}}(x, y) \ge \varepsilon'$ for all $(x, y) \in \sigma^* R \setminus R$. Let $\varepsilon := \min\{\frac{\varepsilon'}{2}, \frac{1}{4}\}$.

We first show that this $\varepsilon$ satisfies (1).

Fix $x, y \in E^{-\infty}$ such that $d_{\mathcal{J}}(x, y) < \varepsilon$. We must show that $\tilde d_{\mathcal{J}}(\tilde\sigma([x]), \tilde\sigma([y])) = 2d_{\mathcal{J}}(x, y)$.

We first show that $d_{\mathcal{J}}(\sigma(x), \sigma(y)) \le 2d_{\mathcal{J}}(x, y)$. To see this, it suffices to show that $d_{\mathcal{J}}(\sigma(x), \sigma(y)) \le 2d_{\mathcal{J}}(x, y) + 2\eta$ for all $\eta < \varepsilon - d_{\mathcal{J}}(x, y)$. So fix $\eta < \varepsilon - d_{\mathcal{J}}(x, y)$. Let $(P, Q)$ be a chain from $x$ to $y$ such that $\sum_{i=0}^k d(p_i, q_i) \le d_{\mathcal{J}}(x, y) + \epsilon$. Consider the chain $(\sigma(P), \sigma(Q))$ from $\sigma(x)$ to $\sigma(y)$. Since $d(p_i, q_i) \le d_{\mathcal{J}}(x, y) + \epsilon \le \beta \le \frac{1}{2}$, we have $d(\sigma(p_i), \sigma(q_i)) = 2d(p_i, q_i)$. Hence, $d_{\mathcal{J}}(\sigma(x), \sigma(y)) \le \sum_{i=0}^k d(\sigma(p_i), \sigma(q_i)) \le 2d_{\mathcal{J}}(x, y) + 2\epsilon$, proving the claim.

Now we show that $d_{\mathcal{J}}(\sigma(x), \sigma(y)) \ge 2d_{\mathcal{J}}(x, y)$. Fix $0 < \eta < \varepsilon - d_{\mathcal{J}}(x, y)$; it suffices to show that $d_{\mathcal{J}}(\sigma(x), \sigma(y)) + \eta \ge 2d_{\mathcal{J}}(x, y)$. Let $(P', Q')$ be a chain from $\sigma(x)$ to $\sigma(y)$ such that $\sum_{i=0}^k d(p_i', q_i') \le d_{\mathcal{J}}(\sigma(x), \sigma(y)) + 2\eta$. Since $d_{\mathcal{J}}(p_0', q_0') \le \frac{1}{2}$, we have $s(p_0') = s(q_0')$. Hence so $p_0 := x$ and $q_0 := q_0' x_{-1}$ are paths in $E^{-\infty}$ with $d(p_0, q_0) = \frac{1}{2} d(p_0', q_0')$. By Proposition 4.5(1), there exists a path $p_1 \sim_{\text{ae}} q_0$ such that $\sigma(p_1) = p_1'$. Again, since $d(p_1', q_1') \le \frac{1}{2}$, $q_1 := q_1'(p_1)_{-1}$ is a path, and $d(p_1, q_1) = \frac{1}{2} d(p_1', q_1')$. Proceeding this way, we obtain a chain $(P, Q)$ from $x$ to a path $q_k$ such that $(\sigma(P), \sigma(Q)) = (P', Q')$ and $\sum_{i=0}^k d_{\mathcal{J}}(p_i, q_i) = \frac{1}{2} \sum_{i=0}^k d_{\mathcal{J}}(p_i', q_i')$.

We claim that $q_k = y$. Since $d(x, q_k) < \varepsilon$ and $d(x, y) < \varepsilon$, the triangle inequality implies that $d_{\mathcal{J}}(y, q_k) < \varepsilon'$. Since $\varepsilon'$ is the lower bound of $d_{\mathcal{J}}$ on $\sigma^* R \setminus R$ and $(y, q_k) \in \sigma^* R$, we have $q_k \sim_{\text{ae}} y$. Proposition 4.5(1) implies that$\sigma : q^{-1}(q(y)) \mapsto q^{-1}(q(\sigma(y))$ is a bijection. So since $q_k \sim_{\text{ae}} y$ and $\sigma(q_k) = \sigma(y)$ we have $q_k = y$ as claimed.

It follows that $(P, Q)$ is a chain from $x$ to $y$ and $2d_{\mathcal{J}}(x, y) \le 2\sum_{i=0}^k d_{\mathcal{J}}(p_i, q_i) = \sum_{i=0}^k d_{\mathcal{J}}(p_i', q_i') \le d_{\mathcal{J}}(\sigma(x), \sigma(y)) + \eta$ as required.

(2) By Lemma 4.6(1), for any $z \in \mathcal{J}$ and $\alpha \le \varepsilon$, there exists $\omega \in E^*$ with $|\omega| \ge n-1$ and $B(\tilde\sigma(z), 2\alpha) \subseteq U_\omega$. By Proposition 4.5(3), there exists $f \in E^1$ such that $s(\omega) = r(f)$ and $z \in U_{\omega f}$. Lemma 4.6(2) then gives $B(z, \alpha) \subseteq U_{\omega e}$ and $\tilde\sigma(B(z, \alpha)) = \tilde\sigma\tilde\delta(B(\tilde\sigma(z), 2\alpha)) = B(\tilde\sigma(z), 2\alpha)$. □

To finish this section, we will show that for strongly-connected graphs, the regularity hypothesis is necessary in the preceding result (we will need to restrict to strongly-connected graphs later in order to apply Kaminker, Putnam and Whittaker's results about $KK$-duality for $C^*$-algebras associated to Smale spaces). Recall that a directed graph $E$ is *strongly connected* if it has at least one edge, and if for all $v, w \in E^0$ the set $vE^*w$ is nonempty.

**Lemma 4.7.** *Let $E$ be a finite directed graph with no sources. Let $(G, E)$ be a contracting self-similar groupoid action with nucleus $\mathcal{N}$. Then there exists $\mu \in E^*$ such that whenever $g \in \mathcal{N}$ satisfies $d(g) = r(\mu)$ and $g \cdot \mu = \mu$, we have $g \cdot \mu\nu = \mu\nu$ for all $\nu$ in $s(\mu)E^*$.*

*Proof.* Suppose first that there is $x \in E^\infty$ such that $g \cdot x \neq x$ for all $g \in \mathcal{N} \setminus E^0$ satisfying $d(g) = r(x)$. Choose $n \in \mathbb{N}$ so that $g \cdot x_1...x_n \neq x_1....x_n$ for all $g$ in $\mathcal{N} \setminus E^0$ satisfying $d(g) = r(x)$. Then $\mu = x_1...x_n$ has the desired property.

Now suppose that for every $x \in E^\infty$, there exists $g \in \mathcal{N} \setminus E^0$ such that $d(g) = r(x)$ and $g \cdot x = x$. Then, $\bigcup_{g \in \mathcal{N} \setminus E^0} \{x \in d(g)E^\infty : g \cdot x = x\} = E^\infty$. Since a finite intersection of open dense sets is itself an open dense set, there exists $x \in E^\infty$ that does not belong to the boundary of $\{x \in d(g)E^\infty : g \cdot x = x\}$ for any $g \in \mathcal{N} \setminus E^0$. So, there is an $n \in \mathbb{N}$ such that for all $g$ in $\mathcal{N} \setminus E^0$ with $d(g) = r(x)$, either $g \cdot x_1...x_n \neq x_1...x_n$ or $g \cdot y = y$ for all $y$ in $Z[x_1...x_n)$. Hence, $\mu = x_1...x_n$ has the desired property. □

**Proposition 4.8.** *Let $E$ be a strongly connected finite directed graph. Let $(G, E)$ be a contracting self-similar groupoid action with nucleus $\mathcal{N}$. If $\tilde{\sigma} : \mathcal{J} \to \mathcal{J}$ is a local homeomorphism, then $(G, E)$ is regular. Hence, $(G, E)$ is regular if and only if $\tilde{\sigma}$ is a local homeomorphism*

*Proof.* Suppose that $(G, E)$ is not regular. It suffices to show that there exists $k \in \mathbb{N}$ such that $\tilde{\sigma}^k$ is not locally injective. Since $(G, E)$ is not regular, there exist $x \in E^\infty$ and $g \in G$ such that $g \cdot x = x$ but $g$ fixes no neighbourhood of $x$. So there is a strictly increasing sequence $(n_j)$ in $\mathbb{N}$ and paths $\alpha_j \in s(x_{n_j})E^*$ such that $g \cdot x_1 \dots x_{n_j}\alpha_j \neq x_1 \dots x_{n_j}\alpha_j$. In particular, the elements $h_j := g|_{x_1 \dots x_j}$ satisfy $h_j \cdot \alpha_j \neq \alpha_j$ for all $j$. By Lemma 3.4 we have $h_j \in \mathcal{N}$ for large $j$. Since $\mathcal{N}$ is finite, by passing to a subsequence, we can assume that $h_j = h_1 =: h$ for all $j$ (and hence $s(x_{n_j}) = d(h)$ for all $j$). So $\alpha := \alpha_1 \in d(h)E^*$ satisfies $h \cdot \alpha \neq \alpha$; let $\beta := h \cdot \alpha$.

For each $j$, fix $y_j \in Z(x_1 \dots x_{n_j}] \subseteq E^{-\infty}$. Since $E^{-\infty}$ is compact, by passing to a subsequence, we can assume that $y_j \to y \in E^{-\infty}$. Since $s(y_j) = s(x_{n_j}) = d(h)$ for all $j$, we have $s(y) = d(h)$, so $y\alpha, y\beta \in E^{-\infty}$. By definition of convergence in $E^{-\infty}$, for each $N \in \mathbb{N}$ there exists $j$ such that $y_{-N} \dots y_{-1} = x_{n_j - N+1} \dots x_{n_j}$. For this $j$, the element $g_N := g|_{x_1 \dots x_{n_j - N+1}}$ satisfies $g_N \cdot y_{-N} \dots y_{-1}\alpha = y_{-N} \dots y_{-1}\beta$. Hence $y\alpha \sim_{\mathrm{ae}} y_\beta$. That is, $[y_\alpha] = [y_\beta] \in \mathcal{J}$. Moreover, $k := |\alpha| = |\beta|$ satisfies $\tilde{\sigma}([y\alpha]) = [y] = \tilde{\sigma}([y\beta])$.

By Lemma 4.7, there exists $\mu \in E^*$ such that every $g \in \mathcal{N}$ that satisfies $g \cdot \mu = \mu$ pointwise fixes $Z[\mu)$. Fix $z \in Z(r(\mu)]$ so that $z\mu \in E^{-\infty}$. Since $E$ is strongly connected, for each $n \in \mathbb{N}$ there exists $\nu_n \in E^*$ such that $r(\nu_n) = s(\mu)$ and $s(\nu_n) = r(y_n)$. So for each $n$ we obtain elements $z\mu\nu_n y_n \dots y_{-1}\alpha$ and $z\mu\nu_n y_n \dots y_{-1}\beta$ of $E^{-\infty}$. Since $\alpha \neq \beta$, our choice of $\mu$ ensures that $g \cdot \mu\nu_n y_n \dots y_{-1}\alpha \neq \mu\nu_n y_n \dots y_{-1}\beta$ for all $g \in \mathcal{N}$, and so Lemma 3.6 shows that $z\mu\nu_n y_n \dots y_{-1}\alpha \not\sim_{\mathrm{ae}} z\mu\nu_n y_n \dots y_{-1}\beta$ for all $n$. We have $z\mu\nu_n y_n \dots y_{-1}\alpha \to y\alpha$ and $z\mu\nu_n y_n \dots y_{-1}\beta \to y_\beta$, and $\tilde{\sigma}^k([z\mu\nu_n y_n \dots y_{-1}\alpha]) = [z\mu\nu_n y_n \dots y_{-1}] = \tilde{\sigma}^k([z\mu\nu_n y_n \dots y_{-1}\beta])$ for all $n$, and therefore $\tilde{\sigma}^k$ is not locally injective. □

## 5. The Smale space of a self-similar groupoid action on a graph

In this section we describe the Wieler Smale space that arises from the locally expanding dynamics described in the preceding section, and show that this Smale space can be realised as the quotient of $E^{\mathbb{Z}}$ by the natural extension of asymptotic equivalence.

We first recall Wieler's axioms, under which the projective limit of a space $V$ under iterates of a given continuous surjection $V \to V$ becomes a Smale space with totally disconnected stable set. Wieler proves more, showing that every Smale space with totally disconnected stable set has this form, but we will not need the full power of her theorem.

For the statement of Wieler's Theorem, recall that if $g : X \to X$ is a continuous self-mapping of a topological space, then $\varprojlim(X, g)$ is the space

$$\varprojlim(X, g) := \{(x_n)_{n=1}^{\infty} \mid x_i \in X \text{ and } g(x_{i+1}) = x_i \text{ for all } i\}.$$

If $\phi : X \to X$ is a continuous self-mapping of a topological space, we say that a point $x \in X$ is *non-wandering* if for every neighbourhood $U$ of $x$ there exists $n \geq 1$ such that $\phi^n(U) \cap U \neq \varnothing$. Finally, recall that the *forward orbit* of $x \in X$ is $\{\phi^n(x) \mid n \geq 0\}$.

**Theorem 5.1** (Wieler [31, Theorem A])**.** *Let $(X, d)$ be a compact metric space, and let $\varphi : X \to X$ be a continuous surjection. Suppose that there exist $\varepsilon > 0, K \in \mathbb{N} \setminus \{0\}$, and $\gamma \in (0, 1)$ such that*

**Axiom 1:** *For all $x, y \in X$ such that $d(x, y) < \varepsilon$, we have*

$$d(\varphi^K(x), \varphi^K(y)) \leq \gamma^K d(\varphi^{2K}(x), \varphi^{2K}(y)).$$

**Axiom 2:** *For all $x \in X$ and $0 < \alpha < \varepsilon$,*

$$\varphi^K(B(\varphi^K(x), \alpha)) \subseteq \varphi^{2K}(B(x, \gamma\alpha)).$$

*Then*

$$d_\infty((x_n), (y_n)) := \sum_{n=1}^{K} \gamma^{-n} \sup_{m \in \mathbb{N}} \gamma^m d(x_{m+n}, y_{m+n}),$$

*defines a metric on $X_\infty := \varprojlim(X, \varphi)$, the formula*

$$\varphi_\infty(x_1, x_2, \dots) = (\varphi(x_1), x_1, x_2, \dots)$$

*defines a homeomorphism $\varphi_\infty : X_\infty \to X_\infty$, and $(X_\infty, \varphi_\infty)$ is a Smale space with totally disconnected stable set. This Smale space is irreducible if and only if every point in $X$ is nonwandering, and there is a point in $X$ whose forward orbit under $\varphi_\infty$ is dense.*

The key point for us is that the results of the preceding two sections show that every contracting, regular self-similar groupoid action on a finite directed graph with no sources gives rise to a dynamical system satisfying Wieler's axioms.

**Lemma 5.2.** *Let $E$ be a finite directed graph with no sources. Let $(G, E)$ be a contracting, regular self-similar groupoid action. Let $\mathcal{J}$ be the limit space of Definition 3.2, let $d_{\mathcal{J}}$ be the equivalence relation metric as in Corollary 3.13, and let $\tilde{\sigma}$ be the local homeomorphism of Theorem 4.3. Let $\varepsilon$ be as in the statement of Theorem 4.3. Then the pair $(\mathcal{J}, \tilde{\sigma})$ satisfies Wieler's axioms for $\gamma = \frac{1}{2}$, $K = 1$ and $\varepsilon' = \frac{\varepsilon}{2}$.*

*Proof.* Theorem 4.3 (1) shows that if $d_{\mathcal{J}}([x], [y]) < \varepsilon'$ then

$$\begin{aligned} d_{\mathcal{J}}(\tilde{\sigma}^K([x]), \tilde{\sigma}^K([y])) &= d_{\mathcal{J}}(\tilde{\sigma}([x]), \tilde{\sigma}([y])) \\ &= \frac{1}{2} d_{\mathcal{J}}(\tilde{\sigma}^2([x]), \tilde{\sigma}^2([y])) = \gamma d_{\mathcal{J}}(\tilde{\sigma}^{2K}([x]), \tilde{\sigma}^{2K}([y])), \end{aligned}$$

establishing Axiom 1.

Theorem 4.3(2) shows that $\tilde{\sigma}^{2K}(B([x], \gamma\alpha)) = \tilde{\sigma}^K(B(\tilde{\sigma}^K([x]), \alpha))$ for all $\alpha \leq \varepsilon'$ and $[x] \in \mathcal{J}$, establishing Axiom 2. □

We now identify the limit space $\mathcal{J}_\infty$ obtained from Theorem 5.1 applied to $(\mathcal{J}, \tilde{\sigma})$ with a quotient of the bi-infinite path space of $E$.

We define asymptotic equivalence on bi-infinite paths just as we define it for right-infinite paths. That is, if $(G, E)$ is a self-similar groupoid action and $x, y \in E^{\mathbb{Z}}$, then $x \sim_{\text{ae}} y$ if there exists a bi-infinite sequence $(g_n)_{n \in \mathbb{Z}}$ in $G$ such that $\{g_n \mid n \in \mathbb{Z}\}$ is a finite set, and such that $g_n \cdot x_n x_{n+1} x_{n+2} \cdots = y_n y_{n+1} y_{n+2} \dots$ for all $n \in \mathbb{Z}$.

The argument of Lemma 3.6 shows that $x, y \in E^{\mathbb{Z}}$ are asymptotically equivalent if and only if there is a sequence $(g_n)_{n\in\mathbb{Z}}$ in $\mathcal{N}$ such that $g_n \cdot x_n x_{n+1} x_{n+2} \cdots = y_n y_{n+1} y_{n+2} \dots$ for all $n$ and such that $g_n|_{x_n} = g_{n+1}$ for all $n$.

**Definition 5.3.** Let $E$ be a finite directed graph with no sources. Let $(G, E)$ be a contracting, self-similar groupoid action. We write $\mathcal{S}$ for the quotient space $E^{\mathbb{Z}}/{\sim_{\text{ae}}}$ and call this the *limit solenoid* of $(G, E)$.

We will need the following notation. Given a directed graph $E$ with no sinks or sources, we will write $\tau : E^{\mathbb{Z}} \to E^{\mathbb{Z}}$ for the translation homeomorphism $\tau(x)_n = x_{n-1}$, $n \in \mathbb{Z}$. For $n \in \mathbb{Z}$ and $x \in E^{-\infty}$ we will write $x(-\infty, n)$ for the element of $E^{-\infty}$ given by $x(-\infty, n) = \dots x_{n-2} x_{n-1} x_n$.

**Proposition 5.4.** *Let $E$ be a finite directed graph with no sinks or sources. Let $(G, E)$ be a contracting, self-similar groupoid action. Let $\mathcal{J}$ be the limit space of $(G, E)$, and let $\tilde{\sigma} : \mathcal{J} \to \mathcal{J}$ be the map defined as in Theorem 4.3. Let $\mathcal{J}_\infty := \varprojlim(\mathcal{J}, \tilde{\sigma})$, and let $\tilde{\sigma}_\infty : \mathcal{J}_\infty \to \mathcal{J}_\infty$ be the homeomorphism of defined as in Theorem 5.1. Let $\mathcal{S} = E^{\mathbb{Z}}/{\sim_{\text{ae}}}$ be the limit solenoid of $(G, E)$. Then there is a homeomorphism $\theta : \mathcal{S} \to \mathcal{J}_\infty$ such that $\theta([x]) = ([x(-\infty, -1)], [x(-\infty, 0)], [x(-\infty, 1)], \dots)$ for all $x \in E^{\mathbb{Z}}$. We have $\theta([\tau(x)]) = \tilde{\sigma}_\infty(\theta([x]))$ for all $x \in E^{\mathbb{Z}}$.*

*Proof.* If $x, y \in E^{\mathbb{Z}}$ satisfy $x \sim_{\text{ae}} y$, then $x(-\infty, n) \sim_{\text{ae}} y(-\infty, n)$ for all $n \in \mathbb{Z}$. So the formula $\theta([x]) = ([x(-\infty, -1)], [x(-\infty, 0)], [x(-\infty, 1)], \dots)$ is well defined and determines a map $\theta : \mathcal{S} \to \prod_{n=1}^{\infty} \mathcal{J}$. By definition of $\tilde{\sigma}$, we have $\tilde{\sigma}([x(-\infty, n)]) = [\sigma(x(-\infty, n))] = [x(-\infty, n-1)]$ for all $n$, and so each $\theta([x]) \in \mathcal{J}_\infty$.

Since $E^{\mathbb{Z}}$ is compact, so is its continuous image $\mathcal{S}$. Projective limits of Hausdorff spaces are Hausdorff, so $\mathcal{J}_\infty$ is Hausdorff. So, to see that $\theta$ is a homeomorphism, it suffices to show that it a continuous bijection.

The maps $E^{\mathbb{Z}} \ni x \mapsto x(-\infty, n)$ indexed by $n \in \mathbb{Z}$ are clearly continuous, and so the maps $x \mapsto [x(-\infty, n)]$ are also continuous because the quotient map from $E^{-\infty}$ to $\mathcal{J}$ is continuous. Hence $\bar{\theta}(x) := ([x(-\infty, -1)], [x(-\infty, 0)], [x(-\infty, 1)], \dots)$ defines a continuous map $\bar{\theta} : E^{\mathbb{Z}} \to \mathcal{J}_\infty$. Since $\theta : \mathcal{S} \to \mathcal{J}_\infty$ is the map induced by $\bar{\theta}$, it is also continuous.

To see that $\theta$ is surjective, fix $(\zeta_1, \zeta_2, \zeta_3, \dots) \in \mathcal{J}_\infty$. For each $j$, choose $x_j \in E^{-\infty}$ such that $[x_j] = \zeta_j$. Since each $\tilde{\sigma}(\zeta_j) = \zeta_{j-1}$, we have $\sigma(x_j) \sim_{\text{ae}} x_{j-1}$ for all $j$. For each $j \geq 1$, consider the sequence $(\sigma^{n-j}(x_n))_{n=j}^{\infty}$ in $E^{-\infty}$. We just saw that each $\sigma^{n-j}(x_n) \sim_{\text{ae}} x_j$, and so Corollary 3.7 shows that this sequence contains just finitely many distinct elements of $E^{-\infty}$. A standard Cantor diagonal argument yields a subsequence $(x_{n_k})$ such that $\big(\sigma^{n_k - j}(x_{n_k})\big)_{n_k \geq j}$ is a constant sequence for each $j$. Define $y \in E^{\mathbb{Z}}$ by $y_j = \lim_k \sigma^{n_k - j}(x_{n_k})$. By construction, $\sigma(y_j) = y_{j-1}$ for all $j$. Also, by construction, each $y(-\infty, n) \sim_{\text{ae}} x_n$ and so $\theta([y]) = (\zeta_1, \zeta_2, \zeta_3, \dots)$.

To show that $\theta$ is injective, suppose that $\theta([x]) = \theta([y])$. We must show that $x \sim_{\text{ae}} y$. For this, fix $m \in \mathbb{Z}$. It suffices to find $g \in \mathcal{N}$ such that $g \cdot x(m, \infty) = y(m, \infty)$. Since $\theta([x]) = \theta([y])$, we have $x(-\infty, n) \sim_{\text{ae}} y(-\infty, n)$ for all $n$. Fix $n \geq m$. Lemma 3.6 shows that there is a sequence $(g_{k,n})_{k \leq n}$ in $\mathcal{N}$ such that $g_{k,n} x_k \dots x_n = y_k \dots y_n$ for all $k$. Taking $k = m \leq n$, we obtain $g_n := g_{m,n} \in \mathcal{N}$, such that $g_n \cdot x_m \dots x_n = y_m \dots y_n$. Since $\mathcal{N}$ is finite, the sequence $(g_n)_{n \geq m}$ has a constant subsequence. The constant value $g$ of this subsequence then satisfies $g \cdot x_m \dots x_n = x_m \dots y_n$ for infinitely many $n \geq m$. It then follows that $g \cdot x(m, \infty) = y(m, \infty)$ as required.

It remains to check that $\theta([\tau(x)]) = \tilde{\sigma}_\infty(\theta([x]))$ for all $x \in E^{\mathbb{Z}}$. This follows from direct calculation: for $x \in E^{\mathbb{Z}}$,

$$\begin{aligned}\theta([\tau(x)]) &= \big([\tau(x)(-\infty,-1)],[\tau(x)(-\infty,0)],[\tau(x)(-\infty,1)],\dots\big)\\ &= \big([x(-\infty,-2)],[x(-\infty,-1)],[x(-\infty,0)],\dots\big)\\ &= \big(\tilde{\sigma}([x(-\infty,-1)]),[x(-\infty,-1)],[x(-\infty,0)],\dots\big)\\ &= \tilde{\sigma}_\infty(\theta([x])).\end{aligned}$$

□

**Corollary 5.5.** *Let $E$ be a finite directed graph with no sinks or sources. Let $(G,E)$ be a contracting, regular self-similar groupoid action. Let $\mathcal{S} := E^{\mathbb{Z}}/\sim_{\mathrm{ae}}$ be the limit solenoid of $(G,E)$. Then there is a homeomorphism $\tilde{\tau} : \mathcal{S} \to \mathcal{S}$ such that $\tilde{\tau}([x]) = [\tau(x)]$ for all $x \in E^{\mathbb{Z}}$. Let $d_{\mathcal{J}}$ be the quotient metric on $\mathcal{J}$ as in Corollary 3.13. There is a metric $d_{\mathcal{S}}$ on $\mathcal{S}$ such that*

$$d_{\mathcal{S}}([x],[y]) = \sup_{m\in\mathbb{N}_0} (\frac{1}{2})^m d_{\mathcal{J}}([x(-\infty,m)],[y(-\infty,m)])$$

*for all $x,y \in E^{\mathbb{Z}}$. There is a constant $\varepsilon_{\mathcal{S}}$ such that $(\mathcal{S}, d_{\mathcal{S}}, \tilde{\tau}, \varepsilon_{\mathcal{S}}, \frac{1}{2})$ is a Smale space with totally disconnected stable set.*

*If $E$ is strongly connected, then $(\mathcal{S},\tilde{\tau})$ is irreducible. If $E$ is primitive, then $(\mathcal{S},\tilde{\tau})$ is topologically mixing.*

*Proof.* Theorem 5.1 shows that $(\mathcal{J}_\infty, \tilde{\sigma}_\infty)$ is a Smale space with totally disconnected stable set, and Proposition 5.4 shows that $(\mathcal{S},\tilde{\tau})$ is conjugate to $(\mathcal{J}_\infty,\tilde{\sigma}_\infty)$. Let $d_\infty$ be the metric of Theorem 5.1. For $x,y \in E^{\mathbb{Z}}$, we have

$$d_\infty(\theta([x]),\theta([y])) = 2\sup_{m\in\mathbb{N}}(\frac{1}{2})^m d(\theta([x])_{m+1},\theta([y])_{m+1}).$$

Since $\theta([x])_{m+1} = [x(-\infty,m)]$, we have $d_\infty = d_{\mathcal{S}} \circ (\theta\times\theta)$. Hence, there is a constant $\varepsilon_{\mathcal{S}} > 0$ such that $(\mathcal{S}, d_{\mathcal{S}}, \tilde{\tau}, \varepsilon_S, \frac{1}{2})$ is a Smale space.

For the irreducibility, by Wieler's Theorem it suffices to show that every point in $\mathcal{J}$ is non-wandering and that $\mathcal{J}$ admits a dense orbit. To see that every point is non-wandering, first observe that if $x \in E^{-\infty}$ is periodic, say $\sigma^n(x) = x$, then $[x] \in \mathcal{J}$ satisfies $\tilde{\sigma}^n([x]) = [\sigma^n(x)] = [x]$, so $[x]$ is also periodic. Since $E$ is strongly connected, for each $\lambda \in E^*$, there exists $\mu$ in $s(\lambda)E^*r(\lambda)$, and then $x := (\lambda\mu)^\infty$ is a periodic point in $Z(\lambda)$. So there is a dense set of periodic points. It follows that the periodic points in $\mathcal{J}$ are dense. So for any $[y] \in \mathcal{J}$ and any open neighbourhood $U$ of $[y]$, we can find $[x] \in U$ and $n \geq 1$ such that $\tilde{\sigma}^n([x]) = [x]$, and so $[x] \in \tilde{\sigma}^n(U) \cap U$. To see that $\mathcal{J}$ has a dense orbit, let $\lambda_1, \lambda_2, \lambda_3 \dots$ be a listing of $E^*$. For each $i \geq 1$, choose $\mu_i \in s(\lambda_i)E^*r(\lambda_{i+1})$. Then $x = \lambda_1\mu_1\lambda_2\mu_2\cdots \in E^\infty$. For each $\lambda \in E^*$, we have $\lambda = \lambda_i$ for some $i$, and then $\sigma^{|\lambda_1\mu_1\dots\lambda_{i-1}\mu_{i-1}|}(x) \in Z(\lambda)$. Hence $\{\sigma^n(x) \mid n \in \mathbb{N}\}$ is dense $E^\infty$. Hence $\{\tilde{\sigma}^n([x]) \mid n \in \mathbb{N}\} = q(\{\sigma^n(x) \mid n \in \mathbb{N}\}$ is a dense forward orbit in $\mathcal{J}$.

If $E$ is primitive, then $\tau : E^{\mathbb{Z}} \mapsto E^{\mathbb{Z}}$ is topologically mixing, see [15, Observation 7.2.2]. Since the quotient map $q : E^{\mathbb{Z}} \mapsto \mathcal{S}$ satisfies $q\circ\tau = \tilde{\tau}\circ q$ and is surjective, $\tau$ being topologically mixing implies $\tilde{\tau}$ is topologically mixing. □

## 6. The $C^*$-algebra of a self-similar groupoid action on a graph

In this section and the next, we will discuss two $C^*$-algebras associated to self-similar groupoid actions. The first of these is the $C^*$-algebra $\mathcal{O}(G,E)$ described by Laca–Raeburn–Ramagge–Whittaker in [17], see also [16, 20, 22]. Our main goal is to provide a groupoid model based on the one developed for self-similar group actions on graphs by Exel and Pardo [7]. This is the subject of the present section. In the next section, we consider the $C^*$-algebra obtained from the Deaconu–Renault groupoid of the dynamics $(\mathcal{J}, \tilde{\sigma})$ of Section 4. Our main result will establish $KK$-duality between these two $C^*$-algebras for contracting, regular self-similar actions.

In [17], the Toeplitz algebra of a self-similar groupoid action is defined as the Toeplitz algebra of an associated Hilbert module. Then Proposition 4.4 of [17] provides an alternative description as the universal $C^*$-algebra for generators and relations. At the beginning of Section 8 of [17], the Cuntz–Pimsner algebra of the self-similar action is defined as the quotient of the Toeplitz algebra by the ideal determined by an additional Cuntz–Krieger-type relation. We follow [17] and define the $C^*$-algebra of a self-similar action in terms of generators and relations.

If $G$ is a discrete groupoid, then a *unitary representation* of $G$ is a function $g \mapsto u_g$ from $G$ to a $C^*$-algebra such that $u_g u_h = \delta_{d(g),c(h)} u_{gh}$ and $u_{g^{-1}} = u_g^*$ for all $g, h \in G$. This is equivalent to the definition presented at the start of [17, Section 4].

If $E$ is a finite directed graph with no sources, and $(G, E)$ is a self-similar groupoid action, then a *covariant representation* of $(G, E)$ in a $C^*$-algebra $A$ is a triple $(u, p, s)$ consisting of a unitary representation $u : g \mapsto u_g$ of $G$ in $A$ and a Cuntz–Krieger $E$-family $(p, s) \in A$ such that $p_v = u_v$ for all $v \in E^0$, and such that

$$u_g s_e = s_{g \cdot e} u_{g|_e} \quad \text{for all } g \in G \text{ and } e \in d(g)E^1.$$

We have $p_v u_g = \delta_{v,c(g)} u_g$ and $u_g p_v = \delta_{d(g),v} u_g$ because $p_v = u_v$. If $d(g) \neq r(e)$, then $u_g s_e = u_g p_{d(g)} p_{r(e)} s_e = 0$. So the relations we have just presented are equivalent to those of [17, Proposition 4.4] combined with the additional relation determining the generators of the ideal $I$ described in [17, Equation (8.1)]. It follows from Proposition 4.4 and the definition of $\mathcal{O}(G, E)$ in [17] that the $C^*$-algebra $\mathcal{O}(G, E)$ is the universal $C^*$-algebra generated by a covariant representation of $(G, E)$.

Our first step is to describe a groupoid model for $\mathcal{O}(G, E)$. Our construction is based on that of [7].

**Lemma 6.1.** *Let $E$ be a finite graph, and let $(G, E)$ be a self-similar groupoid action. The set*

$$S_{G,E} := \{0\} \cup \{(\mu, g, \nu) \in E^* \times G \times E^* \mid s(\mu) = c(g) \text{ and } s(\nu) = d(g)\}$$

*is an inverse semigroup with respect to the multiplication given by*

$$(\alpha, g, \beta)(\mu, h, \nu) = \begin{cases} (\alpha(g \cdot \mu'), g|_{\mu'} h, \nu) & \text{if } \mu = \beta\mu' \\ (\alpha, gh|_{h^{-1} \cdot \beta'}, \nu(h^{-1} \cdot \beta')) & \text{if } \beta = \mu\beta' \\ 0 & \text{otherwise.} \end{cases}$$

*There is an action of $S_{G,E}$ on $E^\infty$ such that* $\operatorname{Dom}(\mu, g, \nu) = Z[\nu]$*, and*

$$(\mu, g, \nu) \cdot \nu x = \mu g \cdot x \quad \text{for all } x \in Z[s(\nu)).$$

*Proof.* It is routine, though tedious, to check that this multiplication is associative. For each $a := (\mu, g, \nu)$, the element $a^* := (\nu, g^{-1}, \mu)$ satisfies $aa^*a = a$ and $a^*aa^* = a^*$. Direct computation shows that the formula $(\mu, g, \nu) \cdot \nu x = \mu g \cdot x$ defines a homeomorphism from $Z[\nu)$ to $Z[\mu)$. A routine calculation very similar to the associativity calculation shows that $a \cdot (b \cdot x) = (ab) \cdot x$ whenever both sides are defined. □

Given any action of an inverse semigroup $S$ on a locally compact Hausdorff space $X$, we can form the associated groupoid of germs $S \ltimes X$ as follows [23, Section 4.3]: we define an equivalence relation on $\{(s, x) \mid s \in S, x \in \operatorname{Dom}(s)\}$ by $(s, x) \sim (t, y)$ if $x = y =: z$ and there is an idempotent $e \in S$ such that $z \in \operatorname{Dom}(e)$, and $se = te$.

The topology has basic open sets $W(s,V) := \{[s,x] \mid x \in V\}$ indexed by pairs $(s,V)$ consisting of an element $s \in S$ and an open set $V \subseteq \mathrm{Dom}(s)$. The unit space of this groupoid is $X$, and the groupoid operations are given by

$$s([t,x]) = x, \quad r([t,x]) = t \cdot x, \quad [t, u \cdot x][u,x] = [tu, x], \quad \text{and} \quad [t,x]^{-1} = [t^*, t \cdot x].$$

Though this groupoid need not be Hausdorff, it is always étale with Hausdorff unit space $X$, and hence locally Hausdorff with a basis of open bisections. The $C^*$-algebra of this groupoid is the completion of the $^*$-algebra

$$\mathcal{C}(S \ltimes X) = \mathrm{span}\{C_c(U) \mid U \text{ is an open bisection}\}$$

in a universal norm. A very nice account of this construction can be found in [6].

**Definition 6.2.** Let $E$ be a finite directed graph, and let $(G,E)$ be a self-similar groupoid action. The groupoid of $(G,E)$, denoted $S_{G,E} \ltimes E^\infty$ is defined to be the groupoid of germs for the action of $S_{G,E}$ on $E^\infty$ as above.

To establish our duality theorem later, we will describe a groupoid equivalence between the groupoid $S_{G,E} \ltimes E^\infty$ and the stable groupoid of the Smale space constructed in Section 5. To do this, it will be helpful first to establish a description of $S_{G,E} \ltimes E^\infty$ as a kind of lag groupoid. A related description for self-similar group actions appears in [7, Section 8], though there the lag takes values in the "sequence group" $\prod_{i=1}^\infty G / \bigoplus_{i=1}^\infty G$. We will give yet another description, which is particularly well suited to our application to $KK$-duality later. We also characterise exactly when this groupoid is Hausdorff, by characterising exactly which pairs of elements (if any) cannot be separated by disjoint open neighbourhoods.

Our groupoid is based on the left-shift map on $E^\infty$ given by $x_1x_2x_3\cdots \mapsto x_2x_3\ldots$. In the graph-algebra literature, it is standard to denote this shift map by $\sigma$, but we have already used that symbol for the right-shift on $E^{-\infty}$. We will instead use $\varsigma$ for the left-shift map.

**Lemma 6.3.** *Let $E$ be a finite directed graph, and let $(G,E)$ be a self-similar groupoid action. There is an equivalence relation $\sim$ on*

$$\{(x,m,g,n,y) \in E^\infty \times \mathbb{N} \times G \times \mathbb{N} \times E^\infty \mid d(g) = r(\varsigma^n(y)) \text{ and } \varsigma^m(x) = g \cdot \varsigma^n(y)\}$$

*such that $(x,m,g,n,y) \sim (w,p,h,q,z)$ if and only if*

- *$x = w$, $y = z$ and $m - n = p - q$, and*
- *there exists $l \geq \max\{n,q\}$ such that $g|_{y(n,l)} = h|_{z(q,l)}$.*

*We write $[x,m,g,n,y]$ for the equivalence class of $(x,m,g,n,y)$ under $\sim$. The set*

$$\mathcal{G}_{G,E} := \{[x,m,g,n,y] \mid d(g) = r(\varsigma^n(y)) \text{ and } \varsigma^m(x) = g \cdot \varsigma^n(y)\}$$

*is an algebraic groupoid with unit space $\{[x,0,r(x),0,x] \mid x \in E^\infty\}$ identified with $E^\infty$, range and source maps $r([x,m,g,n,y]) = x$ and $s([x,m,g,n,y]) = y$, and operations*

$$[x,m,g,n,y][y,p,h,q,z] = [x, m+p, g|_{y(n,n+p)} h|_{z(q,q+n)}, n+q, z], \qquad \text{and}$$
$$[x,m,g,n,y]^{-1} = [y,n,g^{-1},m,x].$$

*There is an injective homomorphism $\iota$ of the graph groupoid $\mathcal{G}_E$ into $\mathcal{G}_{G,E}$ given by $\iota(x, m-n, y) = [x,m,s(x_m),m,y]$ whenever $x,y \in E^\infty$ satisfy $\varsigma^m(x) = \varsigma^n(y)$.*

*Proof.* Reflexivity and symmetry of the relation $\sim$ are clear. For transitivity, suppose that $(x,m,g,n,y) \sim (x',m',g',n',y')$ and $(x',m',g',n',y') \sim (x'',m'',g'',n'',y'')$. Then $x = x' = x''$, $y = y' = y''$ and $m - n = m' - n' = m'' - n''$. Choose $l \geq n, n'$ and

$l' \geq n', n''$ with $g|_{y(n,l)} = g'|_{y(n',l)}$ and $g'|_{y(n',l')} = g''|_{y(n'',l')}$, and put $L = \max\{l, l'\}$. Then $L \geq n, n''$, and we have

$$g|_{y(n,L)} = \big(g|_{y(n,l)}\big)|_{y(l,L)} = \big(g'|_{y(n',l)}\big)_{y(l,L)} = g'|_{y(n',L)}$$
$$= \big(g'|_{y(n',l')}\big)|_{y(l',L)} = \big(g''|_{y(n'',l')}\big)|_{y(l',L)} = g''|_{y(n'',L)}.$$

To show that $\mathcal{G}_{G,E}$ is a groupoid, we first check that if $[x, m, g, n, y]$ and $[y, p, h, q, z]$ belong to $\mathcal{G}_{G,E}$, then so does $[x, m+p, g|_{y(n,n+p)}h|_{z(q,q+n)}, n+q, z]$. For this, just check:

$$(g|_{y(n,n+p)})(h|_{z(q,q+n)}) \cdot \varsigma^{q+n}(z) = (g|_{y(n,n+p)}) \cdot \varsigma^{n}(h \cdot \varsigma^{q}(z))$$
$$= (g|_{y(n,n+p)}) \cdot \varsigma^{n+p}(y) = \varsigma^{p}(g \cdot \varsigma^{n}(y)) = \varsigma^{m+p}(x).$$

The range and source maps are well-defined by definition. We must check that multiplication is well-defined. First suppose that $[x, m, g, n, y] = [x', m', g', n', y']$; so $x = x'$, $y = y'$ and $m - n = m' - n'$, and there exists $l \geq n, n'$ such that $g|_{y(n,l)} = g'|_{y(n',l)}$. Fix $[y, p, h, q, z] \in \mathcal{G}_{G,E}$. We must show that

$$[x, m+p, g|_{y(n,n+p)}h|_{z(q,q+n)}, n+q, z] = [x, m'+p, g'|_{y(n',n'+p)}h|_{z(q,q+n')}, n'+q, z].$$

We have $m + p - (m' + p) = n + q - (n' + q)$. We will show that

$$\big(g|_{y(n,n+p)}h|_{z(q,q+n)}\big)|_{z(q+n,q+l)} = \big(g'|_{y(n',n'+p)}h|_{z(q,q+n')}\big)|_{z(q+n',q+l)}.$$

We have

$$\big(g|_{y(n,n+p)}h|_{z(q,q+n)}\big)|_{z(q+n,q+l)} = \big((g|_{y(n,n+p)})|_{h|_{z(q,q+n)}\cdot z(q+n,q+l)}\big)h|_{z(q,q+l)},$$

and similarly

$$\big(g'|_{y(n',n'+p)}h|_{z(q,q+n')}\big)|_{z(q+n',q+l)} = \big((g'|_{y(n',n'+p)})|_{h|_{z(q,q+n')}\cdot z(q+n',q+l)}\big)h|_{z(q,q+l)}.$$

So we need to check that $(g|_{y(n,n+p)})|_{h|_{z(q,q+n)}\cdot z(q+n,q+l)} = (g'|_{y(n',n'+p)})|_{h|_{z(q,q+n')}\cdot z(q+n',q+l)}$. For this, we observe that $h|_{z(q,q+n)} \cdot z(q+n, q+l) = (h \cdot \varsigma^{q}(z))(n, l)$, which is equal to $\varsigma^{p}(y)(n, l)$ because $[y, p, h, q, z] \in \mathcal{G}_{G,E}$. Hence

$$(g|_{y(n,n+p)})|_{h|_{z(q,q+n)}\cdot z(q+n,q+l)} = (g|_{y(n,n+p)})|_{y(n+p,l+p)} = (g|_{y(n,l)})|_{y(l,l+p)} = (g'|_{y(n',l)})|_{y(l,l+p)}$$
$$= (g'|_{y(n',n'+p)})|_{y(n'+p,l+p)} = (g'|_{y(n',n'+p)})|_{h|_{z(q,q+n')}\cdot z(q+n',q+l)}$$

as required. A very similar calculation shows that if $[y, p, h, q, z] = [y', p', h', q', z']$ and $[x, m, p, n, y] \in \mathcal{G}_{G,E}$, then

$$[x, m+p, g|_{y(n,n+p)}h|_{z(q,q+n)}, n+q, z] = [x, m+p', g|_{y(n,n+p')}h'|_{z(q',q'+n)}, n+q', z],$$

so multiplication in $\mathcal{G}_{G,E}$ is well-defined. It is routine that $[x, 0, r(x), 0, x][x, m, g, n, y] = [x, m, g, n, y] = [x, m, g, n, y][y, 0, r(y), 0, y]$ for all $[x, m, g, n, y] \in \mathcal{G}_{G,E}$, so that $\mathcal{G}_{G,E}$ admits units.

We have

$$[x, m, g, n, y][y, n, g^{-1}, m, x] = [x, m+n, g|_{y(n,2n)}g^{-1}|_{x(m,m+n)}, m+n, x].$$

We calculate:

$$g^{-1}|_{x(m,m+n)} = g^{-1}|_{\varsigma^{m}(x)(0,n)} = g^{-1}|_{g\cdot(\varsigma^{n}(y)(0,n))} = \big(g|_{\varsigma^{n}(y)(0,n)}\big)^{-1}.$$

So $[x, m+n, g|_{y(n,2n)}g^{-1}|_{x(m,m+n)}, m+n, x] = [x, m+n, r(x), m+n, x]$. We now have that $(x, p, r(x), p, x) \sim (x, q, r(x), q, x)$ for all $x, p, q$, and we deduce that $[y, n, g^{-1}, m, x]$ is an inverse for $[x, m, g, n, y]$. Associativity of the multiplication described follows from straightforward calculations like those above, and we deduce that $\mathcal{G}_{G,E}$ is a groupoid. Using the definition of $\sim$, we see that $[x, m, r(x), n, y] \sim [x', m', r(x'), n', y']$ if and only if $x = x'$, $y = y'$ and $m - n = m' - n'$, and it follows that $\iota(x, m - n, y) = [x, m, e, n, y]$ defines a groupoid homomorphism $\mathcal{G}_E \to \mathcal{G}_{G,E}$, which is injective by definition of $\sim$. $\square$

We now describe an algebraic isomorphism of $S_{G,E} \ltimes E^\infty$ onto the groupoid $\mathcal{G}_{G,E}$ of Lemma 6.3, and use it to define an étale topology on $\mathcal{G}_{G,E}$.

**Lemma 6.4.** *Let $E$ be a finite directed graph, and let $(G, E)$ be a self-similar groupoid action. Let $\mathcal{G}_{G,E}$ be the groupoid of Lemma 6.3, and let $S_{G,E} \ltimes E^\infty$ be the groupoid of germs described in Definition 6.2. There is an algebraic isomorphism $\psi : S_{G,E} \ltimes E^\infty \to \mathcal{G}_{G,E}$ such that $\psi([(\mu, g, \nu), \nu x]) = [\mu(g \cdot x), |\mu|, g, |\nu|, \nu x]$ for all $(\mu, g, \nu)$ in $S_{G,E}$ and $x \in Z[s(\nu))$. The sets*

$$Z(\mu, g, \nu) := \{[\mu(g \cdot y), |\mu|, g, |\nu|, \nu y] \mid y \in Z[s(\nu))\}$$

*indexed by triples $(\mu, g, \nu) \in E^* \times G \times E^*$ such that $s(\mu) = g \cdot s(\nu)$ constitute a basis of compact open sets for a locally Hausdorff topology on $\mathcal{G}_{G,E}$ on which the range and source maps are homeomorphisms. Under this topology, $\mathcal{G}_{G,E}$ is an étale groupoid.*

*Proof.* Define

$$\psi^0 : \{\big((\mu, g, \nu), \nu x\big) \mid (\mu, g, \nu) \in S_{E,G}, x \in s(\nu)E^\infty\} \to \mathcal{G}_{G,E}$$

by $\psi^0\big(((\mu, g, \nu), \nu x)\big) = [\mu g \cdot x, |\mu|, g, |\nu|, \nu x]$. We claim that

$$[(\mu, g, \nu), \nu x] = [(\alpha, h, \beta), \beta y] \iff \psi^0\big((\mu, g, \nu), \nu x\big) = \psi^0\big((\alpha, h, \beta), \beta y\big). \tag{6.1}$$

For this, first suppose that $[(\mu, g, \nu), \nu x] = [(\alpha, h, \beta), \beta y]$. Then $\nu x = \beta y$, and there is an idempotent $(\lambda, e, \lambda)$ of $S_{G,E}$ such that $x \in Z[\lambda)$ and $(\mu, g, \nu)(\lambda, e, \lambda) = (\alpha, h, \beta)(\lambda, e, \lambda)$. Without loss of generality, we may assume that $\lambda = x(0, n)$ with $n \geq |\nu|, |\beta|$. So $\lambda = \nu\nu' = \beta\beta'$. Since $(\lambda, e, \lambda) = (\lambda, e, \lambda)(\lambda, e, \lambda) = (\lambda, e^2, \lambda)$, we have $e^2 = e$, so $e = s(\lambda)$. Hence

$$(\mu g \cdot \nu', g|_{\nu'}, \nu\nu') = (\mu, g, \nu)(\lambda, e, \lambda) = (\alpha, h, \beta)(\lambda, e, \lambda) = (\alpha h \cdot \beta', h|_{\beta'}, \beta\beta').$$

In particular, $g|_{(\nu x)(|\nu|, |\lambda|)} = g|_{\nu'} = h|_{\beta'} = h|_{(\alpha y)(|\alpha|, |\lambda|)}$. Also, $\mu(g \cdot x) = \mu(g \cdot \nu')g|_{\nu'} \cdot \varsigma^{|\nu'|}(x) = \alpha(h \cdot \beta')h|_{\beta'} \cdot \varsigma^{|\beta'|}(y) = \alpha(h \cdot y)$. Hence

$$\begin{aligned}
\psi^0\big((\mu, g, \nu), \nu x\big) &= [\mu(g \cdot x), |\mu|, g, |\nu|, \nu x] \\
&= [\mu(g \cdot x), |\mu| + |\nu'|, g|_{(\nu x)(|\nu|, |\nu| + |\nu'|)}, |\nu| + |\nu'|, \nu x] \\
&= [\alpha(h \cdot y), |\alpha| + |\beta'|, h|_{(\beta y)(|\beta|, |\beta| + |\beta'|)}, |\beta| + |\beta'|, \beta y] \\
&= \psi^0\big((\alpha, h, \beta), \beta y\big).
\end{aligned}$$

Conversely suppose that $\psi^0\big((\mu, g, \nu), \nu x\big) = \psi^0\big((\alpha, h, \beta), \beta y\big)$. Then $|\mu| - |\nu| = |\alpha| - |\beta|$, $\nu x = \beta y$, $\mu g \cdot x = \alpha h \cdot y$, and there exists $l \geq |\nu|, |\beta|$ such that $g|_{x(0, l - |\nu|)} = h|_{y(0, l - |\beta|)}$. Let $e = s(y_{l - |\beta|}) = s(x_{l - |\nu|})$. Then

$$\begin{aligned}
&(\alpha, h, \beta)(\beta y(0, l - |\beta|), e, \beta y(0, l - |\beta|)) \\
&\qquad = (\alpha h \cdot y(0, l - |\beta|), h|_{y(0, l - |\beta|)}, (\beta y)(0, l)) \\
&\qquad = (\mu g \cdot x(0, l - |\nu|), g|_{x(0, l - |\nu|)}, (\nu x)(0, l)) \\
&\qquad = (\mu, g, \nu)(\nu x(0, l - |\nu|), e, \nu x(0, l - |\nu|)).
\end{aligned}$$

Since $\nu x(0, l - |\nu|) = (\nu x)(0, l) = (\beta y)(0, l) = \beta y(0, l - |\beta|)$, we obtain

$$(\alpha, h, \beta)(\beta y(0, l - |\beta|), e, \beta y(0, l - |\beta|)) = (\mu, g, \nu)(\beta y(0, l - |\beta|), e, \beta y(0, l - |\beta|)).$$

Hence $[(\mu, g, \nu), \nu x] = [(\alpha, h, \beta), \beta y]$. This completes the proof of (6.1).

It follows that $\psi^0$ descends to an injective map $\psi : S_{G,E} \ltimes E^\infty \to \mathcal{G}_{G,E}$. To see that $\psi$ is surjective, fix $[x, m, g, n, y] \in \mathcal{G}_{G,E}$, let $z := \varsigma^n(y)$, $\nu = y(0, n)$ and $\mu = x(0, m)$, and observe that $[x, m, g, n, y] = [\mu g \cdot z, |\mu|, g, |\nu|, \nu z] = \psi([(\mu, g, \nu), \nu z])$. It is routine to check that $\psi$ is multiplicative, and hence an algebraic isomorphism of groupoids as claimed.

For $(\mu, g, \nu)$ with $s(\mu) = g \cdot s(\nu)$ and an open set $U \subseteq Z[\nu)$ in $E^\infty$, let

$$Z(\mu, g, \nu, U) := \{[\mu g \cdot x, |\mu|, g, |\nu|, \nu x] \mid \nu x \in U\}.$$

Proposition 4.14 of [6] combined with the algebraic isomorphism $\psi$ above shows that the sets $Z(\mu, g, \nu, U)$ are a basis for a topology on $\mathcal{G}_{G,E}$ under which it becomes a topological groupoid. If $[\mu g \cdot x, |\mu|, g, |\nu|, \nu x]$ is in $Z(\mu, g, \nu, U)$, then by definition of the topology on $E^\infty$ there exists $n \in \mathbb{N}$ for which $\nu' := x(0, n)$ is a path such that $\nu x \in Z[\nu\nu') \subseteq U$. Then, we have

$$[\mu g \cdot x, |\mu|, g, |\nu|, \nu x] \in Z(\mu g \cdot \nu', g|_{\nu'}, \nu\nu') = Z(\mu, g, \nu, Z(\nu')) \subseteq Z(\mu, g, \nu, U).$$

So the $Z(\mu, g, \nu)$ are a basis for the same topology as the $Z(\mu, g, \nu, U)$. Now Proposition 4.15 of [6] shows that the range and source maps restrict to homeomorphisms $r : Z(\mu, g, \nu) \to Z[\mu)$ and $s : Z(\mu, g, \nu) \to Z[\nu)$. Since the $Z[\nu)$ are compact and Hausdorff, we deduce that the $Z(\mu, g, \nu)$ are also compact and Hausdorff. It follows that $\mathcal{G}_{G,E}$ is locally Hausdorff and étale as claimed. □

We now show that the $C^*$-algebra of the groupoid $\mathcal{G}_{G,E}$ just constructed coincides with the $C^*$-algebra of the self-similar action $(G, E)$.

Note that for each $\mu \in E^*$ and $g \in G$ with $c(g) = s(\mu)$, we have $(\mu, g, d(g)) \in S_{G,E}$, and $\mathrm{Dom}((\mu, g, d(g))) = Z[d(g))$. Hence $W((\mu, g, d(g)), Z[\lambda))$ is a compact open subset of $S_{G,E} \ltimes E^\infty$ for each $\lambda \in d(g)E^*$.

**Proposition 6.5.** *Let $E$ be a finite directed graph with no sources, and let $(G, E)$ be a self-similar groupoid action. Let $\mathcal{G}_{G,E}$ be the groupoid described in Lemmas 6.3 and 6.4. There is an isomorphism $\pi : \mathcal{O}(G, E) \to C^*(\mathcal{G}_{G,E})$ such that $\pi(u_g) = 1_{Z(c(g),g,d(g))}$ and $\pi(s_e) = 1_{Z(e,s(e),s(e))}$ for all $g \in G$ and $e \in E^1$.*

*Proof.* By Lemma 6.4, it suffices to construct an isomorphism $\pi : \mathcal{O}(G, E) \to C^*(S_{G,E} \ltimes E^\infty)$ such that each $\pi(u_g) = 1_{W((c(g),g,d(g)),Z[d(g)))}$, and each $\pi(s_e) = 1_{W((e,s(e),s(e)),Z[s(e)))}$.

For $g \in G$, $e \in E^1$ and $v \in E^0$, define

$$U_g := 1_{W((c(g),g,d(g)),Z[d(g)))}, \quad S_e := 1_{W((e,s(e),s(e)),Z[s(e)))}, \quad \text{and} \quad P_v := 1_{W((v,v,v),Z[v))}.$$

Elementary calculations using the definition of multiplication in $S_{G,E}$ shows that $(U, P, S)$ is a covariant representation of $(G, E) \in C^*(S_{G,E} \ltimes E^\infty)$. It follows that there is a homomorphism $\pi : \mathcal{O}(G, E) \to C^*(S_{G,E} \ltimes E^\infty)$ satisfying the given formulas. For each $\lambda \in E^*$, write $\lambda = \lambda_1\lambda_2 \ldots \lambda_n$ as a concatenation of edges, and then define $S_\lambda := S_{\lambda_1} \ldots S_{\lambda_n}$. Then

$$1_{Z[\lambda)} = S_\lambda S_\lambda^* \in \pi(\mathcal{O}(G, E)).$$

Since the $Z[\lambda)$ constitute a basis for the topology on $E^\infty$, it follows that $C_0(E^\infty) \subseteq \pi(\mathcal{O}(G, E))$. If $V$ is a compact open bisection in $S_{G,E} \ltimes E^\infty$, we can write it as a finite disjoint union of bisections of the form $W((\mu, g, \nu), Z[\nu\alpha))$. We have

$$1_{W((\mu,g,\nu),Z[\nu\alpha))} = 1_{W((\mu g \cdot \alpha, g|_\alpha, \nu\alpha),Z[s(\alpha)))} = S_{\mu g \cdot \alpha} U_{g|_\alpha} S_{\nu\alpha}^* \in \pi(\mathcal{O}(G, E)),$$

and we deduce that the indicator function of each compact open bisection belongs to the range of $\pi$. For each compact open bisection $V$, indicator functions of this form linearly span a dense sub-algebra of $C_c(V)$. It follows that $\pi$ is surjective.

It remains to show that $\pi$ is injective. To do this, it suffices to construct a right inverse $\rho : C^*(S_{G,E} \ltimes E^\infty) \to \mathcal{O}(G,E)$ for $\pi$. Observe that since $S_{G,E} \ltimes E^\infty$ is the groupoid of germs of the action $\theta$ of the inverse semigroup $S_{G,E}$ on $E^\infty$, [6, Theorem 8.5] shows that $C^*(S_{G,E} \ltimes E^\infty)$ is universal for representations, as defined in [6, Definition 8.1] of $(\theta, S_{G,E}, E^\infty)$. Since the $s_e$ and $p_v$ constitute a Cuntz–Krieger $E$-family in $\mathcal{O}(G,E)$, there is a homomorphism $\rho_0 : C_0(E^\infty) \to \mathcal{O}(G,E)$ such that $\rho_0(1_{Z[\lambda)}) = s_\lambda s_\lambda^*$ for each $\lambda$. For each $(\mu, g, \nu) \in S_{G,E}$ define $S_{(\mu,g,\nu)} := s_\mu u_g s_\nu^*$. If $\lambda = \nu\lambda'$ then the relations in $\mathcal{O}(G,E)$ give

$$S_{(\mu,g,\nu)}\rho_0(1_{Z[\lambda)})S^*_{(\mu,g,\nu)} = s_\mu u_g s_\nu^* s_\lambda s_\lambda^* s_\nu u_{g^{-1}} s_\mu^* = s_{\mu g\cdot\lambda'} s^*_{\mu g\cdot\lambda'},$$

and then linearity and continuity imply that $S_{(\mu,g,\nu)}\rho_0(f)S^*_{(\mu,g,\nu)} = \rho_0(f \circ \theta_{(\mu,g,\nu)^*})$ whenever $f$ is supported on $\mathrm{Dom}(\theta_{(\mu,g,\nu)})$. Routine calculations show that $S_a S_b = S_{ab}$ and $S_{a^*} = S_a^*$ for all $a, b \in S_{G,E}$. So $(\rho_0, S)$ is a representation of $(\theta, S_{G,E}, E^\infty)$ and it follows that there is a homomorphism $\rho : C^*(S_{G,E} \ltimes E^\infty) \to \mathcal{O}(G,E)$ such that $\rho(f) = \rho_0(f)$ for all $f$ in $C_0(E^\infty)$ and $\rho(1_{W((\mu,g,\nu),Z[\nu))}) = s_\mu u_g s_\nu^*$ for all $(\mu,g,\nu) \in S_{G,E}$. In particular, $\rho \circ \pi$ fixes the generators of $\mathcal{O}(G,E)$ and therefore $\rho \circ \pi = \mathrm{id}_{\mathcal{O}(G,E)}$ as required. □

We conclude this section by characterising exactly when $\mathcal{G}_{G,E}$ is Hausdorff.

**Proposition 6.6.** *Let $E$ be a finite directed graph, and let $(G,E)$ be a self-similar groupoid action. Let $\mathcal{G}_{G,E}$ be the groupoid of Lemmas 6.3 and 6.4. Points $[x,m,g,n,y]$ and $[w,p,h,q,z] \in \mathcal{G}_{G,E}$ are distinct but cannot be separated by disjoint open sets if and only if both of the following hold*

1. *$x = w$, $y = z$, $m - n = p - q$ and $g|_{y(n,l)} \neq h|_{y(q,l)}$ for all $l \geq n, q$; and*
2. *for every $l \geq n, q$ there exists $\lambda \in s(y_l)E^*$ such that $g|_{y(n,l)} \cdot \lambda = h|_{y(q,l)} \cdot \lambda$ and $g|_{y(n,l)\lambda} = h|_{y(q,l)\lambda}$.*

*Proof.* First, suppose that $[x,m,g,n,y]$ and $[w,p,h,q,z]$ are distinct but cannot be separated by open neighbourhoods.

The range map $r$, the source map $s$ and the co-cycle map $c : \mathcal{G}_{(G,E)} \mapsto \mathbb{Z}$, defined by $c([x,m,g,n,y]) = m - n$, are continuous mappings onto Hausdorff spaces, so since $[x,m,g,n,y]$ and $[w,p,h,q,z]$ cannot be separated by open neighbourhoods, their images under $r, s$ and $c$ coincide. Hence $x = w$, $y = z$ and $m - n = p - q$. Since $[x,m,g,n,y] \neq [w,p,h,q,z]$, the definition of the equivalence relation of Lemma 6.3 forces $g|_{y(n,l)} \neq h|_{y(q,l)}$ for all $l \geq \max(n,q)$. Therefore, (1) is satisfied.

For (2), let $\xi = y(n,l)$, and $\tau = y(q,l)$. Then $[x,m,g,n,y] \in Z(x(0,m)g\cdot\xi, g|_\xi, y(0,n)\xi)$, and $[x,p,h,q,y]$ is in $Z(x(0,p)h\cdot\tau, h|_\tau, y(0,q)\tau)$. Let $\gamma := y(0,n)\xi = y(0,q)\tau$ and $\omega := x(0,m)g\cdot\xi = x(0,p)h\cdot\tau$. By assumption, $Z(\omega, g|_\xi, \gamma) \cap Z(\omega, h|_\tau, \gamma) \neq \emptyset$, so there exist $u \in s(\gamma)E^\infty$ and $v \in s(\omega)E^\infty$ such that $g|_\xi \cdot u = h|_\tau \cdot u = v$ and $[\omega v, |\omega|, g|_\xi, |\gamma|, \gamma u] = [\omega v, |\omega|, h|_\tau, |\gamma|, \gamma u]$. Hence, there exists $k$ in $\mathbb{N}$ such that $(g|_\xi)|_{u(0,k)} = (h|_\tau|)_{u(0,k)}$. Thus $\lambda := u(0,k) \in s(y_l)E^*$ satisfies (2).

Now suppose that (1) and (2) hold. The last part of (1) and the definition of $\sim$ imply that $[x,m,g,n,y] \neq [w,p,h,q,z]$. Fix neighbourhoods $U \ni [x,m,g,n,y]$ and $V \ni [w,p,h,q,z]$. We must show that $U \cap V \neq \varnothing$. By definition of the topology, there exists $l \geq n, q$ such that $Z(x(0,m-n+l), g|_{y(n,l)}, y(0,l)) \subseteq U$ and $Z(x(0,p-q+l), h|_{y(q,l)}, y(0,l)) \subseteq V$. Fix $\lambda$ as in condition (2) for this $l$. Then

$$\begin{aligned} U \ni &\big[x(0,m-n+l)g|_{y(n,l)} \cdot (\lambda z), m-n+l+|\lambda|, g|_{y(n,l)\lambda}, l+|\lambda|, y(0,l)\lambda z\big] \\ &= \big[x(0,p-q+l)h|_{y(q,l)} \cdot (\lambda z), p-q+l+|\lambda|, h|_{y(q,l)\lambda)}, l+|\lambda|, y(0,l)\lambda z\big] \in V \end{aligned}$$

for all $z \in Z[s(\lambda))$, so $U \cap V \neq \emptyset$. □

For the following corollary, we use the following terminology adapted from [7, Definition 5.2]: if $(G, E)$ is a self-similar groupoid action on a finite graph $E$, we say that a path $\lambda \in E^*$ is *strongly fixed* by an element $g \in Gr(\lambda)$ if $g \cdot \lambda = \lambda$ and $g|_\lambda = s(\lambda)$.

**Corollary 6.7.** *Let $E$ be a finite directed graph, and let $(G, E)$ be a self-similar groupoid action. Then the following are equivalent.*

1. *The groupoid $\mathcal{G}_{G,E}$ is Hausdorff.*
2. *The subgroupoid $\mathcal{G}^{\mathbb{T}}_{G,E} := \{[x, m, g, n, y] \in \mathcal{G}_{G,E} \mid m = n\}$ is Hausdorff.*
3. *The subgroupoid $\{[g \cdot x, 0, g, 0, x] \mid x \in E^\infty$ and $d(g) = r(x)\}$ is Hausdorff.*
4. *If $g \in G$ and $y \in E^\infty$ satisfy $g \cdot y = y$ and $g|_{y(0,n)} \neq s(y_n)$ for all $n$, then there exists $\lambda \in E^*$ such that $y \in Z[\lambda)$ and no element of $\lambda E^*$ is strongly fixed by $g$.*

*In particular, if $(G, E)$ is regular then $\mathcal{G}_{G,E}$ is Hausdorff.*

*Proof.* Since $\mathcal{G}^{\mathbb{T}}_{G,E} \subseteq \mathcal{G}_{G,E}$ and $\mathcal{G}^{\mathbb{T}}_{G,E}$ contains the subgroupoid of (3), we have (1) $\implies$ (2) $\implies$ (3).

For (3) $\implies$ (4), we prove the contrapositive. So suppose that (4) fails with respect to $g \in G$ and $y \in Z[d(g))$. That is $g \cdot y = y$ and $g|_{y(0,n)} \neq s(y_n)$ for all $n$, but for every $\lambda$ such that $y \in Z[\lambda)$ there exists $\mu \in s(\lambda)E^*$ such that $\lambda\mu$ is strongly fixed by $g$. Equivalently, for every $l \geq 0$, there exists $\mu \in s(y_l)E^*$ such that $g \cdot (y(0, l)\mu) = y(0, l)\mu$ and $g|_{y(0,l)\mu} = s(\mu)$. Hence Proposition 6.6 implies that $[y, 0, g, 0, y]$ and $[y, 0, d(g), 0, y]$ are distinct but cannot be separated by open sets. Hence the open subgroupoid $\{[g \cdot x, 0, g, 0, x] \mid x \in E^\infty$ and $d(g) = r(x)\}$ is not Hausdorff.

For (4) $\implies$ (1), suppose that (4) holds. Fix $[x, m, g, n, y]$ and $[w, p, h, q, z] \in \mathcal{G}_{G,E}$. It suffices to show that the conditions of Proposition 6.6 do not both hold for these points. To do this, we suppose that Proposition 6.6(1) holds, and show that Proposition 6.6(2) fails. Let $l := \max(n, q)$, let $k := (g|_{y(n,l)})^{-1}h|_{y(q,l)}$, and put $y' = \varsigma^l(y)$. Since $x = w$, we have $(g|_{y(n,l)}) \cdot y' = h|_{y(q,l)} \cdot y'$ and so $k \cdot y' = y'$. Moreover, $k|_{y'(0,a)} = \big((g|_{y(n,l)})^{-1}h|_{y(q,l)}\big)|_{y'(0,a)} = \big((g|_{y(n,l+a)})^{-1}h|_{y(q,l+a)}\big) \neq s(y_{(l+a)})$ for all $a$. So (4) implies that for large $a \in \mathbb{N}$ and $\lambda \in s(y'_a)E^*$, either $k|_{y'(0,a)} \cdot \lambda \neq \lambda$ or $k|_{y'(0,a)\lambda} \neq s(\lambda)$. That is, for large $a$, for every $\lambda \in s(y_a)E^*$, either $g|_{y(n,l+a)} \cdot \lambda \neq h|_{y(q,l+a)} \cdot \lambda$, or $g|_{y(n,l+a)\lambda} \neq h|_{y(q,l+a)\lambda}$. Thus condition (2) of Proposition 6.6 fails for $[x, m, g, n, y]$ and $[w, p, h, q, z]$ as required.

For the final statement, we show that if $(G, E)$ is regular, then (4) holds. Suppose that $g \in G$ and $y \in E^\infty$ satisfy $g \cdot y = y$. Regularity gives $n \in \mathbb{N}$ such that $g$ pointwise fixes $Z[y(0, n))$. Hence $g|_{y(0,n)}$ pointwise fixes $Z[y(n))$. Since self-similar groupoid actions are, by definition, faithful this implies that $g|_{y(0,n)} = y(n) \in G^{(0)}$. So the hypothesis of (4) is never satisfied, and so (4) holds vacuously. □

We can characterise regularity of $(G, E)$ in terms of of $\mathcal{G}_{G,E}$. Recall that a groupoid $\mathcal{G}$ is *principal* if its isotropy bundle $\mathrm{Iso}(\mathcal{G}) = \{g \in \mathcal{G} : r(g) = s(g)\}$ is equal to the unit space $\mathcal{G}^{(0)}$.

**Proposition 6.8.** *Let $(G, E)$ be a self-similar groupoid action on a finite directed graph $E$. Then, $(G, E)$ is regular if and only if $\mathcal{G}^{\mathbb{T}}_{G,E}$ is principal.*

*Proof.* Suppose that $(G, E)$ is regular. Fix $\gamma \in \mathrm{Iso}(\mathcal{G}_{G,E})$. Then $\gamma = [x, n, g, n, x]$ for some $x \in E^\infty$, $n \geq 0$ and $g \in G$ such that $d(g) = r(\varsigma^n(x))$ and $g \cdot \varsigma^n(x) = \varsigma^n(x)$. By regularity, there exists $k \in \mathbb{N}$ such that $g|_{x(n+1,n+k)} = s(x_{n+k})$. By definition of the equivalence relation $\sim$ defining $\mathcal{G}_{G,E}$ (see Lemma 6.3), we have $[x, n, g, n, x] = [x, n + k, s(x_{n+k}), n + k, x] \in \mathcal{G}^{(0)}_{G,E}$. Therefore, $\mathcal{G}^{\mathbb{T}}_{G,E}$ is principal.

Now, suppose that $\mathcal{G}^{\mathbb{T}}_{G,E}$ is principal. If $g \in G$ and $x \in E^\infty$ satisfy $d(g) = r(x)$ and $g \cdot x = x$, then $[x, 0, g, 0, x] \in \mathrm{Iso}(\mathcal{G}_{G,E})$. Therefore, $(x, 0, g, 0, x) \sim (x, n, s(x_n), n, x)$ for some $n$. Hence there exists $k \geq n$ such that $g|_{x(0,k)} = s(x_n)|_{x(n,k)} = s(x_k)$. Therefore, $(G, E)$ is regular. □

## 7. The dual algebra of a self-similar graph

We now describe a second $C^*$-algebra associated to a contracting, regular self-similar groupoid action on a finite directed graph with no sources; namely the $C^*$-algebra of the Deaconu–Renault groupoid of the homeomorphism $\tilde{\sigma} : \mathcal{J} \to \mathcal{J}$ of Section 4.

Recall that if $X$ is a locally compact Hausdorff space, and $T : X \to X$ is a local homeomorphism, then $\mathcal{G}_{X,T}$ is the set

$$\mathcal{G}_{X,T} := \{(x, m-n, y) \in X \times \mathbb{Z} \times X \mid m, n \in \mathbb{N}_0, T^m(x) = T^n(y)\},$$

endowed with the topology arising from the basic open sets

$$Z(U, m, n, V) := \{(x, m-n, y) \in U \times \{m-n\} \times V \mid T^m(x) = T^n(y)\}.$$

The unit space is $\mathcal{G}^{(0)}_{X,T} := \{(x, 0, x) \mid x \in X\}$ and is identified with $X$. The groupoid structure is given by

$$r(x, p, y) := x, \qquad s(x, p, y) := y,$$
$$(x, p, y)(y, q, z) := x(p+q, z) \qquad \text{and} \qquad (x, p, y)^{-1} := (y, -p, x).$$

It is not hard to check that

$$\{Z(U, m, n, V) \mid T^m|_U \text{ and } T^n|_V \text{ are homeomorphisms and } T^m(U) = T^n(V)\}$$

is a basis of open bisections for the topology, so $\mathcal{G}_{X,T}$ is étale. It is easy to see that it is Hausdorff, and it is locally compact because $X$ is.

**Definition 7.1.** Let $E$ be a finite directed graph with no sources. Let $(G, E)$ be a contracting, regular self-similar groupoid action. Let $\mathcal{J}$ and $\tilde{\sigma}$ be the space and local homeomorphism of Definition 3.2 and Proposition 4.3. We define $\widehat{\mathcal{G}}_{G,E}$ to be the Deaconu–Renault groupoid $\widehat{\mathcal{G}}_{G,E} := \mathcal{G}_{\mathcal{J},\tilde{\sigma}}$, and we define $\widehat{\mathcal{O}}(G, E) := C^*(\widehat{\mathcal{G}}_{G,E})$, and call this the *dual $C^*$-algebra* of $(G, E)$.

Let $\mathcal{G}_\sigma$ be the Deaconu–Renault groupoid associated to $\sigma : E^{-\infty} \mapsto E^{-\infty}$. Recall from Section 3 the quotient map $q : E^{-\infty} \mapsto \mathcal{J}$. Since $q \circ \tilde{\sigma} = \sigma \circ q$, we see that $q$ extends to a groupoid homomorphism $q : \mathcal{G}_\sigma \mapsto \hat{\mathcal{G}}_{G,E}$ which sends $(x, k, y) \in \mathcal{G}_\sigma$ to $q(x, k, y) = (q(x), k, q(y))$. The next result will allow us to deduce properties of $\hat{\mathcal{G}}_{G,E}$ from those of $\mathcal{G}_\sigma$.

**Proposition 7.2.** *Let $E$ be a finite directed graph with no sources and $(G, E)$ a contracting regular self-similar groupoid action. For every $y \in E^{-\infty}$, $q : (\mathcal{G}_\sigma)y \mapsto (\hat{\mathcal{G}}_{G,E})q(y)$ is a bijection, and $q : \mathcal{G}_\sigma \mapsto \hat{\mathcal{G}}_{G,E}$ is proper.*

*Proof.* Suppose $z \in \mathcal{J}$ and $m, n \in \mathbb{N}$ satisfy $\tilde{\sigma}^n(q(y)) = \tilde{\sigma}^m(z) := w$. Proposition 4.5(1) implies that $\sigma^{m+k} : q^{-1}(z) \mapsto q^{-1}(\tilde{\sigma}^k(w))$ and $\sigma^{n+k} : q^{-1}(q(y)) \mapsto q^{-1}(\tilde{\sigma}^k(w))$ are bijective for all $k \geq 0$. Thus there is a unique $x'$ in $q^{-1}(z)$ such that $\sigma^{m+k}(x') = \sigma^{n+k}(y)$, for all $k \geq 0$. Hence, $(x', m-n, y)$ is the unique element of $(\mathcal{G}_\sigma)y$ such that $q((x', m-n, y)) = (z, m-n, q(y))$.

The same uniqueness property shows that if $x, y \in E^{-\infty}$ and $m, n, k \geq 0$ satisfy $\sigma^{m+k}(x) = \sigma^{n+k}(y)$ and $\tilde{\sigma}^m(q(x)) = \tilde{\sigma}^n(q(y))$, then $\sigma^m(x) = \sigma^n(y)$. It follows that $q^{-1}(Z(\mathcal{J}, m, n, \mathcal{J})) = Z(E^{-\infty}, m, n, E^{-\infty})$ for any $m, n \geq 0$. Since these sets form compact open coverings of the respective groupoids, $q$ is proper. □

We investigate when $\widehat{\mathcal{O}}(G,E)$ is simple. A groupoid $\mathcal{G}$ is *minimal* if $\{r(\gamma) \mid s(\gamma)=x\}$ is dense in $\mathcal{G}^{(0)}$ for every $x \in \mathcal{G}^{(0)}$, and that an étale groupoid $\mathcal{G}$ is *effective* if the interior of $\mathrm{Iso}(\mathcal{G}) = \{\gamma \in \mathcal{G} \mid r(\gamma) = s(\gamma)\}$ is $\mathcal{G}^{(0)}$.

**Lemma 7.3.** *Let $E$ be a finite directed graph. Let $(G,E)$ be a contracting, regular self-similar groupoid action. If $E$ is strongly connected and not a simple cycle, then $\widehat{\mathcal{G}}_{G,E}$ is minimal and effective, and $\widehat{\mathcal{O}}(G,E)$ is simple.*

*Proof.* Fix $x \in E^{-\infty}$. Then $\{x\mu \mid \mu \in E^*$ and $s(x) = r(\mu)\} \subseteq \{r(\gamma) \mid \gamma \in \mathcal{G}_\sigma$ and $s(\gamma) = x\}$. Since $E$ is strongly connected, for every $\nu \in E^*$, there is $\mu \in E^*$ such that $s(\mu) = r(\nu)$ and $r(\mu) = s(x)$. Hence $x\mu\nu \in E^{-\infty}$. Therefore, $Z(\nu) \cap \{r(\gamma) \mid s(\gamma) = x\} \neq \emptyset$ and consequently $\mathcal{G}_\sigma$ is minimal.

By Proposition 7.2, for every $x \in E^{-\infty}$, $q : (\mathcal{G}_\sigma)x \mapsto (\hat{\mathcal{G}}_{G,E})q(x)$ is surjective. Hence, $q(r((\mathcal{G}_\sigma)x) = r((\hat{\mathcal{G}}_{G,E})q(x))$. Since $q$ is surjective and continuous, $r((\hat{\mathcal{G}}_{G,E})q(x))$ is dense whenever $r((\mathcal{G}_\sigma)x)$ is dense. Therefore, minimality of $\mathcal{G}_\sigma$ implies minimality of $\hat{\mathcal{G}}_{G,E}$.

To see that $\hat{\mathcal{G}}_{G,E}$ is effective, suppose that $w \in \mathcal{J}$ satisfies $\tilde{\sigma}^m(w) = w$. By Proposition 4.5(1), $\sigma^m$ maps $q^{-1}(w)$ bijectively onto $q^{-1}(w)$. Since $q^{-1}(w)$ is finite, there exists $k \in \mathbb{N}$ such that $\sigma^{mk}(x) = x$ for all $x \in q^{-1}(w)$. Let $P_\sigma$ and $P_{\tilde{\sigma}}$ denote the sets of periodic points for $\sigma$ and $\tilde{\sigma}$ respectively. Then $q^{-1}(P_{\tilde{\sigma}}) = P_\sigma$. Hence, $q^{-1}(\bigcup_{n\geq 0} \tilde{\sigma}^{-n}(P_{\tilde{\sigma}})) = \bigcup_{n\geq 0} \sigma^{-n}(P_\sigma)$. We have $P_\sigma = \bigcup_{\lambda\in E^*} \bigcup_{\mu \in s(\lambda)E^*s(\lambda)\setminus\{\lambda\}} \{\lambda\mu^\infty\}$, and so $P_\sigma$, and hence $\bigcup_{n\geq 0} \tilde{\sigma}^{-n}(P_{\tilde{\sigma}})$, is countable.

If $g \in \hat{\mathcal{G}}_{G,E}\setminus\mathcal{J}$ satisfies $r(g) = s(g)$, then $r(g) \in \bigcup_{n\geq 0} \tilde{\sigma}^{-n}(P_{\tilde{\sigma}})$. Thus $r\big(\mathrm{Iso}(\hat{\mathcal{G}}_{G,E})\setminus\mathcal{J}\big)$ is countable. Since $r$ is an open map, to show $\mathrm{Iso}(\hat{\mathcal{G}}_{G,E})\setminus\mathcal{J}$ has empty interior, it suffices to show no countable set in $\mathcal{J}$ is open. By the Baire Category Theorem, it suffices to show that $\mathcal{J}$ has no isolated points. Since $E$ is strongly connected and not a simple cycle, every open subset of $E^{-\infty}$ is infinite. By continuity and surjectivity of $q$, the preimage of every nonempty open subset of $\mathcal{J}$ is open and hence infinite. Since $q$ is finite-to-one, it follows that no singleton in $\mathcal{J}$ is open. Hence $\mathcal{J}$ has no isolated points, and consequently $\hat{\mathcal{G}}_{G,E}$ is effective.

It now follows from [2, Theorem 5.1] that $\widehat{\mathcal{O}}(G,E) = C^*(\widehat{\mathcal{G}}_{G,E})$ is simple. □

## 8. $KK$-duality via Smale spaces

In this section we establish our $KK$-duality result. We do this using a general result of Kaminker–Putnam–Whittaker [12], which says that the stable and unstable Ruelle algebras of any irreducible Smale space are $KK$-dual. We show that the stable Ruelle algebra of the Smale space of Section 5 is Morita equivalent to the $C^*$-algebra $\widehat{\mathcal{O}}(G,E)$ of Section 7, and that the unstable Ruelle algebra is Morita equivalent to the $C^*$-algebra $\mathcal{O}(G,E)$ of Section 6. This, combined with the duality of the Ruelle algebras, gives our main result.

**Theorem 8.1.** *Let $E$ be a strongly connected finite directed graph. Let $(G,E)$ be a contracting, regular self-similar groupoid action. Then $\mathcal{O}(G,E)$ and $\widehat{\mathcal{O}}(G,E)$ are $KK$-dual in the sense that there are classes $\mu \in KK^1(\mathcal{O}(G,E) \otimes \widehat{\mathcal{O}}(G,E), \mathbb{C})$ and $\beta \in KK^1(\mathbb{C}, \mathcal{O}(G,E) \otimes \widehat{\mathcal{O}}(G,E))$ such that*

$$\beta \mathbin{\widehat{\otimes}}_{\mathcal{O}(G,E)} \mu = \mathrm{id}_{KK(\widehat{\mathcal{O}}(G,E),\widehat{\mathcal{O}}(G,E))} \quad \text{and} \quad \beta \mathbin{\widehat{\otimes}}_{\widehat{\mathcal{O}}(G,E)} \mu = -\,\mathrm{id}_{KK(\mathcal{O}(G,E),\mathcal{O}(G,E))}.$$

*In particular, $K^*(\mathcal{O}(G,E)) \cong K_{*+1}(\widehat{\mathcal{O}}(G,E))$ and $K_*(\mathcal{O}(G,E)) \cong K^{*+1}(\widehat{\mathcal{O}}(G,E))$.*

8.1. **The stable algebra.** We will show that the groupoid $\widehat{\mathcal{G}}_{G,E}$ of Lemmas 6.3 and 6.4 is equivalent to the stable Ruelle groupoid $G^s \rtimes \mathbb{Z}$ of the Smale space $\mathcal{S}$ of Section 5. The idea is to show that in fact $G^s \rtimes \mathbb{Z}$ is equal to the amplification of $\widehat{\mathcal{G}}_{G,E}$ with respect to the surjection $\tilde{\pi} : \mathcal{S} \to \mathcal{J}$ induced by the natural surjection of $E^{\mathbb{Z}}$ onto $E^{-\infty}$. For this, we first need to show that this $\tilde{\pi}$ makes sense and is an open map.

**Lemma 8.2.** *Let $E$ be a finite directed graph with no sources. Let $(G, E)$ be a contracting regular self-similar groupoid action. Let $\mathcal{J}$ be the limit space of Definition 3.2, and let $\mathcal{S}$ be the limit solenoid of Definition 5.3. There is a continuous open surjection $\tilde{\pi} : \mathcal{S} \to \mathcal{J}$ such that $\tilde{\pi}([x]) = [\dots x_{-3}x_{-2}x_{-1}]$ for all $x \in E^{\mathbb{Z}}$.*

*Proof.* Let $\theta : \mathcal{S} \to \mathcal{J}_\infty$ be the homeomorphism of Proposition 5.4. Let $P_1$ be the projection map $P_1 : \mathcal{J}_\infty \to \mathcal{J}$ given by $P_1([x_1], [x_2], [x_3], \dots) = [x_1]$. Then $P_1$ is continuous by definition of the projective-limit topology, and surjective because $\tilde{\sigma}$ is surjective. By definition of the topology on $\mathcal{J}_\infty$, the sets $Z(W, n) := \{([x_1], [x_2], [x_3], \dots) \mid [x_n] \in W\}$ indexed by pairs $(W, n)$ consisting of an open $W \subseteq J$ and an element $n \in \mathbb{N}$ constitute a basis for the topology on $\mathcal{J}_\infty$. Since $\tilde{\sigma}$ is surjective, each $P_1(Z(W, n)) = \tilde{\sigma}^n(W)$, which is open. So $P_1$ is an open map. Hence $\tilde{\pi} := P_1 \circ \theta$ is a continuous open surjection from $\mathcal{S}$ to $\mathcal{J}$. It satisfies $\tilde{\pi}([x]) = [\dots x_{-3}x_{-2}x_{-1}]$ by definition. □

If $X$ is a locally compact Hausdorff space, $\mathcal{G}$ is an étale groupoid and $\pi : X \to \mathcal{G}^{(0)}$ is a continuous open surjection, then we can form the *amplification* of $\mathcal{G}$ by $\pi$ which, as a topological space, is

$$\mathcal{G}^\pi := \{(x, \gamma, y) \in X \times \mathcal{G} \times X \mid r(\gamma) = \pi(x) \text{ and } s(\gamma) = \pi(y)\}$$

under the topology inherited from the product topology. Its unit space is $(\mathcal{G}^\pi)^{(0)} = \{(x, \pi(x), x) \mid x \in X\}$ which we identify with $X$. The range and source maps are given by $r(x, \gamma, y) = x$ and $s(x, \gamma, y) = y$. The multiplication and inversion are given by $(x, \gamma, y)(y, \eta, z) = (x, \gamma\eta, z)$ and $(x, \gamma, y)^{-1} = (y, \gamma^{-1}, x)$. By, for example, [8, (4) $\Longrightarrow$ (1) of Proposition 3.10], the groupoids $\mathcal{G}$ and $\mathcal{G}^\pi$ are equivalent groupoids, and hence $C^*(\mathcal{G})$ and $C^*(\mathcal{G}^\pi)$ are Morita equivalent by [19, Theorem 2.8].

Recall that the stable equivalence relation associated to a Smale space $(\mathcal{S}, d, \tau, \varepsilon_S, \lambda)$ is the equivalence relation

$$G^s := \{(\xi, \eta) \in \mathcal{S} \times \mathcal{S} \mid \lim_{m\to\infty} d(\tau^m(\xi), \tau^m(\eta)) = 0\}.$$

By [25, pg. 179], there exists $\varepsilon'_{\mathcal{S}} \leq \varepsilon_{\mathcal{S}}$ such that for any $\delta \leq \varepsilon'_{\mathcal{S}}$,

$$G^s := \{(\xi, \eta) \in \mathcal{S} \times \mathcal{S} \mid \text{there exists } M \in \mathbb{N} \text{ such that } d(\tau^m(\xi), \tau^m(\eta)) < \delta \text{ for all } m \geq M\} \tag{8.1}$$

For $M \geq 0$ we define

$$G^s_{\varepsilon,M} = \{(\xi, \eta) \mid d(\tau^m(\xi), \tau^m(\eta)) < \varepsilon \text{ for all } m \geq M\},$$

endowed with the subspace topology inherited from $\mathcal{S} \times \mathcal{S}$. We endow $G^s$ with the inductive-limit topology obtained from the inductive limit decomposition $G^s = \bigcup_M (G^s_{\varepsilon,M})$. It is straightforward to check this agrees with the topology on $G^s$ described on [27, pg. 282].

We now give a description of the stable equivalence relation and its topology for the Smale space $(\mathcal{S}, d_{\mathcal{S}}, \tilde{\tau}, \varepsilon_{\mathcal{S}}, \frac{1}{2})$ that will help us in proving the amplification $\widehat{\mathcal{G}}^{\tilde{\pi}}_{G,E}$ and the stable Ruelle groupoid $G^s \rtimes_{\tilde{\tau}} \mathbb{Z}$ are isomorphic.

**Lemma 8.3.** *Let $E$ be a finite directed graph with no sinks or sources and $(G, E)$ be a contracting, regular self-similar groupoid. Let $(\mathcal{S}, d_{\mathcal{S}}, \tilde{\tau}, \varepsilon_{\mathcal{S}}, \frac{1}{2})$ be the Smale space of Corollary 5.5. Let $\varepsilon$ be as in Theorem 4.3 and let $\varepsilon'_{\mathcal{S}}$ be a constant such that (8.1) holds for all $\delta < \varepsilon'_{\mathcal{S}}$. Let $\beta = \min\{\varepsilon, \varepsilon'_{\mathcal{S}}\}$. For each $m \in \mathbb{N}$, let*

$$G^s_m = \{([x], [y]) \in \mathcal{S} \times \mathcal{S} \mid \ [x(-\infty, -m)] = [y(-\infty, -m)]\}.$$

*Then there exists $k \in \mathbb{N}$ such that, for every $m \in \mathbb{N}$, we have $G^s_{\beta,m} \subseteq G^s_m \subseteq G^s_{\beta,m+k}$. Points $[x], [y] \in \mathcal{S}$ are stably equivalent if and only if there exists $m \in \mathbb{N}$ such that $[x(-\infty, -m)] = [y(-\infty, -m)]$. The topology on $G^s$ is equal to the inductive limit topology for the decomposition $G^s = \bigcup_m G^s_m$.*

*Proof.* To see that $G^s_{\beta,m} \subseteq G^s_m$, fix $x, y \in E^{\mathbb{Z}}$ such that $d_{\mathcal{S}}([\tau^n(x)], [\tau^n(y)]) < \beta$ for all $n \geq m$. Fix $n \geq m$. Then

$$\begin{aligned} d([x(-\infty, -n)], [x(-\infty, -n)]) &= d([\tau^n(x)(-\infty, 0)], [\tau^n(x)(-\infty, 0)]) \\ &\leq d_{\mathcal{S}}([\tau^n(x)], [\tau^n(y)]) < \beta < \varepsilon. \end{aligned}$$

By definition of $\epsilon$,

$$\begin{aligned} 2d([x(-\infty, -n)], [x(-\infty, -n)]) &= d(\tilde{\sigma}([x(-\infty, -n)]), \tilde{\sigma}([y(-\infty, -n)])) \\ &= d([x(-\infty, -(n+1))], [x(-\infty, -(n+1))]). \end{aligned}$$

Hence

$$d([x(-\infty, -m)], [x(-\infty, -m)]) = 2^{m-n} d([x(-\infty, -n)], [x(-\infty, -n)]) \leq 2^{m-n}.$$

Since $n \geq m$ was arbitrary, we deduce that $[x(-\infty, -m)] = [y(-\infty, -m)]$.

Fix $k \in \mathbb{N}$ such that $(\frac{1}{2})^k < \beta$. We show that $G^s_m \subseteq G^s_{\beta,m+k}$. Fix $x, y \in E^{\mathbb{Z}}$ such that $[x(-\infty, -m)] = [y(-\infty, -m)]$. Then, $[\tau^n(x)(-\infty, l)] = [\tau^n(y)(-\infty, l)]$ for all $l - n \leq -m$. Therefore,

$$d_{\mathcal{S}}(\tilde{\tau}^n[x], \tilde{\tau}^n[y]) = \sup_{l > n-m} (2^{-l} d([\tau^n(x)(-\infty, l)], [\tau^n(y)(-\infty, l)]) \leq 2^{-(n-m)}.$$

So, whenever $n \geq m + k$, we have $d_{\mathcal{S}}(\tilde{\tau}^n[x], \tilde{\tau}^n[y]) < 2^{-k} < \beta$. Therefore, $([x], [y]) \in G^s_{\beta,m+k}$. □

Recall that the stable Ruelle groupoid is the skew groupoid for the action of $\mathbb{Z}$ on the unit space of $G^s$. That is,

$$G^s \rtimes_{\tilde{\tau}} \mathbb{Z} = \{(\xi, n, \eta) \in \mathcal{S} \times \mathbb{Z} \times \mathcal{S} \mid (\tilde{\tau}^n(\xi), \eta) \in G^s\}.$$

**Theorem 8.4.** *Let $E$ be a finite directed graph with no sinks or sources. Let $(G, E)$ be a contracting, regular self-similar groupoid action. Let $\tilde{\pi} : \mathcal{S} \to \mathcal{J}$ be the continuous open surjection of Lemma 8.2, and let $\tilde{\tau} : \mathcal{S} \to \mathcal{S}$ be the homeomorphism of Corollary 5.5. Then there is an isomorphism $\kappa$ of the stable Ruelle groupoid $G^s \rtimes_{\tilde{\tau}} \mathbb{Z}$ onto the amplification $\widehat{\mathcal{G}}^{\tilde{\pi}}_{G,E}$ of the dual groupoid of $(G, E)$ by $\tilde{\pi}$ satisfying $\kappa([x], n, [y]) = ([x], (\tilde{\pi}([x]), n, \tilde{\pi}([y])), [y])$ for all $([x], n, [y]) \in G^s \rtimes_{\tilde{\tau}} \mathbb{Z}$.*

*Proof.* For $z \in E^{\mathbb{Z}}$ and $n \in \mathbb{N}$, we have $\tilde{\pi}(\tilde{\tau}^n([z])) = [z(-\infty, -n)]$. Hence Lemma 8.3 gives $([x], n, [y]) \in G^s$ if and only if $\tilde{\pi}(\tilde{\tau}^{m+n}([x])) = \tilde{\pi}(\tilde{\tau}^m([y]))$ for some $m \in \mathbb{N}$ such that $m + n \geq 0$. Since $\tilde{\pi} \circ \tilde{\tau}^m = \tilde{\sigma}^m \circ \tilde{\pi}$ for any $m \in \mathbb{N}$,

$$([x], n, [y]) \in G^s \iff ([x], (\tilde{\pi}([x]), n, \tilde{\pi}([y])), [y]) \in \widehat{\mathcal{G}}^{\tilde{\pi}}_{G,E}. \tag{8.2}$$

Hence there is a bijection $\kappa : G^s \rtimes_{\tilde{\tau}} \mathbb{Z} \to \widehat{\mathcal{G}}^{\tilde{\pi}}_{G,E}$ satisfying the desired formula.

We show $\kappa$ is continuous. Extend of $\kappa$ to a map $\tilde{\kappa} : \mathcal{S} \times \mathbb{Z} \times \mathcal{S} \to \mathcal{S} \times \mathcal{J} \times \mathbb{Z} \times \mathcal{J} \times \mathcal{S}$ by $\tilde{\kappa}((\xi, n, \eta)) = (\xi, (\tilde{\pi}(\xi), n, \tilde{\pi}(\eta)), \eta)$. Then $\tilde{\kappa}$ is continuous. For $M \in \mathbb{N}$ and $n \in \mathbb{Z}$ such that $M+n \geq 0$ the restriction of $\kappa$ to $G^s_M \rtimes_{\tilde{\tau}} \{n\} = \{([x], n, [y]) \mid (\tilde{\tau}^n([x]), [y]) \in G^s_M\}$ has co-domain contained in

$$\mathcal{S} * \widehat{\mathcal{G}}^{(M+n,M)}_{G,E} * \mathcal{S} := \{([x], (\tilde{\pi}([x]), n, \tilde{\pi}([y])), [y]) \mid \ \tilde{\sigma}^{M+n}(\tilde{\pi}([x])) = \tilde{\sigma}^M(\tilde{\pi}([y]))\}.$$

The subspace topology of $G^s_M \rtimes_{\tilde{\tau}} \{n\}$ relative to $G^s \rtimes_{\tilde{\tau}} \mathbb{Z}$ is equal to the subspace topology relative to $\mathcal{S} \times \mathbb{Z} \times \mathcal{S}$, and the subspace topology of $\mathcal{S} * \widehat{\mathcal{G}}^{(M+n,M)}_{G,E} * \mathcal{S}$ relative to $\widehat{\mathcal{G}}_{G,E}$ is equal to the subspace topology relative to $\mathcal{S} \times \mathcal{J} \times \mathbb{Z} \times \mathcal{J} \times \mathcal{S}$. So, continuity of $\tilde{\kappa}$ implies that $\kappa : G^s_M \rtimes_{\tilde{\tau}} \{n\} \to \mathcal{S} * \widehat{\mathcal{G}}^{(M+n,M)}_{G,E} * \mathcal{S}$ is continuous. Fix $n \in \mathbb{Z}$. The universal property of the inductive limit topology on $G^s \rtimes_{\tilde{\tau}} \{n\} = \bigcup_{M+n \geq 0} G^s_M \rtimes_{\tilde{\tau}} \{n\}$ implies that $\kappa$ is continuous on the clopen subspace $G^s \rtimes_{\tilde{\tau}} \{n\} \subseteq G^s \rtimes_{\tilde{\tau}} \mathbb{Z}$, for each $n$ in $\mathbb{Z}$. Hence, $\kappa$ is continuous.

Since $\mathcal{S} * \widehat{\mathcal{G}}^{(M+n,M)}_{G,E} * \mathcal{S}$ and $G^s_M \rtimes_{\tilde{\tau}} \{n\}$ are compact, and since $\kappa^{-1}(\mathcal{S} * \widehat{\mathcal{G}}^{(M+n,M)}_{G,E} * \mathcal{S}) \subseteq G^s_M \rtimes_{\tilde{\tau}} \{n\}$, and $\widehat{\mathcal{G}}^{\tilde{\pi}}_{G,E} = \bigcup_{M+n \geq 0} \mathcal{S} * \widehat{\mathcal{G}}^{(M+n,M)}_{G,E} * \mathcal{S}$, the map $\kappa$ is proper. Since proper continuous maps between locally compact Hausdorff spaces are closed, $\kappa$ is a homeomorphism. □

**Corollary 8.5.** *Let $E$ be a finite directed graph with no sinks or sources. Let $(G, E)$ be a contracting, regular self-similar groupoid action. Then the dual $C^*$-algebra $\widehat{\mathcal{O}}(G, E)$ of Definition 7.1 is Morita equivalent to the stable Ruelle algebra $C^*(G^s \rtimes \mathbb{Z})$ of the Smale space $(\mathcal{S}, \tilde{\tau})$ of Corollary 5.5.*

*Proof.* Since $\tilde{\pi} : \mathcal{S} \to \mathcal{J}$ is an open map by Lemma 8.2, [8, Proposition 3.10] shows that $\widehat{\mathcal{G}}^{\tilde{\pi}}_{G,E}$ is groupoid equivalent to $\widehat{\mathcal{G}}_{G,E}$. Therefore [19, Theorem 2.8] shows that $C^*(\widehat{\mathcal{G}}_{G,E})$ is Morita equivalent to $C^*(\widehat{\mathcal{G}}^{\tilde{\pi}}_{G,E})$. Theorem 8.4 shows that $C^*(\widehat{\mathcal{G}}^{\tilde{\pi}}_{G,E})$ is isomorphic to $C^*(G^s \rtimes_{\tilde{\tau}} \mathbb{Z})$, which is precisely the stable Ruelle algebra of $(\mathcal{S}, \tilde{\tau})$. □

8.2. **The unstable algebra.** We now need to show that the unstable Ruelle algebra of the Smale space $(\mathcal{S}, \tilde{\tau})$ is Morita equivalent to the $C^*$-algebra $\mathcal{O}(G, E)$. Our approach again is via groupoid equivalence. We use Putnam and Spielberg's construction of an étale groupoid $G^u_{[x]} \rtimes_{\tilde{\tau}} \mathbb{Z}$ corresponding to a choice of orbit in $\mathcal{S}$. We will show that this groupoid is isomorphic to a suitable amplification of the groupoid $\mathcal{G}_{G,E}$ of Lemma 6.3. To do this, we shall need an alternative description of the unstable equivalence relation and its topology, which we provide in the next lemma.

For $M \in \mathbb{N}$ and $\varepsilon > 0$, let $G^u_{\varepsilon,M} = \{(\eta, \xi) \in \mathcal{S} \times \mathcal{S} \mid d_{\mathcal{S}}(\tilde{\tau}^{-m}(\eta), \tilde{\tau}^{-m}(\xi)) < \varepsilon \text{ for all } m \geq M\}$. By [25, pg. 179], there exists $\varepsilon'_{\mathcal{S}} \leq \varepsilon_{\mathcal{S}}$ such that for every $\varepsilon < \varepsilon'_{\mathcal{S}}$, we have

$$G^u = \bigcup_M G^u_{\varepsilon,M}$$

in the inductive-limit topology. This agrees with the topology on $G^u$ on [27, pg. 282].

**Lemma 8.6.** *Let $E$ be a finite directed graph with no sinks or sources. Let $(G, E)$ be a contracting, regular self-similar groupoid action. Let $\varepsilon'_{\mathcal{S}}$ be as above, and let $\varepsilon$ be as in Theorem 4.3. Let $\beta = \min\{\varepsilon, \varepsilon'_{\mathcal{S}}\}$. Let $F$ be the smallest finite set containing $\mathcal{N} \cup \mathcal{N}^2$ that is closed under restriction. Then, there is an $l \in \mathbb{N}$ such that for every $M \in \mathbb{N}$, we have*
*$G^u_{\beta,M} \subseteq G^u_M \subseteq G^u_{\beta,M+l}$, where*

$$G^u_M = \{([y], [x]) \in \mathcal{S} \times \mathcal{S} \mid \ \exists g \in F : g \cdot x(M+1, \infty) = y(M+1, \infty)\}.$$

*More precisely, if $(\eta, \xi) \in G^u_{\beta,M}$, then for any representatives $y, x$ such that $[y] = \eta$, $[x] = \xi$, there is an element $g \in F$ such that $g \cdot x(M+1, \infty) = y(M+1, \infty)$.*

*In particular, $[x]$ and $[y]$ in $\mathcal{S}$ are unstably equivalent if and only if there is an $M \in \mathbb{N}$ and $g \in G^s(x_M)$ such that $g \cdot x(M+1,\infty) = y(M+1,\infty)$, and the topology on $G^u$ is equal to the inductive limit topology provided by the decomposition $G^u = \bigcup_M G^u_M$.*

*Proof.* Fix $M \in \mathbb{N}$. We first show $G^u_{\beta,M} \subseteq G^u_M$. Suppose $([x],[y]) \in G^u_{\beta,M}$. For $m \in \mathbb{Z}$, let $x^m := [x(-\infty,m)]$ and $y^m := [y(-\infty,m)]$. By definition of the metric $d_{\mathcal{S}}$, for every $m \geq M$,

$$d(x^m,y^m) = d([\tau^{-m}(x)(-\infty,0)],[\tau^{-m}(y)(-\infty,0)]) \leq d_{\mathcal{S}}(\tilde{\tau}^{-m}([x]),\tilde{\tau}^{-m}([y])) \ < \beta.$$

Since $\beta \leq \varepsilon$, Theorem 4.3(1) gives $d(x^m,y^m) = d(\tilde{\sigma}(x^{m+1}),\tilde{\sigma}(y^{m+1})) = 2d(x^{m+1},y^{m+1})$ for all $m \geq M$. So $d(x^{M+s},y^{M+s}) = (\frac{1}{2})^s d(x^M,y^M)$ for all $s \geq 0$. Hence $\alpha := \frac{d(x^M,y^M)+\beta}{2}$, satisfies $y^{M+s} \in B(x^{M+s},(\frac{1}{2})^s\alpha)$ for every $s \in \geq 0$.

Fix $n \in \mathbb{N}$ as in Lemma 4.6 and $k \in \mathbb{N}$ as in Proposition 4.5. Since $\alpha < \varepsilon$, Lemma 4.6(1) yields $\omega \in E^*$ such that $|\omega| \geq n-1$ and $B(x^M,\alpha) \subseteq U_\omega$.

Since $\tilde{\sigma}(x^{M+1}) = x^M$ and $|\omega| \geq n-1 \geq k-1$, Proposition 4.5(3) gives $e \in E^1$ such that $s(\omega) = r(e)$ and $x^{M+1} \in U_{\omega e}$. Since $B(x^M,\alpha) \subseteq U_\omega$, Lemma 4.6 implies that $B(x^{M+1},\frac{1}{2}\alpha) \subseteq U_{\omega e}$.

Applying the argument in the above paragraph inductively gives paths $\{\mu_s\}_{s\in\mathbb{N}}$ such that $\mu_{s+1} = \mu_s e_{s+1}$, for some edge $e_{s+1}$, and that $B(x^{M+s},(\frac{1}{2})^s\alpha) \subseteq U_{\omega\mu_s}$, for all $s \in \mathbb{N}$. Since $y^{M+s} \in B(x^{M+s},(\frac{1}{2})^s\alpha)$ for each $s \in \mathbb{N}$, we obtain $x^{M+s},y^{M+s} \in U_{\omega\mu_s}$ for each $s \in \mathbb{N}$. Therefore, there exist sequences $\{g_s\}_{s\in\mathbb{N}}$, $\{h_s\}_{s\in\mathbb{N}}$ in $\mathcal{N}$ such that $d(g_s) = d(h_s) = r(\mu_s)$, $g_s \cdot \mu_s = x(M+1,M+s)$, and $h_s \cdot \mu_s = y(M+1,M+s)$ for all $s \in \mathbb{N}$. Let $z \in E^\infty$ be the unique element of $\bigcap_s Z[\mu_s)$. Choose an increasing subsequence $\{n_s\}_{s\in\mathbb{N}}$ such that $(g_{n_s})_s$ and $(h_{n_s})_s$ are constant sequence, with constant values $g$ and $h$, say. Then $g \cdot z = x(M+1,\infty)$ and $h \cdot z = y(M+1,\infty)$. So $t := hg^{-1} \in \mathcal{N}^2 \subseteq F$ satisfies $t \cdot x(M+1,\infty) = y(M+1,\infty)$. Therefore, $([y],[x]) \in G^u_M$.

Take $l_1 \in \mathbb{N}$ such that for all $g \in F$ and $\mu \in d(g)E^*$ such that $|\mu| \geq l_1$ we have that $g|_\mu \in \mathcal{N}$. Fix $l_2 \in \mathbb{N}$ such that $\operatorname{diam}(U_\nu) < \frac{1}{2}\beta$ for every path $\nu$ with $|\nu| \geq l_2$. Let $l := l_1 + l_2$. We show that $G^u_M \subseteq G^u_{\beta,M+l}$.

Take $[x],[y] \in \mathcal{S}$ such that there exists $g \in F$ such that $d(g) = s(x_M)$ and $g \cdot x(M+1,\infty) = y(M+1,\infty)$. As above, let $x^m := [x(-\infty,m)]$ and $y^m := [y(-\infty,m)]$ for all $m \in \mathbb{Z}$. Let $h = g|_{x(M+1,M+l_1)}$. By the choice of $l_1$, we have $h \in \mathcal{N}$. Since $h \cdot x(M+l_1,\infty) = y(M+l_1,\infty)$, it follows that $x^{M+l_1+m},y^{M+l_1+m} \in U_{x(M+l_1+1,M+l_1+m)}$ for all $m \in \mathbb{N}$. By the choice of $l_2$, we have $d(x^{M+l_1+m},y^{M+l_1+m}) < \frac{1}{2}\beta$ for all $m \geq l_2$. So, for all $s \geq M+l$ and $k \geq 0$, we have $(\frac{1}{2})^k d(x^{k+s},y^{k+s}) < \frac{1}{2}\beta$. Therefore, $d_{\mathcal{S}}(\tilde{\tau}^{-s}([x]),\tilde{\tau}^{-s}([y])) = \sup_{k\in\mathbb{N}_0}(\frac{1}{2})^k d(x^{k+s},y^{k+s}) \leq \frac{1}{2}\beta < \beta$ for all $s \geq M+l$, forcing $([y],[x]) \in G^u_{\beta,M+l}$. Hence $G^u_M \subseteq G^u_{\beta,M+l}$.

We have established that $[x]$ and $[y]$ are unstably equivalent if and only if there exist $M \in \mathbb{N}$ and $g \in Fs(x_M)$ such that $g \cdot x(M+1,\infty) = y(M+1,\infty)$. If $[x],[y] \in \mathcal{S}$, $N$ in $\mathbb{N}$ and $g \in Gr(x_N)$ satisfy $g \cdot x(N+1,\infty) = y(N+1,\infty)$, then, since $(G,E)$ is contracting, there exists $M \geq N$ such that $h := g|_{x(N+1,M)} \in \mathcal{N} \subseteq F$ and $h \cdot x(M+1,\infty) = y(M+1,\infty)$. This proves the penultimate statement of the lemma.

Since $F$ is closed under restriction, $G^u_M \subseteq G^u_{M+1}$ for all $M \in \mathbb{N}$. Hence the inductive limit topology with respect to the decomposition $G^u = \bigcup_M G^u_M$ is well defined. Since $G^u_{\beta,M} \subseteq G^u_M \subseteq G^u_{\beta,M+l}$ for all $M \in \mathbb{N}$, this topology is equal to the one provide by the decomposition $G^u = \bigcup_M G^u_{\beta,M}$. □

Let $(X,\tau)$ be an irreducible Smale space, and recall that $G^u \rtimes \mathbb{Z} = \{(\eta,l,\xi) \in X \times \mathbb{Z} \times X : (\tau^{-l}(\eta),\xi) \in G^u\}$. In [27], Putnam and Spielberg show that, given any point $x \in X$, the groupoid $G^u_x \rtimes \mathbb{Z}$ defined as

$$G^u_x \rtimes \mathbb{Z} := \{(\eta, n, \varepsilon) \in G^u \rtimes \mathbb{Z} \mid (x, \eta), (x, \varepsilon) \in G^s\}, \tag{8.3}$$

endowed with a suitable topology, is an étale groupoid that is equivalent to $G^u \rtimes \mathbb{Z}$ when $(X, \tau)$ is mixing, and use this to study the unstable $C^*$-algebra of a Smale space up to Morita equivalence. We will make use of the same technique here.

Let $E$ be a finite directed graph with no sinks or sources, and let $(G, E)$ be a contracting, regular self-similar groupoid action. Let $(\mathcal{S}, d_{\mathcal{S}}, \tilde{\tau}, \varepsilon_{\mathcal{S}}, \frac{1}{2})$ be the Smale space of Corollary 5.5. Let $\beta > 0$ and $k \in \mathbb{N}$ be as in Lemma 8.3. Consider $x \in E^{\mathbb{Z}}$. In line with [27], the global stable equivalence class

$$\mathcal{S}^s([x]) := \{\xi \in \mathcal{S} : ([x], \xi) \in G^s\} \tag{8.4}$$

is endowed with the inductive-limit topology coming from the decomposition $\mathcal{S}^s([x]) = \bigcup_M \mathcal{S}^s_{\beta,M}([x])$, where

$$\mathcal{S}^s_{\beta,M}([x]) = \{\xi \in \mathcal{S} \mid d_{\mathcal{S}}(\tilde{\tau}^n([x]), \tilde{\tau}^n(\xi)) < \beta, \ \forall\, n \geq M\}$$

is given the subspace topology relative to $\mathcal{S}$. For $M \in \mathbb{N}$, define

$$\mathcal{S}^s_M([x]) = \{[y] \in \mathcal{S} : [x(-\infty, M)] = [y(-\infty, M)]\}.$$

Then Lemma 8.3 implies that $\mathcal{S}^s_{\beta,M}([x]) \subseteq \mathcal{S}^s_M([x]) \subseteq \mathcal{S}^s_{\beta,M+k}([x])$. Hence, the inductive-limit topology on $\mathcal{S}^s([x])$ is equivalent to the inductive-limit topology for the decomposition $\mathcal{S}^s([x]) = \bigcup_M \mathcal{S}^s_M([x])$. Note that $\mathcal{S}^s([x])$ is not compact in this topology even though $\mathcal{S}$ is compact.

We equip the groupoid $G^u_{[x]} \rtimes \mathbb{Z} = (G^u \rtimes \mathbb{Z}) \cap (\mathcal{S}^s([x]) \times \mathbb{Z} \times \mathcal{S}^s([x]))$ with the topology with sub-basis

$$\big\{U \cap r^{-1}(V) \cap s^{-1}(W) \mid U \subseteq G^u \rtimes \mathbb{Z} \text{ and } V, W \subseteq \mathcal{S}^s([x]) \text{ are open}\big\}.$$

Fix a periodic orbit $P = \{\tilde{\tau}^l(p_0) : l \in \mathbb{Z}\} = \{\tilde{\tau}^l(p_0) : 0 \leq l < N\}$ of $\tilde{\tau}$. Then $\mathcal{S}^s(p) \cap S^s(q) = \emptyset$ for distinct $p, q \in P$. So $\mathcal{S}^s(P) := \bigcup_{p \in P} \mathcal{S}^s(p)$ is the topological disjoint union of the sets $\mathcal{S}^s(p)$. Consider the groupoid $G^u(P) \rtimes \mathbb{Z} := (G^u \rtimes \mathbb{Z}) \cap (\mathcal{S}^s(P) \times \mathbb{Z} \times \mathcal{S}^s(P))$. As in the preceding paragraph, we give $G^u(P) \rtimes \mathbb{Z}$ the topology with sub-basis

$$\big\{U \cap r^{-1}(V) \cap s^{-1}(W) \mid U \subseteq G^u \rtimes \mathbb{Z} \text{ and } V, W \subseteq \mathcal{S}^s(P) \text{ are open}\big\}.$$

Then $G^u(P) \rtimes \mathbb{Z}$ is a locally compact étale Hausdorff groupoid [12, Section 3], and $G^u(P) \rtimes \mathbb{Z}$ is groupoid equivalent to $G^u \rtimes \mathbb{Z}$ [26, Section 2].

We claim that for any $p \in P$, the subset $G^u_p \rtimes \mathbb{Z}$ is also a locally compact Hausdorff groupoid that is equivalent to $G^u \rtimes \mathbb{Z}$. Indeed, the open subgroupoid $G^u(P) \rtimes \mathbb{Z}|_{\mathcal{S}^s(p)} = \{g \in G^u(P) \rtimes \mathbb{Z} : r(g), s(g) \in \mathcal{S}^s(p)\}$ is equal to $G^u_p \rtimes \mathbb{Z}$, and the relative topology of $G^u_p \rtimes \mathbb{Z}$ inherited from $G^u(P) \rtimes \mathbb{Z}$ is equal to the topology on $G^u_p \rtimes \mathbb{Z}$ described above. Hence $G^u_p \rtimes \mathbb{Z}$ is a locally compact étale Hausdorff groupoid.

We show $G^u_p \rtimes \mathbb{Z}$ is equivalent to $G^u(P) \rtimes \mathbb{Z}$, and hence to $G^u \rtimes \mathbb{Z}$. Since $H = \{g \in G^u(P) \rtimes \mathbb{Z} : r(g) \in \mathcal{S}^s(p)\}$ is a clopen subset, it follows from [19, Example 2.7] that $H$ implements a groupoid equivalence between $G^u_p \rtimes \mathbb{Z}$ and $G^u(P) \rtimes \mathbb{Z}$ if and only if the source map restricted to $H$ surjects onto $\mathcal{S}^s(P)$. Fox $q \in P$ and $z \in \mathcal{S}^s(q)$. There exists $n \in \mathbb{N}$ such that $\tilde{\tau}^n(q) = p$. So $(\tilde{\tau}^n(z), n, z) \in H$, proving surjectivity.

We describe an amplification of $\mathcal{G}_{G,E}$ which we will prove in Theorem 8.12 is Morita equivalent to $G^u \rtimes \mathbb{Z}$.

For $x \in E^{\mathbb{Z}}$, we write

$$E^{\mathbb{Z}}_x := \{y \in E^{\mathbb{Z}} \mid y(-\infty, -M) = x(-\infty, -M) \text{ for some } M \in \mathbb{N}\}. \tag{8.5}$$

For $M \in \mathbb{N}$, let

$$E_x^{\mathbb{Z}}(M) := \{y \in E^{\mathbb{Z}} : y(-\infty, -M) = x(-\infty, -M)\},$$

endowed with the relative topology inherited from $E^{\mathbb{Z}}$. We endow $E_x^{\mathbb{Z}}$ with the inductive-limit topology determined by this decomposition.

The map $\pi_x : E_x^{\mathbb{Z}} \mapsto E^\infty$ sending $z \in E_x^{\mathbb{Z}}$ to $\pi_x(z) = z(1, \infty)$ is an open continuous map (in fact, it is a local homeomorphism onto an open set in $E^\infty$). It is a surjection whenever $E$ is strongly connected. We show that the amplification $\mathcal{G}_{G,E}^{\pi_x}$ is isomorphic to $G_{[x]}^u \rtimes \mathbb{Z}$. We start by analysing the space $E_x^{\mathbb{Z}}$.

We first prove that an element $x \in E^{\mathbb{Z}}$ is completely determined by its class $[x]$ in the limit solenoid $\mathcal{S}$ of Definition 5.3 together with the tail $\dots x_{n-3}x_{n-2}x_{n-1}$ for any $n \in \mathbb{Z}$.

**Lemma 8.7.** *Let $E$ be a finite directed graph with no sinks or sources. Let $(G, E)$ be a contracting, regular self-similar groupoid action. Let $\mathcal{S}$ be the limit solenoid of Definition 5.3. Suppose that $x, y \in E^{\mathbb{Z}}$ satisfy $[x] = [y] \in \mathcal{S}$, and suppose that there exists $n$ in $\mathbb{Z}$ such that $x_m = y_m$ for all $m \leq n$. Then $x = y$.*

*Proof.* Fix $n$ satisfying $x_m = y_m$ for all $m \leq n$. Let $k$ be as in Lemma 4.4, with respect to the finite set $F' = \mathcal{N} \cup G^0$. Since $x \sim_{\text{ae}} y$, Lemma 3.6 shows that there is a sequence $(g_m)_{m\in\mathbb{Z}}$ in $\mathcal{N}$ such that $g_m \cdot x_m x_{m+1} \cdots = y_m y_{m+1} \dots$ for all $m \in \mathbb{Z}$ and $g_m|_{x_m \cdots x_{l-1}} = g_l$ for all $m \leq l$. In particular,

$$g_m \cdot x_{-n-k} \dots x_{-n-1} = y_{-n-k} \dots y_{-n-1} = x_{-n-k} \dots x_{-n-1}.$$

Since $v := r(x_{-n-k}) \in G^{(0)} \subseteq F'$ and also satisfies $v \cdot x_{-n-k} \dots x_{-n-1} = x_{-n-k} \dots x_{-n-1}$, the choice of $k$ guarantees that $g_m|_{x_{-n-k} \cdots x_{-n-1}} = v|_{x_{-n-k} \cdots x_{-n-1}} = r(x_n)$. We then have

$$\begin{aligned} y_{n+1}y_{n+2} \dots &= g_{n-1} \cdot x_{n+1}x_{n+2} \dots \\ &= g_m|_{x_{-n-k} \cdots x_{-n-1}} \cdot x_{n+1}x_{n+2} \cdots = v \cdot x_{n+1}x_{n+2} \cdots = x_{n+1}x_{n+2} \dots \end{aligned}$$

Since we already have $x_m = y_m$ for $m \leq n$, we conclude that $x = y$. □

Lemma 8.7 shows that for $x \in E^{\mathbb{Z}}$ the quotient map $y \mapsto [y]$ from $E^{\mathbb{Z}}$ to $\mathcal{S}$ restricts to an injection $E_x^{\mathbb{Z}} \to \mathcal{S}^s([x])$. We show that this is a homeomorphism with respect to the inductive-limit topologies.

**Lemma 8.8.** *Let $E$ be a finite directed graph with no sinks or sources, and let $(G, E)$ be a contracting, regular self-similar groupoid action. Fix $x \in E^{\mathbb{Z}}$. The map $y \mapsto [y]$ is a homeomorphism from $E_x^{\mathbb{Z}}$ onto $\mathcal{S}^s([x])$ with respect to the inductive-limit topologies described at* (8.4) *and* (8.5).

*Proof.* Fix $n \in \mathbb{N}$. Then $y \mapsto [y]$ restricts to a bijection of $E_x^{\mathbb{Z}}(n)$ onto $\mathcal{S}_n^s([x])$. By definition of the inductive-limit topologies it suffices to show that this map is continuous and open. For fixed $n$ the set $E_x^{\mathbb{Z}}(n)$ is compact because it is closed in $E^{\mathbb{Z}}$, and the set $\mathcal{S}_n^s([x])$ is Hausdorff because $\mathcal{S}$ is. Since $y \mapsto [y]$ is the quotient map, it is continuous on $E_x^{\mathbb{Z}}(n)$, and we deduce that it is a homeomorphism. □

We now analyse the topology of $G_{[x]}^u \rtimes \mathbb{Z}$, for any $x \in E^{\mathbb{Z}}$.

*Remark* 8.9. For any $m, M \in \mathbb{N}$ such that $m \leq M$, we have

$$G_{\beta,m}^u = G_{\beta,M}^u \bigcap_{k : m \leq k \leq M} \{(\eta, \xi) \in \mathcal{S} \times \mathcal{S} : d_{\mathcal{S}}(\tau^{-k}(\eta), \tau^{-k}(\xi)) < \beta\}.$$

Therefore, $G_{\beta,m}^u$ is open in $G^u$, and consequently $G_{\beta,m}^u \rtimes \{l\} := \{(\eta, l, \xi) : (\tilde{\tau}^{-l}(\eta), \xi) \in G_{\beta,m}^u\}$ is open in $G^u \rtimes \mathbb{Z}$.

**Lemma 8.10.** *Let $E$ be a finite directed graph with no sources, and let $(G,E)$ be a contracting, regular self-similar action. Fix $x \in E^{\mathbb{Z}}$. For $k,m \in \mathbb{N}$ and $l \in \mathbb{Z}$, let*

$$X_{k,l,m} := \{([z],l,[y]) \in G^u_{\beta,m} \rtimes \{l\} : x(-\infty,-m) = y(-\infty,-m) = z(-\infty,-m)\}.$$

*Then, $X_{k,l,m}$ is an open subset of $G^u_{[x]} \rtimes \mathbb{Z}$, and the relative topology it inherits from $G^u_{[x]} \rtimes \mathbb{Z}$ coincides with the relative topology inherited from $\mathcal{S} \times \mathbb{Z} \times \mathcal{S}$.*

*Proof.* We have $X_{k,l,m} = G^u_{\beta,k} \rtimes \{l\} \bigcap r^{-1}(q(E^{\mathbb{Z}}_x(m))) \cap s^{-1}(q(E^{\mathbb{Z}}_x(m)))$. The first factor is open in $G^u \rtimes \mathbb{Z}$ by Remark 8.9. The image $q(E^{\mathbb{Z}}_x(m))$ is open in $\mathcal{S}^s([x])$ by Lemma 8.8. Therefore, $X_{k,l,m}$ is open in $G^u_{[x]} \rtimes \mathbb{Z}$. The topologies on $G^u_{\beta,k} \rtimes \{l\}$ and $q(E^{\mathbb{Z}}_x(m))$ are the subspace topologies relative to $\mathcal{S} \times \mathbb{Z} \times \mathcal{S}$ and to $\mathcal{S}$ respectively. Hence, for all triples of open sets $U \subseteq G^u \rtimes \mathbb{Z}$ and $V, W \subseteq \mathcal{S}^s([x])$, there exist open sets $U' \subseteq \mathcal{S} \times \mathbb{Z} \times \mathcal{S}$ and $V', W' \subseteq \mathcal{S}$ such that

$$\begin{aligned} X_{k,l,m} &\cap U \cap r^{-1}(V) \cap s^{-1}(W) \\ &= \big(U \cap G^u_{\beta,k} \rtimes \{l\}\big) \cap \big(r^{-1}(V) \cap r^{-1}(q(E^{\mathbb{Z}}_x(m)))\big) \cap \big(s^{-1}(W) \cap s^{-1}(q(E^{\mathbb{Z}}_x(m)))\big) \\ &= X_{k,l,m} \cap U' \cap r^{-1}(V') \cap s^{-1}(W'). \end{aligned}$$

Since $r, s$ are continuous with respect to the subspace topologies, $U' \cap r^{-1}(V') \cap s^{-1}(W')$ is open in $\mathcal{S} \times \mathbb{Z} \times \mathcal{S}$. Hence $X_{k,l,m}$ has the subspace topology relative to $\mathcal{S} \times \mathbb{Z} \times \mathcal{S}$. □

To lighten notation, for all $z, y \in E^{\mathbb{Z}}_x$, all $m, n \in \mathbb{N}$ and all $g \in G$ such that $g \cdot \sigma^n(\pi_x(y)) = \sigma^m(\pi_x(z))$, we define

$$[z,m,g,n,y] := (z, [\pi_x(z), m, g, n, \pi_x(y)], y) \in \mathcal{G}^{\pi_x}_{G,E}. \tag{8.6}$$

**Notation 8.11.** Give finite paths $\mu_-, \mu_+, \nu_-, \nu_+ \in E^*$ such that $s(\mu_-) = r(\mu_+)$, $s(\nu_-) = r(\nu_+)$, $|\mu_-| = |\nu_-|$ and an element $g \in G$ with $s(\nu_+) = d(g)$, $s(\mu_+) = c(g)$, we define

$$\begin{aligned} \mathcal{Z}(\mu_-\mu_+) = \{z \in E^{\mathbb{Z}}_x : z(-|\mu_-|, |\mu_+|) &= \mu_-\mu_+ \\ \text{and } z(-\infty, -|\mu_-|-1) &= x(-\infty, -|\mu_-|-1)\}, \end{aligned}$$

and

$$\mathcal{Z}(\mu_-\mu_+, g, \nu_-\nu_+) = \{[z, |\mu_+|, g, |\nu_-|, y] \in \mathcal{G}^{\pi_x}_{G,E} : z \in \mathcal{Z}(\mu_-\mu_+), y \in \mathcal{Z}(\nu_-\nu_+)\}.$$

These two collections form a basis for the topologies on $E^{\mathbb{Z}}_x$, $\mathcal{G}^{\pi_x}_{G,E}$, respectively.

We show that $\mathcal{G}^{\pi_x}_{G,E}$ and $G^u_{[x]} \rtimes \mathbb{Z}$ are isomorphic.

**Theorem 8.12.** *Let $E$ be a finite directed graph with no sinks or sources, and let $(G,E)$ be a contracting, regular self-similar groupoid action. Fix $x \in E^{\mathbb{Z}}$. The map $\theta : \mathcal{G}^{\pi_x}_{G,E} \to G^u_{[x]} \rtimes \mathbb{Z}$ such that $\theta([z,m,g,n,y]) = ([z], m-n, [y])$ is an isomorphism of topological groupoids.*

*Proof.* If $[z,m,g,n,y] = [z',m',g',n'y']$, then $z = z'$ and $y = y'$ by (8.6), and $m - n = m' - n'$ by definition of the equivalence relation defining $\mathcal{G}_{G,E}$. So there is a well-defined map $\theta$ satisfying $\theta([z,m,g,n,y]) = ([z], m-n, [y])$.

If $[z,m,h,n,y] \in \mathcal{Z}(\mu_-\mu_+, g, \nu_-\nu_+)$ for $g$ in the nucleus $\mathcal{N}$, then $g \cdot y(|\nu_+|+1, \infty) = z(|\nu_+|+1+(m-n), \infty)$. Lemma 8.6 yields $l \in \mathbb{N}$ such that $([z], m-n, [y]) \in G^u_{\beta,|\nu_+|+l} \rtimes \{n-m\}$. We have $([z], m-n, [y]) \in r^{-1}(q(\mathcal{Z}(\mu_-\mu_+))) \cap s^{-1}(q(\mathcal{Z}(\nu_-\nu_+)))$, so $([z], m-n, [y]) \in X_{|\nu_+|+l, |\mu_+|-|\nu_+|, |\nu_-|}$. Hence,

$$\theta(\mathcal{Z}(\mu_-\mu_+, g, \nu_-\nu_+)) \subseteq X_{|\nu_+|+l, |\mu_+|-|\nu_+|, |\nu_-|}, \ g \in \mathcal{N}. \tag{8.7}$$

Hence $\theta(\mathcal{G}_{G,E}^{\pi_x}) \subseteq G_{[x]}^u \rtimes \mathbb{Z}$.

It is straightforward that $\theta$ is a groupoid homomorphism.

We show that $\theta$ is continuous. It is enough to show that $\theta$ restricts to a continuous map from $\mathcal{Z}(\mu_-\mu_+, g, \nu_-\nu_+)$ to $X_{|\nu_+|+l,|\mu_+|-|\nu_+|,|\nu_-|}$ for all $\mu_+, \mu_-, \nu_+, \nu_-$ as in Notation 8.11. By Lemma 8.10, $X_{|\nu_+|+l,|\mu_+|-|\nu_+|,|\nu_-|}$ has the subspace topology inherited from $\mathcal{S} \times \mathbb{Z} \times \mathcal{S}$, so it suffices to show that $\theta : \mathcal{Z}(\mu_-\mu_+, g, \nu_-\nu_+) \mapsto \mathcal{S} \times \mathbb{Z} \times \mathcal{S}$ is continuous.

Let $([z_\lambda, m_\lambda, h_\lambda, n_\lambda, y_\lambda])_{\lambda\in\Lambda}$ be a net in $\mathcal{Z}(\mu_-\mu_+, g, \nu_-\nu_+)$ converging to $[z, m, h, n, y]$. Since $r, s$ are continuous, it follows that $z_\lambda \to z$ in $\mathcal{Z}(\mu_-\mu_+)$ and $y_\lambda \to y$ in $\mathcal{Z}(\nu_-\nu_+)$. Since $\mathcal{Z}(\mu_-\mu_+)$ and $\mathcal{Z}(\nu_-\nu_+)$ carry the subspace topologies relative to $E^{\mathbb{Z}}$, we have $y_n \to y$ and $z_n \to z$ in $E^{\mathbb{Z}}$. Since the quotient map $q : E^{\mathbb{Z}} \mapsto \mathcal{S}$ is continuous,

$$([z_\lambda], m_\lambda - n_\lambda, [y_\lambda]) \to ([z], m_\lambda - n_\lambda, [y]) = ([z], m - n, [y])$$

in $\mathcal{S} \times \mathbb{Z} \times \mathcal{S}$. Hence, $\theta$ is continuous.

Now, we show that $\theta$ is an open map. It suffices to show that each $\theta(\mathcal{Z}(\mu_-\mu_+, g, \nu_-\nu_+))$ is open.

By (8.7) and Lemma 8.10, it is enough to show that $\theta(\mathcal{Z}(\mu_-\mu_+, g, \nu_-\nu_+))$ is an open subset of $X_{|\nu_+|+l,|\mu_+|-|\nu_+|,|\nu_-|}$. Write $n = |\nu_+|$, $m = |\mu_+|$ and $k = |\nu_-|$. Let

$$\begin{aligned}\tilde{X} = \{([z], m-n, [y]) \mid \exists h \in F : h \cdot z(m+l+1, \infty) = y(n+l+1, \infty) \\ \text{and } x(-\infty, -k) = z(-\infty, -k) = y(-\infty, -k)\}.\end{aligned}$$

Then Lemma 8.6 gives $X_{n+l,m-n,k} \subseteq \tilde{X}$. So it suffices to show that $\theta(\mathcal{Z}(\mu_-\mu_+, g, \nu_-\nu_+))$ is open in the relative topology inherited from $\mathcal{S}_k^s([x]) \times \{m-n\} \times \mathcal{S}_k^s([x])$.

Let $c$ be the number from Lemma 4.4 applied to the set $F = E^0 \cup \mathcal{N} \cup \mathcal{N}^2$. Let $L = \max\{(m+l+1+c), k, (n+l+1+c)\}$. Fix $(u, m, g, n, v) \in \mathcal{Z}(\mu_-\mu_+, g, \nu_-\nu_+)$. Let $W = q(\mathcal{Z}(u(-L,-1)u(1,L))) \times \{m-n\} \times q(\mathcal{Z}(v(-L,-1)v(l,L)))$. Then $W$ is open in $\mathcal{S}_k^s([x]) \times \{m-n\} \times \mathcal{S}_k^s([x])$. We show that $W \cap \tilde{X} \subseteq \theta(\mathcal{Z}(\mu_-\mu_+, g, \nu_-\nu_+))$.

Fix $a, b \in E_x^{\mathbb{Z}}$, $m, n \in \mathbb{N}$ and $h \in F$ such that $h \cdot a(m+l+1, \infty) = b(n+l+1, \infty)$ and $x(-\infty, -k) = a(-\infty, -k) = b(-\infty, -k)$. Suppose that $([a], m-n, [b]) \in W \cap \tilde{X}$. For $L \geq m, n, k$, the paths $\mu_-\mu_+, \nu_-\nu_+$ are subwords of $u(-L,-1)u(1,L)$, $v(-L,-1)v(1,L)$, respectively. Therefore, since $q : E_x^{\mathbb{Z}} \mapsto \mathcal{S}^s([x])$ is bijective (Lemma 8.8), that $([a], m-n, [b]) \in W$ implies that $a \in \mathcal{Z}(\mu_-\mu_+)$ and $b \in \mathcal{Z}(\nu_-\nu_+)$. It only remains to show that $g \cdot a(m+1, \infty) = b(n+1, \infty)$.

By the choice of $L$, we have $a(m+l+1, m+l+c) = u(m+l+1, m+l+c) =: p$ and $g|_{u(m+1,m+l+1)} \cdot p = h \cdot p$. By the choice of $c$, we have $g|_{u(m+1,m+l+c)} = h|_{a(m+l+1,m+l+c)}$. Therefore,

$$\begin{aligned}g \cdot a(m+1, \infty) &= v(n+1, n+l+c)(g|_{u(m+1,m+l+c)}) \cdot a(m+l+1+c, \infty) \\ &= b(n+1, n+l+c)(h|_{a(m+l+1,m+l+c)}) \cdot a(m+l+1+c, \infty) \\ &= b(n+1, \infty).\end{aligned}$$

We have shown that the open neighbourhood $W \cap \tilde{X}$ of $\theta([u, m, g, n, v])$ is contained in $\theta(\mathcal{Z}(\mu_-\mu_+, g, \nu_-\nu_+))$. Therefore, $\theta(\mathcal{Z}(\mu_-\mu_+, g, \nu_-\nu_+))$ is open, and hence $\theta$ is an open map.

Now, we show that $\theta$ is a bijection. We first show injectivity. Since $q : E_x^{\mathbb{Z}} \mapsto \mathcal{S}^s([x])$ is a bijection, $\theta([z, m, g, n, y]) = \theta([z', m', g', n', y'])$ implies $z = z'$ and $y = y'$. Since $m - n = m' - n' = l$, for $k$ large enough, we have $g|_{y(n,k)} \cdot y(k+1, \infty) = g'|_{y(n',k)} \cdot y(k+1, \infty)$. By regularity, $g|_{y(n,K)} = g'|_{y(n,K)}$ for some $K \geq k$. Therefore, $[z, m, g, n, y] = [z', m', g', n', y']$.

Finally, we show surjectivity. Fix $([z], l, [y]) \in G^u_{[x]} \rtimes \mathbb{Z}$. By Lemma 8.6, there exist $M \in \mathbb{N}$ and $g \in F$ such that $M + l \geq 0$ and $g \cdot y(M+1, \infty) = z(M+l+1, \infty)$. Hence $[z, M+l, g, M, y] \in \mathcal{G}^{\pi_x}_{G,E}$ satisfies $\theta([z, M+l, g, M, y]) = ([z], l, [y])$. □

**Corollary 8.13.** *Let $E$ be a finite strongly connected directed graph, and let $(G, E)$ be a contracting, regular self-similar groupoid action. Then the $C^*$-algebra $\mathcal{O}(G, E)$ of Section 6 is Morita equivalent to the unstable Ruelle algebra $C^*(G^u \rtimes \mathbb{Z})$ of the Smale space $(\mathcal{S}, \tilde{\tau})$ of Corollary 5.5.*

*Proof.* Fix $x \in E^{\mathbb{Z}}$ such that $[x]$ is periodic under the action of $\tilde{\tau}$. Since $E$ is assumed strongly connected, Corollary 5.5 implies $(\mathcal{S}, \tilde{\tau})$ is irreducible. Thus, as shown above Lemma 8.10, $G^u \rtimes \mathbb{Z}$ is groupoid equivalent to $G^u_{[x]} \rtimes \mathbb{Z}$. Hence $C^*(G^u \rtimes \mathbb{Z})$ is Morita equivalent to $C^*(G^u_{[x]} \rtimes \mathbb{Z})$. Theorem 8.12 implies that $C^*(G^u_{[x]} \rtimes \mathbb{Z}) \cong C^*\big(\mathcal{G}^{\pi_x}_{G,E}\big)$. Since $E$ is strongly connected, $\pi_x$ is an open surjection, so [8, Proposition 3.10] implies that $C^*(\mathcal{G}_{G,E})$ is Morita equivalent to $C^*(\mathcal{G}^{\pi_x}_{G,E})$, and Proposition 6.5 shows that $\mathcal{O}(G, E) \cong C^*(\mathcal{G}_{G,E})$. Stringing these isomorphisms and Morita equivalences together gives the desired Morita equivalence $C^*(G^u \rtimes \mathbb{Z}) \sim_{\mathrm{Me}} \mathcal{O}(G, E)$. □

*Remark* 8.14. If $E$ is not strongly connected, then $\pi_x$ is not necessarily surjective, and $\mathcal{G}^{\pi_x}_{G,E}$ is only groupoid equivalent to the reduction $\mathcal{G}_{G,E}|_{\pi_x(E^{\mathbb{Z}}_x)}$ of $\mathcal{G}_{G,E}$ to the image of $\pi_x$, which is open. However, if $\mathcal{G}_{G,E}$ is minimal, then $\mathcal{G}_{G,E}|_{\pi_x(E^{\mathbb{Z}}_x)}$ is still groupoid equivalent to $\mathcal{G}_{G,E}$.

We can now prove our main theorem.

*Proof of Theorem 8.1.* Corollary 5.5 shows that $(\mathcal{S}, \tilde{\tau})$ is an irreducible Smale space. So Theorem 1.1 of [12] shows that there are classes $\delta$ in $KK^1\big(\mathbb{C}, C^*(G^s \rtimes \mathbb{Z}) \otimes C^*(G^u \rtimes \mathbb{Z})\big)$ and $\Delta$ in $KK^1\big(C^*(G^s \rtimes \mathbb{Z}) \otimes C^*(G^u \rtimes \mathbb{Z}), \mathbb{C}\big)$ such that $\delta \widehat{\otimes}_{C^*(G^u \rtimes \mathbb{Z})} \Delta = \mathrm{id}_{\mathbb{K}(C^*(G^s \rtimes \mathbb{Z}), C^*(G^s \rtimes \mathbb{Z}))}$ and $\delta \widehat{\otimes}_{C^*(G^s \rtimes \mathbb{Z})} \Delta = -\,\mathrm{id}_{\mathbb{K}(C^*(G^u \rtimes \mathbb{Z}), C^*(G^u \rtimes \mathbb{Z}))}$.

Corollary 8.5 gives a Morita equivalence bimodule between $\widehat{\mathcal{O}}(G, E)$ and $C^*(G^s \rtimes \mathbb{Z})$, which induces a $KK$-equivalence $\hat{\alpha} \in KK^0\big(\widehat{\mathcal{O}}(G, E), C^*(G^s \rtimes \mathbb{Z})\big)$. Likewise Corollary 8.13 gives a $KK$-equivalence $\alpha$ in $KK\big(\mathcal{O}(G, E), C^*(G^u \rtimes \mathbb{Z})\big)$. So the Kasparov products

$$\beta := (\hat{\alpha} \otimes \alpha) \mathbin{\widehat{\otimes}} \delta \mathbin{\widehat{\otimes}} (\hat{\alpha} \otimes \alpha)^{-1} \quad \text{and} \quad \mu := (\hat{\alpha} \otimes \alpha) \mathbin{\widehat{\otimes}} \Delta \mathbin{\widehat{\otimes}} (\hat{\alpha} \otimes \alpha)^{-1}$$

implement the desired duality. □

## References


[1] C. Anantharaman-Delaroche and J.N. Renault, Amenable groupoids, With a foreword by Georges Skandalis and Appendix B by E. Germain, L'Enseignement Mathématique, Geneva, 2000, 196.

[2] J.H. Brown, L.O. Clark, C. Farthing and A. Sims, *Simplicity of algebras associated to étale groupoids*, Semigroup Forum **88** (2014), 433–452.

[3] D. Burago, Y. Burago, S. Ivanov, *A course in metric geometry*, Graduate Studies in Mathematics, Vol 31, American Mathematical Society, Providence, 2001.

[4] A. Connes, *Noncommutative Geometry*, Academic Press, London and San Diego, 1994.

[5] R. Engelking, *General topology*, Sigma Series in Pure Mathematics, vol. 6, Heldermann Verlag, Berlin, 1989.

[6] R. Exel, *Inverse semigroups and combinatorial $C^*$-algebras*, Bull. Braz. Math. Soc. **39** (2008), 191–313.

[7] R. Exel and E. Pardo, *Self-similar graphs a unified treatment of Katsura and Nekrashevych $C^*$-algebras*, Adv. Math. **306** (2017), 1046–1129.

[8] C. Farsi, A. Kumjian, D. Pask and A. Sims, *Ample groupoids: equivalence, homology, and Matui's HK-conjecture*, Münster J. Math. **12** (2019), 411–451.

[9] Gelfand, I. and Naĭmark, M. *On the imbedding of normed rings into the ring of operators in Hilbert space*, Rec. Math. [Mat. Sbornik] N.S. **12** (1943), 197–213.
[10] R. I. Grigorchuk, *On Burnside's problem on periodic groups*, Func. Anal. Appl. **14** (1980), 41–43.
[11] J. B. Hume, *The K-theory of a rational map*, in preparation.
[12] J. Kaminker, I.F. Putnam, and M.F. Whittaker, *K-theoretic duality for hyperbolic dynamical systems*, J. reine angew. Math. **730** (2017), 263–299.
[13] J. Kaminker, C. L. Schochet, *Spanier-Whitehead K-duality for $C^*$-algebras*, J. Topol. Anal. **11** (2019), 21–52.
[14] T. Katsura, *A construction of actions on Kirchberg algebras which induce given actions on their K-groups*, J. reine angew. Math. **617** (2008), 27–65.
[15] B. Kitchens, Symbolic Dynamics: One-sided, Two-sided and Countable State Markov Shifts, Universitext, Springer, 1998.
[16] M. Laca, I. Raeburn, J. Ramagge and M.F. Whittaker, *Equilibrium states on the Cuntz–Pimsner algebras of self-similar actions*, J. Funct. Anal. **266** (2014), 6619–6661.
[17] M. Laca, I. Raeburn, J. Ramagge, and M.F. Whittaker, *Equilibrium states on operator algebras associated to self-similar actions of groupoids on graphs*, Adv. Math. **331** (2018), 268–325.
[18] D. Maharam, *On homogeneous measure algebras*, Proc. Nat. Acad. Sci. U.S.A. **28** (1942), 108–111.
[19] P.S. Muhly, J.N. Renault and D.P. Williams, *Equivalence and isomorphism for groupoid $C^*$-algebras*, J. Operator Th. **17** (1987), 3–22.
[20] V. Nekrashevych, *Cuntz–Pimsner algebras of group actions*, J. Operator Th. **52** (2004), 223–249.
[21] V. Nekrashevych, Self-Similar Groups, Math. Surveys and Monographs vol. 117, Amer. Math. Soc., Providence, 2005.
[22] V. Nekrashevych, *$C^*$-algebras and self-similar groups*, J. reine angew. Math. **630** (2009), 59–123.
[23] A.L.T. Paterson, Groupoids, inverse semigroups, and their operator algebras, Progress in Mathematics vol. 170, Birkhäuser Boston, MA, 1999.
[24] V. Proietti and M. Yamashita, *Homology and K-theory of Dynamical Systems III. Beyond Totally Disconnected Case*, preprint, arXiv 2207.03118, 1–22.
[25] I.F. Putnam, *$C^*$-Algebras from Smale Spaces*, Canad. J. Math. **48** (1996), 175-195.
[26] I.F. Putnam, *Functoriality of the $C^*$-algebras associated with hyperbolic dynamical systems,* J. Lond. Math. Soc. **62**(3) (2000), 873–884.
[27] I.F. Putnam and J. Spielberg *The structure of $C^*$-algebras associated with hyperbolic dynamical systems*, J. Func. Anal. **163** (1999), 279–299.
[28] I. Raeburn, *Graph algebras*, CBMS Regional Conference Series in Mathematics, vol. 103, Amer. Math. Soc., Providence, 2005.
[29] D. Ruelle, *Noncommutative algebras for hyperbolic diffeomorphisms*, Invent. Math. **93** (1988), 1–13.
[30] I.E. Segal, *Equivalences of measure spaces*, American J. Math. **73** (1951), 275–313.
[31] S. Wieler, *Smale spaces via inverse limits*, Erg. Th. Dynam. Sys. **34** (2014), 2066–2092.
[32] I. Yi, *K-theory of $C^*$-algebras from one-dimensional generalized solenoids*, J. Operator Th. **50** (2000), 283–295.

Nathan Brownlowe, School of Mathematics and Statistics, University of Sydney
*Email address*: nathan.brownlowe@sydney.edu.au

Alcides Buss and Daniel Gonçalves, Departamento de Matemática, Universidade Federal de Santa Catarina
*Email address*: alcides.buss@ufsc.br, daemig@gmail.com

Jeremy B. Hume and Michael F. Whittaker, School of Mathematics and Statistics, University of Glasgow
*Email address*: jeremybhume@gmail.com, Mike.Whittaker@glasgow.ac.uk

Aidan Sims, School of Mathematics and Applied Statistics, University of Wollongong
*Email address*: asims@uow.edu.au