

RECONSTRUCTION OF TWISTED STEINBERG ALGEBRAS

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ABSTRACT. We show how to recover a discrete twist over an ample Hausdorff groupoid from a pair consisting of an algebra and what we call a *quasi-Cartan subalgebra*. We identify precisely which twists arise in this way (namely, those that satisfy the *local bisection hypothesis*), and we prove that the assignment of twisted Steinberg algebras to such twists and our construction of a twist from a quasi-Cartan pair are mutually inverse. We identify the algebraic pairs that correspond to effective groupoids and to principal groupoids. We also indicate the scope of our results by identifying large classes of twists for which the local bisection hypothesis holds automatically.

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1. INTRODUCTION

In this paper we provide a concrete representation theorem for a large class of abstract algebras by establishing a correspondence between algebraic twists over ample Hausdorff groupoids and a corresponding class of pairs (A, B) of abstract algebras.

When trying to understand an algebra, it is often helpful to describe it in terms of a concrete representation. For example: Stone duality for commutative algebras generated by idempotents; the duality between commutative von Neumann algebras and hyperstonean spaces; Gelfand duality for commutative C^* -algebras; and the Gelfand–Naimark theorem for noncommutative C^* -algebras.

A more recent example comes from Leavitt path algebras, whose advent [1, 3] has sparked substantial activity around interactions between abstract algebra and C^* -algebras over the last decade or so. It was discovered early on that the complex Leavitt path algebra of a graph embeds in the graph C^* -algebra, providing a concrete representation of any given Leavitt path algebra by bounded operators on a Hilbert space. But a clear understanding of the striking structural similarities between Leavitt path algebras over general rings and the corresponding graph C^* -algebras was only achieved through the development of convolution algebras of functions on groupoids [45, 15], now called Steinberg algebras, as a unifying framework. It emerged that both Leavitt path algebras and graph C^* -algebras can be realised as algebras of functions on the underlying graph groupoid [15]. Moreover, the groupoid can be recovered from either the graph C^* -algebra together with its abelian subalgebra generated by the range projections of its generating partial isometries [11], or from the Leavitt path algebra (over a very broad class of rings) and its corresponding abelian subalgebra [10, 12, 47]. This led to deep connections between abstract algebra and symbolic dynamics [13], and these have been significantly extended through recent reconstruction results that show how increasingly broad classes of groupoids can be reconstructed both from their C^* -algebras [43, 14] and from their Steinberg algebras [2, 47]. Analogously to the case for groupoid C^* -algebras, there is a disintegration theorem [46] for realising modules over Steinberg algebras as sheaves over the groupoid, and Morita equivalence of Steinberg algebras can often be seen at the groupoid level [17, 46].

The upshot is that there are significant advantages to being able to recognise a given algebra as the Steinberg algebra, or a twisted Steinberg algebra [4], of a groupoid. Thus it is important to understand exactly which algebras admit such a model.

This question was first answered for von Neumann algebras by Feldman and Moore [19, 20, 21]. They showed that every Borel 2-cocycle on a Borel equivalence relation gives rise to a von Neumann algebra and a so-called Cartan subalgebra (roughly, an abelian subalgebra whose normaliser generates the whole algebra); and that the equivalence relation and 2-cocycle can be recovered (up to cohomology) from the pair of algebras. A corresponding theorem for C^* -algebras was developed by Renault and by Kumjian first for principal groupoids [25] and then for effective groupoids [43], using Kumjian’s notion of a *twist* over an étale groupoid in place of a 2-cocycle. It was these results that Matsumoto and Matui subsequently used [32] to prove that flow equivalence of shift spaces is characterised by diagonal-preserving isomorphism of the associated Cuntz–Krieger algebras, leading to the surge of activity around reconstruction of groupoids that are not even effective from the C^* -algebras discussed above.

Renault’s results beg an extension to abstract algebras, following the program of “algebraisation” of operator-theoretic ideas going back to von Neumann and Kaplansky (see [23]) described in [16]: in the words of Berberian [6] “if all the functional analysis is

stripped away . . . what remains should stand firmly as a substantial piece of algebra, completely accessible through algebraic avenues.” The groundwork for such an extension was laid recently in [4], where a notion of an algebraic twist over an ample groupoid, suitable for constructing twisted Steinberg algebras, was developed, and the structure of the resulting twisted Steinberg algebras was analysed.

In this paper we provide such an extension. We identify a class of abelian subalgebras, called *algebraic quasi-Cartan subalgebras*, in abstract algebras over indecomposable commutative rings. Our main theorems establish a correspondence between algebraic quasi-Cartan pairs and twists over ample Hausdorff étale groupoids that satisfy a *local bisection hypothesis* (analogous to that of [47]) that generalises the no-nontrivial-units condition for twisted group rings. Indeed, we show that the local bisection hypothesis follows from the no-nontrivial-units condition for sufficiently many of the twisted group rings over the interior of the isotropy of the groupoid. This guarantees that our results apply to many important examples, as we detail in the final section of the paper—carrying our results for algebras beyond the current state of the art for C^* -algebras. We also pick out from amongst all algebraic quasi-Cartan pairs the algebraic Cartan pairs that correspond to the effective groupoids of Renault’s setting, and the algebraic diagonal pairs that correspond to the principal groupoids of Kumjian’s setting.

This opens interesting new avenues of study for algebras admitting quasi-Cartan subalgebras, and provides a wealth of invariants for algebraic quasi-Cartan pairs: for example, the topological full-group of the underlying groupoid G , with intriguing links to the Thompson groups [39, 33]; the homology groups of G , which have promising connections to computations of K -theory in the C^* -algebraic setting [34, 35, 36]; and the twist itself, which behaves like a second cohomology class for the orbit space of the groupoid that can be used as a classifying invariant for Fell algebras [22].

Related work. In the final stages of preparation of this manuscript we became aware of Bice’s independent work [7] on ringoid bundles, and in particular on Steinberg rings and Steinberg bundles. We thank Bice bringing his work to our attention, and for sharing his preprint.

Bice’s results deal, at their most general, with topological categories appropriately fibred over étale groupoids. His main results are about *Steinberg bundles* which, loosely speaking, bear the same relationship to our discrete R -twists as Fell bundles bear to twists in the C^* -algebraic setting; in particular, in Bice’s setting, the fibres of $q^{-1}(G)$ can vary, and need not be commutative. Bice shows that every Steinberg bundle determines a *Steinberg ring*: a ring A with distinguished subsemigroups $Z \subseteq S \subseteq A$ (corresponding to $I(B) \subseteq N(B) \subseteq A$), and a suitable notion of a conditional expectation $\Phi: A \rightarrow S$ such that $Z \subseteq \Phi(A)$, satisfying an additional support condition. He demonstrates that every Steinberg ring determines a corresponding Steinberg bundle (via an ultrafilter construction) whose Steinberg ring is the original (A, S, Z, Φ) . However, the same pair (A, Z) could admit multiple Steinberg-ring structures, yielding different Steinberg bundles.

Our set-up is less general, but requires less algebraic data to describe the algebraic objects of study: the quasi-Cartan pair (A, B) is all the information required, and then $N(B)$ and P can be recovered. We then obtain a complete and concrete correspondence between quasi-Cartan pairs and R -twists satisfying the local bisection hypothesis.

Outline of the paper. The paper is organised as follows. In [Section 2](#), we introduce preliminaries on abelian R -algebras without torsion (this is a standing assumption on the abelian subalgebras B in our quasi-Cartan pairs), on groupoids and discrete R -twists, on twisted Steinberg algebras, and on inverse semigroups of normalisers of abelian subalgebras.

In [Section 3](#), we introduce our three notions of algebraic pairs (see [Definition 3.3](#)), and we prove in [Lemma 3.5](#) that every algebraic diagonal pair is an algebraic Cartan pair. We also show that for a given quasi-Cartan pair (A, B) , there is only one conditional expectation from A to B that satisfies a key defining algebraic condition ([AQP](#)). To finish the section, we show that if (A, B) is an algebraic Cartan pair, then there is a unique conditional expectation from A to B , and we show in [Lemma 3.6](#) that every algebraic Cartan pair is an algebraic quasi-Cartan pair.

In [Section 4](#), we study the pairs of algebras arising from discrete R -twists, and demonstrate that such a pair is an algebraic quasi-Cartan pair if and only if the twist satisfies the local bisection hypothesis. We also provide tools for checking the local bisection hypothesis, and we finish the section by showing in [Proposition 4.8](#) that twists over principal groupoids yield algebraic diagonal pairs, and twists over effective groupoids yield algebraic Cartan pairs.

In [Section 5](#), we show how to construct a twist from a pair of algebras. We have made an effort to be clear about which hypotheses are required for each part of the construction. In [Section 5.1](#), we begin with a pair (A, B) such that B is without torsion and the idempotents of B are a set of local units for A , and we describe an associated groupoid Σ of ultrafilters of normalisers of B and some of its key properties. In [Section 5.2](#) we additionally assume that B is the R -linear span of its idempotent elements. With this hypothesis, we construct a quotient G of Σ by the action of R^\times , and show that G is an ample groupoid. In [Section 5.3](#), we show that under the same hypotheses we obtain a sequence $G^{(0)} \times R^\times \hookrightarrow \Sigma \twoheadrightarrow G$ that has all of the properties of a discrete R -twist over G except, potentially, Hausdorffness. We show that if (A, B) is an algebraic quasi-Cartan pair, then G (and hence also Σ) is indeed Hausdorff—this is all contained in [Theorem 5.6](#) and [Proposition 5.10](#).

In [Section 6](#), we prove our first main result. We start with an algebraic quasi-Cartan pair (A, B) , and consider the twist Σ over G constructed in the preceding section. Our main theorem, [Theorem 6.6](#), shows that there is an explicit isomorphism of A onto the twisted Steinberg algebra $A_R(G; \Sigma)$ that restricts to the standard isomorphism of B onto $A_R(G^{(0)}; q^{-1}(G^{(0)})) \cong A_R(G^{(0)})$ arising from Stone duality as in [[24](#), Théorème 1].

In [Section 7](#), we explore the relationship between properties of a twist Σ over G satisfying the local bisection hypothesis and properties of the associated Steinberg algebra. Specifically, we prove that if (A, B) is an algebraic quasi-Cartan pair, then it is a Cartan pair if and only if the associated groupoid G is effective (so that our terminology is consistent with Renault’s in [[43](#)]) and that it is a diagonal pair if and only if G is principal (so that our terminology is consistent with Kumjian’s in [[25](#)]).

We see in [Section 6](#) that passing from an algebraic quasi-Cartan pair to the associated twist and then to the associated twisted Steinberg algebra and its diagonal subalgebra recovers the original pair. In [Section 8](#) we consider the dual approach where we start with a twist, pass to the associated pair of algebras, and then construct the associated sequence of groupoids. We prove in [Proposition 8.4](#) that given any twist Σ over an ample Hausdorff groupoid G , if the algebraic pair (A, B) consists of a Steinberg algebra and a diagonal subalgebra, then the groupoid Σ' of ultrafilters of normalisers of B admits a continuous open embedding of Σ that restricts to a homeomorphism of unit spaces. Then in [Theorem 8.7](#), we show that this embedding is an isomorphism if and only if (A, B) is an algebraic quasi-Cartan pair (equivalently, if and only if $\Sigma \rightarrow G$ satisfies the local bisection hypothesis). We deduce in [Corollary 8.10](#) that given twists $\Sigma_1 \rightarrow G_1$ and $\Sigma_2 \rightarrow G_2$ such that $\Sigma_1 \rightarrow G_1$ satisfies the local bisection hypothesis, the twists are isomorphic (which implies in particular that $\Sigma_2 \rightarrow G_2$ also satisfies the local bisection hypothesis) if and only if the associated algebraic pairs are isomorphic. We then deduce in [Corollary 8.11](#)

that if G_1 is effective, then the twists are isomorphic (which implies in particular that G_2 is effective) if and only if there is an isomorphism $A_R(G_1; \Sigma_1) \xrightarrow{\sim} A_R(G_2; \Sigma_2)$ of twisted Steinberg algebras that carries the diagonal subalgebra of $A_R(G_1; \Sigma_1)$ into the diagonal subalgebra of $A_R(G_2; \Sigma_2)$.

Finally, in [Section 9](#) we demonstrate the applicability of our results by showing that there are substantial classes of twists that satisfy the local bisection hypothesis, and therefore correspond to algebraic quasi-Cartan pairs. We consider groups H with the *unique product property*; that is, that products AB of finite subsets of H always contain an element with a unique factorisation of the form ab with $a \in A$ and $b \in B$. (Examples include right-orderable groups, and in particular torsion-free abelian groups.) By adapting the approach used in [\[47\]](#) for untwisted group rings, we show that if R is reduced and indecomposable and H is a group with the unique product property, then every twisted group R -algebra of H has no nontrivial units. Combining this with our results in [Section 4](#), we deduce that for reduced and indecomposable R , all R -twists over groupoids G such that the fibres of the interior of the isotropy of G have the unique product property satisfy the local bisection hypothesis (the reduced assumption can be dropped if the groupoid is effective). In particular, this includes all twists over ample Deaconu–Renault groupoids; we give a specific application to twisted Kumjian–Pask algebras.

2. PRELIMINARIES

2.1. Abelian R -algebras without torsion. Throughout this article, R denotes a commutative ring with identity, R^\times is the group of units of R , and A is an R -algebra. We always assume that $B \subseteq A$ is a commutative (also called abelian) subalgebra with idempotents $I(B)$. We also ask that B is *without torsion* with respect to R in the sense used in [\[24\]](#); specifically,

$$\text{if } e \in I(B) \setminus \{0\} \text{ and } t \in R \text{ satisfy } te = 0, \text{ then } t = 0. \quad (\text{WT})$$

[Condition \(WT\)](#) holds automatically if R is a field (and also if R is an integral domain and B is a torsion-free R -module, as explained below). Observe that if B has at least one nonzero idempotent, then [condition \(WT\)](#) also implies that R is *indecomposable*¹ in the sense that its only idempotents are 0 and 1: if $t \in R$ satisfies $t^2 = t$, then $t^2e = te$ for any $e \in I(B) \setminus \{0\}$, and then $t(1-t)e = 0$, forcing $t = 0$ or $t = 1$ (since $(1-t)e \in I(B)$). Since we will be working exclusively with algebras B that are generated by their idempotent elements, it follows that R is generally indecomposable. This is also the condition on R used in [\[47\]](#). The condition of a ring R being indecomposable is a natural one in commutative ring theory. For instance, recall that the Zariski (or prime) spectrum $\text{Spec}(R)$ of R is the set of prime ideals of R with the topology whose basic open subsets are of the form $D(r) = \{\mathfrak{p} \in \text{Spec}(R) : r \notin \mathfrak{p}\}$ with $r \in R$. It is well known that $\text{Spec}(R)$ is a connected space if and only if R is indecomposable.

We will frequently use without comment the following observation. If B is spanned as an R -module by $I(B)$, then each element $b \in B$ can be written as $b = \sum_{i=1}^n t_i e_i$ with $e_1, \dots, e_n \in I(B)$ mutually orthogonal and $t_1, \dots, t_n \in R$. Indeed, if $b = \sum_{e \in F} t_e e$ with $F \subseteq I(B)$ finite, then F generates a finite Boolean algebra, where we recall that the join of commuting idempotents e and f is $e \vee f = e + f - ef$, their meet is $e \wedge f = ef$, and their relative complement is $e \setminus f = e - ef$. We can then take e_1, \dots, e_n to be the atoms (minimal idempotents) of this Boolean algebra.

An R -module M is called *torsion-free* if $rm = 0$ implies that either $m = 0$ or r is a zero divisor, for $r \in R$ and $m \in M$. Notice that if $I(B)$ generates B as an R -algebra,

¹For us, indecomposable implicitly implies unital.

then [condition \(WT\)](#) implies that B is a torsion-free R -module. Indeed, if $0 \neq b \in B$ and $tb = 0$, we can write $b = \sum_{i=1}^n t_i e_i$ with $0 \neq t_i \in R$ and $e_i \in I(B)$ with e_1, \dots, e_n mutually orthogonal. Then $0 = tbe_1 = tt_1 e_1$, and so $tt_1 = 0$ by [\(WT\)](#), whence t is a zero divisor. In particular, if R is an integral domain and B is generated by $I(B)$, then B is torsion-free if and only if it satisfies [condition \(WT\)](#).

2.2. Groupoids. We use G to mean a locally compact Hausdorff topological groupoid with unit space $G^{(0)}$, composable pairs $G^{(2)} \subseteq G \times G$, and range and source maps $\mathbf{r}, \mathbf{s}: G \rightarrow G^{(0)}$. We will refer to such groupoids as *Hausdorff groupoids*. We evaluate composition of groupoid elements from right to left, which means that $\gamma\gamma^{-1} = \mathbf{r}(\gamma)$ and $\gamma^{-1}\gamma = \mathbf{s}(\gamma)$, for all $\gamma \in G$.

For each $x \in G^{(0)}$, we define

$$G_x := \mathbf{s}^{-1}(x), \quad G^x := \mathbf{r}^{-1}(x), \quad \text{and} \quad G_x^x := G_x \cap G^x.$$

For any two subsets U and V of a groupoid G , we define

$$UV := \{\alpha\beta : (\alpha, \beta) \in (U \times V) \cap G^{(2)}\} \quad \text{and} \quad U^{-1} := \{\alpha^{-1} : \alpha \in U\}.$$

We call a subset B of G a *bisection* if there exists an open subset U of G such that $B \subseteq U$, and $\mathbf{r}|_U$ and $\mathbf{s}|_U$ are homeomorphisms onto open subsets of $G^{(0)}$. We say that G is *étale* if \mathbf{r} (or, equivalently, \mathbf{s}) is a local homeomorphism. If G is étale, then $G^{(0)}$ is open, and both G_x and G^x are discrete in the subspace topology for any $x \in G^{(0)}$. We recall that G is étale if and only if G has a basis of open bisections. We say that G is *ample* if it has a basis of *compact* open bisections. If G is étale, then G is ample if and only if its unit space $G^{(0)}$ is totally disconnected (see [18, Proposition 4.1]). It is well known that an étale groupoid is Hausdorff if and only if its unit space is closed.

If B and D are compact open bisections of an ample groupoid, then B^{-1} and BD are also compact open bisections. The collection of compact open bisections is an inverse semigroup under these operations (see [42, Proposition 2.2.4]).

The *isotropy* of a groupoid G is the set

$$\text{Iso}(G) := \{\gamma \in G : \mathbf{r}(\gamma) = \mathbf{s}(\gamma)\} = \bigcup_{x \in G^{(0)}} G_x^x.$$

We say that G is *principal* if $\text{Iso}(G) = G^{(0)}$, and that G is *effective* if the topological interior of $\text{Iso}(G)$ is equal to $G^{(0)}$. So every principal étale groupoid is effective.

Remark 2.1. In full generality, effectiveness should not be confused with several related notions. There is the notion of a groupoid being topologically principal, which means that points with trivial isotropy (that is, $G_x^x = \{x\}$) are dense in $G^{(0)}$, and the weaker notion of a groupoid being topologically free, which means that $\text{Iso}(G) \setminus G^{(0)}$ has empty interior. Both effective and topologically principal étale groupoids are topologically free. For second-countable étale groupoids, the notions of topologically principal and topologically free coincide, whereas for Hausdorff étale groupoids, being effective is equivalent to being topologically free. Hence for second-countable Hausdorff étale groupoids, all three notions agree. However, there are examples of minimal ample groupoids showing that there are no other implications between these notions.

For the majority of this paper, the groupoids we work with are ample Hausdorff groupoids, but we will make these assumptions explicit throughout.

2.3. Discrete twists. We next recall [4, Definition 4.1]. Let G be a Hausdorff étale groupoid, let R be a commutative unital ring, and let T be a subgroup of R^\times . A *discrete twist* over G is a sequence

$$G^{(0)} \times T \xrightarrow{i} \Sigma \xrightarrow{q} G, \quad (2.1)$$

where the groupoid $G^{(0)} \times T$ is regarded as a trivial group bundle with fibres T , Σ is a groupoid with $\Sigma^{(0)} = i(G^{(0)} \times \{1\})$, and i and q are continuous groupoid homomorphisms that restrict to homeomorphisms of unit spaces, such that the following conditions hold:

- (DT1) The sequence is exact, in the sense that $i(\{x\} \times T) = q^{-1}(x)$ for every $x \in G^{(0)}$, i is injective, and q is surjective.
- (DT2) The groupoid Σ is a locally trivial G -bundle, in the sense that for each $\alpha \in G$, there is an open bisection B_α of G containing α , and a continuous map $P_\alpha: B_\alpha \rightarrow \Sigma$ such that
 - (i) $q \circ P_\alpha = \text{id}_{B_\alpha}$;
 - (ii) the map $(\beta, t) \mapsto i(\mathbf{r}(\beta), t) P_\alpha(\beta)$ is a homeomorphism from $B_\alpha \times T$ to $q^{-1}(B_\alpha)$.
- (DT3) The image of i is *central* in Σ , in the sense that $i(\mathbf{r}(\sigma), t) \sigma = \sigma i(\mathbf{s}(\sigma), t)$ for all $\sigma \in \Sigma$ and $t \in T$.

We denote a discrete twist over G by (Σ, i, q) , or by $\Sigma \rightarrow G$, or simply by Σ . We identify $\Sigma^{(0)}$ with $G^{(0)}$ via i (or via $q|_{\Sigma^{(0)}}$). Note that [4, Definition 4.1] also requires Σ to be Hausdorff, but we show in [Corollary 2.3](#) below that this is automatic.

Recall from [4, Page 14] that there is a continuous free action of T on Σ given by

$$t \cdot \sigma = i(\mathbf{r}(\sigma), t) \sigma \text{ for all } t \in T \text{ and } \sigma \in \Sigma.$$

In this paper, we will only be interested in the setting where $T = R^\times$. We will sometimes emphasise the dependence on R^\times by calling Σ a *discrete R -twist over G* , or sometimes just an *R -twist over G* .

We begin by observing that if Σ is a discrete twist, then it, too, is an étale groupoid, and the map q is a quotient map. In fact, we have the following characterisations of discrete twists, some of which are easier to check than others.

Proposition 2.2. *Consider an exact sequence of groupoids as in (2.1) with G étale but not necessarily Hausdorff. Suppose that conditions (DT1) and (DT3) in the definition of a discrete twist are satisfied. If q is a local homeomorphism, then Σ is étale. The following are equivalent.*

- (1) Condition (DT2) in the definition of a discrete twist is satisfied.
- (2) The map i is open and q is a covering map.
- (3) The map i is open and the map q is a local homeomorphism.
- (4) The maps i and q are open, and Σ is étale.

Proof. If q is a local homeomorphism, then since the source map for Σ is the composition of q with the source map for G , and since local homeomorphisms are closed under composition, Σ is étale.

(1) \implies (2). Condition (DT2) says that Σ is a locally trivial fibre bundle with discrete fibres T , and hence q is a covering map (and so a local homeomorphism). Thus Σ is étale by the previous assertion. Now [4, Lemma 4.3(b)] (which formally assumes that Σ and G are Hausdorff but only uses that they are étale and that conditions (DT1)–(DT3) are satisfied) implies that i is open.

(2) \implies (3) is immediate: every covering map is a local homeomorphism.

(3) \implies (4) is also immediate: if q is a local homeomorphism then it is open, and Σ is étale by the first assertion of the proposition.

(4) \implies (1). Suppose that i and q are open and Σ is étale. We show (DT2), that is, that Σ is locally trivial. Fix $\alpha \in G$ and $\sigma \in \Sigma$ with $q(\sigma) = \alpha$, and choose an open bisection $V \subseteq \Sigma$ with $\sigma \in V$ using that Σ is étale. Note that $q|_V$ is injective, because if $q(\beta) = q(\beta')$ with $\beta, \beta' \in V$, then $\mathbf{s}(\beta) = \mathbf{s}(\beta')$, and so $\beta = \beta'$ since V is a bisection. Put $B_\alpha := q(V)$ and $P_\alpha := (q|_V)^{-1}$. Then P_α is a homeomorphism since q is open by hypothesis. Moreover, B_α is an open bisection, because V is an open bisection and q is an open map that respects \mathbf{r} and \mathbf{s} . It remains to show that the map $\phi_{P_\alpha}: (\beta, t) \mapsto i(\mathbf{r}(\alpha), t) P_\alpha(\beta)$ is a homeomorphism from $B_\alpha \times T$ to $q^{-1}(B_\alpha)$. Since i, \mathbf{r}, P_α , and the multiplication in an étale groupoid are all continuous and open, it follows that ϕ_{P_α} is continuous and open. We now show that it is bijective. If $i(\mathbf{r}(\beta), t) P_\alpha(\beta) = i(\mathbf{r}(\gamma), t') P_\alpha(\gamma)$, then applying q to both sides shows that $\beta = \gamma$. We then obtain that $i(\mathbf{r}(\beta), t) = i(\mathbf{r}(\beta), t')$, and so $t = t'$ by the injectivity of i . Finally, to see that ϕ_{P_α} is surjective, fix $\beta \in q^{-1}(B_\alpha)$. Then $q(\beta) = q(P_\alpha(q(\beta)))$, and so $\beta P_\alpha(q(\beta))^{-1} \in q^{-1}(G^{(0)})$. Thus $\beta = i(\mathbf{r}(\beta), t) P_\alpha(q(\beta))$ for some $t \in T$. This completes the proof. \square

The only time we apply Proposition 2.2 to non-Hausdorff groupoids is in the proof of Theorem 5.6, where we only use the implication (4) \implies (1), which does not rely on [4, Lemma 4.3(b)] (stated only for Hausdorff groupoids).

Corollary 2.3. *Let (Σ, i, q) be a discrete twist over a Hausdorff étale groupoid G . Then Σ is a Hausdorff étale groupoid, and q is a local homeomorphism. In particular, G has the quotient topology.*

Proof. The map q is a covering map, and hence a local homeomorphism, by Proposition 2.2. Thus Σ is Hausdorff, because any covering space of a Hausdorff space is Hausdorff. \square

In [4, Lemma 4.3(c)] it is shown that if (Σ, i, q) is a discrete twist, then the sets B_α and maps P_α of condition (DT2) can be chosen such that $P_\alpha(B_\alpha \cap G^{(0)}) \subseteq \Sigma^{(0)}$. A continuous map $P_\alpha: B_\alpha \rightarrow \Sigma$ defined as in condition (DT2) that satisfies this additional condition is called a (*continuous*) *local section*. If G is ample, then the open bisections from condition (DT2) can be chosen to be compact.

If (Σ, i, q) is a discrete twist, then by definition, every element of G has an open bisection neighbourhood that admits a local trivialisation. To finish this section, we observe that the reduction of Σ to any countable union of compact open subsets of G is topologically trivial. The proof of the following result is essentially the proof of [4, Theorem 4.10].

Lemma 2.4. *Let G be an ample Hausdorff groupoid, and let (Σ, i, q) be a discrete R -twist over G . Suppose that $U \subseteq G$ is a countable union of compact open subsets. Then there exists a continuous section $S: U \rightarrow \Sigma$ that trivialises Σ over U , in the sense that $(\gamma, t) \mapsto t \cdot S(\gamma)$ is a homeomorphism of $U \times R^\times$ onto $q^{-1}(U)$.*

In fact, in Lemma 2.4 it is enough for $U \setminus G^{(0)}$ to be paracompact rather than U being a countable union of compact open sets, since it is well known that a paracompact locally compact space can be partitioned into clopen σ -compact subspaces.

In [4], the authors studied discrete twists arising from continuous 2-cocycles (see [4, Example 4.5]), and showed that every discrete R -twist (Σ, i, q) admitting a continuous global section for q is induced by a continuous 2-cocycle (see [4, Proposition 4.8]). A continuous 2-cocycle on a Hausdorff étale groupoid G is a continuous map $\sigma: G^{(2)} \rightarrow R^\times$ that satisfies the *2-cocycle identity*: $\sigma(\alpha, \beta) \sigma(\alpha\beta, \gamma) = \sigma(\alpha, \beta\gamma) \sigma(\beta, \gamma)$ for all composable $\alpha, \beta, \gamma \in G$; and is *normalised*, in the sense that $\sigma(\mathbf{r}(\gamma), \gamma) = \sigma(\gamma, \mathbf{s}(\gamma)) = 1$ for all $\gamma \in G$.

Example 2.5. We now sketch an argument showing that there are discrete R -twists over principal ample groupoids that do not admit a continuous global section, and hence are

not equivalent to a twist coming from a continuous 2-cocycle, for a fairly broad class of rings R . Kumjian shows in [25, Proposition 4.4] that if X is a locally compact space and G is the principal groupoid corresponding to the equivalence relation $X \times X$, then twists (in the classical sense) over G are in bijection with principal circle bundles over X . Moreover, it is observed in [37] that a twist over G admits a continuous global section if and only if the corresponding circle bundle is trivial. The same construction can be applied *mutatis mutandis* in our setting to show that if X is a totally disconnected locally compact Hausdorff space, then discrete R -twists over $X \times X$ are in bijection with principal R^\times -bundles over X (where R^\times is viewed as a discrete abelian group), and the twist admits a continuous global section if and only if the corresponding bundle is trivial. The construction just relies on the fact that for a suitable topological abelian group A , the isomorphism classes of principal A -bundles over X form an abelian group under the ‘‘Baer sum’’, with inverse given by inverting the action. (In particular, one needs to replace complex conjugation by inversion in the construction given in [25].)

So it suffices to give an example of a totally disconnected locally compact Hausdorff space X that admits a nontrivial R^\times -bundle for some ring R . We will show that this can be done for a ring R whose group of units is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times A$ for some abelian group A . This includes \mathbb{Z} , and more generally any ordered ring (for example, a subring of an ordered field like \mathbb{R}), and finite fields of order congruent to 3 modulo 4.

In [50, Example 5.3], a totally disconnected locally compact Hausdorff space X is given with a covering \mathcal{U} by compact open sets such that the first Čech cohomology group of X with respect to this covering and with coefficients in some sheaf of abelian groups does not vanish. In the footnote to the example, the author points out that one can, in fact, prove that the first cohomology group $H^1(\mathcal{U}; \mathbb{Z}/2\mathbb{Z})$ with coefficients in the constant sheaf with value $\mathbb{Z}/2\mathbb{Z}$ does not vanish (with respect to this covering or any covering by compact open sets); some further details are provided in [44]. By the theory of principal bundles, this is equivalent to the existence of a nontrivial principal $(\mathbb{Z}/2\mathbb{Z})$ -bundle over X . In fact, if $R^\times \cong \mathbb{Z}/2\mathbb{Z} \times A$ for an abelian group A , then since $\mathbb{Z}/2\mathbb{Z}$ is a retract of R^\times , we see that $H^1(\mathcal{U}; \mathbb{Z}/2\mathbb{Z})$ is a retract of $H^1(\mathcal{U}; R^\times)$ and hence the latter is nontrivial. Therefore, the space X admits a nontrivial principal R^\times -bundle for any such ring R .

This then shows that the principal groupoid $X \times X$ admits a discrete R -twist (R as above) with no continuous global sections; that is, a twist that does not come from a continuous 2-cocycle.

2.4. Twisted Steinberg algebras. In [4], the authors defined the *twisted Steinberg algebra* arising from a discrete R -twist (Σ, i, q) over G . This was accomplished in the setting where Σ is *topologically trivial* (that is, Σ admits a continuous global section for q , and is thus induced by a continuous 2-cocycle). The original definition also allowed for twists over groupoids with fibres a subgroup $T \leq R^\times$. We include the definition which we use below.

Given a topological space X and a ring R , we write $C(X, R)$ for the R -module of locally constant maps from X to R . For $f \in C(X, R)$, we define

$$\text{supp}(f) := \{x \in X : f(x) \neq 0\},$$

which is a clopen set. We also define

$$C_c(X, R) := \{f \in C(X, R) : \text{supp}(f) \text{ is compact}\},$$

which is an R -submodule of $C(X, R)$.

Definition 2.6. Let G be an ample Hausdorff groupoid, and let (Σ, i, q) be a discrete R -twist over G . We say that $f \in C(\Sigma, R)$ is *R^\times -contravariant* if $f(t \cdot \sigma) = t^{-1}f(\sigma)$ for all

$t \in R^\times$ and $\sigma \in \Sigma$,² and we define

$$A_R(G; \Sigma) := \{f \in C(\Sigma, R) : f \text{ is } R^\times\text{-contravariant and } q(\text{supp}(f)) \text{ is compact}\}.$$

As a notational convenience, given a discrete R -twist (Σ, i, q) over G , for $f \in A_R(G; \Sigma)$, we define

$$\text{supp}_G(f) := q(\text{supp}(f)) \subseteq G.$$

We then have $\text{supp}(f) = q^{-1}(\text{supp}_G(f))$ because of the R^\times -contravariance of f . Since q is a quotient map and $\text{supp}(f)$ is clopen, $\text{supp}_G(f)$ is also clopen. Thus the condition that $q(\text{supp}(f))$ is compact and the condition that $\overline{q(\text{supp}(f))}$ is compact (as in [4, Definition 4.17]) are equivalent.

In [4], the authors were mainly concerned with topologically trivial twists, and in particular proved that $A_R(G; \Sigma)$ has a well-defined multiplication under this assumption. Here we show that $A_R(G; \Sigma)$ is an R -algebra even when the twist is not topologically trivial.

We now establish that elements of $A_R(G; \Sigma)$ can be expressed in terms of their restriction to any ‘‘slice’’ of their support with respect to the action of R^\times on Σ .

Lemma 2.7. *Let G be an ample Hausdorff groupoid and let (Σ, i, q) be a discrete twist over G . Suppose that X is an open subset of Σ such that $q|_X$ is injective. Let $f: X \rightarrow R$ be a locally constant compactly supported function. Then there is a unique element \tilde{f} of $A_R(G; \Sigma)$ such that $\text{supp}(\tilde{f}) \subseteq R^\times \cdot X$ and $\tilde{f}|_X = f$.*

Proof. We have $q(t \cdot \sigma) = q(\sigma)$ for all $t \in R^\times$ and $\sigma \in \Sigma$, and so since $q|_X$ is injective, X intersects each R^\times -orbit at no more than one point. So there is a well-defined function $f': R^\times \cdot \text{supp}(f) \rightarrow R$ satisfying $f'(t \cdot \sigma) = t^{-1}f(\sigma)$ for all $\sigma \in \text{supp}(f)$ and $t \in R^\times$. Since f is locally constant and $\sigma \mapsto t \cdot \sigma$ is a homeomorphism for each fixed $t \in R^\times$, the function f' is also locally constant. It follows from q being an open map, $q(\text{supp}(f))$ being compact, and G being Hausdorff that $q(\text{supp}(f))$ is clopen, and hence $R^\times \cdot \text{supp}(f)$ is also clopen. Therefore, the function \tilde{f} given by $\tilde{f}|_{R^\times \cdot \text{supp}(f)} = f'$ and $\tilde{f}|_{\Sigma \setminus R^\times \cdot \text{supp}(f)} = 0$ is locally constant. Since $q(\text{supp}(\tilde{f})) = q(\text{supp}(f))$, we have that $q(\text{supp}(\tilde{f}))$ is compact, and \tilde{f} is R^\times -contravariant by definition. The uniqueness is clear: if $g \in A_R(G; \Sigma)$ agrees with f on X then it agrees with f' on $R^\times \cdot X$, and so if in addition $\text{supp}(g) \subseteq R^\times \cdot X$, then $g = \tilde{f}$. \square

Suppose that $X \subseteq \Sigma$ is a compact open bisection. Then in particular, $q|_X$ is injective because q respects \mathbf{r} and \mathbf{s} and restricts to a homeomorphism of unit spaces. We will denote by $\tilde{1}_X$ the element of $A_R(G; \Sigma)$ obtained by applying Lemma 2.7 to the constant function $1_X: X \rightarrow R$ that maps every element of X to 1.

Proposition 2.8. *Let G be an ample Hausdorff groupoid, and let (Σ, i, q) be a discrete R -twist. Then for each $f \in A_R(G; \Sigma)$, there exist a finite set \mathcal{F} of compact open bisections of Σ with mutually disjoint images in G , and elements r_U of R , for each $U \in \mathcal{F}$, such that $f = \sum_{U \in \mathcal{F}} r_U \tilde{1}_U$.*

Proof. We apply Lemma 2.4 to find a section $S: \text{supp}_G(f) \rightarrow \text{supp}(f)$ that trivialises Σ over $\text{supp}_G(f)$. Then $X := S(\text{supp}_G(f)) \cong \text{supp}_G(f) \times \{1\}$ is compact and open, and $f|_X$ belongs to the (untwisted) Steinberg algebra $A_R(\Sigma)$. Consequently, by [45, Proposition 4.14], we can write $f|_X = \sum_{U \in \mathcal{F}} r_U \tilde{1}_U$ as a finite R -linear combination of

²In [4, Definition 4.17], the authors consider R^\times -equivariant functions $f \in C(\Sigma, R)$, which satisfy $f(t \cdot \sigma) = tf(\sigma)$ for all $t \in R^\times$ and $\sigma \in \Sigma$. We remark that the results in [4, Section 4.3] go through with Definition 2.6 as well. See [4, Remark 4.24]. Our use of the term ‘‘contravariant’’ is not related to the notion of a ‘‘contravariant functor’’.

characteristic functions of mutually disjoint compact open bisections. Since q is injective on X , the images of the elements of \mathcal{F} under q are also mutually disjoint.

By Lemma 2.7, each $\tilde{1}_U$ belongs to $A_R(G; \Sigma)$, and so $\sum_{U \in \mathcal{F}} r_U \tilde{1}_U \in A_R(G; \Sigma)$ because $A_R(G; \Sigma)$ is an R -module. Now, since the functions f and $\sum_{U \in \mathcal{F}} r_U \tilde{1}_U$ agree on X and both belong to $A_R(G; \Sigma)$ with support contained in $\text{supp}(f) = R^\times \cdot X$, Lemma 2.7 implies that $f = \sum_{U \in \mathcal{F}} r_U \tilde{1}_U$. \square

Proposition 2.9. *Let G be an ample Hausdorff groupoid, and let (Σ, i, q) be a discrete R -twist over G . Given $f, g \in A_R(G; \Sigma)$, let $S: \text{supp}_G(f) \rightarrow \text{supp}(f)$ be any section (not necessarily continuous) for q on $\text{supp}_G(f)$. The formula*

$$(f * g)(\sigma) := \sum_{\alpha \in G^{r(\sigma)} \cap \text{supp}_G(f)} f(S(\alpha)) g(S(\alpha)^{-1}\sigma)$$

does not depend on the choice of S . This formula defines an associative multiplication on $A_R(G; \Sigma)$, making it into an R -algebra.

Proof. It is straightforward to check that $A_R(G; \Sigma)$ is an R -module. Fix $f, g \in A_R(G; \Sigma)$. For each $\sigma \in \Sigma$, define $F^{r(\sigma)} := G^{r(\sigma)} \cap \text{supp}_G(f)$. Since each $F^{r(\sigma)}$ is compact and discrete, and hence finite, the convolution formula makes sense.

Suppose that S and S' are two sections for q on $\text{supp}_G(f)$. Then there is a unique (not necessarily continuous) function $\alpha \mapsto t_\alpha$ from $\text{supp}_G(f)$ to R^\times such that $S(\alpha) = t_\alpha \cdot S'(\alpha)$ for all $\alpha \in \text{supp}_G(f)$. Fix $\sigma \in \Sigma$. Using the centrality of $i(G^{(0)} \times R^\times)$ and the R^\times -contravariance of f and g for the second equality, we obtain

$$\begin{aligned} \sum_{\alpha \in F^{r(\sigma)}} f(S(\alpha)) g(S(\alpha)^{-1}\sigma) &= \sum_{\alpha \in F^{r(\sigma)}} f(t_\alpha \cdot S'(\alpha)) g((t_\alpha \cdot S'(\alpha))^{-1}\sigma) \\ &= \sum_{\alpha \in F^{r(\sigma)}} t_\alpha^{-1} f(S'(\alpha)) t_\alpha g(S'(\alpha)^{-1}\sigma) \\ &= \sum_{\alpha \in F^{r(\sigma)}} f(S'(\alpha)) g(S'(\alpha)^{-1}\sigma), \end{aligned}$$

and thus the convolution formula does not depend on S .

By Proposition 2.8, any $f \in A_R(G; \Sigma)$ can be written in the form $f = \sum_{U \in \mathcal{F}} r_U \tilde{1}_U$, where \mathcal{F} is a finite collection of compact open bisections of Σ . As the convolution product is clearly R -bilinear by definition, to show both that $A_R(G; \Sigma)$ is closed under convolution and that the convolution product is associative, it suffices to show that $\tilde{1}_U * \tilde{1}_V = \tilde{1}_{UV}$ for compact open bisections U and V of Σ (as the compact open bisections form a semi-group). Note that $q|_U$ and $q|_V$ are injective. The definition of the convolution shows that $\text{supp}(\tilde{1}_U * \tilde{1}_V) \subseteq (R^\times \cdot U)(R^\times \cdot V) = R^\times \cdot UV$. For $(\tau, \rho) \in (U \times V) \cap \Sigma^{(2)}$, choosing a section $\zeta: G \rightarrow \Sigma$ such that $\zeta(q(\tau)) = \tau$ gives that if $\sigma = \tau\rho$, then $(\tilde{1}_U * \tilde{1}_V)(\sigma) = \sum_{\alpha \in G^{r(\sigma)}} \tilde{1}_U(\zeta(\alpha)) \tilde{1}_V(\zeta(\alpha)^{-1}\sigma) = \tilde{1}_U(\tau) \tilde{1}_V(\rho) = 1$, and so $\tilde{1}_U * \tilde{1}_V$ agrees with $\tilde{1}_{UV}$ on UV . Thus $\tilde{1}_U * \tilde{1}_V = \tilde{1}_{UV}$ by Lemma 2.7. \square

Note that when the intended meaning is clear, we sometimes write fg rather than $f * g$ for the convolution product of two functions $f, g \in A_R(G, \sigma)$.

The following corollary is a straightforward consequence of Proposition 2.8 and the proof of Proposition 2.9.

Corollary 2.10. *Let \mathcal{F} and \mathcal{G} be finite collections of compact open bisections of Σ with mutually disjoint images in G , and suppose that $f, g \in A_R(G; \Sigma)$ satisfy $f = \sum_{U \in \mathcal{F}} r_U \tilde{1}_U$*

and $g = \sum_{V \in \mathcal{G}} t_V \tilde{1}_V$. Then

$$f * g = \sum_{U \in \mathcal{F}, V \in \mathcal{G}} r_{UV} t_V \tilde{1}_{UV}. \quad (2.2)$$

In particular, each $\sigma \in \Sigma$ is contained in a compact open bisection X of Σ , and

$$(\tilde{1}_X * f)(\mathbf{r}(\sigma')) = f(\sigma') = (f * \tilde{1}_X)(\mathbf{s}(\sigma')) \text{ for all } \sigma' \in X.$$

Proof. Equation (2.2) follows from the formula $\tilde{1}_U * \tilde{1}_V = \tilde{1}_{UV}$ established in the proof of Proposition 2.9, and from the bilinearity of multiplication. The second claim follows easily from equation (2.2). \square

An important feature of the twisted Steinberg algebra $A_R(G; \Sigma)$ is that, even though it does not contain the algebra $A_R(\Sigma^{(0)})$ of locally constant functions on the unit space of Σ (because such functions are not R^\times -contravariant), it does contain an algebra canonically isomorphic to $A_R(\Sigma^{(0)})$, and the actions of elements of this algebra on $A_R(G; \Sigma)$ by multiplication are exactly as one would expect.

Proposition 2.11. *Let G be an ample Hausdorff groupoid and let (Σ, i, q) be a discrete R -twist over G . The map $f \mapsto \tilde{f}$ obtained from Lemma 2.7 applied to elements $f \in C_c(\Sigma^{(0)}, R) \cong A_R(G^{(0)})$ defines an injective R -algebra homomorphism of $C_c(\Sigma^{(0)}, R)$ into $A_R(G; \Sigma)$ whose image is precisely $A_R(G^{(0)}; q^{-1}(G^{(0)}))$. For $f \in C_c(\Sigma^{(0)}, R)$ and $a \in A_R(G; \Sigma)$, we have $(\tilde{f} * a)(\sigma) = f(\mathbf{r}(\sigma))a(\sigma)$ and $(a * \tilde{f})(\sigma) = a(\sigma)f(\mathbf{s}(\sigma))$ for all $\sigma \in \Sigma$.*

Proof. That $f \mapsto \tilde{f}$ is an injective R -module map that has image contained in $A_R(G^{(0)}; q^{-1}(G^{(0)}))$ is straightforward. For the reverse containment, observe that if $g \in A_R(G^{(0)}; q^{-1}(G^{(0)}))$, then $f := g|_{\Sigma^{(0)}} \in C_c(\Sigma^{(0)}, R)$, and since \tilde{f} and g are elements of $A_R(G^{(0)}; q^{-1}(G^{(0)}))$ that agree on $\Sigma^{(0)}$, they are equal by Lemma 2.7.

That this map is multiplicative and hence is an R -algebra homomorphism is a particular case of the final assertion (where $a \in C_c(\Sigma^{(0)}, R)$), so we now just have to prove the final assertion. Fix $f \in C_c(\Sigma^{(0)}, R)$ and $a \in A_R(G; \Sigma)$. Let $S: q(\text{supp}(f)) \rightarrow \text{supp}(f)$ be the identity map. Then S is a section from $\text{supp}_G(\tilde{f})$ to $\text{supp}(f)$, and so Proposition 2.9 shows that for all $\sigma \in \Sigma$,

$$(\tilde{f} * a)(\sigma) = \sum_{\beta \in G^{\mathbf{r}(\sigma)} \cap \text{supp}_G(\tilde{f})} \tilde{f}(S(\beta)) a(S(\beta)^{-1}\sigma).$$

The only term in the sum is $\beta = \mathbf{r}(\sigma)$, which yields $S(\beta) = \mathbf{r}(\sigma)$, and so the sum collapses to $(\tilde{f} * a)(\sigma) = \tilde{f}(\mathbf{r}(\sigma))a(\sigma) = f(\mathbf{r}(\sigma))a(\sigma)$ since \tilde{f} extends f by definition.

For the second equality, fix $\sigma \in \text{supp}(a)$ and choose a section $S: \text{supp}_G(a) \rightarrow \text{supp}(a)$ such that $S(q(\sigma)) = \sigma$. Then

$$(a * \tilde{f})(\sigma) = \sum_{\beta \in G^{\mathbf{r}(\sigma)} \cap \text{supp}_G(a)} a(S(\beta)) \tilde{f}(S(\beta)^{-1}\sigma).$$

The only nonzero terms in the sum are those corresponding to $\beta \in G^{\mathbf{r}(\sigma)} \cap \text{supp}_G(a)$ such that

$$S(\beta)^{-1}\sigma \in \text{supp}(\tilde{f}) \subseteq i(G^{(0)} \times R^\times).$$

Since $\text{supp}_G(\tilde{f}) \cap G^{\mathbf{r}(\sigma)} = \{\mathbf{r}(\sigma)\}$, the only possible nonzero summand is such that $q(S(\beta)^{-1}\sigma) = \mathbf{r}(\sigma)$; that is, $\beta = q(\sigma)$. Since $\sigma \in \text{supp}(a)$, we have $S(\beta) = \sigma$ by our choice of S , and so the sum collapses to $(a * \tilde{f})(\sigma) = a(\sigma)\tilde{f}(\sigma^{-1}\sigma) = a(\sigma)f(\mathbf{s}(\sigma))$. If $\sigma \in \Sigma \setminus \text{supp}(a)$, then both $(a * \tilde{f})(\sigma)$ and $a(\sigma)f(\mathbf{s}(\sigma))$ are zero. \square

When R is indecomposable, the idempotents of $A_R(G^{(0)})$ are the characteristic functions of compact open subsets of $G^{(0)}$. Given that indecomposability of R is implied by [condition \(WT\)](#) when $I(B)$ is nontrivial, the R -algebra homomorphism from [Proposition 2.11](#) gives a better perspective on the elements of $I(B)$.

Corollary 2.12. *Let G be an ample Hausdorff groupoid, let R be an indecomposable commutative ring, and let (Σ, i, q) be a discrete R -twist over G . Then the map $U \mapsto \tilde{1}_U$ gives a lattice isomorphism between compact open subsets of $\Sigma^{(0)}$ and $I(B)$.*

2.5. Normalisers and inverse semigroups. An inverse semigroup is a semigroup \mathcal{S} such that, for each $s \in \mathcal{S}$, there is a unique element s^\dagger with $ss^\dagger s = s$ and $s^\dagger ss^\dagger = s^\dagger$. We have $(st)^\dagger = t^\dagger s^\dagger$ and $(s^\dagger)^\dagger = s$. The idempotents of an inverse semigroup form a commutative subsemigroup, and are precisely the elements of the form $s^\dagger s$. Details can be found in [\[28\]](#).

Definition 2.13. Let R be a commutative unital ring, let A be an R -algebra, and let $B \subseteq A$ be a commutative R -subalgebra. Suppose that the set $I(B)$ of idempotents of B is a set of *local units* for A : that is, for any $\{a_1, \dots, a_n\} \subseteq A$, there exists $e \in I(B)$ with $ea_i = a_i = a_i e$ for all $i \in \{1, \dots, n\}$. We define the *normaliser* of B to be the set

$$N(B) := \{n \in A : \text{there exists } k \in A \text{ with } knk = k, nkn = n, \text{ and } kBn \cup nBk \subseteq B\}.$$

We use the notation $N_A(B)$ for $N(B)$ if we need to specify the ambient algebra A . We begin by establishing that $N(B)$ is an inverse semigroup.

Lemma 2.14. *For each $n \in N(B)$, there exists a unique $k \in N(B)$ such that*

$$knk = k, \quad nkn = n, \quad \text{and} \quad kBn \cup nBk \subseteq B. \quad (2.3)$$

Proof. Fix $n \in N(B)$. If k satisfies [equation \(2.3\)](#) for n , then k is itself in $N(B)$ because n satisfies [equation \(2.3\)](#) for k . Suppose that k_1 and k_2 both satisfy [equation \(2.3\)](#) for n . We claim that since $I(B)$ forms a set of local units for A , we have

$$k_1 n, k_2 n, n k_1, n k_2 \in B.$$

Indeed, there exists $e \in I(B)$ with $k_1, k_2, n \in eAe$, and so $k_i n = k_i e n \in B$ and $n k_i = n e k_i \in B$ for $i \in \{1, 2\}$. Using this and that B is commutative for the third and sixth equalities, we obtain

$$k_1 = k_1 n k_1 = k_1 n k_2 n k_1 = k_1 n k_1 n k_2 = k_1 n k_2 = k_1 n k_2 n k_2 = k_2 n k_1 n k_2 = k_2 n k_2 = k_2. \quad \square$$

For each $n \in N(B)$, we write n^\dagger for the unique element $k \in N(B)$ satisfying [equation \(2.3\)](#). So $a \in A$ belongs to $N(B)$ if and only if a^\dagger exists. Notice that the above proof shows that $aa^\dagger, a^\dagger a \in I(B)$ for all $a \in N(B)$. Also note that $I(B) \subseteq N(B)$ since $e^\dagger = e$ for each $e \in I(B)$.

Lemma 2.15. *The normaliser $N(B)$ is an inverse semigroup with inverse $n \mapsto n^\dagger$ and set of idempotents $I(B)$.*

Proof. To see that $N(B)$ is closed under multiplication, fix $n, m \in N(B)$. Then $nmBm^\dagger n^\dagger \subseteq nBn^\dagger \subseteq B$, and similarly, $m^\dagger n^\dagger Bnm \subseteq B$. As observed above, $mm^\dagger, n^\dagger n \in I(B)$, and so mm^\dagger and $n^\dagger n$ commute. Hence

$$(nm)(m^\dagger n^\dagger)(nm) = nn^\dagger nmm^\dagger m = nm.$$

Similarly, $m^\dagger n^\dagger (nm)m^\dagger n^\dagger = m^\dagger n^\dagger$, and so $nm \in N(B)$. Thus $N(B)$ is a semigroup. [Lemma 2.14](#) therefore implies that $N(B)$ is an inverse semigroup with inversion given by $n \mapsto n^\dagger$. The final statement follows because if $e \in N(B)$ is an idempotent, then $e = e^\dagger e \in I(B)$, as we saw earlier. \square

Recall that there is a partial order on any inverse semigroup given by $s \leq t$ if and only if $s = ss^\dagger t$ (or, equivalently, if and only if $s = ts^\dagger s$). The partial order is preserved by multiplication and inversion, and the product of two idempotents is their meet. Furthermore, $s \leq t$ if and only if there is an idempotent e such that $s = te$, and this is equivalent to the existence of an idempotent f such that $s = ft$. See [28] for details. Note that \leq reduces to the usual order relation on commuting idempotents; that is, $e \leq f$ if and only if $ef = e$. In Lemmas 2.16 and 2.17, we present various additional properties of \leq that we use throughout the paper. The first of these results is straightforward, so we omit the proof.

Lemma 2.16. *Suppose that A is an R -algebra, and that $B \subseteq A$ is an abelian subalgebra such that $I(B)$ forms a set of local units for A . Fix $t \in R^\times$.*

- (a) *If $n \in N(B)$, then $tn \in N(B)$ with $(tn)^\dagger = t^{-1}n^\dagger$.*
- (b) *If $n, m \in N(B)$ with $n \leq m$, then $tn \leq tm$.*

Lemma 2.17. *Suppose that A is an R -algebra, and that $B \subseteq A$ is an abelian subalgebra satisfying *condition (WT)* such that $I(B)$ forms a set of local units for A . Then the following hold.*

- (a) *If $n \in N(B)$ and $e \in I(B)$ satisfy $n \leq e$, then $n \in I(B)$.*
- (b) *For $f, e \in I(B)$ and $t \in R^\times$, if $0 \neq f \leq te$, then $t = 1$.*
- (c) *For $n, m \in N(B)$, we have $nn^\dagger m = n \iff mn^\dagger n = n$.*
- (d) *For all $n \in N(B)$ and $e \in I(B)$, we have $en, ne \leq n$.*
- (e) *If $e \in I(B)$ and $n \in N(B)$, then $n - ne, n - en \in N(B)$, with $(n - ne)^\dagger = n^\dagger - en^\dagger$ and $(n - en)^\dagger = n^\dagger - n^\dagger e$, and $n - ne, n - en \leq n$.*

Proof. Part (a) is [28, Proposition 1.4.7(5)], and parts (c) and (d) are contained in [28, Lemma 1.4.6]. For part (b), suppose that $0 \neq f \leq te$. Then $tfe = f \neq 0$ implies that $fe \neq 0$ and

$$t^2 fe = (tfe)^2 = f^2 = f = tfe.$$

Multiplying both sides by t^{-1} then gives $(1 - t)fe = 0$. Hence $t = 1$, as B satisfies (WT).

Part (e) follows from (d) once we observe that $n^\dagger ne \leq n^\dagger n$ and hence $n^\dagger n - n^\dagger ne \in I(B) \subseteq N(B)$. Thus $n - ne = n(n^\dagger n - n^\dagger ne) \leq n$, and $n - ne$ is an element of $N(B)$ with inverse $(n^\dagger n - n^\dagger ne)n^\dagger = n^\dagger - en^\dagger$. A similar argument proves the claim about $n - en$. \square

We define *free normalisers* by analogy (with suitable modifications) with the definition given by Kumjian in the setting of C^* -algebras.

Definition 2.18 (cf. [25, Def. 1]). Let R be a commutative unital ring and let A be an R -algebra, with a commutative subalgebra $B \subseteq A$ whose idempotents form a set of local units for A . We say that $n \in N(B)$ is a *free normaliser* if either $n \in B$ or $(n^\dagger n)(nn^\dagger) = 0$ (this latter condition is equivalent to $n^2 = 0$, as $n^2 = n(n^\dagger nnn^\dagger)n$).

3. ALGEBRAIC QUASI-CARTAN PAIRS

In this section we introduce our main objects of study—algebraic quasi-Cartan pairs. Our main theorem shows that these are precisely the pairs of algebras that arise from discrete R -twists satisfying an appropriate local bisection hypothesis. We also define two additional types of pairs of algebras: algebraic diagonal pairs, and algebraic Cartan pairs. We prove later in this section that every diagonal pair is an algebraic Cartan pair and every algebraic Cartan pair is an algebraic quasi-Cartan pair. The diagonal pairs correspond to twists over principal groupoids, while the Cartan pairs correspond to twists over effective groupoids.

Before defining these pairs of algebras, we need to introduce an appropriate notion of a conditional expectation from an algebra onto an abelian subalgebra.

Definition 3.1. Let R be an indecomposable commutative ring, let A be an R -algebra, and let $B \subseteq A$ be an abelian subalgebra such that $I(B)$ is a set of local units for A . A map $P: A \rightarrow B$ is called a *conditional expectation* if

- (i) P is R -linear;
- (ii) $P|_B = \text{id}_B$; and
- (iii) $P(bab') = bP(a)b'$ for $a \in A$ and $b, b' \in B$.

The conditional expectation $P: A \rightarrow B$ is *faithful* if, for every $a \in A \setminus \{0\}$, there exists $n \in N(B)$ such that $P(na) \neq 0$.

Note that since B contains local units for A , condition (iii) of Definition 3.1 implies that P is both left and right B -linear.

Remark 3.2. Watatani [49] defines a conditional expectation $P: A \rightarrow B$ to be *nondegenerate* if for every nonzero $a \in A$ there exists $a' \in A$ such that $P(a'a) \neq 0$. Since we will assume that A is spanned by $N(B)$, the notion of a nondegenerate conditional expectation is equivalent to our definition of a faithful conditional expectation in our setting.

We now define algebraic diagonal pairs, Cartan pairs, and quasi-Cartan pairs.

Definition 3.3. Let R be a unital ring, let A be an R -algebra, and let $B \subseteq A$ be an abelian subalgebra satisfying condition (WT) and with the following properties.

- (i) The set $I(B)$ forms a set of local units for A .
- (ii) $B = \text{span}(I(B))$.
- (iii) $A = \text{span}(N(B))$.
- (iv) There exists a faithful conditional expectation $P: A \rightarrow B$.

Then we say that the pair (A, B) is:

- (ADP) an *algebraic diagonal pair* if A is spanned by the free normalisers of B ;
- (ACP) an *algebraic Cartan pair* if B is a maximal abelian subalgebra of A ; and
- (AQP) an *algebraic quasi-Cartan pair* if there is a faithful conditional expectation $P: A \rightarrow B$ such that for each $n \in N(B)$ there exists $e \in I(B)$ such that $P(n) = en = ne$.

Remarks 3.4.

- (1) To construct the inverse semigroup $N(B)$ in Definition 2.13, we required B to be abelian and to satisfy property (i) of Definition 3.3. In fact, B being abelian and property (i) are almost enough to build our discrete twist Σ over G in the next two sections. However, we need property (AQP) of Definition 3.3 in Proposition 5.10 to show that G is Hausdorff.
- (2) In the definition of an algebraic quasi-Cartan pair, note that if $n \in N(B)$ satisfies $P(n) = 0$, then $e = 0$ satisfies $P(n) = en = ne$. So we could equivalently require just that whenever $P(n) \neq 0$ there exists $e \in I(B)$ such that $P(n) = en = ne$.

The simplest example of an algebraic diagonal pair is the following. Let R be an indecomposable commutative ring, let $A = M_n(R)$ be the ring of $n \times n$ matrices over R , and let B be the subalgebra of diagonal matrices. Then B is spanned by the diagonal matrix units E_{ii} , which are idempotent, and A is spanned by the elementary matrix units E_{ij} , which are free normalisers. The faithful conditional expectation is given by making all non-diagonal entries zero. This example motivates the terminology.

A more complicated, but illustrative, class of examples is the class of Leavitt path algebras associated to directed graphs. Recall that if E is a directed graph and $L_R(E)$ is its Leavitt path algebra over a ring R , then it admits a commutative subalgebra

$D_R(E)$ —the subalgebra generated by the idempotents of the form $s_\mu s_{\mu^*}$ for finite paths μ . [Example 9.5](#) shows that $(A_R(E), D_R(E))$ is an algebraic quasi-Cartan pair. [Proposition 7.1](#) together with known structure theory for the groupoid of a directed graph show that $(A_R(E), D_R(E))$ is an algebraic Cartan pair if and only if every cycle in E has an entrance, and is an algebraic diagonal pair if and only if E contains no cycles.

Lemma 3.5. *Every algebraic diagonal pair is an algebraic Cartan pair.*

Proof. Suppose that (A, B) is an algebraic diagonal pair and that $a \in A$ commutes with every element of B ; we must show that $a \in B$. Since (A, B) is an algebraic diagonal pair, we can express a as a finite linear combination of free normalisers. Every free normaliser n either belongs to B or satisfies $n^\dagger n n n^\dagger = 0$; so there exist $b \in B$, a finite set $F \subseteq N(B)$ such that $n^\dagger n n n^\dagger = 0$ for all $n \in F$, and coefficients $\alpha_n \in R$ for each $n \in F$ such that

$$a = b + \sum_{n \in F} \alpha_n n.$$

Thus it suffices to show that $a' := \sum_{n \in F} \alpha_n n$ is zero. Since a and b are in the commutant (centraliser) of B , their difference $a' = a - b$ is also in the commutant of B . Let M be the set of minimal nonzero idempotents in the Boolean algebra X generated by $\{n^\dagger n, n n^\dagger : n \in F\}$. Then each $e \in M$ commutes with a' . Since $\sum_{e \in M} e$ is the maximum element of X , we have $n \sum_{e \in M} e = n n^\dagger n \sum_{e \in M} e = n n^\dagger n = n$. Hence $a' = \sum_{e \in M} a' e = \sum_{e \in M} e a' e = \sum_{n \in F} \alpha_n \sum_{e \in M} e n e$. So it suffices to show that each $e n e = 0$. Fix $n \in F$ and $e \in M$. If $n^\dagger n e = 0$, then $e n e = e n n^\dagger n e = 0$, so we may suppose that $n^\dagger n e \neq 0$. Since e is a minimal nonzero idempotent in X , we deduce that $e n^\dagger n = e$, and hence $e n n^\dagger = e n^\dagger n n n^\dagger = 0$, since $n \in F$. Thus $e n e = e n n^\dagger n e = 0$, as required. \square

We next show that every algebraic Cartan pair is an algebraic quasi-Cartan pair, so that we have [\(ADP\)](#) implies [\(ACP\)](#) implies [\(AQP\)](#).

Lemma 3.6. *Every algebraic Cartan pair is an algebraic quasi-Cartan pair. Moreover, every conditional expectation on an algebraic Cartan pair satisfies [\(AQP\)](#).*

Proof. Suppose that (A, B) is an algebraic Cartan pair with conditional expectation $P: A \rightarrow B$. To establish that (A, B) is an algebraic quasi-Cartan pair, suppose that $n \in N(B)$ satisfies $P(n) \neq 0$. Write $P(n) = \sum_{i=1}^k t_i e_i$, where the e_i are mutually orthogonal elements of $I(B)$ and each $t_i \in R \setminus \{0\}$. Let

$$e(n) := \sum_{i=1}^k e_i.$$

Then $e(n) \in I(B)$. We show that $e(n)n = n e(n) = P(n)$. First, we claim that if $f \leq e(n)$ with $f \in I(B)$, then $n^\dagger f n = f$. To prove the claim, notice that

$$P(nf) = P(n)f = \sum_{i=1}^k t_i e_i f,$$

and since $nfn^\dagger \in nBn^\dagger \subseteq B$,

$$P(nf) = P(nn^\dagger nf) = P(nfn^\dagger n) = nfn^\dagger P(n) = P(n)nfn^\dagger = \sum_{i=1}^k t_i e_i nfn^\dagger. \quad (3.1)$$

Since $P(nf)f = P(nf)$, we deduce after multiplying both sides of [equation \(3.1\)](#) on the right by f that $\sum_{i=1}^k t_i e_i f = \sum_{i=1}^k t_i e_i nfn^\dagger f$. For any $j \in \{1, \dots, k\}$, multiplying both sides by e_j on the left yields $t_j e_j f = t_j e_j nfn^\dagger f$, and so each $t_j(e_j f - e_j nfn^\dagger f) = 0$. Since $t_j \neq 0$ and $e_j f - e_j nfn^\dagger f = e_j f - e_j f nfn^\dagger$ is an idempotent, we deduce from

condition (WT) that each $e_j f = e_j n f n^\dagger f$. Summing over all $j \in \{1, \dots, k\}$ and using that $f \leq e(n)$ yields

$$f = e(n)f = \sum_{i=1}^k e_i f = \sum_{i=1}^k e_i n f n^\dagger f = e(n) n f n^\dagger f \leq n f n^\dagger.$$

Therefore, $n^\dagger f n \leq n^\dagger n f n^\dagger n \leq f$. To prove the reverse inequality, note that

$$P(nf) = P(n)f = fP(n) = P(fn) = P(nn^\dagger fn) = P(n)n^\dagger fn = \sum_{i=1}^k t_i e_i n^\dagger fn.$$

Therefore, for any $j \in \{1, \dots, k\}$, we have $t_j e_j f = t_j e_j n^\dagger fn$, forcing

$$t_j (e_j f - e_j n^\dagger fn) = 0.$$

Hence $e_j f = e_j n^\dagger fn$ by (WT), as $n^\dagger f n \leq f$ implies that $e_j f - e_j n^\dagger fn$ is an idempotent. Once again summing over j , and using that $f \leq e(n)$, yields $f = e(n)n^\dagger fn \leq n^\dagger f n$, and so $f = n^\dagger f n$ for all $f \leq e(n)$, proving the claim.

It follows that if $f \leq e(n)$ in $I(B)$, then

$$nf = nn^\dagger fn = f n n^\dagger n = fn. \quad (3.2)$$

Now fix $g \in I(B)$. Applying equation (3.2) to $f = ge(n) \leq e(n)$ at the first step, and to $f = e(n)$ at the final step, we obtain

$$g(e(n)n) = nge(n) = ne(n)g = (e(n)n)g.$$

Since B is spanned by $I(B)$, we deduce that $e(n)n$ commutes with all elements of B . Since (A, B) is an algebraic Cartan pair, B is a maximal abelian subalgebra, and so $e(n)n \in B$. We then have $e(n)n = P(e(n)n) = e(n)P(n) = P(n)$ by the definition of $e(n)$. As we already observed, equation (3.2) gives $e(n)n = ne(n)$, and so (A, B) is an algebraic quasi-Cartan pair. \square

In Section 6, we will use the expectation $P: A \rightarrow B$ satisfying condition (AQP) to construct an isomorphism of A onto the twisted Steinberg algebra of an associated twist. So *a priori* the isomorphism would depend upon the choice of conditional expectations satisfying (AQP). We finish this section by showing that in fact such a conditional expectation is uniquely determined by the algebraic structure of (A, B) .

Proposition 3.7. *Suppose that (A, B) is an algebraic quasi-Cartan pair. For each normaliser $n \in N(B)$, the set $\{e \in I(B) : e \leq n^\dagger n \text{ and } ne \in B\}$ contains a maximum element $e(n)$. If P is a conditional expectation satisfying (AQP), then $P(n) = ne(n) = e(n)n$. In particular, there is a unique conditional expectation $P: A \rightarrow B$ that satisfies (AQP).*

Proof. Fix $n \in N(B)$ and suppose that $P: A \rightarrow B$ is a conditional expectation satisfying (AQP). Write $P(n) = \sum_{i=1}^k t_i e_i$, where the e_i are mutually orthogonal idempotents and each $t_i \in R \setminus \{0\}$. Let $e(n) := \sum_{i=1}^k e_i \in I(B)$. Then we have $P(n)e(n) = P(n)$. We claim that $e(n)$ is the smallest such element of $I(B)$. To see this, suppose that $f \in I(B)$ satisfies $P(n)f = P(n)$. Then, for each $j \in \{1, \dots, k\}$, we have

$$t_j f e_j = P(n) f e_j = P(n) e_j = t_j e_j,$$

and hence $t_j (e_j - f e_j) = 0$, whence $e_j = f e_j$ by (WT). Thus $e(n) = f e(n) \leq f$, proving the claim. Since P satisfies (AQP), there exists $f \in I(B)$ such that $fn = nf = P(n)$. Thus

$$ne(n) = n f e(n) = P(n) e(n) = P(n) = e(n) P(n) = e(n) f n = f e(n) n = e(n) n.$$

Notice that $e(n)$ is an element of the set $\{e \in I(B) : e \leq n^\dagger n \text{ and } ne \in B\}$: we have shown that $ne(n) = P(n) \in B$, and since $P(n)(n^\dagger n) = P(n)$, we have $e(n) \leq n^\dagger n$ by our previous argument showing that $e(n)$ is the smallest such element. It remains to show that $e(n)$ is a maximum. Fix $g \in \{e \in I(B) : e \leq n^\dagger n \text{ and } ne \in B\}$. We show that $g \leq e(n)$. Using that $ng, g \in B$, we compute $ng = P(ng) = P(n)g = ne(n)g = nge(n)$, and so $g = n^\dagger ng = n^\dagger nge(n) = ge(n) \leq e(n)$. \square

Corollary 3.8. *Suppose that (A, B) is an algebraic Cartan pair. Then there is a unique conditional expectation from A to B .*

Proof. Lemma 3.6 implies that every conditional expectation from A to B satisfies (AQP), and Proposition 3.7 implies that there is a unique such expectation. \square

We conclude this section with an example showing that (AQP) is a generalisation of the much studied no-nontrivial-units condition on twisted group rings. Recall that if R is a commutative unital ring, G is a group with identity e , and $c: G \times G \rightarrow R^\times$ is a normalised 2-cocycle, then the *twisted group ring* $R(G, c)$ is the R -algebra of finitely supported functions $f: G \rightarrow R$ with multiplication given by

$$(f * g)(\beta) = \sum_{\alpha \in G} c(\alpha, \alpha^{-1}\beta) f(\alpha) g(\alpha^{-1}\beta).$$

It is a unital ring with identity the point-mass function δ_e . Note that twisted group rings are precisely the twisted Steinberg algebras of twists over discrete groups (viewed as ample groupoids); see Lemma 4.3.

Example 3.9. Let G be a discrete group with identity e , let R be an indecomposable commutative ring, and let $c: G \times G \rightarrow R^\times$ be a normalised 2-cocycle. Let $A := R(G, c)$ be the corresponding twisted group ring, and let $B := R\delta_e \cong R$. Then $I(B) = \{0, \delta_e\}$ spans B (which trivially satisfies (WT)), and $N(B) = A^\times \cup \{0\}$, which clearly contains $\{\delta_g : g \in G\}$ and hence spans A . We claim that (A, B) is an algebraic quasi-Cartan pair if and only if every unit of A is of the form $t\delta_g$ for some $t \in R^\times$ and $g \in G$; that is, A only has trivial units.

In this setup, a conditional expectation is an R -linear map $P: A \rightarrow R\delta_e$ that fixes $R\delta_e$. We claim that the only possible conditional expectation satisfying (AQP) is given on the basis by fixing δ_e and annihilating δ_g if $g \neq e$. Indeed, δ_g is a normaliser and hence $P(\delta_g) = f\delta_e$ for some $f \in I(B) = \{0, \delta_e\}$. But since P has image B , we must choose $f = 0$ unless $g = e$, in which case we must choose $f = \delta_e$. This formula defines a conditional expectation P . It is faithful because if $a \in A \setminus \{0\}$ has a nonzero coefficient of δ_g , then $P(\delta_{g^{-1}}a) \neq 0$.

It remains to verify that P satisfies (AQP) if and only if A only has trivial units. If A has only trivial units and $u \in N(A) = A^\times \cup \{0\}$, then $P(u) = u = u\delta_e = \delta_e u$ if $u \in R^\times \delta_e$, or $P(u) = 0 = u0 = 0u$, otherwise, and so P satisfies (AQP). Conversely, if P satisfies (AQP) and $u \in A^\times$, then we can find $g \in G$ with e in the support of $u\delta_{g^{-1}}$. Then $P(u\delta_{g^{-1}}) = r\delta_e$ for some $r \neq 0$. On the other hand, $P(u\delta_{g^{-1}}) = u\delta_{g^{-1}}f$ with $f = 0$ or $f = \delta_e$. We conclude that $r\delta_e = P(u\delta_{g^{-1}}) = u\delta_{g^{-1}}$, and so $r\delta_g = u \in A^\times$ and $r \in R^\times$.

4. BUILDING AN ALGEBRAIC QUASI-CARTAN PAIR FROM A TWIST

In this section we show that a twist gives rise to an algebraic quasi-Cartan pair if and only if it satisfies an appropriate analogue of the local bisection hypothesis of [47]. We then also follow the arguments of [47] to see that an algebraic twist satisfies the local bisection hypothesis if and only if the sub-twist corresponding to the interior of the isotropy does so, and that a sufficient condition for this is that there is a dense set of units x for which

the twisted group algebra corresponding to the fibre of the interior of the isotropy over x has no nontrivial units. Finally, we demonstrate that if the underlying groupoid is effective, then the twist gives rise to an algebraic Cartan pair, and if it is principal, then we obtain an algebraic diagonal pair.

Definition 4.1. Let R be a commutative unital ring. We say that a discrete R -twist (Σ, i, q) over G satisfies the local bisection hypothesis if for every normaliser n of $A_R(G^{(0)}; q^{-1}(G^{(0)}))$ in $A_R(G; \Sigma)$, the set $\text{supp}_G(n)$ is a bisection of G .

Our interest in twists satisfying the local bisection hypothesis is that they give rise to algebraic quasi-Cartan pairs. The following lemma can be seen as a generalisation of [Example 3.9](#).

Lemma 4.2. Let R be an indecomposable commutative ring and let (Σ, i, q) be a discrete R -twist over G . Let $A = A_R(G; \Sigma)$, and let $B = A_R(G^{(0)}; q^{-1}(G^{(0)})) \subseteq A$. The pair (A, B) satisfies conditions (i)–(iv) of [Definition 3.3](#) with faithful conditional expectation $P: A \rightarrow B$ given by restriction of functions from Σ to $q^{-1}(G^{(0)})$. If (A, B) is an algebraic quasi-Cartan pair, then P satisfies (AQP). The pair (A, B) is an algebraic quasi-Cartan pair if and only if (Σ, i, q) satisfies the local bisection hypothesis.

Proof. Since R is indecomposable, the idempotents of $C_c(G^{(0)}, R)$ are precisely the characteristic functions of compact open subsets of $G^{(0)}$, and hence B satisfies [condition \(WT\)](#), because $B \cong C_c(G^{(0)}, R)$ by [Proposition 2.11](#). A routine argument using [Corollary 2.10](#) shows that for each compact open bisection U of Σ , $\tilde{1}_U$ is a normaliser of B in A with inverse $\tilde{1}_U^\dagger = \tilde{1}_{U^{-1}}$.

Properties (i) and (ii) of [Definition 3.3](#) follow immediately from [Proposition 2.11](#), and property (iii) follows from [Proposition 2.8](#). For (iv), it is straightforward to see that P is a conditional expectation, and for faithfulness, note that if $f(\sigma) \neq 0$ for some $\sigma \in \Sigma$, then for any compact open bisection $X \subseteq \Sigma$ containing σ , [Corollary 2.10](#) implies that $P(\tilde{1}_X f)(\mathbf{r}(\sigma)) = (\tilde{1}_X f)(\mathbf{r}(\sigma)) = f(\sigma) \neq 0$.

Suppose that (A, B) is an algebraic quasi-Cartan pair. Then by [Proposition 3.7](#), there is a unique faithful conditional expectation $P': A \rightarrow B$ satisfying (AQP). We claim that $P = P'$. Since the functions $\tilde{1}_V$ with $V \subseteq \Sigma$ a compact open bisection span A as an R -module by [Proposition 2.8](#), it suffices by the linearity of P and P' to prove that for each such V , $P'(\tilde{1}_V) = \tilde{1}_V|_{q^{-1}(G^{(0)})} = \tilde{1}_W$, where $W := V \cap q^{-1}(G^{(0)})$, which is a compact open bisection contained in V . Recall from [Corollary 2.12](#) that $U \mapsto \tilde{1}_U$ is a lattice isomorphism between compact open subsets of $\Sigma^{(0)}$ and $I(B)$. By [Proposition 3.7](#) (and several applications of [Corollary 2.10](#)), there is a maximum element $\tilde{1}_U$ of $I(B)$ satisfying $\tilde{1}_U \leq \tilde{1}_V^\dagger \tilde{1}_V = \tilde{1}_{\mathbf{s}(V)}$ such that $\tilde{1}_V \tilde{1}_U \in B$, and $P'(\tilde{1}_V) = \tilde{1}_V \tilde{1}_U = \tilde{1}_{VU}$. It follows that $VU \subseteq W$ by the definition of B (and since $U \subseteq \Sigma^{(0)}$). On the other hand, $\mathbf{s}(W) \subseteq \mathbf{s}(V)$ and $\tilde{1}_V \tilde{1}_{\mathbf{s}(W)} = \tilde{1}_W \in B$, and so it follows by maximality that $\mathbf{s}(W) \subseteq U$. Thus $W = V\mathbf{s}(W) \subseteq VU$, and so $W = VU$. We conclude that $P'(\tilde{1}_V) = \tilde{1}_W = P(\tilde{1}_V)$, as required.

Now, we show that (Σ, i, q) satisfies the local bisection hypothesis. Suppose that n is a normaliser and that $\sigma \in \text{supp}(n)$. We show that if $\tau \in \Sigma_{\mathbf{s}(\sigma)} \setminus (R^\times \cdot \sigma)$, then $n(\tau) = 0$; that the same holds if $\tau \in \Sigma^{\mathbf{r}(\sigma)} \setminus (R^\times \cdot \sigma)$ follows by a similar argument. By choosing a continuous local section from G to Σ , we can find a compact open bisection $U \subseteq \Sigma$ containing σ such that $(r, \epsilon) \mapsto r \cdot \epsilon$ is a homeomorphism of $R^\times \times U$ onto $q^{-1}(q(U))$. Then we have $P(n\tilde{1}_{U^{-1}}) = (n\tilde{1}_{U^{-1}})|_{q^{-1}(G^{(0)})}$, and in particular, by [Corollary 2.10](#),

$$P(n\tilde{1}_{U^{-1}})(\mathbf{r}(\sigma)) = n(\sigma)\tilde{1}_{U^{-1}}(\sigma^{-1}) = n(\sigma) \neq 0.$$

Since $P(n\tilde{\mathbf{1}}_{U^{-1}}) = n\tilde{\mathbf{1}}_{U^{-1}}\tilde{\mathbf{1}}_V$ for some compact open subset $V \subseteq G^{(0)}$, and we deduce that $\mathbf{r}(\sigma) \in V$. Using again that $P(n\tilde{\mathbf{1}}_{U^{-1}}) = (n\tilde{\mathbf{1}}_{U^{-1}})|_{q^{-1}(G^{(0)})}$, we see that

$$n(\tau) = (n\tilde{\mathbf{1}}_{U^{-1}})(\tau\sigma^{-1}) = (n\tilde{\mathbf{1}}_{U^{-1}}\tilde{\mathbf{1}}_V)(\tau\sigma^{-1}) = P(n\tilde{\mathbf{1}}_{U^{-1}})(\tau\sigma^{-1}).$$

Since $\tau \in \Sigma_{\mathbf{s}(\sigma)} \setminus (R^\times \cdot \sigma)$, we have $\tau\sigma^{-1} \notin q^{-1}(G^{(0)})$, so we deduce that $n(\tau) = 0$.

Finally, suppose that (Σ, i, q) satisfies the local bisection hypothesis. We must show that (A, B) is an algebraic quasi-Cartan pair. Fix a normaliser $n \in N(B)$. Then $\text{supp}_G(n)$ is a bisection. Let $U := \text{supp}(n) \cap \Sigma^{(0)}$, and let $e := \tilde{\mathbf{1}}_U \in I(B)$. Then $ne = n|_{\mathbf{s}^{-1}(U)}$. If $n(\sigma) \neq 0$ and $\sigma \notin R^\times \cdot U = \text{supp}(n) \cap q^{-1}(G^{(0)})$, then $\mathbf{s}(\sigma) \notin U$ because $\text{supp}_G(n)$ is a bisection. It follows by a routine calculation that $ne = n|_{\mathbf{s}^{-1}(U)} = P(n)$, and similarly $en = P(n)$, as required. \square

Since twists satisfying the local bisection hypothesis give examples of algebraic quasi-Cartan pairs, we are interested in identifying conditions that guarantee the local bisection hypothesis. In the next few results, we follow the analysis used in [47] to give a sufficient condition in terms of twisted group algebras associated to the fibres of the interior of the isotropy in the twist. We begin by describing these algebras. This is standard, and our proof follows ideas from [4, Section 4.3].

Given a discrete R -twist (Σ, i, q) over a groupoid G , let \mathcal{I} denote the interior of the isotropy in G , and let $\mathcal{J} := q^{-1}(\mathcal{I})$, which is the interior of the isotropy in Σ . Notice that the image of i is contained in \mathcal{J} and we can restrict q to \mathcal{J} and obtain a map $q|_{\mathcal{J}}: \mathcal{J} \rightarrow \mathcal{I}$. It is straightforward to check that $(\mathcal{J}, i, q|_{\mathcal{J}})$ is again a twist and that $A_R(\mathcal{I}; \mathcal{J})$ can be seen as a subalgebra of $A_R(G; \Sigma)$ by extending an element of $A_R(\mathcal{I}; \mathcal{J})$ to be zero outside \mathcal{J} . Similarly, if we fix $x \in G^{(0)}$, then $\mathcal{I}_x = \mathcal{I}^x = \mathcal{I}_x^x$, and $\mathcal{J}_x = \mathcal{J}^x = \mathcal{J}_x^x$, and we can build a discrete R -twist $(\mathcal{J}_x, i_x, q_x)$ over \mathcal{I}_x , where $i_x = i|_{\{x\} \times R^\times}$ and $q_x = q|_{\mathcal{J}_x}$.

Lemma 4.3. *Let R be a commutative unital ring and let (Σ, i, q) be a discrete R -twist over G . Let \mathcal{I} denote the interior of the isotropy in G and let $\mathcal{J} := q^{-1}(\mathcal{I})$, which is the interior of the isotropy in Σ . Fix $x \in G^{(0)}$, and let $\zeta: \mathcal{I}_x \rightarrow \mathcal{J}_x$ be a section for $q|_{\mathcal{J}_x}$ such that $\zeta(x) = x$. Define $c_x: \mathcal{I}_x \times \mathcal{I}_x \rightarrow R^\times$ by $\zeta(\alpha)\zeta(\beta) = c_x(\alpha, \beta) \cdot \zeta(\alpha\beta)$. Then c_x is a normalised 2-cocycle on \mathcal{I}_x , and there is an isomorphism $\rho_\zeta: A_R(\mathcal{I}_x; \mathcal{J}_x) \rightarrow R(\mathcal{I}_x, c_x)$ of R -algebras such that $\rho_\zeta(f)(\beta) = f(\zeta(\beta))$ for all $f \in A_R(\mathcal{I}_x; \mathcal{J}_x)$ and $\beta \in \mathcal{I}_x$.*

Proof. It is routine to verify that each c_x is a normalised 2-cocycle using associativity of multiplication in Σ and that ζ preserves the identity. Certainly there is an R -linear map ρ_ζ satisfying the given formula. To see that ρ_ζ is injective, suppose that $\rho_\zeta(f) = 0$. Then for each $\sigma \in \mathcal{J}_x$, we have $\sigma = t \cdot \zeta(q(\sigma))$ for some $t \in R^\times$, and then $f(\sigma) = f(t \cdot \zeta(q(\sigma))) = t^{-1}f(\zeta(q(\sigma))) = t^{-1}\rho_\zeta(f)(q(\sigma)) = 0$. To see that ρ_ζ is surjective, fix $h \in R(\mathcal{I}_x, c_x)$. For each $\sigma \in \mathcal{J}_x$, there is a unique $t_\sigma \in R^\times$ such that $\sigma = t_\sigma \cdot \zeta(q(\sigma))$ and $t_\sigma = 1$ for $\sigma \in \zeta(\mathcal{I}_x)$. Define $f: \mathcal{J}_x \rightarrow R$ by $f(\sigma) = t_\sigma^{-1}h(q(\sigma))$ for all $\sigma \in \mathcal{J}_x$. Then $f \in A_R(\mathcal{I}_x; \mathcal{J}_x)$, and $\rho_\zeta(f) = h$.

It remains to check that ρ_ζ is multiplicative. Fix $f, g \in A_R(\mathcal{I}_x; \mathcal{J}_x)$ and $\beta \in \mathcal{I}_x$. Then

$$\rho_\zeta(f * g)(\beta) = (f * g)(\zeta(\beta)).$$

By the definition of convolution in $A_R(\mathcal{I}_x; \mathcal{J}_x)$, the convolution product $(f * g)(\zeta(\beta))$ can be computed by choosing any section from \mathcal{I}_x to \mathcal{J}_x , so we obtain

$$\rho_\zeta(f * g)(\beta) = \sum_{\alpha \in \mathcal{I}_x} f(\zeta(\alpha)) g(\zeta(\alpha)^{-1}\zeta(\beta)).$$

For each $\alpha \in \mathcal{I}_x$, we have $\zeta(\alpha)\zeta(\alpha^{-1}) = c_x(\alpha, \alpha^{-1})\zeta(\alpha\alpha^{-1}) = c_x(\alpha, \alpha^{-1}) \cdot x$, and rearranging this shows that $\zeta(\alpha)^{-1} = c_x(\alpha, \alpha^{-1})^{-1}\zeta(\alpha^{-1})$. Therefore, since $\zeta(\alpha^{-1})\zeta(\beta) =$

$c_x(\alpha^{-1}, \beta) \cdot \zeta(\alpha^{-1}\beta)$ for all $\alpha \in \mathcal{I}^x$, we have

$$\begin{aligned} \rho_\zeta(f * g)(\beta) &= \sum_{\alpha \in \mathcal{I}^x} c_x(\alpha, \alpha^{-1}) f(\zeta(\alpha)) g(\zeta(\alpha^{-1})\zeta(\beta)) \\ &= \sum_{\alpha \in \mathcal{I}^x} c_x(\alpha, \alpha^{-1}) c_x(\alpha^{-1}, \beta)^{-1} f(\zeta(\alpha)) g(\zeta(\alpha^{-1}\beta)). \end{aligned}$$

The 2-cocycle identity and that c_x is normalised give

$$c_x(\alpha^{-1}, \beta) c_x(\alpha, \alpha^{-1}\beta) = c_x(\alpha, \alpha^{-1}) c_x(\alpha\alpha^{-1}, \beta) = c_x(\alpha, \alpha^{-1}).$$

Thus, rearranging gives $c_x(\alpha, \alpha^{-1}) c_x(\alpha^{-1}, \beta)^{-1} = c_x(\alpha, \alpha^{-1}\beta)$. Hence

$$\begin{aligned} \rho_\zeta(f * g)(\beta) &= \sum_{\alpha \in \mathcal{I}^x} c_x(\alpha, \alpha^{-1}\beta) f(\zeta(\alpha)) g(\zeta(\alpha^{-1}\beta)) \\ &= \sum_{\alpha \in \mathcal{I}^x} c_x(\alpha, \alpha^{-1}\beta) \rho_\zeta(f)(\alpha) \rho_\zeta(g)(\alpha^{-1}\beta) \\ &= (\rho_\zeta(f) * \rho_\zeta(g))(\beta). \end{aligned} \quad \square$$

Changing the section ζ in [Lemma 4.3](#) results in a cohomologous normalised 2-cocycle and hence it does not change the algebra (up to isomorphism), nor does it affect whether or not the twisted group ring has only trivial units. In fact, a twisted group ring is graded by the underlying group, and the property of having only trivial units means that all the units are in the homogeneous component of the identity; the isomorphism of algebras corresponding to replacing a 2-cocycle by a cohomologous one does not change the grading.

Remark 4.4. Let R be an indecomposable commutative ring and let (Σ, i, q) be a discrete R -twist over G . The argument of [[47](#), Proposition 4.5] shows that if n is a normaliser of $A_R(G^{(0)}; q^{-1}(G^{(0)}))$ in $A_R(G; \Sigma)$, then $n^\dagger n \in A_R(G^{(0)}; q^{-1}(G^{(0)}))$ is equal to $\tilde{1}_{\mathfrak{s}(\text{supp}(n))}$ and $nn^\dagger = \tilde{1}_{\mathfrak{r}(\text{supp}(n))}$.

Proposition 4.5. *Let R be an indecomposable commutative ring and let (Σ, i, q) be a discrete R -twist over G . Let \mathcal{I} denote the interior of the isotropy in G and let $\mathcal{J} := q^{-1}(\mathcal{I})$, which is the interior of the isotropy in Σ . Let $x \in G^{(0)}$, let \mathcal{I}_x and c_x be as in [Lemma 4.3](#), and suppose that the twisted group ring $R(\mathcal{I}_x, c_x)$ has only trivial units. Then for any normaliser n of $A_R(G^{(0)}; q^{-1}(G^{(0)}))$ in $A_R(\mathcal{I}; \mathcal{J})$, we have $|\text{supp}_{\mathcal{I}}(n) \cap \mathcal{I}_x| \leq 1$.*

Proof. Suppose that $n|_{\mathcal{J}_x} \neq 0$. We must show that $n|_{\mathcal{J}_x}$ is supported on $q^{-1}(\mu)$ for some $\mu \in \mathcal{I}$. Fix $\alpha \in \mathcal{J}_x$ with $n(\alpha) \neq 0$. [Remark 4.4](#) gives $n^\dagger n = 1_{\mathfrak{s}(\text{supp}(n))} = 1_{\mathfrak{r}(\text{supp}(n))} = nn^\dagger$. So we obtain $(n^\dagger n)(\mathfrak{s}(\alpha)) = 1 = (nn^\dagger)(\mathfrak{s}(\alpha))$. Fix a section $\zeta: \mathcal{I}_x \rightarrow \mathcal{J}_x$ and let $\rho_\zeta: A_R(\mathcal{I}_x; \mathcal{J}_x) \rightarrow R(\mathcal{I}_x, c_x)$ be the isomorphism of [Lemma 4.3](#). Since $\mathcal{J} = q^{-1}(\mathcal{I})$ consists of isotropy, the convolution product $n^\dagger n$ evaluated at x is the same as $n^\dagger|_{\mathcal{J}_x} * n|_{\mathcal{J}_x}$ at x (and similarly for nn^\dagger), and we obtain $n^\dagger|_{\mathcal{J}_x} * n|_{\mathcal{J}_x} = 1_{A_R(\mathcal{I}_x; \mathcal{J}_x)} = n|_{\mathcal{J}_x} * n^\dagger|_{\mathcal{J}_x}$. It follows that $\rho_\zeta(n|_{\mathcal{J}_x})$ is a unit, and therefore it has the form $t\delta_\mu$ for some $\mu \in \mathcal{I}_x$. So $n|_{\mathcal{J}_x} = t\rho_\zeta^{-1}(\delta_\mu)$ is supported on $R^\times \cdot \zeta(\mu) = q^{-1}(\mu)$, as required. \square

The following lemma can be viewed as a far-reaching extension of [Example 3.9](#).

Lemma 4.6. *Let R be an indecomposable commutative ring and let (Σ, i, q) be a discrete R -twist over G . Let \mathcal{I} denote the interior of the isotropy in G and $\mathcal{J} := q^{-1}(\mathcal{I})$, which is the interior of the isotropy in Σ . For each $x \in G^{(0)}$, let \mathcal{I}_x and c_x be as in [Lemma 4.3](#). Suppose that the set of units $x \in G^{(0)}$ for which $R(\mathcal{I}_x, c_x)^\times = \{t\delta_\alpha : t \in R^\times \text{ and } \alpha \in \mathcal{I}_x\}$ is dense in $G^{(0)}$. Then $(\mathcal{J}, i, q|_{\mathcal{J}})$ satisfies the local bisection hypothesis.*

Proof. Suppose that n is a normaliser of $A_R(G^{(0)}; q^{-1}(G^{(0)}))$ in $A_R(\mathcal{I}; \mathcal{J})$. We must show that $\text{supp}_{\mathcal{I}}(n)$ is a bisection. Suppose for contradiction that this is not the case. Then we can find $\alpha, \beta \in \text{supp}_{\mathcal{I}}(n)$ with $\mathbf{s}(\alpha) = \mathbf{s}(\beta)$ but $\alpha \neq \beta$. Since \mathcal{I} is Hausdorff and $\text{supp}_{\mathcal{I}}(n)$ is open, we can find disjoint compact open bisections U and V of \mathcal{I} with $\alpha \in U$, $\beta \in V$, and $U, V \subseteq \text{supp}_{\mathcal{I}}(n)$. By assumption, $\mathbf{s}(U) \cap \mathbf{s}(V) \neq \emptyset$ as $\mathbf{s}(\alpha) = \mathbf{s}(\beta)$ is in the intersection. Since $\mathbf{s}(U) \cap \mathbf{s}(V)$ is an open subset of $G^{(0)}$, there exists $x \in \mathbf{s}(U) \cap \mathbf{s}(V)$ such that the twisted group ring $R(\mathcal{I}_x, c_x)$ has only trivial units. Take $\alpha' \in U$ and $\beta' \in V$ with $\mathbf{s}(\alpha') = x = \mathbf{s}(\beta')$. Then $\alpha' = \beta'$ by [Proposition 4.5](#), contradicting that U and V are disjoint. \square

The next lemma shows that the local bisection hypothesis can be checked on the interior of the isotropy.

Lemma 4.7. *Let R be an indecomposable commutative ring and let (Σ, i, q) be a discrete R -twist over G . Let \mathcal{I} denote the interior of the isotropy in G and let $\mathcal{J} := q^{-1}(\mathcal{I})$, which is the interior of the isotropy in Σ . Let $A := A_R(G; \Sigma)$, let $A' := A_R(\mathcal{I}; \mathcal{J})$, and let $B := A_R(G^{(0)}; q^{-1}(G^{(0)}))$. Then*

(a) *the normaliser $N_{A'}(B)$ of B in A' satisfies*

$$N_{A'}(B) = N_A(B) \cap A'; \text{ and}$$

(b) *(Σ, i, q) satisfies the local bisection hypothesis if and only if $(\mathcal{J}, i, q|_{\mathcal{J}})$ does.*

Proof. The proof very closely follows the arguments of [[47](#), Proposition 4.7, Corollary 4.8, and Proposition 4.10], with just minor adjustments to incorporate the generality of twists.

Claim 1 (cf. [[47](#), Proposition 4.7]): if $n \in N_A(B)$ and $\sigma, \tau \in \text{supp}(n)$, then $\mathbf{s}(\sigma) = \mathbf{s}(\tau)$ if and only if $\mathbf{r}(\sigma) = \mathbf{r}(\tau)$. We suppose that $x = \mathbf{s}(\sigma) = \mathbf{s}(\tau)$ and prove that $\mathbf{r}(\sigma) = \mathbf{r}(\tau)$; the converse follows from a symmetric argument. Suppose for contradiction that $y := \mathbf{r}(\sigma) \neq \mathbf{r}(\tau) =: z$. Fix disjoint compact open neighbourhoods $y \in U$ and $z \in V$ in $\Sigma^{(0)}$, and let $p := \tilde{1}_U$ and $q := \tilde{1}_V$. Then $pn, qn \in N_A(B)$.

[Remark 4.4](#) gives $(n^\dagger pn)(x) = 1$ because $(pn)(\sigma) \neq 0$ and $(pn)^\dagger(pn) = n^\dagger pn$. Similarly, $(n^\dagger qn)(x) = 1$ because $(qn)(\tau) \neq 0$. Since $p + q = \tilde{1}_{U \cup V} \in I(B)$ by the disjointness of U and V , we have by the same reasoning that $(n^\dagger(p + q)n)(x) = 1$ because $((p + q)n)(\sigma) \neq 0$. Hence

$$2 = (n^\dagger pn)(x) + (n^\dagger qn)(x) = (n^\dagger(p + q)n)(x) = 1,$$

which is a contradiction.

Claim 2 (cf. [[47](#), Corollary 4.8]): if $n \in N_A(B)$, then $\text{supp}(n)\text{supp}(n)^{-1}$ and $\text{supp}(n)^{-1}\text{supp}(n)$ are contained in \mathcal{J} . For this, suppose that $\sigma \in \text{supp}(n)$ and $\tau \in \text{supp}(n)^{-1}$ and $\mathbf{s}(\sigma) = \mathbf{r}(\tau)$. Then $\tau^{-1} \in \text{supp}(n)$ and $\mathbf{s}(\sigma) = \mathbf{s}(\tau^{-1})$, so Claim 1 shows that $\mathbf{r}(\sigma) = \mathbf{r}(\tau^{-1})$. So $\sigma\tau$ is in the isotropy of Σ . It follows that the open set $\text{supp}(n)\text{supp}(n)^{-1}$ is contained in the isotropy, and hence $\text{supp}(n)\text{supp}(n)^{-1} \subseteq \mathcal{J}$. A symmetric argument shows that $\text{supp}(n)^{-1}\text{supp}(n) \subseteq \mathcal{J}$.

Now for part (a), note that $N_{A'}(B)$ is clearly contained in $N_A(B) \cap A'$. For the reverse containment, it suffices to show that if $n \in N_A(B) \cap A'$, then $\text{supp}(n^\dagger) \subseteq \mathcal{J}$. So suppose that $\sigma \in \text{supp}(n^\dagger)$. Then [Remark 4.4](#) shows that $(nn^\dagger)(\mathbf{s}(\sigma)) = 1$. So any section $\zeta: G_{\mathbf{s}(\sigma)} \rightarrow \Sigma_{\mathbf{s}(\sigma)}$ satisfies

$$1 = (nn^\dagger)(\mathbf{s}(\sigma)) = \sum_{\alpha \in G_{\mathbf{s}(\sigma)}} n(\zeta(\alpha)^{-1}) n^\dagger(\zeta(\alpha)).$$

Since the sum is nonzero, at least one term, say corresponding to $\alpha_0 \in G_{\mathbf{s}(\sigma)}$, satisfies $\zeta(\alpha_0)^{-1} \in \text{supp}(n)$ and $\zeta(\alpha_0) \in \text{supp}(n^\dagger)$. Applying Claim 1 to n^\dagger , we see that $\mathbf{r}(\alpha_0) = \mathbf{r}(\sigma)$. Since $\text{supp}(n) \subseteq \mathcal{J}$ by hypothesis on n , we have $\alpha_0^{-1} \in \mathcal{I}$, and so we deduce that $\mathbf{s}(\sigma) = \mathbf{s}(\alpha_0) = \mathbf{r}(\alpha_0) = \mathbf{r}(\sigma)$, as required.

For part (b), the “only if” implication is clear, so we suppose that $(\mathcal{J}, i, q|_{\mathcal{J}})$ satisfies the local bisection hypothesis, and establish that (Σ, i, q) does too. Fix $n \in N_A(B)$. Suppose that $\sigma, \tau \in \text{supp}(n)$ satisfy $\mathbf{s}(\sigma) = \mathbf{s}(\tau)$. Then $\mathbf{r}(\sigma) = \mathbf{r}(\tau)$ by Claim 1. We must show that $q(\sigma) = q(\tau)$. Since n is locally constant, there is a compact open bisection U containing σ such that $n|_U$ is constant. Claim 2 implies that $U^{-1} \text{supp}(n) \subseteq \mathcal{J}$. Let $m := \tilde{1}_{U^{-1}}$. Then mn belongs to $N_A(B) \cap A'$, and so part (a) shows that $mn \in N_{A'}(B)$. Since $(\mathcal{J}, i, q|_{\mathcal{J}})$ satisfies the local bisection hypothesis, we deduce that $\text{supp}_G(mn) = \text{supp}_{\mathcal{I}}(mn)$ is a bisection. Let $x = \mathbf{s}(\sigma)$. We claim that $x, \sigma^{-1}\tau \in \text{supp}(mn)$. To see this, fix a section $S: \text{supp}_G(m) = q(U^{-1}) \rightarrow \text{supp}(m) = R^\times \cdot U^{-1}$ with $S(q(\sigma)) = \sigma$. Since U is a bisection, we have

$$(mn)(x) = \sum_{\gamma \in G^x} m(S(\gamma)) n(S(\gamma)^{-1}) = n(S(q(\sigma))) = n(\sigma) \neq 0,$$

and

$$(mn)(\sigma^{-1}\tau) = \sum_{\gamma \in G^x} m(S(\gamma)) n(S(\gamma)^{-1}\sigma^{-1}\tau) = n(\tau) \neq 0,$$

as claimed. Since $q(\text{supp}(mn))$ is a bisection and $\mathbf{s}(x) = x = \mathbf{s}(\sigma^{-1}\tau)$, we deduce that $q(\sigma^{-1}\tau) = q(x)$, and hence $q(\sigma) = q(\tau)$. \square

We have seen that twists satisfying the local bisection hypothesis give rise to algebraic quasi-Cartan pairs. To finish the section we show that if the underlying groupoid G is effective, then we obtain an algebraic Cartan pair, and if it is principal, then we obtain an algebraic diagonal pair.

Proposition 4.8. *Let R be an indecomposable commutative ring and let (Σ, i, q) be a discrete R -twist over G . If G is effective then $(A, B) := (A_R(G; \Sigma), A_R(G^{(0)}; q^{-1}(G^{(0)})))$ is an algebraic Cartan pair, and if G is principal then (A, B) is an algebraic diagonal pair.*

Proof. If G is effective, then Lemma 4.7(b) implies that the local bisection hypothesis holds. Then by Lemma 4.2, (A, B) is an algebraic quasi-Cartan pair with conditional expectation $P: A \rightarrow B$ given by restriction of functions from Σ to $q^{-1}(G^{(0)})$. So, we just have to show that if G is effective, then (A, B) satisfies (ACP) and that if G is principal then (A, B) satisfies (ADP).

First suppose that G is effective. Then for $f \in A \setminus B$, we must show that f does not commute with B . Since $f \notin B$, there exists $\sigma \in \Sigma \setminus q^{-1}(G^{(0)})$ such that $f(\sigma) \neq 0$. Then $U := f^{-1}(f(\sigma))$ is open. Since G is effective, $q(U)$ is open, and $q(\sigma) \notin G^{(0)}$, [9, Lemma 3.1(3)] implies that there exists $\tau \in U$ such that $\mathbf{r}(\tau) \neq \mathbf{s}(\tau)$. Fix a compact open set $V \subseteq G^{(0)}$ that contains $\mathbf{s}(\tau)$ but not $\mathbf{r}(\tau)$, and let $e_V := \tilde{1}_V \in B$. Then $(fe_V)(\tau) = f(\tau) = f(\sigma) \neq 0 = (e_V f)(\tau)$. So $fe_V \neq e_V f$, and hence f does not commute with B , as required.

Now suppose that G is principal. Fix a compact open bisection $U \subseteq \Sigma$. By Proposition 2.8, it suffices to show that $\tilde{1}_U$ is a linear combination of free normalisers. Let $U_0 := U \cap q^{-1}(G^{(0)})$ and $V := U \setminus U_0$. Both U_0 and V are compact open bisections, and we have $\tilde{1}_U = \tilde{1}_{U_0} + \tilde{1}_V$. Since $\tilde{1}_{U_0} \in B$ is a free normaliser, we just have to show that $\tilde{1}_V$ is a linear combination of free normalisers. Since $q(V) \cap G^{(0)}$ is empty and G is principal, we have $\mathbf{r}(\sigma) \neq \mathbf{s}(\sigma)$ for all $\sigma \in V$. Thus, for each $\sigma \in V$, we can find a compact open bisection $V_\sigma \subseteq V$ containing σ with $V_\sigma^2 = \emptyset$ by choosing disjoint compact open neighbourhoods $U, U' \subseteq G^{(0)}$ of $\mathbf{r}(\sigma)$ and $\mathbf{s}(\sigma)$, respectively, and putting $V_\sigma := UVU' \subseteq V$. By compactness, we can cover V by finitely many compact open bisections V_1, \dots, V_n such that $V_i^2 = \emptyset$ for $i \in \{1, \dots, n\}$. Then by putting $V'_i = V_i \setminus \bigcup_{j=1}^{i-1} V'_j$ for each $i \in \{1, \dots, n\}$, we obtain a refined cover of V consisting of mutually disjoint sets. Now each $\tilde{1}_{V'_i}$ is a free normaliser, and $\tilde{1}_V = \sum_{i=1}^n \tilde{1}_{V'_i}$. \square

5. BUILDING A TWIST FROM A PAIR OF ALGEBRAS

Throughout this section we assume that A is an R -algebra with B an abelian subalgebra without torsion in the sense of [condition \(WT\)](#) (that is, for $e \in I(B)$ and $t \in R$, if $te = 0$, then $t = 0$ or $e = 0$), and satisfying property [\(i\)](#) of [Definition 3.3](#) (that is, $I(B)$ is a set of local units for A).

5.1. The groupoid Σ . In this section we use the partial order \leq on $N(B)$ defined in [Section 2.5](#) (on [page 14](#)) to define the groupoid of ultrafilters on $N(B)$, which will play the same role as Renault's Weyl twist in [\[43\]](#).

If $U \subseteq N(B)$, then the *upclosure* of U is the set

$$U^\uparrow := \{m \in N(B) : \text{there exists } n \in U \text{ with } n \leq m\}.$$

A *filter* of $N(B)$ is a subset $U \subseteq N(B) \setminus \{0\}$ such that $U = U^\uparrow$ and whenever $m, n \in U$ there exists $p \in U$ such that $p \leq m, n$. An *ultrafilter* is a maximal filter. The collection Σ of all ultrafilters of $N(B)$ forms a groupoid with the following structure (see [\[28, Proposition 9.2.1\]](#) and [\[29, Proposition 2.13\]](#)). For each $U \in \Sigma$,

$$U^{-1} := \{n^\dagger : n \in U\}.$$

A pair (U, V) of ultrafilters of $N(B)$ is composable if and only if $m^\dagger m n n^\dagger \neq 0$ (or, equivalently, if and only if $m n \neq 0$) for all $m \in U$ and $n \in V$, and then

$$UV := \{mn : m \in U, n \in V\}^\uparrow.$$

The definition of an ultrafilter ensures that a pair (U, V) of ultrafilters is composable if and only if $\mathbf{s}(U) := U^{-1}U$ is equal to $\mathbf{r}(V) := VV^{-1}$. It follows that

$$\Sigma^{(0)} = \{U \in \Sigma : U \cap I(B) \neq \emptyset\}.$$

For $n \in N(B)$, we write

$$\mathcal{V}_n := \{U \in \Sigma : n \in U\}.$$

The collection $\{\mathcal{V}_n : n \in N(B)\}$ forms a basis of open bisections for a topology on Σ , making it an étale groupoid. We have $\mathcal{V}_n \mathcal{V}_m = \mathcal{V}_{nm}$ and $\mathcal{V}_n^{-1} = \mathcal{V}_{n^\dagger}$ for all $m, n \in N(B)$. For more details, see [\[5, Lemma 3.2\]](#). By [\[30, Lemma 2.22\]](#) (see also [\[5, Propositions 2.2 and 4.4\]](#)), $\Sigma^{(0)}$ is a Hausdorff subspace of Σ . In [Lemma 5.3](#), we show how scalar multiplication in the algebra interacts with the groupoid structure.

Remark 5.1. Note that [\[5, Proposition 2.2 and Proposition 4.4\]](#) say that $U \mapsto U \cap I(B)$ is a homeomorphism from the set of ultrafilters of $N(B)$ that contain an element of $I(B)$ to the set of ultrafilters of $I(B)$, and so we often identify elements of $\Sigma^{(0)}$ with ultrafilters of $I(B)$ when convenient. In particular, since $I(B)$ is a Boolean algebra, Stone duality implies that \mathcal{V}_e is compact for each $e \in I(B)$.

We will frequently use the following standard fact about ultrafilters. This fact, and much of what we say about the groupoid Σ as an ample groupoid, would immediately follow from the results of [\[31\]](#) if one proved that $N(B)$ is a Boolean inverse semigroup (called a weakly Boolean inverse semigroup in [\[31\]](#)). However, to prove this would go too far afield.

Lemma 5.2. *Suppose that A is an R -algebra, and that $B \subseteq A$ is an abelian subalgebra satisfying [condition \(WT\)](#) and [\(i\)](#) of [Definition 3.3](#). Let Σ be the groupoid of ultrafilters of $N(B)$. Suppose that $n \in U \in \Sigma$ and that $e_1, \dots, e_k \in I(B)$ are mutually orthogonal idempotents satisfying $n = \sum_{i=1}^k n e_i$. Then there exists a unique $i \in \{1, \dots, k\}$ such that $n e_i \in U$. Dually, if $n = \sum_{i=1}^k f_i n$ for mutually orthogonal idempotents $f_1, \dots, f_k \in I(B)$, then $f_i n \in U$ for a unique $i \in \{1, \dots, k\}$.*

Proof. Without loss of generality, we may assume that $n^\dagger ne_i = e_i$ for each $i \in \{1, \dots, k\}$, and hence $n^\dagger n = \sum_{i=1}^k e_i$. Since $n^\dagger n \in U^{-1}U \cap I(B)$, which is an ultrafilter of the Boolean algebra $I(B)$ by [Remark 5.1](#), it follows that there is a unique $i \in \{1, \dots, k\}$ with $e_i \in U^{-1}U \cap I(B)$ (since the characteristic function of an ultrafilter on a Boolean algebra is a Boolean algebra homomorphism). Then $ne_i \in U(U^{-1}U) = U$. Moreover, if $ne_j \in U$ for some $j \in \{1, \dots, k\}$, then $e_j = (ne_j)^\dagger ne_j \in U^{-1}U \cap I(B)$, and so $j = i$. This proves the first statement.

For the dual result, note that $n = \sum_{i=1}^k f_i n = \sum_{i=1}^k n(n^\dagger f_i n)$, and that the $n^\dagger f_i n$ are mutually orthogonal idempotents in $I(B)$. Therefore, $f_i n = n(n^\dagger f_i n) \in U$ for a unique $i \in \{1, \dots, k\}$ by the previous paragraph. \square

Lemma 5.3. *Suppose that A is an R -algebra, and that $B \subseteq A$ is an abelian subalgebra satisfying [condition \(WT\)](#) and satisfying property (i) of [Definition 3.3](#). Let Σ be the groupoid of ultrafilters of $N(B)$. For $t, s \in R^\times$ and $n \in N(B)$, we have the following.*

- (a) If $U \in \Sigma$, then $tU := \{tm : m \in U\} \in \Sigma$.
- (b) If $U \in \Sigma$, then $(tU)^{-1} = t^{-1}U^{-1}$.
- (c) If $(U, W) \in \Sigma^{(2)}$, then $(tU, sW) \in \Sigma^{(2)}$ and $(tU)(sW) = (ts)(UW)$.
- (d) If $U \in \Sigma$, then $\mathbf{s}(U) = \mathbf{s}(tU)$ and $\mathbf{r}(U) = \mathbf{r}(tU)$.
- (e) $t\mathcal{V}_n := \{tU : U \in \mathcal{V}_n\} = \mathcal{V}_{tn}$.
- (f) If $t \neq 1$, then $\mathcal{V}_{tn} \cap \mathcal{V}_n = \emptyset$.
- (g) If $U, W \in \Sigma^{(0)}$ with $tU = sW$, then $t = s$ and $U = W$.

Proof. For part (a), notice that $tU \subseteq N(B)$ by [Lemma 2.16\(a\)](#). We first show that tU is a filter. Fix $tm, tn \in tU$. Since U is a filter, there exists $k \in U$ such that $k \leq m$ and $k \leq n$. Then $tk \in tU$, and [Lemma 2.16\(b\)](#) implies that $tk \leq tm$ and $tk \leq tn$. Next suppose that $p \in N(B)$ and that there exists $tm \in tU$ with $tm \leq p$. Then $m \leq t^{-1}p$ by [Lemma 2.16\(b\)](#) again. Again using that U is a filter, we have $t^{-1}p \in U$, and hence $p = tt^{-1}p \in tU$. Also, $0 \notin tU$, for otherwise $0 = t^{-1}0 \in U$. Hence tU is a filter.

To see that tU is an ultrafilter, suppose that $tU \subseteq W$, where W is a filter. Then $U \subseteq t^{-1}W$. Since U is an ultrafilter and $t^{-1}W$ is a filter by our previous argument, $U = t^{-1}W$, and hence $tU = W$.

For part (b), [Lemma 2.16\(a\)](#) implies that

$$(tU)^{-1} = \{(tm)^\dagger : m \in U\} = \{t^{-1}m^\dagger : m \in U\} = t^{-1}U^{-1}.$$

For part (c), fix $(U, W) \in \Sigma^{(2)}$. We begin by verifying that tU and sW are composable. Suppose that $m \in tU$ and $n \in sW$. Then $m = tm'$ with $m' \in U$ and $n = sn'$ with $n' \in W$. Since $(U, W) \in \Sigma^{(2)}$, we have that $m'n' \neq 0$. Therefore, $mn = (ts)m'n' \neq 0$ since $ts \in R^\times$. Thus $(tU, sW) \in \Sigma^{(2)}$. Since $(ts)(UW)$ is an ultrafilter by part (a), it suffices to show that $(ts)(UW) \subseteq (tU)(sW)$. Fix $k \in (ts)(UW)$. Then there exists $m \in UW$ such that $k = (ts)m$. Therefore, there exist $n \in U$ and $p \in W$ such that $np \leq m$, and so $(tn)(sp) = (ts)(np) \leq (ts)m = k$. Hence $k \in (tU)(sW)$, and so $(tU)(sW) = (ts)(UW)$.

Item (d) follows easily from parts (b) and (c).

For part (e), the containment $t\mathcal{V}_n \subseteq \mathcal{V}_{tn}$ follows from part (a). For the reverse containment, fix $W \in \mathcal{V}_{tn}$. Then $tn \in W \in \Sigma$, and hence $n \in t^{-1}W \in \Sigma$ by part (a). So $t^{-1}W \in \mathcal{V}_n$, and $W = t(t^{-1}W) \in t\mathcal{V}_n$.

For part (f), we prove the contrapositive. For this, suppose that $U \in \mathcal{V}_{tn} \cap \mathcal{V}_n$. Then $n \neq 0$ and $tn, n \in U$, and so there exists $m \in U$ such that $0 \neq m \leq tn, n$. Thus $0 \neq mm^\dagger \leq tnn^\dagger$, and so $t = 1$ by [Lemma 2.17\(b\)](#).

For part (g), suppose that $U, W \in \Sigma^{(0)}$ satisfy $tU = sW$. Then $U = (t^{-1}s)W$, and so it suffices to show that $t^{-1}s = 1$. For this, fix $e_1, e_2 \in I(B)$ with $e_1 \in U$ and $e_2 \in W$.

Then $t^{-1}se_2 \in U$, and so there exists $f \in U$ such that $0 \neq f \leq e_1, t^{-1}se_2$. Now by [Lemma 2.17\(a\)](#), we have $f \in I(B)$, and so [Lemma 2.17\(b\)](#) implies that $t^{-1}s = 1$. \square

5.2. The groupoid G . In this section we assume that A is an R -algebra, and that B is an abelian subalgebra satisfying [condition \(WT\)](#) and satisfying properties [\(i\)](#) and [\(ii\)](#) of [Definition 3.3](#).

We begin by defining a relation \simeq on Σ , the groupoid of ultrafilters of the inverse semigroup $N(B)$, by

$$U \simeq W \iff \text{there exists } t \in R^\times \text{ such that } U = tW. \quad (5.1)$$

Then \simeq is an equivalence relation. We define G to be the quotient Σ/\simeq endowed with the quotient topology, and we denote the corresponding quotient map by $q: \Sigma \rightarrow G$. In the following lemma we show that G is a groupoid that inherits its structure from Σ .

Lemma 5.4. *Suppose that A is an R -algebra, and that $B \subseteq A$ is an abelian subalgebra satisfying [condition \(WT\)](#) and satisfying properties [\(i\)](#) and [\(ii\)](#) of [Definition 3.3](#). Let Σ be the groupoid of ultrafilters of $N(B)$, and let $q: \Sigma \rightarrow G$ be the quotient map defined above.*

- (a) *The quotient $G = \{q(U) : U \in \Sigma\}$ is a groupoid with inversion given by $q(U)^{-1} := q(U^{-1})$, composable pairs $G^{(2)} := \{(q(U), q(W)) : (U, W) \in \Sigma^{(2)}\}$, and composition given by $q(U)q(W) := q(UW)$ for all $(U, W) \in \Sigma^{(2)}$. We have $\mathbf{s}(q(U)) = q(\mathbf{s}(U))$ and $\mathbf{r}(q(U)) = q(\mathbf{r}(U))$ for all $U \in \Sigma$, and so $G^{(0)} = q(\Sigma^{(0)})$.*
- (b) *The quotient map $q: \Sigma \rightarrow G$ is a groupoid homomorphism.*
- (c) *The quotient map restricts to a bijection $q|_{\Sigma^{(0)}}: \Sigma^{(0)} \rightarrow G^{(0)}$.*

Proof. For part [\(a\)](#), note that [Lemma 5.3\(b\)](#) implies that the inverse is well-defined. Composability and the product are well-defined by [Lemma 5.3\(c\)](#). The remainder of the groupoid structure comes from the groupoid structure of Σ , and part [\(b\)](#) follows as well. For part [\(c\)](#), first note that $q(\Sigma^{(0)}) = G^{(0)}$ by part [\(a\)](#). Injectivity of $q|_{\Sigma^{(0)}}$ follows from [Lemma 5.3\(g\)](#) with $t = 1$. \square

Lemma 5.5. *Suppose that A is an R -algebra, and that $B \subseteq A$ is an abelian subalgebra satisfying [condition \(WT\)](#) and satisfying properties [\(i\)](#) and [\(ii\)](#) of [Definition 3.3](#). Let Σ be the groupoid of ultrafilters of $N(B)$, and let $q: \Sigma \rightarrow G$ be the quotient map defined above.*

- (a) *The collection $\{\mathcal{V}_n : n \in N(B)\}$ forms a basis of compact open bisections for the topology on Σ . In particular, Σ is an ample groupoid.*
- (b) *The quotient map is open and restricts to a homeomorphism of unit spaces.*
- (c) *The collection $\{q(\mathcal{V}_n) : n \in N(B)\}$ forms a basis of compact open bisections for the quotient topology on G . In particular, G is an ample groupoid.*

Proof. For part [\(a\)](#), to see that Σ is ample, we show that \mathcal{V}_n is compact for each $n \in N(B)$. Fix $n \in N(B)$. Then \mathcal{V}_n is homeomorphic to $\mathbf{r}(\mathcal{V}_n)$ because \mathcal{V}_n is a bisection. Now

$$\mathbf{r}(\mathcal{V}_n) = \mathcal{V}_n \mathcal{V}_n^\dagger = \mathcal{V}_{nn^\dagger},$$

which is compact by [Remark 5.1](#).

For part [\(b\)](#), since G is endowed with the quotient topology by definition, to see that q is open, it suffices to show that $q^{-1}(q(\mathcal{V}_n))$ is open in Σ for any $n \in N(B)$. Using [Lemma 5.3\(e\)](#) for the last equality, we calculate

$$q^{-1}(q(\mathcal{V}_n)) = \{tU : t \in R^\times \text{ and } U \in \mathcal{V}_n\} = \bigcup_{t \in R^\times} t\mathcal{V}_n = \bigcup_{t \in R^\times} \mathcal{V}_{tn},$$

which is open in Σ . That q restricts to a homeomorphism of unit spaces follows from [Lemma 5.4\(c\)](#).

For part (c), compactness of each $q(\mathcal{V}_n)$ follows from part (a), because q is continuous. Since $\{\mathcal{V}_n : n \in N(B)\}$ is a basis for Σ and q is surjective, the collection $\{q(\mathcal{V}_n) : n \in N(B)\}$ covers G . Part (b) shows that each $q(\mathcal{V}_n)$ is open in G . So to see that $\{q(\mathcal{V}_n) : n \in N(B)\}$ is a basis for the quotient topology, it suffices to show that for each open subset O of G and each element $q(U) \in O$, there exists $n \in N(B)$ such that

$$q(U) \in q(\mathcal{V}_n) \subseteq O.$$

Since O is open in the quotient topology on G , $q^{-1}(O)$ is open in Σ . Since $U \in q^{-1}(O)$ and $\{\mathcal{V}_n : n \in N(B)\}$ is a basis for Σ , there exists $n \in N(B)$ such that $q(U) \in q(\mathcal{V}_n) \subseteq O$, and so $\{q(\mathcal{V}_n) : n \in N(B)\}$ is a basis. To see that the sets $q(\mathcal{V}_n)$ are open bisections, we need to show that the source and range maps are injective on each $q(\mathcal{V}_n)$. But this follows from parts (a) and (c) of [Lemma 5.4](#), since \mathcal{V}_n is an open bisection. Now [[8](#), Proposition 6.6] implies that G is a topological groupoid, and hence G is ample. \square

5.3. The twist of an algebraic quasi-Cartan pair. The main theorem of this section is that if (A, B) is an algebraic quasi-Cartan pair, then (Σ, i, q) is a discrete R -twist.

Theorem 5.6. *Suppose that A is an R -algebra, and that $B \subseteq A$ is an abelian subalgebra satisfying [condition \(WT\)](#) and satisfying properties (i) and (ii) of [Definition 3.3](#). Let Σ be the groupoid of ultrafilters of $N(B)$, let G be the quotient of Σ by the equivalence relation given in [equation \(5.1\)](#), and let $q: \Sigma \rightarrow G$ be the quotient map. Define $i: G^{(0)} \times R^\times \rightarrow \Sigma$ by $i(q(U), t) = tU$ for $U \in \Sigma^{(0)}$ and $t \in R^\times$. Then the sequence*

$$G^{(0)} \times R^\times \xrightarrow{i} \Sigma \xrightarrow{q} G$$

satisfies all of the properties of a discrete R -twist except, perhaps, that G is Hausdorff. If (A, B) is an algebraic quasi-Cartan pair, then the sequence is a discrete R -twist.

Since B is without torsion as in [condition \(WT\)](#), [[24](#), Théorème 1] shows that there is an isomorphism $\phi: B \rightarrow A_R(\Sigma^{(0)})$ that satisfies

$$\phi(e) = 1_{\mathcal{V}_e} \text{ for all } e \in I(B). \quad (5.2)$$

To start the proof of [Theorem 5.6](#), we show that if (A, B) is an algebraic quasi-Cartan pair then G is Hausdorff (see [Proposition 5.10](#)). For this, we need the following lemma.

Lemma 5.7. *Suppose that (A, B) is an algebraic quasi-Cartan pair. If $n \in N(B) \cap B$, then $n^\dagger \in B$, and $\phi(n)(U) \in R^\times \cup \{0\}$ for all $U \in \Sigma^{(0)}$.*

Proof. Suppose that $n \in N(B) \cap B$. Then

$$n^\dagger = n^\dagger n n^\dagger = P(n^\dagger n) n^\dagger = P(n^\dagger) n n^\dagger = P(n^\dagger n n^\dagger) = P(n^\dagger) \in B,$$

proving the first statement. Since $n^\dagger n$ is an idempotent element of B and is a right inverse for n , and since R is indecomposable, we have $\phi(n^\dagger n) = 1_W$ for some compact open set W containing the support of $\phi(n)$. Since ϕ is multiplicative, we deduce that $\phi(n^\dagger)(U) \phi(n)(U) = 1$ for all $U \in \text{supp}(\phi(n))$, and so each $\phi(n^\dagger)(U)$ is an inverse for $\phi(n)(U)$ (because R is commutative). \square

The following consequence of [Lemma 5.7](#) is used frequently, often without comment.

Corollary 5.8. *Suppose that (A, B) is an algebraic quasi-Cartan pair. Let $n \in N(B) \cap B$, and suppose that $n = \sum_{i=1}^k t_i e_i$, where e_1, \dots, e_k are mutually orthogonal idempotents and $t_1, \dots, t_k \in R \setminus \{0\}$. Then $t_i \in R^\times$ for each $i \in \{1, \dots, k\}$.*

Proof. Choose an ultrafilter $W \in \mathcal{V}_{e_i}$. Then $e_j \in W$ if and only if $j = i$, and so $\phi(n)(W) = t_i \in R^\times$ by [Lemma 5.7](#). \square

Remark 5.9. If (A, B) is an algebraic quasi-Cartan pair, then since $B = \text{span}(I(B))$, every $n \in B$ can be expressed in the form $n = \sum_{i=1}^k t_i e_i$, where $e_1, \dots, e_k \in I(B)$ are mutually orthogonal idempotents and $t_1, \dots, t_k \in R$. It follows (using [Corollary 5.8](#)) that $n \in N(B) \cap B$ if and only if $t_1, \dots, t_k \in R^\times \cup \{0\}$. Thus

$$N(B) \cap B = \{b \in B : \phi(b) \in R^\times \cup \{0\}\}.$$

Proposition 5.10. *Suppose that (A, B) is an algebraic quasi-Cartan pair. Then G is Hausdorff.*

Proof. We know that G is ample by [Lemma 5.5\(c\)](#), and an ample groupoid is Hausdorff if and only if its unit space is closed. So we prove that $G^{(0)}$ is closed. Since q is a quotient map, it suffices to show that $q^{-1}(G \setminus G^{(0)})$ is open. This set consists of all ultrafilters U of $N(B)$ such that U contains no elements of the form te with $e \in I(B)$ and $t \in R^\times$. Let U be such an ultrafilter, and fix $n \in U$. By [\(AQP\)](#), there exists $f \in I(B)$ such that $P(n) = nf = fn \in N(B) \cap B$. Take mutually orthogonal idempotents $e_1, \dots, e_k \in I(B)$ and coefficients $t_1, \dots, t_k \in R \setminus \{0\}$ such that $nf = P(n) = \sum_{i=1}^k t_i e_i$, and note that $t_i \in R^\times$ for each $i \in \{1, \dots, k\}$, by [Corollary 5.8](#). Also, $nfe_i = t_i e_i$ for all $i \in \{1, \dots, k\}$, and so $nf = \sum_{i=1}^k nfe_i$.

Note that $n - nf \in N(B)$ by [Lemma 2.17\(e\)](#). Thus, since $n = n(n^\dagger n - n^\dagger nf) + nf$, we deduce that either $nf \in U$ or $n(n^\dagger n - n^\dagger nf) = n - nf \in U$ by [Lemma 5.2](#). But if $nf \in U$, then since $nf = \sum_{i=1}^k nfe_i$, we must have $t_i e_i = nfe_i \in U$ for some $i \in \{1, \dots, k\}$ by [Lemma 5.2](#), which is a contradiction to our hypothesis on U . Thus we must have $n - nf \in U$, and so $U \in \mathcal{V}_{n-nf}$.

We claim that if $V \in \mathcal{V}_{n-nf}$, then $V \notin q^{-1}(G^{(0)})$. To see this, suppose for contradiction that $tg \in V$ for some $t \in R^\times$ and $g \in I(B)$. Then tg and $n - nf$ have a common lower bound m in V , and so $t^{-1}m \leq g$. Hence $t^{-1}m \in I(B)$ by [Lemma 2.17\(a\)](#), and so $m = th$ for some idempotent $h \in I(B)$. Since $th = m \leq n - nf \leq n$, we have that $th = tht^{-1}hn = hn$. So

$$th = P(th) = P(hn) = hP(n) = hnf = thf.$$

But then $m = th = thf \leq (n - nf)f = 0$, and so $m = 0$, which contradicts $m \in V$. Thus $V \notin q^{-1}(G^{(0)})$, as required. We conclude that $q^{-1}(G \setminus G^{(0)})$ is open and hence $G^{(0)}$ is closed, whence G is Hausdorff. \square

Proof of [Theorem 5.6](#). [Lemma 5.4\(b\)](#) shows that q is a groupoid homomorphism. It is continuous and surjective by definition. By [Lemma 5.5\(b\)](#), q restricts to a homeomorphism of unit spaces. In what follows, we identify $G^{(0)}$ with $\Sigma^{(0)}$ for cleaner notation.

To see that i is a groupoid homomorphism, fix a composable pair $((U, t), (U, s)) \in (G^{(0)} \times R^\times)^{(2)}$, where $U \in \Sigma^{(0)}$. Then

$$i((U, t)(U, s)) = i(U, ts) = tsU = tsUU.$$

An application of [Lemma 5.3\(c\)](#) shows that this is equal to $tUsU = i(U, t)i(U, s)$, and hence i is a groupoid homomorphism.

To see that i is continuous, fix a basic open set $\mathcal{V}_n \subseteq \Sigma$. Then

$$\begin{aligned} i^{-1}(\mathcal{V}_n) &= \{(U, t) \in \Sigma^{(0)} \times R^\times : tU \in \mathcal{V}_n\} \\ &= \{(U, t) \in \Sigma^{(0)} \times R^\times : U \in \mathcal{V}_{t^{-1}n}\} \quad \text{by [Lemma 5.3\(e\)](#)} \\ &= \bigcup_{t \in R^\times} (\mathcal{V}_{t^{-1}n} \cap \Sigma^{(0)}) \times \{t\}. \end{aligned}$$

Since $\mathcal{V}_{t^{-1}n} \cap \Sigma^{(0)}$ is open in $\Sigma^{(0)} \cong G^{(0)}$, we deduce that $i^{-1}(\mathcal{V}_n)$ is open in $G^{(0)} \times R^\times$.

Lemma 5.3(g) shows that i is injective, and clearly $i(G^{(0)} \times \{1\}) = \Sigma^{(0)}$. For the exactness condition **(DT1)**, fix $U \in G^{(0)}$, and note that

$$i(\{U\} \times R^\times) = \{tU : t \in R^\times\} = q^{-1}(U).$$

We next check condition **(DT3)**, which requires that the image of i is central in Σ . Fix $U \in \Sigma$ and $t \in R^\times$. Then we have

$$i(\mathbf{r}(U), t)U = t(UU^{-1})U = tU = U(t(U^{-1}U)) = Ui(\mathbf{s}(U), t)$$

by **Lemma 5.3(c)**.

We use the implication **(4) \implies (1)** of **Proposition 2.2** to complete the proof that the sequence satisfies **(DT1)–(DT3)** (and hence is a discrete R -twist when G is Hausdorff). We have that Σ is ample and that the map q is open by parts **(a)** and **(b)** of **Lemma 5.5**. We show that i is open. Fix a basic open set $\mathcal{V}_e \times \{t\}$ with $e \in I(B)$ and $t \in R^\times$. Then, using **Lemma 5.3(e)** for the last equality, we see that

$$i(\mathcal{V}_e \times \{t\}) = t\mathcal{V}_e = \mathcal{V}_{te}.$$

Thus i is an open map, as required.

The final statement now follows by **Proposition 5.10**. □

6. THE ISOMORPHISM $A \cong A_R(G; \Sigma)$

Let (A, B) be an algebraic quasi-Cartan pair and let Σ be the groupoid of ultrafilters of $N(B)$. Let $C(\Sigma, R)$ denote the R -module of continuous (or equivalently, locally constant) functions from Σ to R with pointwise operations. In this section we build a map from A to $C(\Sigma, R)$ using both the faithful conditional expectation $P: A \rightarrow B$ satisfying **(AQP)** and the isomorphism $\phi: B \rightarrow A_R(\Sigma^{(0)})$ from **equation (5.2)** that satisfies $\phi(e) = 1_{\mathcal{V}_e}$ for all $e \in I(B)$, and we prove that this map is in fact an isomorphism of A onto the twisted Steinberg algebra $A_R(G; \Sigma)$.

Proposition 6.1. *Suppose that (A, B) is an algebraic quasi-Cartan pair, and let G and Σ be the groupoids constructed in **Section 5**. Let $\phi: B \rightarrow A_R(\Sigma^{(0)})$ be the isomorphism from **equation (5.2)** that satisfies $\phi(e) = 1_{\mathcal{V}_e}$ for all $e \in I(B)$. For each $a \in A$, there is a function $\widehat{a}: \Sigma \rightarrow R$ such that for any ultrafilter $U \in \Sigma$ and any $n \in U$,*

$$\widehat{a}(U) = \phi(P(n^\dagger a))(\mathbf{s}(U)).$$

Furthermore,

- (a) \widehat{a} is continuous;
- (b) the map $a \mapsto \widehat{a}$ from A to $C(\Sigma, R)$ is R -linear;
- (c) the map $a \mapsto \widehat{a}$ from A to $C(\Sigma, R)$ is injective;
- (d) $\widehat{a}(tU) = t^{-1}\widehat{a}(U)$ for every $t \in R^\times$ and $U \in \Sigma$;
- (e) for $b \in B$, we have $\widehat{b}|_{\Sigma^{(0)}} = \phi(b)$, and $\text{supp}(\widehat{b}) \subseteq i(G^{(0)} \times R^\times)$.

Proof. To see that there exists a (well-defined) function \widehat{a} satisfying the given formula, we must show that $\phi(P(n^\dagger a))(\mathbf{s}(U))$ is independent of the choice of $n \in U$. So fix $a \in A$, $U \in \Sigma$, and $n, m \in U$. We must show that

$$\phi(P(n^\dagger a))(\mathbf{s}(U)) - \phi(P(m^\dagger a))(\mathbf{s}(U)) = 0.$$

Observe that if $m, n \in U$, then there exists $k \in U$ such that $k \leq m, n$, whence $k^\dagger \leq m^\dagger, n^\dagger$. So

$$k^\dagger k m^\dagger = k^\dagger = k^\dagger k n^\dagger,$$

and moreover,

$$\phi(k^\dagger k)(\mathbf{s}(U)) = 1_{\mathcal{V}_{k^\dagger k}}(U^{-1}U) = 1.$$

Now, using that ϕ is multiplicative and that P is a conditional expectation, we compute

$$\begin{aligned} \phi(P(n^\dagger a))(\mathbf{s}(U)) - \phi(P(m^\dagger a))(\mathbf{s}(U)) &= \phi(k^\dagger k)(\mathbf{s}(U)) \phi(P(n^\dagger a - m^\dagger a))(\mathbf{s}(U)) \\ &= \phi(P(k^\dagger k n^\dagger a - k^\dagger k m^\dagger a))(\mathbf{s}(U)) \\ &= \phi(P(k^\dagger a - k^\dagger a))(\mathbf{s}(U)) \\ &= 0. \end{aligned}$$

For part (a), fix $a \in A$. We show that \widehat{a} is locally constant. Let $U \in \Sigma$ and choose $n \in U$. The function $\phi(P(n^\dagger a)) \in A_R(\Sigma^{(0)})$ is locally constant, and so there is an idempotent $e \in \mathbf{s}(U)$ such that for each ultrafilter $W \in \Sigma^{(0)}$ containing e , we have $\phi(P(n^\dagger a))(W) = \Phi(P(n^\dagger a))(\mathbf{s}(U))$. Let $m \in U$ with $m^\dagger m \leq e$. Then m and n have a common lower bound $k \in U$, and hence $k^\dagger k \leq m^\dagger m \leq e$. Thus $k = nk^\dagger k \leq ne$. So $ne \in U$, and hence \mathcal{V}_{ne} is an open neighbourhood of U . If $V \in \mathcal{V}_{ne}$, then $n \in V$, because $ne \leq n$ (by Lemma 2.17(d)), and $e \in \mathbf{s}(V)$ because $(ne)^\dagger ne = en^\dagger ne \leq e$. Therefore,

$$\widehat{a}(V) = \phi(P(n^\dagger a))(\mathbf{s}(V)) = \phi(P(n^\dagger a))(\mathbf{s}(U)) = \widehat{a}(U),$$

by the choice of e . Thus \widehat{a} is locally constant, and hence is continuous.

Part (b) follows from the R -linearity of P and ϕ .

For part (c), suppose that $\widehat{a} = 0$. Then $\widehat{a}(U) = 0$ for all $U \in \Sigma$. We claim that $\phi(P(na)) = 0$ for all $n \in N(B)$. Fix $n \in N(B)$ and $U \in \Sigma^{(0)}$. We show that $\phi(P(na))(U) = 0$ by considering two cases. First, suppose that $nn^\dagger \in U$. Then

$$U \in \mathcal{V}_{nn^\dagger} = \mathcal{V}_n \mathcal{V}_{n^\dagger} = \mathbf{s}(\mathcal{V}_{n^\dagger}),$$

and so we can find an ultrafilter $W \in \mathcal{V}_{n^\dagger}$ such that $\mathbf{s}(W) = U$. We then have

$$\phi(P(na))(U) = \phi(P(na))(\mathbf{s}(W)) = \widehat{a}(W) = 0.$$

For the second case, suppose that $nn^\dagger \notin U$. Then $U \notin \mathcal{V}_{nn^\dagger}$, and since P is a conditional expectation, we have

$$\phi(P(na))(U) = \phi(P(nn^\dagger na))(U) = \phi(nn^\dagger)(U) \phi(P(na))(U) = 1_{\mathcal{V}_{nn^\dagger}}(U) \phi(P(na))(U) = 0,$$

as claimed. Since ϕ is injective, we deduce that $P(na) = 0$ for all $n \in N(B)$. Now, since P is faithful (property (iv) of Definition 3.3), we deduce that $a = 0$.

For part (d), fix $U \in \Sigma$ and $t \in R^\times$. Then $\mathbf{s}(U) = \mathbf{s}(tU)$ by Lemma 5.3(d). Let $n \in U$. Then $tn \in tU$ and $(tn)^\dagger = t^{-1}n^\dagger$ by Lemma 2.16(a). Therefore,

$$\widehat{a}(tU) = \phi(P((tn)^\dagger a))(\mathbf{s}(tU)) = \phi(P(t^{-1}n^\dagger a))(\mathbf{s}(U)) = t^{-1} \phi(P(n^\dagger a))(\mathbf{s}(U)) = t^{-1} \widehat{a}(U),$$

as required.

For part (e), fix $b \in B$ and $U \in \Sigma^{(0)}$ with $e \in U \cap I(B)$, and note that $e = e^\dagger$. Then

$$\widehat{b}(U) = \phi(P(eb))(\mathbf{s}(U)) = \phi(eb)(U) = \phi(e)(U) \phi(b)(U) = 1_{\mathcal{V}_e}(U) \phi(b)(U) = \phi(b)(U).$$

Finally, we show that the support of \widehat{b} is contained in $i(G^{(0)} \times R^\times)$. Since B is spanned by $I(B)$ and the map $a \mapsto \widehat{a}$ is R -linear by part (b), it suffices to consider the case where $b = e \in I(B)$. Fix $U \in \Sigma$ such that $\widehat{e}(U) \neq 0$. We claim that $U \in R^\times \cdot \Sigma^{(0)}$. To see this, fix $n \in U$. Then

$$\phi(P(n^\dagger e))(\mathbf{s}(U)) = \widehat{e}(U) \neq 0.$$

Since (A, B) is an algebraic quasi-Cartan pair, there exists $f \in I(B)$ such that $fn^\dagger = n^\dagger f = P(n^\dagger)$, and so

$$fen^\dagger = efn^\dagger = eP(n^\dagger) = P(n^\dagger)e = n^\dagger fe.$$

Also, $P(n^\dagger e) = P(n^\dagger)e = n^\dagger fe$. We then have

$$\begin{aligned} 0 \neq \phi(P(n^\dagger e))(\mathbf{s}(U)) &= \phi(n^\dagger fe)(\mathbf{s}(U)) \\ &= \phi(n^\dagger fefe)(\mathbf{s}(U)) \\ &= \phi(n^\dagger fe)(\mathbf{s}(U)) \phi(fe)(\mathbf{s}(U)) \\ &= \phi(n^\dagger fe)(\mathbf{s}(U)) 1_{\mathcal{V}_{fe}}(\mathbf{s}(U)). \end{aligned}$$

Thus $fe \in \mathbf{s}(U)$, and so $nfe \in U\mathbf{s}(U) = U$. Therefore, $fen^\dagger \in U^{-1}$. Since $P(n^\dagger e) = n^\dagger fe \in N(B) \cap B$, [Remark 5.9](#) implies that $fen^\dagger = n^\dagger fe = P(n^\dagger e)$ can be expressed as a finite sum $\sum_{i=1}^k t_i e_i$, where $e_1, \dots, e_k \in I(B)$ are mutually orthogonal idempotents and $t_1, \dots, t_k \in R^\times$. Notice that for each $i \in \{1, \dots, k\}$, we have $t_i e_i = fen^\dagger e_i$, and so

$$\sum_{i=1}^k fen^\dagger e_i = fen^\dagger \in U^{-1}.$$

Since U^{-1} is an ultrafilter, it follows from [Lemma 5.2](#) that there exists (a unique) $i \in \{1, \dots, k\}$ such that $t_i e_i = fen^\dagger e_i \in U^{-1}$. Hence $t_i^{-1} e_i = (t_i e_i)^\dagger \in U$ by [Lemma 2.16\(a\)](#), which forces $V := t_i U \in \Sigma^{(0)}$. Therefore, $U = t_i^{-1} V \in q^{-1}(G^{(0)}) = i(G^{(0)} \times R^\times)$. \square

We now prove that if (A, B) is an algebraic quasi-Cartan pair, and G and Σ are as constructed in [Section 5](#), then there is an isomorphism of A onto $A_R(G; \Sigma)$ that carries B to the canonical subalgebra isomorphic to $A_R(\Sigma^{(0)})$. We need the following technical lemma.

In what follows, we write $\widehat{P}(a) := \widehat{P(a)}$ for each $a \in A$, where $a \mapsto \widehat{a}: A \rightarrow C(\Sigma, R)$ is the map from [Proposition 6.1](#).

Lemma 6.2. *Let R be an indecomposable commutative ring. Suppose that (A, B) is an algebraic quasi-Cartan pair with faithful conditional expectation $P: A \rightarrow B$ satisfying (AQP). Let G and Σ be the groupoids constructed in [Section 5](#). Let $m, n \in N(B)$ and $U \in \Sigma^{(0)}$. If $\widehat{P}(mn)(U) \neq 0$, then there exists $f \in U \cap I(B)$ such that $fmn = mnf = \widehat{P}(mn)(U)f \in B$.*

Proof. Suppose that $\widehat{P}(mn)(U) \neq 0$. Then $P(mn) \neq 0$ by [Proposition 6.1\(b\)](#), and by (AQP), there exists $e_0 \in I(B)$ such that $e_0 mn = mne_0 = P(mn)$. Since $P(mn) \in B$, [Proposition 6.1\(e\)](#) implies that

$$\begin{aligned} 0 \neq \widehat{P}(mn)(U) &= \phi(P(mn))(U) = \phi(mm^\dagger mne_0^2)(U) \\ &= \phi(mm^\dagger)(U) \phi(mne_0)(U) \phi(e_0)(U) = 1_{\mathcal{V}_{mm^\dagger}}(U) \phi(mne_0)(U) 1_{\mathcal{V}_{e_0}}(U), \end{aligned}$$

and so $mm^\dagger, e_0 \in U$. Hence $e := e_0 mm^\dagger \in U \cap I(B)$ satisfies

$$emn = ee_0 mn = eP(mn) = P(mn)e = mne_0 e = mne. \quad (6.1)$$

Since $eP(mn) \in B$, there is a finite set $F \subseteq I(B)$ of mutually orthogonal idempotents and nonzero coefficients $r_f \in R$ such that $eP(mn) = \sum_{f \in F} r_f f$; without loss of generality we may assume that $f \leq e$ for each $f \in F$. Thus, using [Proposition 6.1\(e\)](#) and [equation \(5.2\)](#), we see that

$$0 \neq \widehat{P}(mn)(U) = \widehat{e}(U) \widehat{P}(mn)(U) = \sum_{f \in F} r_f \widehat{f}(U) = \sum_{f \in F} r_f 1_{\mathcal{V}_f}(U).$$

Hence we deduce that $U \in \bigcup_{f \in F} \mathcal{V}_f$. But the mutual orthogonality of the idempotents in F implies that the \mathcal{V}_f are pairwise disjoint, and so there is a unique $f \in F \cap U$. Therefore, $r_f = \widehat{P}(mn)(U)$. Since $f \leq e$, a similar argument to the one used in [equation \(6.1\)](#) shows that f commutes with mn , and $fmn = femn = feP(mn) = r_f f = \widehat{P}(mn)(U)f \in B$. \square

Lemma 6.3. *Let (A, B) be an algebraic quasi-Cartan pair, and let G and Σ be the groupoids constructed in Section 5. Then for $n \in N(B)$ and $U \in \Sigma$, we have*

$$\widehat{n}(U) = \begin{cases} t^{-1} & \text{if } U \in \mathcal{V}_{tn} \text{ for some } t \in R^\times \\ 0 & \text{otherwise.} \end{cases}$$

In particular, \widehat{n} is equal to the function $\tilde{1}_{\mathcal{V}_n} \in A_R(G; \Sigma)$ of Lemma 2.7.

Proof. For the first case, suppose that $U \in \mathcal{V}_{tn}$ for some $t \in R^\times$, and let $\phi: B \rightarrow A_R(\Sigma^{(0)})$ be the isomorphism of equation (5.2) that satisfies $\phi(e) = 1_{\mathcal{V}_e}$ for all $e \in I(B)$. Then $(tn)^\dagger = t^{-1}n^\dagger$ by Lemma 2.16(a), and so $n^\dagger n = (tn)^\dagger tn \in \mathfrak{s}(U)$. Therefore, we have

$$\widehat{n}(U) = \phi(P(t^{-1}n^\dagger n))(\mathfrak{s}(U)) = \phi(t^{-1}n^\dagger n)(\mathfrak{s}(U)) = t^{-1}\phi(n^\dagger n)(\mathfrak{s}(U)) = t^{-1}.$$

For the second case, we prove the contrapositive. Suppose that $\widehat{n}(U) \neq 0$, and fix $m \in U$. Then by Proposition 6.1(e) and the definition of \widehat{n} , we have

$$\widehat{P}(m^\dagger n)(\mathfrak{s}(U)) = \phi(P(m^\dagger n))(\mathfrak{s}(U)) = \widehat{n}(U) \neq 0.$$

Hence Lemma 6.2 shows that there exists an idempotent $f \in \mathfrak{s}(U)$ such that $fm^\dagger n = m^\dagger nf = \widehat{n}(U)f$. Since $\widehat{n}(U)f = fm^\dagger n \in N(B) \cap B$, Lemma 5.7 implies that $t := \widehat{n}(U)$ belongs to R^\times . Since $f \in \mathfrak{s}(U)$ and $m \in U$, we have $mf \in U$. Note that $t(mf) = m(tf) = mm^\dagger nf \leq n$ by Lemma 2.17(d). Thus $mf \leq t^{-1}n$ by Lemma 2.16(b), and so $t^{-1}n \in U$. That is, $U \in \mathcal{V}_{t^{-1}n}$, as required. \square

For the surjectivity in the main theorem (Theorem 6.6), we need to know that each element of $A_R(G; \Sigma)$ can be written as a finite sum of elements of the form $t\widehat{n}$ (where $t \in R$ and $n \in N(B)$); we do this in the following two results, the first of which is standard.

Lemma 6.4. *Let H be an ample groupoid and let D be a compact open bisection of H . Let \mathcal{B} be an inverse semigroup of compact open bisections that form a basis for H and whose idempotents are closed under relative complement and disjoint union. Then D can be expressed as a finite disjoint union of elements of \mathcal{B} .*

Proof. Since \mathcal{B} is a basis and D is compact, we can certainly write $D = \bigcup_{i=1}^N B_i$ with $B_i \in \mathcal{B}$. Put $B'_1 = B_1$. Assume inductively that we have found $B'_1, \dots, B'_j \in \mathcal{B}$ pairwise disjoint with $\bigcup_{i=1}^j B_i = \bigsqcup_{i=1}^j B'_i$ for $1 \leq j < n$. Then put $B'_{j+1} = B_{j+1}(\mathfrak{s}(B_{j+1}) \setminus \bigcup_{i=1}^j \mathfrak{s}(B'_i))$. By assumption on \mathcal{B} , we have that $B'_{j+1} \in \mathcal{B}$ as $B'_i, B'_k \subseteq D$ disjoint implies that $\mathfrak{s}(B'_i)$ and $\mathfrak{s}(B'_k)$ are disjoint because D is a bisection. Note that B'_{j+1} is disjoint from B'_1, \dots, B'_j by construction.

Since $B'_{j+1} \subseteq B_{j+1}$, by the inductive assumption it suffices to show that $B_{j+1} \subseteq \bigsqcup_{i=1}^{j+1} B'_i$. Fix $\gamma \in B_{j+1}$. If $\mathfrak{s}(\gamma) \notin \mathfrak{s}(B'_1) \cup \dots \cup \mathfrak{s}(B'_j)$, then trivially $\gamma \in B'_{j+1}$. If $\mathfrak{s}(\gamma) \in \mathfrak{s}(B'_i)$ for some $i \in \{1, \dots, j\}$, then since $B'_i, B_{j+1} \subseteq D$ and D is a bisection, we must have that $\gamma \in B'_i$. This completes the proof. \square

Proposition 6.5. *Let (A, B) be an algebraic quasi-Cartan pair, and let G and Σ be the groupoids constructed in Section 5. For any $f \in A_R(G; \Sigma)$, there exist $n_1, \dots, n_M \in N(B)$ and $t_1, \dots, t_M \in R$ such that*

$$f = \sum_{j=1}^M t_j \widehat{n}_j.$$

Proof. By Proposition 2.8, it suffices to prove the claim for $f = \tilde{1}_D$, where D is a compact open bisection of Σ . The collection $\mathcal{B} = \{\mathcal{V}_n : n \in N(B)\}$ is an inverse semigroup of compact open bisections forming a basis for the topology on Σ . Moreover, if $e, f \in I(B)$, then $\mathcal{V}_e \setminus \mathcal{V}_f = \mathcal{V}_{e-ef}$, and $\mathcal{V}_e \cap \mathcal{V}_f = \emptyset$ if and only if $ef = 0$, in which case, $\mathcal{V}_e \sqcup \mathcal{V}_f = \mathcal{V}_{e+f}$,

by Stone duality applied to $I(B)$. Thus, by Lemma 6.4, we can express D a finite disjoint union of basic compact open sets, say, $D = \bigsqcup_{j=1}^M \mathcal{V}_{n_j}$. Then $g := \sum_{j=1}^M \widehat{n}_j$ is an element of $A_R(G; \Sigma)$ satisfying $\text{supp}(g) \subseteq R^\times \cdot D$ and $g|_D \equiv 1$, by Lemma 5.3(e) and Lemma 6.3. Therefore, Lemma 2.7 gives $g = \tilde{1}_D$. \square

Theorem 6.6. *Suppose that (A, B) is an algebraic quasi-Cartan pair. Let G and Σ be the groupoids constructed in Section 5. Then the map $\varphi: a \mapsto \widehat{a}$ from A to $C(\Sigma, R)$ defined in Proposition 6.1 is an isomorphism of A onto $A_R(G; \Sigma)$ that takes B to $A_R(G^{(0)}; q^{-1}(G^{(0)}))$, which is isomorphic to $A_R(\Sigma^{(0)})$, and hence also to $A_R(G^{(0)})$.*

Proof. We begin by showing that $\widehat{a} \in A_R(G; \Sigma)$ for each $a \in A$. If $n \in N(B)$, then $\widehat{n} \in A_R(G; \Sigma)$ by Lemma 6.3. Since each $a \in A$ can be expressed as an R -linear combination of elements of $N(B)$ (by property (iii) of Definition 3.3), it follows that $\widehat{a} \in A_R(G; \Sigma)$, because φ is R -linear by Proposition 6.1(b).

Proposition 6.1 implies that φ is an injective R -linear map, and Proposition 6.5 implies that φ is surjective.

To complete the proof that φ is an isomorphism, we must show that $\widehat{aa'} = \widehat{a} * \widehat{a'}$ for all $a, a' \in A$. Since A is the R -linear span of $N(B)$, it suffices to prove this for $a = n, a' = m \in N(B)$. But this follows from Lemma 6.3 and Corollary 2.10, since $\mathcal{V}_n \mathcal{V}_m = \mathcal{V}_{nm}$.

Finally, we check the statement about B . If $e \in I(B)$, then $\varphi(e) = \widehat{e} = \tilde{1}_{\mathcal{V}_e}$ by Lemma 6.3. Since $I(B)$ spans B and $\{\tilde{1}_{\mathcal{V}_e} : e \in I(B)\}$ spans $A_R(G^{(0)}; q^{-1}(G^{(0)}))$ (by Proposition 2.11), it follows that φ carries B isomorphically to $A_R(G^{(0)}; q^{-1}(G^{(0)}))$, completing the proof. \square

7. ALGEBRAIC INFORMATION FROM THE ISOTROPY STRUCTURE OF G

In this section we describe the properties of the groupoid G (from the previous section) that identify the algebraic diagonal pairs and the algebraic Cartan pairs amongst all algebraic quasi-Cartan pairs.

Proposition 7.1. *Suppose that (A, B) is an algebraic quasi-Cartan pair. Then*

- (a) (A, B) is an algebraic Cartan pair if and only if G is effective, and
- (b) (A, B) is an algebraic diagonal pair if and only if G is principal.

The proof will follow easily once we establish the following two lemmas.

Lemma 7.2. *Let (A, B) be an algebraic Cartan pair, and let $n \in N(B)$. Suppose that $\mathcal{V}_n \subseteq \text{Iso}(\Sigma)$. Then $n \in B$.*

Proof. Let $P: A \rightarrow B$ be the faithful conditional expectation satisfying (AQP), and let $\phi: B \rightarrow A_R(\Sigma^{(0)})$ be the isomorphism from equation (5.2) that satisfies $\phi(e) = 1_{\mathcal{V}_e}$ for all $e \in I(B)$. We claim that $nb = bn$ for every $b \in B$. Since B is spanned by $I(B)$ (property (ii) of Definition 3.3), it suffices to fix $e \in I(B)$ and show that $en = ne$. We show that $\widehat{ne} = \widehat{en}$; the claim then follows by the injectivity of $a \mapsto \widehat{a}$ (Proposition 6.1(c)). Fix $U \in \Sigma$. If $\widehat{n}(U) = 0$, then by multiplicativity of $a \mapsto \widehat{a}$ (Theorem 6.6), we have

$$\widehat{en}(U) = \widehat{e}(U) \widehat{n}(U) = 0 = \widehat{n}(U) \widehat{e}(U) = \widehat{ne}(U).$$

So it suffices to consider $U \in \text{supp}(\widehat{n}) = R^\times \cdot \mathcal{V}_n$ by [Lemma 6.3](#); so, in particular, $\mathbf{r}(U) = \mathbf{s}(U)$ since $\mathcal{V}_n \in \text{Iso}(\Sigma)$. Now fix $m \in U$. Then

$$\begin{aligned} \widehat{ne}(U) &= \phi(P(m^\dagger ne))(\mathbf{s}(U)) \\ &= \phi(P(m^\dagger n))(\mathbf{s}(U)) \phi(e)(\mathbf{s}(U)) \\ &= \begin{cases} \phi(P(m^\dagger n))(\mathbf{s}(U)) & \text{if } e \in \mathbf{s}(U) \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} \widehat{n}(U) & \text{if } e \in \mathbf{s}(U) \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

In the first case, we have $e \in \mathbf{s}(U) = \mathbf{r}(U)$. Since $\mathbf{r}(U)U = U$, we have $em \in U$, and hence

$$\widehat{en}(U) = \phi(P(m^\dagger en))(\mathbf{s}(U)) = \phi(P((em)^\dagger n))(\mathbf{s}(U)) = \widehat{n}(U) = \widehat{ne}(U).$$

For the second case, suppose that $e \notin \mathbf{s}(U) = \mathbf{r}(U)$. Let $e_1 := mm^\dagger - emm^\dagger$ and let $e_2 := emm^\dagger$. These are orthogonal idempotents in $I(B)$, and $m = e_1m + e_2m$. Therefore, either $e_1m = m - em$ or $e_2m = em$ belongs to U , by [Lemma 5.2](#). But if $em \in U$, then since $em(me)^\dagger = emm^\dagger e \leq e$, we have $e \in \mathbf{r}(U)$, which is a contradiction. So we must have $m - em \in U$. Let $k := (m - em)^\dagger$. Then $k = m^\dagger - m^\dagger e$ by [Lemma 2.17\(e\)](#), and so $ke = 0$. Therefore, $\widehat{en}(U) = \phi(P(ken))(\mathbf{s}(U)) = 0$, as required. This completes the proof of the claim.

By the claim, n commutes with all of B . Since B is maximal abelian, this forces $n \in B$. \square

Lemma 7.3. *Suppose that (A, B) is an algebraic quasi-Cartan pair. If $n \in N(B) \cap B$, then $\mathcal{V}_n \subseteq q^{-1}(G^{(0)})$.*

Proof. The result is trivial if $n = 0$, and so we assume that $n \neq 0$. By [Remark 5.9](#), there exist mutually orthogonal idempotents $e_1, \dots, e_k \in I(B)$ and coefficients $t_1, \dots, t_k \in R^\times$ such that $n = \sum_{i=1}^k t_i e_i$. Note that $ne_i = te_i$ for each $i \in \{1, \dots, k\}$, and so $n = \sum_{i=1}^k ne_i$. Fix $V \in \mathcal{V}_n$. Then [Lemma 5.2](#) implies that $ne_i \in V$ for some (unique) $i \in \{1, \dots, k\}$. Thus $t_i e_i = ne_i \in V$, and so $e_i \in t_i^{-1}V$. Therefore, $t_i^{-1}V \in \Sigma^{(0)}$, and so $V = t_i(t_i^{-1}V) \in R^\times \cdot \Sigma^{(0)} = q^{-1}(G^{(0)})$, as required. \square

Proof of Proposition 7.1. For part (a), first suppose that (A, B) is an algebraic Cartan pair. Fix an open set O contained in the interior of the isotropy of G . Without loss of generality, we may assume that $O = q(\mathcal{V}_n)$ for some $n \in N(B)$. Then \mathcal{V}_n is contained in the isotropy of Σ . [Lemma 7.2](#) implies that $n \in B$, and [Lemma 7.3](#) gives $q(\mathcal{V}_n) \subseteq G^{(0)}$.

Now suppose that G is effective. Then the algebraic quasi-Cartan pair (A, B) is isomorphic to $(A_R(G; \Sigma), A_R(G^{(0)}, q^{-1}(G^{(0)}))$ (by [Theorem 6.6](#)), which is an algebraic Cartan pair by [Proposition 4.8](#).

For part (b), first suppose that (A, B) is an algebraic diagonal pair. Fix $U \in \Sigma$ such that $\mathbf{r}(U) = \mathbf{s}(U)$. We must show that $q(U) \in G^{(0)}$. For this, fix $n \in U$. Since A is spanned by the free normalisers of B , we can write $n = n_0 + \sum_{i=1}^k n_i$, where $n_0 \in B$ and each n_i is a normaliser satisfying $n_i^2 = 0$. Since $\mathbf{s}(U)$ is an ultrafilter, we can find an idempotent $e \in \mathbf{s}(U) = \mathbf{r}(U)$ such that for every $i \in \{1, \dots, k\}$ we have $e \leq n_i^\dagger n_i$ whenever $n_i^\dagger n_i \in \mathbf{s}(U)$, and $en_i^\dagger n_i = 0$ whenever $n_i^\dagger n_i \notin \mathbf{s}(U)$. Since $e \in \mathbf{s}(U) = \mathbf{r}(U)$, we have $\sum_{i=0}^k en_i e = ene \in U$. If $i \in \{1, \dots, k\}$ satisfies $n_i^\dagger n_i \notin \mathbf{s}(U)$, then $en_i e = en_i n_i^\dagger n_i e = 0$; and if $i \in \{1, \dots, k\}$ satisfies $n_i^\dagger n_i \in \mathbf{s}(U)$, then $e \leq n_i^\dagger n_i$, and so $en_i e = e(n_i^\dagger n_i)(n_i e) = 0$

as well. So we obtain $ene = en_0e \in U \cap B \cap N(B)$. Hence $U \in \mathcal{V}_{ene} \subseteq q^{-1}(G^{(0)})$ by [Lemma 7.3](#).

Now suppose that G is principal. Then $(A, B) \cong (A_R(G; \Sigma), A_R(G^{(0)}, q^{-1}(G^{(0)})))$ is the algebra of a twist over a principal groupoid (by [Theorem 6.6](#)), and hence is an algebraic diagonal pair by [Proposition 4.8](#). \square

8. RECOVERING A TWIST FROM ITS QUASI-CARTAN PAIR

In this section we show that if (Σ, i, q) is a discrete R -twist over a groupoid G , then we can construct a natural embedding of Σ into the groupoid of ultrafilters Σ' obtained from [Theorem 5.6](#) applied to the pair $(A_R(G; \Sigma), A_R(G^{(0)}, q^{-1}(G^{(0)})))$. We then show that this map is an isomorphism if and only if (Σ, i, q) satisfies the local bisection hypothesis. We start with some technical results. Throughout, we will identify $\Sigma'^{(0)}$ with ultrafilters of $I(B)$, where $B := A_R(G^{(0)}, q^{-1}(G^{(0)}))$, via the homeomorphism described in [Remark 5.1](#).

Lemma 8.1. *Let (Σ, i, q) be a discrete R -twist over G , let $A := A_R(G; \Sigma)$, and let $B := A_R(G^{(0)}, q^{-1}(G^{(0)}))$. If U is an ultrafilter of $I(B)$ and $n \in N(B)$ satisfies $n^\dagger n \in U$, then the upclosure $V := (nU)^\dagger$ of $nU \subseteq N(B) \setminus \{0\}$ is an ultrafilter of $N(B)$ containing n .*

Proof. Regarding U as an ultrafilter in $\Sigma'^{(0)}$, we have $U \in \mathcal{V}_{n^\dagger n} = \mathbf{s}(\mathcal{V}_n)$, and so there exists an ultrafilter $V \in \mathcal{V}_n$ with $\mathbf{s}(V) = U$. Then [[5](#), Lemma 3.1(a)] implies that $V = (nU)^\dagger$. Since $n^\dagger n \in U$, we have $n = nn^\dagger n \in nU \subseteq V \subseteq N(B) \setminus \{0\}$. \square

Lemma 8.2. *Let (Σ, i, q) be a discrete R -twist over G , let $A := A_R(G; \Sigma)$, and let $B := A_R(G^{(0)}, q^{-1}(G^{(0)}))$. Let $X \subseteq \Sigma$ be a compact open bisection, and let $n \in N(B)$. Then*

- (a) $\tilde{1}_X \leq n$ if and only if $n|_X \equiv 1$ and $q(X) = \mathbf{s}^{-1}(\mathbf{s}(q(X))) \cap \text{supp}_G(n)$;
- (b) if $n \leq \tilde{1}_X$, then $n = \tilde{1}_Y$ for some compact open bisection $Y \subseteq X$; and
- (c) given a compact open bisection $Y \subseteq \Sigma$, we have $\tilde{1}_Y \leq \tilde{1}_X$ if and only if $Y \subseteq X$.

Proof. For part (a), suppose first that $\tilde{1}_X \leq n$; that is, that $\tilde{1}_X = n\tilde{1}_X^\dagger \tilde{1}_X = n\tilde{1}_{\mathbf{s}(X)}$. [Proposition 2.11](#) implies that for all $\sigma \in X$,

$$1 = \tilde{1}_X(\sigma) = (n\tilde{1}_{\mathbf{s}(X)})(\sigma) = n(\sigma) 1_{\mathbf{s}(X)}(\mathbf{s}(\sigma)) = n(\sigma).$$

The containment $q(X) \subseteq \mathbf{s}^{-1}(\mathbf{s}(q(X))) \cap \text{supp}_G(n)$ follows from the above equality. For the reverse containment, suppose that $\sigma \in \Sigma$ satisfies $q(\sigma) \in \mathbf{s}^{-1}(\mathbf{s}(q(X))) \cap \text{supp}_G(n)$. Then $\mathbf{s}(q(\sigma)) \in \mathbf{s}(q(X))$, and so there exists $\gamma \in X$ such that $q(\mathbf{s}(\gamma)) = \mathbf{s}(q(\gamma)) = \mathbf{s}(q(\sigma)) = q(\mathbf{s}(\sigma))$. This implies that there exists a unique $t \in R^\times$ such that $\mathbf{s}(\sigma) = t \cdot \mathbf{s}(\gamma)$. By the equality $\tilde{1}_X = n\tilde{1}_{\mathbf{s}(X)}$, we obtain

$$\tilde{1}_X(\sigma) = (n\tilde{1}_{\mathbf{s}(X)})(\sigma) = n(\sigma) \tilde{1}_{\mathbf{s}(X)}(\mathbf{s}(\sigma)) = t^{-1}n(\sigma) \neq 0.$$

This implies that $\sigma = u \cdot \sigma'$ for some $\sigma' \in X$ and $u \in R^\times$. Hence $q(\sigma) = q(\sigma') \in q(X)$, and the containment $\mathbf{s}^{-1}(\mathbf{s}(q(X))) \cap \text{supp}_G(n) \subseteq q(X)$ follows.

For the converse, notice that for $\gamma \in \Sigma$, applying [Proposition 2.11](#) at the first step gives

$$\begin{aligned} (n\tilde{1}_{\mathbf{s}(X)})(\gamma) \neq 0 &\iff n(\gamma) \tilde{1}_{\mathbf{s}(X)}(\mathbf{s}(\gamma)) \neq 0 \\ &\iff \gamma \in \text{supp}(n) \text{ and } \mathbf{s}(q(\gamma)) = q(\mathbf{s}(\gamma)) \in q(\mathbf{s}(X)) = \mathbf{s}(q(X)) \\ &\iff q(\gamma) \in \mathbf{s}^{-1}(\mathbf{s}(q(X))) \cap \text{supp}_G(n) = q(X) \\ &\iff \gamma \in R^\times \cdot X. \end{aligned}$$

Since $(n\tilde{1}_{\mathbf{s}(X)})(\sigma) = 1$ for all $\sigma \in X$, we deduce that $n\tilde{1}_{\mathbf{s}(X)} = \tilde{1}_X$ by [Lemma 2.7](#), and hence $\tilde{1}_X \leq n$.

For part (b), let $Y := X \cap \mathbf{s}^{-1}(\mathbf{s}(\text{supp}(n)))$. Note that $\mathbf{s}(\text{supp}(n)) = \mathbf{s}(\text{supp}_G(n))$ (under the usual identification of $G^{(0)}$ and $\Sigma^{(0)}$) is compact and open, and hence is clopen since $G^{(0)}$ is Hausdorff. Thus Y is a clopen subset of the compact open bisection X , and so Y is a compact open bisection. Since $n \leq \tilde{\mathbf{1}}_X$ by hypothesis, [Remark 4.4](#) implies that

$$n = \tilde{\mathbf{1}}_X n^\dagger n = \tilde{\mathbf{1}}_X \tilde{\mathbf{1}}_{\mathbf{s}(\text{supp}(n))} = \tilde{\mathbf{1}}_Y.$$

For part (c), let Y be a compact open bisection of Σ . If $\tilde{\mathbf{1}}_Y \leq \tilde{\mathbf{1}}_X$, then since $\tilde{\mathbf{1}}_Y \in N(B)$, part (b) implies that $\tilde{\mathbf{1}}_Y = \tilde{\mathbf{1}}_Z$ for some compact open bisection $Z \subseteq X$. This forces $Y = Z$, so that $Y \subseteq X$. On the other hand, if $Y \subseteq X$, then $X\mathbf{s}(Y) = Y$, and so $\tilde{\mathbf{1}}_Y = \tilde{\mathbf{1}}_X \tilde{\mathbf{1}}_{\mathbf{s}(Y)} \leq \tilde{\mathbf{1}}_X$ by [Lemma 2.17\(d\)](#) and [Corollary 2.10](#) (since $\tilde{\mathbf{1}}_{\mathbf{s}(Y)} \in I(B)$ and $\tilde{\mathbf{1}}_X \in N(B)$). \square

Lemma 8.3. *Let (Σ, i, q) be a discrete R -twist over G , let $A := A_R(G; \Sigma)$, and let $B := A_R(G^{(0)}; q^{-1}(G^{(0)}))$. Fix $\sigma \in \Sigma$. The set $U_{\mathbf{s}(\sigma)} := \{e \in I(B) : e(\mathbf{s}(\sigma)) = 1\}$ is an ultrafilter of $I(B)$.*

- (a) *If X and Y are compact open bisections such that $\sigma \in X \cap Y$, then $(\tilde{\mathbf{1}}_X U_{\mathbf{s}(\sigma)})^\dagger = (\tilde{\mathbf{1}}_Y U_{\mathbf{s}(\sigma)})^\dagger$.*
- (b) *If X and Y are compact open bisections such that $\sigma \in X$ and $\tilde{\mathbf{1}}_Y \in (\tilde{\mathbf{1}}_X U_{\mathbf{s}(\sigma)})^\dagger$, then $\sigma \in Y$.*
- (c) *If X is a compact open bisection such that $\sigma \in X$ and $n \in (\tilde{\mathbf{1}}_X U_{\mathbf{s}(\sigma)})^\dagger$, then there exists a compact open set $W \subseteq \mathbf{s}(X)$ such that $\sigma \in XW$ and $\tilde{\mathbf{1}}_{XW} \leq n$.*

Proof. A routine argument shows that $U_{\mathbf{s}(\sigma)}$ is an ultrafilter of $I(B)$.

For part (a), it suffices by symmetry to show that $(\tilde{\mathbf{1}}_X U_{\mathbf{s}(\sigma)})^\dagger \subseteq (\tilde{\mathbf{1}}_Y U_{\mathbf{s}(\sigma)})^\dagger$. Since the right-hand side is closed upwards, it suffices to show that it contains $\tilde{\mathbf{1}}_X U_{\mathbf{s}(\sigma)}$. But the right-hand side is also closed under right multiplication by $U_{\mathbf{s}(\sigma)}$. Thus it suffices to show that $\tilde{\mathbf{1}}_X \in (\tilde{\mathbf{1}}_Y U_{\mathbf{s}(\sigma)})^\dagger$. Since $X \cap Y$ is open, there is a compact open set $W \subseteq X \cap Y$ with $\sigma \in W$, and clearly W is a compact open bisection. So $\tilde{\mathbf{1}}_W \leq \tilde{\mathbf{1}}_X, \tilde{\mathbf{1}}_Y$ by [Lemma 8.2\(c\)](#), and $\tilde{\mathbf{1}}_{\mathbf{s}(W)} \in U_{\mathbf{s}(\sigma)}$ since $\sigma \in W$. Note that $\tilde{\mathbf{1}}_W = \tilde{\mathbf{1}}_Y \tilde{\mathbf{1}}_{\mathbf{s}(W)}$ by [Corollary 2.10](#). Therefore, $\tilde{\mathbf{1}}_Y \tilde{\mathbf{1}}_{\mathbf{s}(W)} = \tilde{\mathbf{1}}_W \leq \tilde{\mathbf{1}}_X$, and so $\tilde{\mathbf{1}}_X \in (\tilde{\mathbf{1}}_Y U_{\mathbf{s}(\sigma)})^\dagger$, as required.

For part (b), note that since $\tilde{\mathbf{1}}_Y \in (\tilde{\mathbf{1}}_X U_{\mathbf{s}(\sigma)})^\dagger$, there exists a compact open set $W \subseteq \Sigma^{(0)}$ such that $\tilde{\mathbf{1}}_W \in U_{\mathbf{s}(\sigma)}$ and $\tilde{\mathbf{1}}_{XW} = \tilde{\mathbf{1}}_X \tilde{\mathbf{1}}_W \leq \tilde{\mathbf{1}}_Y$ by [Corollary 2.10](#). Thus $\sigma = \sigma\mathbf{s}(\sigma) \in XW$, and so $\tilde{\mathbf{1}}_Y(\sigma) = 1$ by [Lemma 8.2\(a\)](#). Hence $\sigma \in Y$.

For part (c), note that since $n \in (\tilde{\mathbf{1}}_X U_{\mathbf{s}(\sigma)})^\dagger$, there exists a compact open set $W \subseteq \Sigma^{(0)}$ such that $\tilde{\mathbf{1}}_W \in U_{\mathbf{s}(\sigma)}$ and $\tilde{\mathbf{1}}_X \tilde{\mathbf{1}}_W \leq n$. By replacing W with $W \cap \mathbf{s}(X)$, we may assume that $W \subseteq \mathbf{s}(X)$. Hence $\tilde{\mathbf{1}}_{XW} = \tilde{\mathbf{1}}_X \tilde{\mathbf{1}}_W \leq n$. Since $W \in U_{\mathbf{s}(\sigma)}$, we have $\mathbf{s}(\sigma) \in W$, and so $\sigma = \sigma\mathbf{s}(\sigma) \in XW$. \square

We are now in a position to show that for any discrete R -twist (Σ, i, q) over G , there is a continuous open groupoid homomorphism Φ of Σ into the twist obtained from the pair $(A_R(G; \Sigma), A_R(G^{(0)}; q^{-1}(G^{(0)})))$, and that this induces a morphism of twists. We first establish the existence of Φ .

Proposition 8.4. *Let (Σ, i, q) be a discrete R -twist over G , let $A := A_R(G; \Sigma)$, and let $B := A_R(G^{(0)}; q^{-1}(G^{(0)}))$. Let Σ' be the groupoid of ultrafilters of $N(B)$ defined in [Section 5.1](#), and let G' be the corresponding quotient of Σ' by the action of R^\times . For each $x \in \Sigma^{(0)}$, let U_x be the ultrafilter $\{e \in I(B) : e(x) = 1\}$ of $I(B)$ corresponding to x . For $\sigma \in \Sigma$, let X be a compact open bisection containing σ , and let*

$$\Phi(\sigma) := (\tilde{\mathbf{1}}_X U_{\mathbf{s}(\sigma)})^\dagger \in \Sigma'.$$

Then $\Phi(\sigma)$ does not depend on the choice of X . The map $\Phi: \Sigma \rightarrow \Sigma'$ is an R^\times -equivariant continuous open embedding of topological groupoids such that $\Phi(\Sigma^{(0)}) = \Sigma'^{(0)}$.

Proof. Lemma 8.3(a) shows that $\Phi(\sigma)$ does not depend on the choice of X . To see that Φ is injective, suppose that σ and τ are distinct elements of Σ . Since Σ is Hausdorff, there is a compact open bisection $X \subseteq \Sigma$ such that $\sigma \in X$ and $\tau \notin X$. In particular, $\tilde{\Gamma}_X \tilde{\Gamma}_{s(X)} = \tilde{\Gamma}_X \in \Phi(\sigma) \setminus \Phi(\tau)$ by Corollary 2.10 and Lemma 8.3(b), and so $\Phi(\sigma) \neq \Phi(\tau)$.

Fix $x \in \Sigma^{(0)}$. By taking any compact open neighbourhood $X \subseteq \Sigma^{(0)}$ of x , we see that $\Phi(x) = (\tilde{\Gamma}_X U_x)^\dagger = (U_x)^\dagger$ is the unique ultrafilter of $N(B)$ containing U_x . By Stone duality, it follows that Φ restricts to a homeomorphism from $\Sigma^{(0)}$ to $\Sigma'^{(0)}$.

Fix $\sigma \in \Sigma$ and let $X \subseteq \Sigma$ be a compact open bisection containing σ . Put $V := (\tilde{\Gamma}_X U_{s(\sigma)})^\dagger = \Phi(\sigma)$. Then $\mathbf{s}(\Phi(\sigma)) = V^{-1}V$. If $e \in U_{s(\sigma)}$, then $e \geq (\tilde{\Gamma}_X e)^\dagger (\tilde{\Gamma}_X e) \in V^{-1}V$, and so $e \in V^{-1}V$. Therefore, $\Phi(\mathbf{s}(\sigma)) = (U_{s(\sigma)})^\dagger \subseteq V^{-1}V$, and so by the maximality property of ultrafilters, we have $\Phi(\mathbf{s}(\sigma)) = V^{-1}V = \mathbf{s}(\Phi(\sigma))$. Suppose that $e \in U_{r(\sigma)} \subseteq \Phi(\mathbf{r}(\sigma))$. Then $(\tilde{\Gamma}_X^\dagger e \tilde{\Gamma}_X)(\mathbf{s}(\sigma)) = 1$ by Proposition 2.11, and so $\tilde{\Gamma}_X^\dagger e \tilde{\Gamma}_X \in U_{s(\sigma)} = V^{-1}V$. Therefore, $e \geq \tilde{\Gamma}_X (\tilde{\Gamma}_X^\dagger e \tilde{\Gamma}_X) \tilde{\Gamma}_X^\dagger \in V(V^{-1}V)V^{-1} = VV^{-1}$, and so $e \in VV^{-1}$. Thus $\Phi(\mathbf{r}(\sigma)) = (U_{r(\sigma)})^\dagger \subseteq VV^{-1} = \mathbf{r}(\Phi(\sigma))$, and so $\Phi(\mathbf{r}(\sigma)) = \mathbf{r}(\Phi(\sigma))$ since these are ultrafilters. It follows that Φ carries composable pairs to composable pairs.

To see that Φ is multiplicative, fix $(\sigma, \tau) \in \Sigma^{(2)}$, and choose a compact open bisection $W \subseteq \Sigma$ containing $\sigma\tau$. By the continuity of multiplication, there exist compact open bisections X and Y of Σ such that $\sigma \in X$, $\tau \in Y$, and $XY \subseteq W$. Then $\tilde{\Gamma}_X \tilde{\Gamma}_Y = \tilde{\Gamma}_{XY} \leq \tilde{\Gamma}_W$ by Corollary 2.10 and Lemma 8.2(c). It follows that $\tilde{\Gamma}_W \in \Phi(\sigma)\Phi(\tau)$. Since $\mathbf{s}(\tau) = \mathbf{s}(\sigma\tau)$, it follows that $\tilde{\Gamma}_W U_{s(\sigma\tau)} \subseteq \Phi(\sigma)\Phi(\tau)U_{s(\sigma\tau)} \subseteq \Phi(\sigma)\Phi(\tau)$, and so $\Phi(\sigma\tau) \subseteq \Phi(\sigma)\Phi(\tau)$. Hence $\Phi(\sigma\tau) = \Phi(\sigma)\Phi(\tau)$ by the maximality property of ultrafilters.

To see that Φ is continuous, fix $n \in N(B)$, and let $\sigma \in \Phi^{-1}(\mathcal{V}_n)$. Choose a compact open bisection $X \subseteq \Sigma$ containing σ . Then $(\tilde{\Gamma}_X U_{s(\sigma)})^\dagger = \Phi(\sigma) \in \mathcal{V}_n$, and so by the definition of \mathcal{V}_n , we have $n \in (\tilde{\Gamma}_X U_{s(\sigma)})^\dagger$. It follows from Lemma 8.3(c) that there is a compact open set $W \subseteq \mathbf{s}(X)$ such that $\sigma \in XW$ and $\tilde{\Gamma}_{XW} \tilde{\Gamma}_{s(X)} = \tilde{\Gamma}_{XW} \leq n$. Hence $\Phi(\tau) \in \mathcal{V}_n$ for each $\tau \in XW$, and so XW is an open neighbourhood of σ contained in $\Phi^{-1}(\mathcal{V}_n)$.

To see that Φ is open, it suffices to fix a compact open bisection X of Σ , and show that $\Phi(X) = \mathcal{V}_{\tilde{\Gamma}_X}$. If $\sigma \in X$, then certainly $\tilde{\Gamma}_X \in \Phi(\sigma)$, and so $\Phi(\sigma) \in \mathcal{V}_{\tilde{\Gamma}_X}$. This gives $\Phi(X) \subseteq \mathcal{V}_{\tilde{\Gamma}_X}$. For the reverse containment, suppose that $U \in \mathcal{V}_{\tilde{\Gamma}_X}$. Then $\tilde{\Gamma}_X \in U$, and so $\tilde{\Gamma}_{s(X)} = \tilde{\Gamma}_X^\dagger \tilde{\Gamma}_X \in \mathbf{s}(U)$. We already know that $\mathbf{s}(U) = (U_x)^\dagger$ for some $x \in \Sigma^{(0)}$, and hence $(\tilde{\Gamma}_{s(X)})(x) = 1$, which implies that $x \in \mathbf{s}(X)$. Let σ be the unique element of X such that $\mathbf{s}(\sigma) = x$. We claim that $U = \Phi(\sigma)$. Indeed, since $\mathbf{s}(U) = (U_x)^\dagger = (U_{s(\sigma)})^\dagger$ and $\tilde{\Gamma}_X \in U$, we have $\tilde{\Gamma}_X U_{s(\sigma)} \subseteq U$, and hence $\Phi(\sigma) \subseteq U$. Thus $U = \Phi(\sigma) \subseteq \Phi(X)$ by the maximality property of ultrafilters, and so Φ is an open map.

It remains to prove R^\times -equivariance. Fix $t \in R^\times$ and $\sigma \in \Sigma$. Fix a compact open bisection $X \subseteq \Sigma$ containing σ . Then $t \cdot X$ is a compact open bisection containing $t \cdot \sigma$, and $\tilde{\Gamma}_{t \cdot X} = t \tilde{\Gamma}_X$. Since $\mathbf{s}(t \cdot \sigma) = \mathbf{s}(\sigma)$, we have $\Phi(t \cdot \sigma) = (\tilde{\Gamma}_{t \cdot X} U_{s(\sigma)})^\dagger$. Now

$$\tilde{\Gamma}_{t \cdot X} U_{s(\sigma)} = (t \tilde{\Gamma}_X) U_{s(\sigma)} \subseteq t \cdot \Phi(\sigma),$$

and thus $\Phi(t \cdot \sigma) \subseteq t \cdot \Phi(\sigma)$. Hence $\Phi(t \cdot \sigma) = t \cdot \Phi(\sigma)$ since they are ultrafilters. \square

It is now relatively straightforward to show that Φ induces a morphism of discrete R -twists.

Corollary 8.5. *Let (Σ, i, q) be a discrete R -twist over G , let $A := A_R(G; \Sigma)$, and let $B := A_R(G^{(0)}; q^{-1}(G^{(0)}))$. Let*

$$G'^{(0)} \times R^\times \xrightarrow{i'} \Sigma' \xrightarrow{q'} G'$$

be the sequence obtained from (A, B) via [Theorem 5.6](#). Let $\Phi: \Sigma \rightarrow \Sigma'$ be the map of [Proposition 8.4](#). Then there is a (well-defined) map $\Phi_G: G \rightarrow G'$ given by $\Phi_G(q(\sigma)) = q'(\Phi(\sigma))$, and the following diagram commutes.

$$\begin{array}{ccccc} G^{(0)} \times R^\times & \xleftarrow{i} & \Sigma & \xrightarrow{q} & G \\ \downarrow \Phi_G \times \text{id} & & \downarrow \Phi & & \downarrow \Phi_G \\ G'^{(0)} \times R^\times & \xleftarrow{i'} & \Sigma' & \xrightarrow{q'} & G' \end{array}$$

Proof. [Proposition 8.4](#) shows that Φ is R^\times -equivariant. Thus Φ_G is well-defined, and the right-hand square of the diagram commutes. To see that the left-hand square commutes, observe that for all $x \in \Sigma^{(0)}$, we have

$$\Phi(i(q(x), t)) = \Phi(t \cdot x) = t \cdot \Phi(x) = i'(q'(\Phi(x)), t) = i'(\Phi_G(q(x)), t). \quad \square$$

To proceed, we need the following observation about normalisers $n \in N(B)$ with $\text{supp}_G(n)$ a bisection.

Proposition 8.6. *Fix $n \in N(B)$. Then $\text{supp}_G(n)$ is a bisection if and only if $n = \tilde{1}_X$ for a (necessarily unique) compact open bisection X of Σ .*

Proof. Suppose that $X \subseteq \Sigma$ is a compact open bisection of Σ . Then $\text{supp}_G(\tilde{1}_X) = q(X)$ is a compact open bisection of G , since q is a continuous open map that restricts to a homeomorphism of unit spaces. Moreover, $\tilde{1}_X^{-1}(1) = X$, and so uniqueness is clear.

For the converse, suppose that $\text{supp}_G(n)$ is a compact open bisection of G . Note that $n^\dagger n = \tilde{1}_{\mathbf{s}(\text{supp}(n))}$ by [Remark 4.4](#). Fix $\sigma \in \text{supp}(n)$. Then $1 = (n^\dagger n)(\mathbf{s}(\sigma)) = n^\dagger(\sigma^{-1})n(\sigma)$ since $\text{supp}_G(n)$ is a bisection, and so $n(\sigma) \in R^\times$. It follows that n takes values in R^\times , and so by the R^\times -contravariance of n , we have $\text{supp}(n) = R^\times \cdot X$, where $X := n^{-1}(1)$. Since n is continuous, X is open. Moreover, $q(X) = \text{supp}_G(n)$ by the above observation. We claim that $q|_X$ is injective. To see this, suppose that $q(\sigma) = q(\tau)$ for $\sigma, \tau \in X$. Then $\tau = t \cdot \sigma$ for some $t \in R^\times$, and so $n(\sigma) = 1 = n(\tau) = t^{-1}$. Hence $t = 1$, and so $\sigma = \tau$. Therefore, X is homeomorphic to $q(X) = \text{supp}_G(n)$, and hence X is compact. To see that X is a bisection, note that if $\mathbf{s}(\sigma) = \mathbf{s}(\tau)$ for $\sigma, \tau \in X$, then $q(\sigma) = q(\tau)$ since $\text{supp}_G(n)$ is a bisection, and so $\sigma = \tau$ by the injectivity of $q|_X$. Similarly, $\mathbf{r}|_X$ is injective. Since $\tilde{1}_X$ and n are supported on $R^\times \cdot X$ and agree on X , we deduce that $n = \tilde{1}_X$ by [Lemma 2.7](#). \square

It follows from [Proposition 8.6](#) that $X \mapsto \tilde{1}_X$ is an isomorphism of the inverse semi-group of compact open bisections of Σ onto $N(B)$ if the twist $\Sigma \rightarrow G$ satisfies the local bisection hypothesis (using [Corollary 2.10](#)). Therefore, we deduce by so-called noncommutative Stone duality [[31](#)] that $\Sigma \cong \Sigma'$. Moreover, the map Φ is the isomorphism that noncommutative Stone duality provides. The next result gives a more precise statement; namely, that the map Φ of [Proposition 8.4](#) is surjective precisely when the pair of algebras associated to G is an algebraic quasi-Cartan pair.

Theorem 8.7. *Let (Σ, i, q) be a discrete R -twist over G , let $A := A_R(G; \Sigma)$, and let $B := A_R(G^{(0)}; q^{-1}(G^{(0)}))$. Let Σ' be the groupoid of ultrafilters of $N(B)$ defined in [Section 5.1](#). Let $\Phi: \Sigma \rightarrow \Sigma'$ be the map from [Proposition 8.4](#). The following are equivalent.*

- (1) *The pair (A, B) is an algebraic quasi-Cartan pair.*
- (2) *The twist (Σ, i, q) satisfies the local bisection hypothesis.*
- (3) *The map Φ is surjective and, in particular, it is an isomorphism of topological groupoids.*

Proof. [Lemma 4.2](#) gives (1) \iff (2).

To see that (2) \implies (3), let $U \in \Sigma'$ and $n \in U$. Then $\text{supp}_G(n)$ is a bisection of G , and so by [Proposition 8.6](#), there is a compact open bisection X of Σ such that $n = \tilde{1}_X$. For

each $x \in \Sigma^{(0)}$, let U_x be the ultrafilter $\{e \in I(B) : e(x) = 1\}$ of $I(B)$. By [Proposition 8.4](#), we have $\mathbf{s}(U) = (U_x)^\uparrow$ for some $x \in \Sigma^{(0)}$, and since $\tilde{\mathbf{I}}_{\mathbf{s}(X)} = n^\dagger n \in \mathbf{s}(U)$, there exists $e \in U_x$ such that $e \leq \tilde{\mathbf{I}}_{\mathbf{s}(X)}$. Since $e(x) = 1$, [Lemma 8.2\(c\)](#) implies that $x \in \mathbf{s}(X)$. Since X is a bisection, there is a unique $\sigma \in X$ with $\mathbf{s}(\sigma) = x$. Note that $\tilde{\mathbf{I}}_X U_x = nU_x \subseteq U\mathbf{s}(U) = U$, and hence $\Phi(\sigma) = (\tilde{\mathbf{I}}_X U_{\mathbf{s}(\sigma)})^\uparrow \subseteq U$. Thus $U = \Phi(\sigma)$ since these are ultrafilters. Therefore, Φ is surjective and hence is an isomorphism by [Proposition 8.4](#).

To see that [\(3\)](#) \implies [\(2\)](#), fix $n \in N(B) \setminus \{0\}$ and $x \in \mathbf{s}(\text{supp}_G(n))$. Note that $n^\dagger n = \tilde{\mathbf{I}}_{\mathbf{s}(\text{supp}(n))}$ by [Remark 4.4](#), and hence $\mathbf{s}(\text{supp}_G(n)) = \text{supp}_G(n^\dagger n)$. Therefore, $n^\dagger n \in U_x$, and so [Lemma 8.1](#) implies that $(nU_x)^\uparrow$ is an ultrafilter of $N(B)$ containing n , with source U_x . Since Φ is surjective and respects the source map, there exists $\tau \in \Sigma$ with $\mathbf{s}(\tau) = x$ such that $\Phi(\tau) = (nU_x)^\uparrow$. So $n \in \Phi(\tau)$. Let X be a compact open bisection of Σ containing τ , so that $\Phi(\tau) = (\tilde{\mathbf{I}}_X U_{\mathbf{s}(\tau)})^\uparrow$. By [Lemma 8.3\(c\)](#), there exists a compact open set $W \subseteq \mathbf{s}(X)$ such that $\tau \in XW$ and $\tilde{\mathbf{I}}_{XW} \leq n$. Since $XW \subseteq X$ is a bisection, [Lemma 8.2\(a\)](#) implies that $q(\tau)$ is the unique element of $\text{supp}_G(n)$ with $\mathbf{s}(q(\tau)) = x$. Since $x \in \mathbf{s}(\text{supp}_G(n))$ was arbitrary, it follows that \mathbf{s} is injective on $\text{supp}_G(n)$. A symmetric argument shows that \mathbf{r} is injective on $\text{supp}_G(n)$, and so $\text{supp}_G(n)$ is a bisection. \square

To finish off, we deduce that the local bisection hypothesis is encoded algebraically in the sense that if the algebraic pairs corresponding to two twists are isomorphic, then either neither of the two twists satisfies the local bisection hypothesis, or they both do.

Definition 8.8. Let (Σ_1, i_1, q_1) and (Σ_2, i_2, q_2) be discrete R -twists over ample Hausdorff groupoids G_1 and G_2 , respectively. We say that an isomorphism $\Psi: A_R(G_1; \Sigma_1) \rightarrow A_R(G_2; \Sigma_2)$ is *diagonal-preserving* if $\Psi(A_R(G_1^{(0)}; q_1^{-1}(G_1^{(0)}))) = A_R(G_2^{(0)}; q_2^{-1}(G_2^{(0)}))$.

Remark 8.9. Let (Σ, i, q) be a discrete R -twist over G , let $A := A_R(G; \Sigma)$, and let $B := A_R(G^{(0)}; q^{-1}(G^{(0)}))$. As in the untwisted case, we call B the *diagonal subalgebra* of A , even though [Proposition 7.1](#) says that (A, B) is an algebraic diagonal pair if and only if G is principal.

Corollary 8.10. *Let (Σ_1, i_1, q_1) and (Σ_2, i_2, q_2) be discrete R -twists over ample Hausdorff groupoids G_1 and G_2 , respectively. Suppose that $\Sigma_1 \rightarrow G_1$ satisfies the local bisection hypothesis. The following are equivalent.*

- (1) *The twists (Σ_1, i_1, q_1) and (Σ_2, i_2, q_2) are isomorphic.*
- (2) *There exists a diagonal-preserving isomorphism of R -algebras $\Psi: A_R(G_1; \Sigma_1) \rightarrow A_R(G_2; \Sigma_2)$.*

Proof. The implication [\(1\)](#) \implies [\(2\)](#) is immediate. We prove the implication [\(2\)](#) \implies [\(1\)](#). Write (A_1, B_1) and (A_2, B_2) for the algebraic pairs associated with $\Sigma_1 \rightarrow G_1$ and $\Sigma_2 \rightarrow G_2$, respectively. We first show that (A_2, B_2) is an algebraic quasi-Cartan pair. Define $\tilde{P}: A_2 \rightarrow B_2$ by $\tilde{P}(a_2) = \Psi(P_1(\Psi^{-1}(a_2)))$, where P_1 is the unique faithful conditional expectation satisfying [\(AQP\)](#) for the pair (A_1, B_1) . Since Ψ is diagonal-preserving, it is straightforward to prove that $\tilde{P}|_{B_2} = \text{id}_{B_2}$, that $\tilde{P}(b_2 a_2 b_2') = b_2 \tilde{P}(a_2) b_2'$ for all $a_2 \in A_2$ and $b_2, b_2' \in B_2$, and that $\Psi(N(B_1)) = N(B_2)$. The latter implies that \tilde{P} is faithful and that it satisfies condition [\(AQP\)](#). Since Ψ and P_1 are R -linear, \tilde{P} is R -linear. Hence the pair (A_2, B_2) with \tilde{P} is an algebraic quasi-Cartan pair. By [Lemma 4.2](#), (Σ_2, i_2, q_2) satisfies the local bisection hypothesis, and by [Proposition 3.7](#) and [Lemma 4.2](#), \tilde{P} coincides with the conditional expectation of (A_2, B_2) given by restriction of functions.

We now build an isomorphism between the twists. By [Theorem 8.7\(3\)](#) and [Corollary 8.5](#), we can identify each Σ_i with the groupoid of ultrafilters in $N(B_i)$, and we can identify each G_i with the corresponding quotient. Since the partial order on each $N(B_i)$ is defined in terms of the multiplication, it follows that if U is a filter of

$N(B_1)$, then $\Psi(U)$ is a filter of $N(B_2)$, and if V is a filter of $N(B_2)$, then $\Psi^{-1}(V)$ is a filter of $N(B_1)$. Hence the map $\psi: \Sigma_1 \rightarrow \Sigma_2$ given by $\psi(U) := \Psi(U)$ is a well-defined bijection. Moreover, since the multiplication and topology on each Σ_i depend only on the multiplicative structure of $N(B_i)$, it is straightforward to show that ψ is a topological groupoid isomorphism. Since Ψ is R -linear, we have $\psi(tU) = t\psi(U)$ for all $t \in R^\times$ and $U \in \Sigma_1$, and so ψ is R^\times -equivariant. Hence there is a (well-defined) topological groupoid isomorphism $\psi_G: G_1 \rightarrow G_2$ given by $\psi_G(q_1(U)) := q_2(\psi(U))$. It follows that the diagram

$$\begin{array}{ccccc} G_1^{(0)} \times R^\times & \xrightarrow{i_1} & \Sigma_1 & \xrightarrow{q_1} & G_1 \\ \downarrow \psi_G \times \text{id} & & \downarrow \psi & & \downarrow \psi_G \\ G_2^{(0)} \times R^\times & \xrightarrow{i_2} & \Sigma_2 & \xrightarrow{q_2} & G_2 \end{array}$$

gives an isomorphism between of the twists (Σ_1, i_1, q_1) and (Σ_2, i_2, q_2) . \square

If at least one of the two twists is effective, then we can strengthen the previous result by asking only for an isomorphism of twisted Steinberg algebras that restricts to an inclusion of diagonal subalgebras; this implies that in fact it restricts to an isomorphism of diagonal subalgebras.

Corollary 8.11. *Let (Σ_1, i_1, q_1) and (Σ_2, i_2, q_2) be discrete R -twists over ample Hausdorff groupoids G_1 and G_2 , respectively. Suppose that G_1 is effective. The following are equivalent.*

- (1) *The twists (Σ_1, i_1, q_1) and (Σ_2, i_2, q_2) are isomorphic.*
- (2) *There exists a diagonal-preserving isomorphism of R -algebras $\Psi: A_R(G_1; \Sigma_1) \rightarrow A_R(G_2; \Sigma_2)$.*
- (3) *There exists an isomorphism of R -algebras $\Psi: A_R(G_1; \Sigma_1) \rightarrow A_R(G_2; \Sigma_2)$ such that $\Psi(A_R(G_1^{(0)}; q_1^{-1}(G_1^{(0)}))) \subseteq A_R(G_2^{(0)}; q_2^{-1}(G_2^{(0)}))$.*

Proof. Since G_1 is effective, [Lemmas 4.2](#) and [3.6](#) and [Proposition 4.8](#) together imply that (Σ_1, i_1, q_1) satisfies the local bisection hypothesis. The equivalence (1) \iff (2) follows from [Corollary 8.10](#). The implication (2) \implies (3) is clear. We prove that (3) \implies (2). Let Ψ be an isomorphism as in (3). Since G_1 is effective, [Proposition 4.8](#) implies that $B_1 := A_R(G_1^{(0)}; q_1^{-1}(G_1^{(0)}))$ is a maximal abelian subalgebra of $A_R(G_1; \Sigma_1)$, which implies that $\Psi(B_1)$ is a maximal abelian subalgebra of $A_R(G_2; \Sigma_2)$. Since $A_R(G_2^{(0)}; q_2^{-1}(G_2^{(0)}))$ is an abelian subalgebra containing $\Psi(B_1)$, they are equal. \square

9. EXAMPLES SATISFYING THE LOCAL BISECTION HYPOTHESIS

In this short section we show that our theorems apply to a relatively large class of groupoids and twists. In particular, all twists over Deaconu–Renault groupoids with totally disconnected unit spaces satisfy the local bisection hypothesis.

The starting point is to show that there are plenty of groups whose twisted group algebras over large classes of indecomposable rings have no nontrivial units. Most of what we write here is well known.

Recall that a group G is *right orderable* if there is a total order $<$ on G such that $x < y$ implies that $xz < yz$ for all $x, y, z \in G$. If, in addition, the order satisfies $zx < zy$ for all $x, y, z \in G$, then G is said to be *orderable*. Note that a group G is (right) orderable if and only if all of its finitely generated subgroups are too [[41](#), Lemma 13.2.2].

A group G is said to have the *unique product property* if, given two finite nonempty subsets A and B of G , there is an element of AB that can be uniquely written as a

product ab with $a \in A$ and $b \in B$. This property is satisfied by any right-orderable group [41, Theorem 13.1.7] and so, in particular, by any torsion-free abelian group, torsion-free nilpotent group [41, Lemma 13.1.6], and free group. Note that locally indicable groups are right orderable. Here, a group G is *locally indicable* if every finitely generated subgroup admits a nontrivial homomorphism to an infinite cyclic group.

It was shown by Strojnowski [48] that if G has the unique product property and A and B are finite subsets of G with $|A| + |B| > 2$, then there are two distinct elements $g, h \in AB$ which have unique representations $g = ab$ and $h = a'b'$ with $a, a' \in A$ and $b, b' \in B$. Note that every group satisfying the unique product property is torsion-free, and that there are groups satisfying the unique product property that are not right orderable.

If R is an integral domain and G has the unique product property, then it is well known that the group ring RG has no nontrivial units. The same argument applies *mutatis mutandis* to twisted group rings. A more general result for group graded rings can be found in [40, Theorem 3.4]. We include the elementary proof for completeness.

Lemma 9.1. *Let G be a group satisfying the unique product property, let R be a unital integral domain, and let c be a normalised R^\times -valued 2-cocycle on G . Then the only units of the twisted group ring $R(G, c)$ are the elements of the set $\{t\delta_g : t \in R^\times, g \in G\}$.*

Proof. Let e denote the identity of G . Suppose that $a, b \in R(G, c)$ satisfy $ab = ba = 1_{R(G, c)} = \delta_e$. Write $a = \sum_{g \in G} a_g \delta_g$ and $b = \sum_{h \in G} b_h \delta_h$. Let $A := \{g : a_g \neq 0\}$ and $B := \{h : b_h \neq 0\}$. These are finite sets. If a and b are not both trivial units, then $|A| + |B| > 2$. So there exist distinct $g, h \in AB$ which have unique factorisations $g = xy$ and $h = x'y'$ with $x, x' \in A$ and $y, y' \in B$. It follows that the coefficients of δ_g and δ_h in ab are $a_x b_y c(x, y)$ and $a_{x'} b_{y'} c(x', y')$, respectively, and these are both nonzero since R is an integral domain. This contradicts $ab = \delta_e$. Therefore, $|A| + |B| \leq 2$, and so $a = a_g \delta_g$ and $b = b_h \delta_h$ with $g, h \in G$. We then have $a_g b_h c(g, h) = 1$, and so $a_g, b_h \in R^\times$, as required. \square

We call G a *trivial-units-only group* if the twisted group ring $k(G, c)$ has no nontrivial units for every field k and every normalised k^\times -valued 2-cocycle c on G . For example, a group satisfying the unique product property is a trivial-units-only group by Lemma 9.1.

Recall that a commutative unital ring R is *reduced* if it has no nonzero nilpotent elements. This is equivalent to the intersection of all prime ideals of R being trivial. Every integral domain is reduced and indecomposable. However, the class of reduced and indecomposable commutative rings is much larger and includes the coordinate rings of connected (but not necessarily irreducible) affine varieties over algebraically closed fields. For instance, $\mathbb{C}[x, y]/(xy)$, which is the coordinate ring of the union of the coordinate axes in \mathbb{C}^2 , is reduced and indecomposable but not an integral domain.

The following proposition is a variation of [47, Proposition 2.1], which is based on [38, Theorem 3]. It generalises Lemma 9.1.

Proposition 9.2. *Let G be a nontrivial trivial-units-only group as defined above (for example, a group satisfying the unique product property). Let R be a commutative unital ring and let c be an R^\times -valued normalised 2-cocycle on G . Then $R(G, c)$ has no nontrivial units if and only if R is reduced and indecomposable.*

Proof. Let e denote the identity of G . We begin with necessity; this argument applies to all nontrivial groups G . Fix $g \in G \setminus \{e\}$. Suppose first that R is not indecomposable, and let $f \in R \setminus \{0, 1\}$ be an idempotent. Then $a = f\delta_e + (1 - f)\delta_g$ is a nontrivial unit with inverse $b = f\delta_e + (1 - f)c(g, g^{-1})^{-1}\delta_{g^{-1}}$. Indeed, using that c is normalised, we have that $ab = (f + (1 - f)c(g, g^{-1})c(g, g^{-1})^{-1})\delta_e = \delta_e$, and similarly, $ba = \delta_e$, using that $c(g, g^{-1}) = c(g^{-1}, g)$, since c is a normalised 2-cocycle.

Next assume that R is not reduced. If $n \in R \setminus \{0\}$ is nilpotent, then $n\delta_g$ is nilpotent, and so $\delta_e - n\delta_g$ is a nontrivial unit with inverse $\delta_e + n\delta_g$.

For sufficiency, we use some algebraic geometry, following [38]. Since R is indecomposable, its prime (Zariski) spectrum $\text{Spec}(R)$ is connected. If $\mathfrak{p} \in \text{Spec}(R)$, then the residue field at \mathfrak{p} is the field of fractions $\kappa(\mathfrak{p})$ of R/\mathfrak{p} . For each $r \in R$, we write $r(\mathfrak{p})$ for the image of r in $\kappa(\mathfrak{p})$ under the composition $R \rightarrow R/\mathfrak{p} \rightarrow \kappa(\mathfrak{p})$. We have an induced $\kappa(\mathfrak{p})^\times$ -valued normalised 2-cocycle $c_{\mathfrak{p}}$ on G obtained by putting $c_{\mathfrak{p}}(g, h) = c(g, h)(\mathfrak{p})$. Moreover, there is a canonical surjective homomorphism $\pi_{\mathfrak{p}}: R(G, c) \rightarrow \kappa(\mathfrak{p})(G, c_{\mathfrak{p}})$ sending $\sum_{g \in G} a_g \delta_g$ to $a(\mathfrak{p}) := \sum_{g \in G} a_g(\mathfrak{p}) \delta_g$.

Let $a \in R(G, c)$ be a unit, say $a = \sum_{g \in G} a_g \delta_g$. Then $a(\mathfrak{p})$ is a unit of $\kappa(\mathfrak{p})(G, c_{\mathfrak{p}})$ and hence is a trivial unit by the hypothesis that G is a trivial-units-only group. It follows that there is a unique element $h \in G$ with $a_h(\mathfrak{p}) \neq 0$. Denote this element by $g(\mathfrak{p})$. Then $g: \mathfrak{p} \mapsto g(\mathfrak{p})$ is a map from $\text{Spec}(R)$ to G . We claim that g is locally constant. Indeed, if $g(\mathfrak{p}) = h$, then $a_h \notin \mathfrak{p}$. Let $D(a_h)$ denote the basic open set of all prime ideals not containing a_h . For $\mathfrak{q} \in D(a_h)$, we have $a_h(\mathfrak{q}) \neq 0$, and so $g(\mathfrak{q}) = h$. Therefore, g is constant on $D(a_h)$, and so g is locally constant, as claimed. Since $\text{Spec}(R)$ is connected, we deduce that g is constant. Hence there exists $h \in G$ such that $a_h(\mathfrak{p}) \neq 0$ for all $\mathfrak{p} \in \text{Spec}(R)$; and for all $h' \neq h$, we have $a_{h'}(\mathfrak{p}) = 0$ for all $\mathfrak{p} \in \text{Spec}(R)$. Since R is reduced, the intersection of all of its prime ideals is zero, and so $a_{h'} = 0$ for $h' \neq h$. And, since a_h belongs to no prime ideal, and hence to no maximal ideal, it is a unit. Thus $a = a_h \delta_h$ is a trivial unit. \square

Corollary 9.3. *Let R be a reduced and indecomposable commutative ring, and let (Σ, i, q) be a discrete R -twist over G . Let \mathcal{I} denote the interior of the isotropy of G , and suppose that there is a dense set $X \subseteq G^{(0)}$ such that for each $x \in X$, the group \mathcal{I}_x satisfies the unique product property (for example, it might be a right-orderable group). Let $A := A_R(G; \Sigma)$ and $B := A_R(G^{(0)}; q^{-1}(G^{(0)})) \subseteq A$. Then (A, B) is an algebraic quasi-Cartan pair, and the map Φ of Proposition 8.4 from Σ to the twist Σ' is an isomorphism of twists.*

Proof. Lemma 9.1 and Proposition 9.2 show that twisted group rings of groups satisfying the unique product property have no nontrivial units, and so the hypothesis of Lemma 4.6 is satisfied. Therefore, Lemma 4.6 and Lemma 4.7(b) together show that (Σ, i, q) satisfies the local bisection hypothesis. The result follows from Corollary 8.5 and Theorem 8.7. \square

Since free abelian groups are orderable and hence have the unique product property, we have the following corollary of Corollary 9.3.

Corollary 9.4. *Let X be a totally disconnected locally compact Hausdorff space, and let $T: n \mapsto T_n$ be an action of \mathbb{N}^k on X by local homeomorphisms. Let G be the associated Deaconu–Renault groupoid, let R be a reduced and indecomposable commutative ring, and let (Σ, i, q) be a discrete R -twist over G . Let $(A, B) := (A_R(G; \Sigma), A_R(G^{(0)}; q^{-1}(G^{(0)})))$ be the associated algebraic pair. Then (A, B) is an algebraic quasi-Cartan pair, and Theorem 8.7 recovers the groupoid G and the discrete twist Σ from (A, B) .*

Example 9.5. Let Λ be a countable row-finite higher-rank graph with no sources as in [26], let R be a reduced and indecomposable commutative ring, and let ω be normalised R^\times -valued 2-cocycle on Λ as in [27]. The arguments of [27, Section 6] show how to construct a normalised locally constant R^\times -valued 2-cocycle c_ω on the k -graph groupoid G_Λ . Since this groupoid is a Deaconu–Renault groupoid, our results show that the twist $G_\Lambda \times_{c_\omega} R^\times$ over G_Λ obtained from c_ω can be recovered from the twisted Steinberg algebra $A_R(G_\Lambda; G_\Lambda \times_{c_\omega} R^\times)$ and its canonical abelian subalgebra (isomorphic to) $C_c(\Lambda^\infty, R)$.

Moreover, $A_R(G_\Lambda; G_\Lambda \times_{c_\omega} R^\times) \cong A_R(G_\Lambda, c_\omega)$ as in [4, Theorem 4.23], and one can recover the cohomology class of c_ω from the twist $G_\Lambda \times_{c_\omega} R^\times$ over G_Λ as in [4, Corollary 4.16]. So the groupoid G_Λ and the cohomology class of c_ω are invariants of the pair $(A_R(G_\Lambda, c_\omega), C_c(\Lambda^\infty, R))$.

REFERENCES

- [1] G. Abrams and G. Aranda Pino, *The Leavitt path algebra of a graph*, J. Algebra **293** (2005), 319–334. MR 2172342
- [2] P. Ara, J. Bosa, R. Hazrat, and A. Sims, *Reconstruction of graded groupoids from graded Steinberg algebras*, Forum Math. **29** (2017), 1023–1037. MR 3692025
- [3] P. Ara, M.A. Moreno, and E. Pardo, *Nonstable K-theory for graph algebras*, Algebr. Represent. Theory **10** (2007), 157–178. MR 2310414
- [4] B. Armstrong, L.O. Clark, K. Courtney, Y.-F. Lin, K. McCormick, and J. Ramagge, *Twisted Steinberg algebras*, preprint, 2020, arXiv:1910.13005v2 [math.RA].
- [5] B. Armstrong, L.O. Clark, A. an Huef, M. Jones, and Y.-F. Lin, *Filtering germs: groupoids associated to inverse semigroups*, preprint, 2020, arXiv:2010.16113v1 [math.RA].
- [6] S.K. Berberian, *Baer *-rings*, Springer-Verlag, New York-Berlin, 1972, Die Grundlehren der mathematischen Wissenschaften, Band 195. MR 0429975
- [7] T. Bice, *Representing rings on ringoid bundles*, preprint, 2020, arXiv:2012.03006v1 [math.RA].
- [8] T. Bice and C. Starling, *General non-commutative locally compact locally Hausdorff Stone duality*, Adv. Math. **341** (2019), 40–91. MR 3872844
- [9] J.H. Brown, L.O. Clark, C. Farthing, and A. Sims, *Simplicity of algebras associated to étale groupoids*, Semigroup Forum **88** (2014), 433–452.
- [10] J.H. Brown, L.O. Clark, and A. an Huef, *Diagonal-preserving ring *-isomorphisms of Leavitt path algebras*, J. Pure Appl. Algebra **221** (2017), 2458–2481. MR 3646311
- [11] N. Brownlowe, T.M. Carlsen, and M.F. Whittaker, *Graph algebras and orbit equivalence*, Ergodic Theory Dynam. Systems **37** (2017), 389–417. MR 3614030
- [12] T.M. Carlsen, **-isomorphism of Leavitt path algebras over \mathbb{Z}* , Adv. Math. **324** (2018), 326–335. MR 3733888
- [13] T.M. Carlsen and J. Rout, *Diagonal-preserving graded isomorphisms of Steinberg algebras*, Commun. Contemp. Math. **20** (2018), 1–25. MR 3848066
- [14] T.M. Carlsen, E. Ruiz, A. Sims, and M. Tomforde, *Reconstruction of groupoids and C*-rigidity of dynamical systems*, Adv. Math., to appear, arXiv:1711.01052v1 [math.OA].
- [15] L.O. Clark, C. Farthing, A. Sims, and M. Tomforde, *A groupoid generalization of Leavitt path algebras*, Semigroup Forum **89** (2014), 501–517.
- [16] L.O. Clark and R. Hazrat, *étale groupoids and steinberg algebras, a concise introduction*, preprint, 2019, arXiv:1901.01612v1 [math.RA].
- [17] L.O. Clark and A. Sims, *Equivalent groupoids have Morita equivalent Steinberg algebras*, J. Pure Appl. Algebra **219** (2015), no. 6, 2062–2075. MR 3299719
- [18] R. Exel, *Reconstructing a totally disconnected groupoid from its ample semigroup*, Proc. Amer. Math. Soc. **138** (2010), 2991–3001. MR 2644910
- [19] J. Feldman and C.C. Moore, *Ergodic equivalence relations, cohomology, and von Neumann algebras*, Bull. Amer. Math. Soc. **81** (1975), 921–924. MR 425075
- [20] ———, *Ergodic equivalence relations, cohomology, and von Neumann algebras. I*, Trans. Amer. Math. Soc. **234** (1977), 289–324. MR 578656
- [21] ———, *Ergodic equivalence relations, cohomology, and von Neumann algebras. II*, Trans. Amer. Math. Soc. **234** (1977), 325–359. MR 578730
- [22] A. an Huef, A. Kumjian, and A. Sims, *A Dixmier–Douady theorem for Fell algebras*, J. Funct. Anal. **260** (2011), 1543–1581.
- [23] I. Kaplansky, *Rings of operators*, W.A. Benjamin, Inc., New York-Amsterdam, 1968. MR 0244778
- [24] K. Keimel, *Algèbres commutatives engendrées par leurs éléments idempotents*, Canadian J. Math. **22** (1970), 1071–1078. MR 272765
- [25] A. Kumjian, *On C*-diagonals*, Canad. J. Math. **38** (1986), 969–1008. MR 854149
- [26] A. Kumjian and D. Pask, *Higher rank graph C*-algebras*, New York J. Math. **6** (2000), 1–20. MR 1745529

- [27] A. Kumjian, D. Pask, and A. Sims, *On twisted higher-rank graph C^* -algebras*, Trans. Amer. Math. Soc. **367** (2015), no. 7, 5177–5216. MR 3335414
- [28] M.V. Lawson, *Inverse semigroups*, World Scientific Publishing Co., Inc., River Edge, NJ, 1998, The theory of partial symmetries. MR 1694900
- [29] ———, *A noncommutative generalization of Stone duality*, J. Aust. Math. Soc. **88** (2010), 385–404. MR 2827424
- [30] ———, *Non-commutative Stone duality: inverse semigroups, topological groupoids and C^* -algebras*, Internat. J. Algebra Comput. **22** (2012), 1–47. MR 2974110
- [31] M.V. Lawson and D.H. Lenz, *Pseudogroups and their étale groupoids*, Adv. Math. **244** (2013), 117–170. MR 3077869
- [32] K. Matsumoto and H. Matui, *Continuous orbit equivalence of topological Markov shifts and Cuntz–Krieger algebras*, Kyoto J. Math. **54** (2014), 863–877. MR 3276420
- [33] ———, *Full groups of Cuntz–Krieger algebras and Higman–Thompson groups*, Groups Geom. Dyn. **11** (2017), no. 2, 499–531. MR 3668049
- [34] H. Matui, *Homology and topological full groups of étale groupoids on totally disconnected spaces*, Proc. Lond. Math. Soc. (3) **104** (2012), no. 1, 27–56. MR 2876963
- [35] ———, *Topological full groups of one-sided shifts of finite type*, J. Reine Angew. Math. **705** (2015), 35–84. MR 3377390
- [36] ———, *étale groupoids arising from products of shifts of finite type*, Adv. Math. **303** (2016), 502–548. MR 3552533
- [37] P.S. Muhly and D.P. Williams, *Continuous trace groupoid C^* -algebras. II*, Math. Scand. **70** (1992), no. 1, 127–145. MR 1174207
- [38] E. Neher, *Invertible and nilpotent elements in the group algebra of a unique product group*, Acta Appl. Math. **108** (2009), no. 1, 135–139. MR 2540962
- [39] V.V. Nekrashevych, *Cuntz–Pimsner algebras of group actions*, J. Operator Theory **52** (2004), no. 2, 223–249. MR 2119267
- [40] J. Öinert, *Units, zero-divisors and idempotents in rings graded by torsion-free groups*, preprint, 2019, arXiv:1904.04847v2 [math.RA].
- [41] D.S. Passman, *The algebraic structure of group rings*, Pure and Applied Mathematics, Wiley-Interscience [John Wiley & Sons], New York-London-Sydney, 1977. MR 470211
- [42] A.L.T. Paterson, *Groupoids, inverse semigroups, and their operator algebras*, Progress in Mathematics, vol. 170, Birkhäuser Boston, Inc., Boston, MA, 1999. MR 1724106
- [43] J. Renault, *Cartan subalgebras in C^* -algebras*, Irish Math. Soc. Bull. (2008), no. 61, 29–63. MR 2460017
- [44] W. Sawin, *A set theoretic question arising from trying to understand a sheaf cohomology question*, MathOverflow, URL: <https://mathoverflow.net/q/381534> (version: 2021-01-18).
- [45] B. Steinberg, *A groupoid approach to discrete inverse semigroup algebras*, Adv. Math. **223** (2010), no. 2, 689–727. MR 2565546
- [46] ———, *Modules over étale groupoid algebras as sheaves*, J. Aust. Math. Soc. **97** (2014), no. 3, 418–429. MR 3270778
- [47] ———, *Diagonal-preserving isomorphisms of étale groupoid algebras*, J. Algebra **518** (2019), 412–439. MR 3873946
- [48] A. Strojnowski, *A note on u.p. groups*, Comm. Algebra **8** (1980), no. 3, 231–234. MR 558112
- [49] Y. Watatani, *Index for C^* -subalgebras*, Mem. Amer. Math. Soc. **83** (1990), no. 424, vi+117. MR 996807
- [50] R. Wiegand, *Sheaf cohomology of locally compact totally disconnected spaces*, Proc. Amer. Math. Soc. **20** (1969), 533–538. MR 253324

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