Connes' Integral Formula of 1915

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So Long and Thanks for All the Fish

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Summary of this talk

1. The Year 1915: Szegő's limit theorem
2. The Year 1979: Widom's argument
3. The Year 1988: Connes' integral formula
4. The Year 2025: NCG Quantum Ergodicity

This talk is based on joint work with Ed McDonald (Penn State).

The Protagonists

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Szegő Limit Theorem (1915)

Let $0 < w \in C(\mathbb{S}^1)$, and define the Toeplitz matrices

$$T_n := \begin{pmatrix} w_0 & w_1 & w_2 & \cdots & w_n \\ w_{-1} & w_0 & w_1 & \cdots & w_{n-1} \\ w_{-2} & w_{-1} & w_0 & \cdots & w_{n-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ w_{-n} & w_{-n+1} & w_{-n+2} & \cdots & w_0 \end{pmatrix},$$

where w_j are the Fourier coefficients of w. Then,

$$\lim_{n\to\infty} (\det T_n)^{\frac{1}{n}} = \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \log(w(\theta))\, d\theta\right).$$

The Protagonists II

Connes' Integral Formula (1988)

Let (M, g) be a d-dimensional compact orientable Riemannian manifold. Then,

$$\mathrm{Tr}_\omega(M_f(1-\Delta_g)^{-\frac{d}{2}}) = C_d \int_M f\, d\nu_g, \quad f \in C(M),$$

where $M_f : g \mapsto fg$ is a multiplication operator on $L_2(M)$, Δ_g is the Laplace–Beltrami operator, ν_g the Riemannian volume form, and C_d a constant depending on the dimension d.

Part 1: The Year 1915

Toeplitz Matrices

Let $\{e_n\}_{n\in\mathbb{Z}}$ be the standard basis of $L_2(\mathbb{S}^1)$. The matrix elements of the multiplication operator M_w are

$$\langle e_n, M_w e_m\rangle = \frac{1}{2\pi}\int_0^{\theta} e^{-in\theta} w(\theta) e^{im\theta}\, d\theta = w_{m-n}.$$

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The matrix elements of T_n are also of the form w_{m-n}.

We therefore have

$$T_n = P_n M_w P_n,$$

where where P_n is the orthogonal projection in $L_2(\mathbb{S}^1)$ onto the Fourier modes $\{e_0, e_1, \ldots, e_n\}$.

Reworking Szegő's theorem

Szegő's limit theorem gives

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We have

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hence a different way to put Szegő's theorem is

$$\frac{1}{n+1}\mathrm{Tr}(\log(P_n M_w P_n)) \xrightarrow{n\to\infty} \int_{\mathbb{S}^1}\log(w(\theta))\,d\nu(\theta),$$

where P_n is the orthogonal projection in $L_2(\mathbb{S}^1)$ onto the Fourier modes $\{e_0, e_1, \ldots, e_n\}$.

Szegő's even better limit theorem

In fact, Szegő proved a stronger statement.

Szegő's even better limit theorem (1915)

For $w \in C(\mathbb{S}^1)$ real-valued,

$$\frac{1}{n+1}\mathrm{Tr}(f(P_n M_w P_n)) \xrightarrow{n\to\infty} \frac{1}{2\pi}\int_0^{2\pi} f(w(\theta))\, d\theta, \quad f \in C(\mathbb{R}).$$

Part 2: The Year 1979

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Widom's Szegő's limit theorem (1979)

Let (M, g) be a compact Riemannian manifold, $w \in C(M)$ real-valued. Then

$$\frac{\mathrm{Tr}(f(P_\lambda M_w P_\lambda))}{\mathrm{Tr}(P_\lambda)} \xrightarrow{\lambda\to\infty} \int_M f(w(x))\, d\nu_g(x), \quad f \in C(\mathbb{R}),$$

where $P_\lambda = \chi_{[-\lambda,\lambda]}(\Delta_g)$.

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where $P_\lambda = \chi_{[-\lambda,\lambda]}(\Delta_g)$.

Note: Widom only proved his result for homogeneous, but his arguments can easily be used for all compact Riemannian manifolds. Also he gave the microlocal version.

Idea

Widom's proof is quite short.

- The first step is to prove the case where $f(x) = x$, i.e.

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- The next step is to do polynomials, by proving that

$$\frac{\mathrm{Tr}((P_\lambda M_w P_\lambda)^n - P_\lambda M_w^n P_\lambda)}{\mathrm{Tr}(P_\lambda)} \xrightarrow{\lambda\to\infty} 0.$$

Part 3: The Year 1988

Traces

Let $\mathcal{H}$ be a Hilbert space. An eigenvalue sequence of a compact operator $A \in B(\mathcal{H})$ is a sequence $\{\lambda(k, A)\}_{k\in\mathbb{N}}$ of the eigenvalues of A listed with multiplicity, such that $\{|\lambda(k, A)|\}_{k\in\mathbb{N}}$ is non-increasing.

The usual operator trace Tr can be characterised for trace class operators $A \in \mathcal{L}_1 \subset B(\mathcal{H})$ as

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The Dixmier trace is defined on so-called weak trace class operators $A \in \mathcal{L}_{1,\infty} \subset B(\mathcal{H})$ by

$$\mathrm{Tr}_\omega(A) = \omega\text{-}\lim_{n\to\infty} \frac{1}{\log(2+n)} \sum_{k=1}^{n} \lambda(k, A),$$

where $\omega \in \ell_\infty(\mathbb{N})^*$ is an extended limit. Note that $\mathcal{L}_1 \subset \mathcal{L}_{1,\infty}$, but if $A \in \mathcal{L}_1$, $\mathrm{Tr}_\omega(A) = 0$.

Connes' integral formula

Connes proved the following.

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Let (M, g) be a d-dimensional compact Riemannian manifold, $f \in C(M)$. Then,

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Also here, Connes actually gave a microlocal version. Even more, he gave a version without assuming a Riemannian structure.

Part 4: The Year 2025

Comparison

Now compare the first step in Widom's proof, the formula

$$\frac{\mathrm{Tr}(P_\lambda M_w P_\lambda)}{\mathrm{Tr}(P_\lambda)} \xrightarrow{\lambda\to\infty} \int_M w(x)\, d\nu_g(x),$$

where $P_\lambda = \chi_{[-\lambda,\lambda]}(\Delta_g)$, with Connes' formula

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$$\mathrm{Tr}_\omega(M_w(1-\Delta_g)^{-\frac{d}{2}}) = C_d \int_M w(x)\, d\nu_g(x).$$

H.–McDonald

Let $\mathcal{H}$ be a separable Hilbert space, $A \in B(\mathcal{H})$, D self-adjoint with compact resolvent, $P_\lambda := \chi_{[-\lambda,\lambda]}(D)$. If D satisfies Weyl's law $\lambda(k,|D|) \sim Ck^{\frac{1}{d}}$, then for all extended limits $\omega \in \ell_\infty^*$,

$$\frac{\mathrm{Tr}_\omega(A(1+D^2)^{-\frac{d}{2}})}{\mathrm{Tr}_\omega((1+D^2)^{-\frac{d}{2}})} = \omega \circ M\left(\frac{\mathrm{Tr}(P_{\lambda_n} A P_{\lambda_n})}{\mathrm{Tr}(P_{\lambda_n})}\right).$$

Here, $M : \ell_\infty(\mathbb{N}) \to \ell_\infty(\mathbb{N})$ is defined by $M(x)_n = \frac{1}{\log(n+2)} \sum_{k=0}^{n} \frac{x_k}{k}$.

Truncated Spectral Triples

If we have a spectral triple $(\mathcal{A}, \mathcal{H}, D)$, some noncommutative geometers are interested in truncated triples $(P_\lambda A P_\lambda, P_\lambda \mathcal{H}, P_\lambda D)$ (e.g. Connes–Van Suijlkeom).

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Our result shows that if $(\mathcal{A}, \mathcal{H}, D)$ is d-dimensional and D satisfies Weyl's law, then

$$P_\lambda A P_\lambda \mapsto \frac{\mathrm{Tr}(P_\lambda A P_\lambda)}{\mathrm{Tr}(P_\lambda)}$$

is a reasonable approximation of the noncommutative integral

$$A \mapsto \frac{\mathrm{Tr}_\omega(A(1+D^2)^{-\frac{d}{2}})}{\mathrm{Tr}_\omega((1+D^2)^{-\frac{d}{2}})}.$$

Noncommutative Szegő theorem

With Widom's argument, we have a Szegő formula for NCG as well.

H.–McDonald

Let $\mathcal{H}$ be a separable Hilbert space, $A \in B(\mathcal{H})_{sa}$, D self-adjoint with compact resolvent. If D satisfies Weyl's law $\lambda(k, |D|) \sim Ck^{\frac{1}{d}}$, and if $[D, A]$ extends to a bounded operator, then for all extended limits $\omega \in \ell_\infty^*$,

$$\frac{\mathrm{Tr}_\omega(f(A)(1+D^2)^{-\frac{d}{2}})}{\mathrm{Tr}_\omega((1+D^2)^{-\frac{d}{2}})} = \omega \circ M\left(\frac{\mathrm{Tr}(f(P_{\lambda_n} A P_{\lambda_n}))}{\mathrm{Tr}(P_{\lambda_n})}\right), \quad f \in C_c(\mathbb{R}), f(0) = 0.$$

Thanks

So long!