

# Poincaré duality for Cuntz–Pimsner algebras of bimodules

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## Abstract

We present a new approach to Poincaré duality for Cuntz–Pimsner algebras. We provide sufficient conditions under which Poincaré self-duality for the coefficient algebra of a Hilbert bimodule lifts to Poincaré self-duality for the associated Cuntz–Pimsner algebra.

With these conditions in hand, we can constructively produce fundamental classes in  $K$ -theory for a wide range of examples. We can also produce  $K$ -homology fundamental classes for the important examples of Cuntz–Krieger algebras (following Kaminker–Putnam) and crossed products of manifolds by isometries, and their non-commutative analogues.

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# 1 Introduction

In this paper we explore a new approach to Poincaré duality for Cuntz–Pimsner algebras. Our approach emphasises the interaction between the dynamics defined by a bimodule and the topology of its coefficient algebra.

Our motivation was to understand existing proofs of Poincaré duality for  $C^*$ -algebras associated to dynamical systems in a more geometric light: specifically, the proofs of Poincaré duality for Cuntz–Krieger algebras [21],  $k$ -graph algebras [32] and Smale spaces [22]. Other examples of Poincaré duality, such as [9] and [14] are somewhat different, but there is overlap in the algebras treated, if not in the techniques. Overviews of  $C^*$ -algebraic Poincaré duality appear in [4, 23].

Our approach has several elements. Given a suitable coefficient algebra  $A$  satisfying Poincaré self-duality (in the sense that  $A$  is Poincaré dual to its opposite algebra  $A^{\text{op}}$ ), we consider an  $A$ – $A$ -correspondence  $E$ . Our aim is to lift the Poincaré self-duality for  $A$  to a duality for the Cuntz–Pimsner algebra  $\mathcal{O}_E$ .

In the first instance, based on Connes’ work [9], we seek a Poincaré duality between  $\mathcal{O}_E$  and  $\mathcal{O}_E^{\text{op}}$ . On the other hand, the results of Kaminker–Putnam suggest that, when  $E$  is a bi-Hilbertian bimodule, we might also (or instead) expect duality between  $\mathcal{O}_E$  and  $\mathcal{O}_{E^{\text{op}}}$ . It turns out that when  $E$  is an invertible bimodule (that is, a self-Morita-equivalence bimodule), the algebras  $\mathcal{O}_{E^{\text{op}}}$  and  $\mathcal{O}_E^{\text{op}}$  coincide, so the question of which potential dual algebra to consider is moot. But no such isomorphism exists in general, so we explore both possible dual algebras. To aid our study of duality with  $\mathcal{O}_E^{\text{op}}$ , we establish that  $\mathcal{O}_E^{\text{op}}$  is isomorphic to  $\mathcal{O}_{\overline{E}^{\text{op}}}$ , where  $\overline{E}^{\text{op}}$  is the dual module  $\overline{E}$  regarded as a right  $A^{\text{op}}$ -module.

Our first main result provides checkable conditions, for both possible dual algebras, on potential  $K$ -theory and  $K$ -homology fundamental classes that guarantee that they implement a Poincaré duality for  $\mathcal{O}_E$ . The conditions involve the interaction of the dynamics defined by  $E$  with the fundamental classes witnessing the Poincaré self-duality of  $A$ , and the Kasparov class of the defining short exact sequence for  $\mathcal{O}_E$ .

In fact we can obtain significant information from the existence of either “half” of a Poincaré duality pair. We describe how either a  $K$ -theory or a  $K$ -homology fundamental class, even in the absence of its counterpart, provides non-trivial information: isomorphisms of  $K$ -theory groups for

one algebra with  $K$ -homology groups for the other. This is important, because our next main result provides an explicit construction of a  $K$ -theory fundamental class for both possible dual algebras under very mild hypotheses on the bimodule  $E$ . In many cases we obtain explicit representatives of these classes which can be compared directly with known examples.

Examples to which our methods for the  $K$ -theory fundamental class apply directly are: the Wiener solenoids discussed in [11]; crossed products of compact  $\text{spin}^c$  manifolds by isometries; and more generally topological graphs over manifolds and crossed products by injective endomorphisms. We also recover Kaminker and Putnam’s  $K$ -theory fundamental class for Cuntz–Krieger algebras and extend it to more general graph algebras.

Establishing the existence of a  $K$ -homology fundamental class turns out to be more challenging and we have not discovered a general method. We do show that Kaminker and Putnam’s  $K$ -homology fundamental class can be obtained via our approach, and extend their construction to a broader class of graph algebras. We also provide sufficient conditions for the crossed product  $C(M) \rtimes_{\alpha} \mathbb{Z}$  of a compact  $\text{spin}^c$  manifold by a  $\text{spin}^c$ -structure-preserving isometry to satisfy Poincaré duality. This construction extends to  $\theta$ -deformations of manifolds.

Our formulation of Poincaré duality is not always suitable for non-unital algebras. A compactly supported version of  $K$ -homology and the attendant exact sequences etc are required, [25, 35]. This requires either  $RKK$  [25], for  $C(X)$ -algebras, or significant work to establish the required exact sequences for any proposed compactly supported  $K$ -homology. For the most part we leave these issues to future work, focussing on the unital case.

The paper is organised as follows. The coefficient algebras, correspondences, bimodules and Cuntz–Pimsner algebras we consider are discussed in Section 2. In particular we tease out some of the relationships between the two possible algebras that may play the role of Poincaré dual algebra to  $\mathcal{O}_E$ . The two contenders are the opposite algebra  $\mathcal{O}_E^{\text{op}}$  and the Cuntz–Pimsner algebra  $\mathcal{O}_{E^{\text{op}}}$  of the opposite module. Of course this choice influences what we mean by self-duality for  $\mathcal{O}_E$ .

Our sufficient conditions for lifting Poincaré self-duality of the coefficient algebra  $A$  to the Cuntz–Pimsner algebra are described in Section 3. Again, we discuss criteria for both possible dual algebras. Section 4 covers the construction of the  $K$ -theory fundamental class for  $\mathcal{O}_E$ . This uses mapping cone techniques, from [1, 7, 33]. We produce explicit representatives of this class in our main examples. The construction of the  $K$ -homology class for crossed products and for Cuntz–Krieger algebras is described in Section 5.

Our techniques apply to more general pairs of Poincaré dual algebras, as we describe in Appendix A. Finally, in Appendix B we discuss the relationships between the extension classes. These extension classes play a central role throughout the paper, and the fine structure of our Poincaré duality classes (such as summability, real structures etc) will depend on these relationships.

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## 2 Cuntz–Pimsner algebras and associated Kasparov classes

In all the following, we suppose that  $A$  is a separable,  $C^*$ -algebra. Given a right Hilbert  $C^*$ - $A$ -module  $E$  (written  $E_A$  when we want to emphasise the coefficient algebra), we denote the  $C^*$ -algebra of adjointable operators on  $E$  by  $\text{End}_A(E)$ . For  $e, f \in E$  we write  $\Theta_{e,f} \in \text{End}_A(E)$  for the rank-one operator  $\Theta_{e,f}(g) = e \cdot (f | g)$ . We write  $\text{End}_A^0(E)$  for the closed 2-sided ideal

$$\text{End}_A^0(E) := \overline{\text{span}}\{\Theta_{e,f} : e, f \in E\} \subseteq \text{End}_A(E)$$

of generalised compact operators on  $E$ .

We denote by  $\overline{E}$  a copy  $\{\bar{e} : e \in E\}$  of  $E$  as a set with vector-space structure given by  $\lambda\bar{e} + \bar{f} = \overline{\lambda e + f}$  for  $\lambda \in \mathbb{C}$  and  $\bar{e}, \bar{f} \in \overline{E}$ . The vector space  $\overline{E}$  is a left-Hilbert  $A$ -module with  $a \cdot \bar{e} = \overline{e \cdot a^*}$  and  $A(\bar{e} | \bar{f}) = (e | f)_A$ . We call  $\overline{E}$  with this structure the *conjugate module* of  $E$ .

Given a  $C^*$ -algebra  $A$ , we write  $\ell^2(A)$  for the standard  $C^*$ -module

$$\ell^2(A) := \left\{ (a_n)_{n=1}^\infty \in \prod_{\mathbb{N}} A \mid \sum_n a_n^* a_n \text{ converges in } A \right\}$$

endowed with the diagonal right action of  $A$  and with inner product  $((a_n) | (b_n)) = \sum_n a_n^* b_n$ .

We make regular use of frames. A countable frame  $\{e_j\}_{j \geq 1} \subset E_A$  for a countably generated right Hilbert  $C^*$ - $A$ -module  $E$  is a sequence such that

$$\sum_{j \geq 1} \Theta_{e_j, e_j}(e) = e \quad \text{for all } e \in E; \tag{2.1}$$

that is,  $\sum_j \Theta_{e_j, e_j}$  converges strictly to  $\text{Id}_E$  in  $\text{End}_A(E) = \text{Mult}(\text{End}_A^0(E))$ . Frames provide a stabilisation map in the sense of Kasparov as follows. Let  $\{e_j\}$  be a frame for  $E_A$ . A quick calculation using (2.1) shows that there is an isometry

$$v : E_A \rightarrow \ell^2(A) \quad \text{such that} \quad v(e) = ((e_j | e)_A)_{j \geq 1} \quad \text{for all } e \in E. \tag{2.2}$$

So  $p := vv^* \in \text{End}_A(\ell^2(A))$  is a projection and  $E \cong p\ell^2(A)$ . Writing  $\{\delta_i\}$  for the orthonormal basis of the separable Hilbert space  $\ell^2$ , and writing  $\Theta_{i,j}$  for the rank-one operator  $h \mapsto \delta_i(\delta_j | h)$  on  $\ell^2$ , we can express  $p$  as the strict limit  $p := \sum_{i,j} \Theta_{i,j} \otimes (e_i | e_j)_A$ .

### 2.1 Toeplitz–Pimsner and Cuntz–Pimsner algebras

**Definition 2.1.** Let  $A$  be a  $C^*$ -algebra. An  $A$ - $A$ -*correspondence*, or a *correspondence over  $A$* , is a right  $C^*$ - $A$ -module together with a homomorphism  $\phi : A \rightarrow \text{End}_A(E)$ , which we regard as defining a left action of  $A$  on  $E$ . Given  $a \in A$  and  $e \in E$ , we frequently write  $a \cdot e$  for  $\phi(a)e$ .

*For all correspondences  $E$  in this paper, we assume that  $\phi$  is injective and takes values in  $\text{End}_A^0(E)$ .*

The latter is automatic when  $A$  is unital and  $E$  is finitely generated. These hypotheses are not necessary for constructing the Toeplitz–Cuntz–Pimsner algebra (often just called the Toeplitz algebra) and the Cuntz–Pimsner algebra of  $E$  [26], but we will need them for our later results.

Given correspondences  $E, F$  over  $A$ , the formula  $(e \otimes f | e' \otimes f')_A = (f | (e | e')_A \cdot f')_A$  determines a positive-semidefinite sesquilinear form on  $E \odot F$ . Taking the quotient by the subspace of

vectors of length zero and then completing yields the balanced tensor product  $E \otimes_A F$ , which is a correspondence over  $A$  with left action  $a \cdot (e \otimes f) = (a \cdot e) \otimes f$ : see [27, Proposition 4.5].

If  $(e_i)$  and  $(f_j)$  are frames for  $E$  and  $F$ , then  $(e_i \otimes f_j)_{i,j}$  is a frame for  $E \otimes_A F$ : this is immediate for finitely generated modules, but needs a little thought in general. If the left actions on  $E$  and  $F$  are implemented by injective homomorphisms into the compacts, then so is the left action on  $E \otimes_A F$ .

We define  $E^{\otimes 0} := {}_A A_A$ ,  $E^{\otimes 1} := E$ , and  $E^{\otimes n+1} := E^{\otimes n} \otimes_A E$  for  $n \geq 1$ . The *Fock module* of  $E$  is the  $\ell^2$ -direct sum  $\mathcal{F}_E := \bigoplus_{n=0}^{\infty} E^{\otimes n}$  regarded as a correspondence over  $A$  with diagonal left action.

As in [30], the *Toeplitz algebra*  $\mathcal{T}_E$  is the  $C^*$ -subalgebra of  $\text{End}_A(\mathcal{F}_E)$  generated by the creation operators  $T_e$ ,  $e \in E$  given by

$$T_e(e_1 \otimes e_2 \otimes \cdots \otimes e_k) := e \otimes e_1 \otimes e_2 \otimes \cdots \otimes e_k.$$

The adjoint  $T_e^*$  of  $T_e$  is called the annihilation operator associated to  $e$  and satisfies  $T_e^*(e_1 \otimes \cdots \otimes e_k) = (e \mid e_1)_A \cdot e_2 \otimes \cdots \otimes e_k$  for  $k \geq 1$ , and  $T_e^*|_{E^{\otimes 0}} = 0$ . We let  $T_a$  be the operator of left multiplication by  $a \in A$  given on simple tensors by  $T_a(e_1 \otimes \cdots \otimes e_k) = a \cdot e_1 \otimes \cdots \otimes e_k$ .

By [30, Remark 1.2(4)], that each  $\phi(a) \in \text{End}_A^0(E)$  ensures that  $\text{End}_A^0(\mathcal{F}_E) \subseteq \mathcal{T}_E$ . The *Cuntz–Pimsner algebra*  $\mathcal{O}_E$  is defined to be the quotient  $\mathcal{T}_E / \text{End}_A^0(\mathcal{F}_E)$ . Thus we have an exact sequence

$$0 \rightarrow \text{End}_A^0(\mathcal{F}_E) \longrightarrow \mathcal{T}_E \xrightarrow{q} \mathcal{O}_E \rightarrow 0. \quad (2.3)$$

Pimsner shows that  $\mathcal{T}_E$  and  $\mathcal{O}_E$  each enjoy a natural universal property, which we now describe. A *representation* of  $E$  in a  $C^*$ -algebra  $B$  is a pair  $(\psi, \pi)$  consisting of a linear map  $\psi : E \rightarrow B$  and a homomorphism  $\pi : A \rightarrow B$  such that  $\psi(a \cdot e) = \pi(a)\psi(e)$ ,  $\psi(e \cdot a) = \psi(e)\pi(a)$  and  $\pi((e \mid f)_A) = \psi(e)^* \psi(f)$  for all  $e, f \in E$  and  $a \in A$ . The maps  $e \mapsto T_e$  and  $a \mapsto T_a$  constitute a representation of  $E$  whose image generates  $\mathcal{T}_E$ . This representation is universal, meaning that for any representation  $(\psi, \pi)$  of  $E$  in a  $C^*$ -algebra  $B$ , there is a homomorphism  $\psi \times \pi : \mathcal{T}_E \rightarrow B$  such that  $\psi \times \pi(T_e) = \psi(e)$  and  $\psi \times \pi(T_a) = \pi(a)$  for all  $e \in E$  and  $a \in A$  (see [30, Theorem 3.4]).

To describe the universal property of  $\mathcal{O}_E$ , recall from [30] that each representation  $(\psi, \pi)$  of  $E$  in a  $C^*$ -algebra  $B$  determines a homomorphism  $\psi^{(1)} : \text{End}_A^0(E) \rightarrow B$  such that  $\psi^{(1)}(\Theta_{e,f}) = \psi(e)\psi(f)^*$  for all  $e, f \in E$ . Under our assumption that each  $\phi(a) \in \text{End}_A^0(E)$ , the pair  $(\psi, \pi)$  is called *covariant* if  $\psi^{(1)} \circ \phi = \pi$ . The Cuntz–Pimsner algebra  $\mathcal{O}_E$  is generated by the covariant representation of  $E$  given by  $E \ni e \mapsto S_e := q(T_e)$  and  $A \ni a \mapsto S_a := q(T_a)$  that is universal in the following sense. For every covariant representation  $(\psi, \pi)$  of  $E$  in a  $C^*$ -algebra  $B$  there is a homomorphism  $\psi \times \pi : \mathcal{O}_E \rightarrow B$  such that  $(\psi \times \pi)(S_e) = \psi(e)$  and  $(\psi \times \pi)(S_a) = \pi(a)$  for  $e \in E$  and  $a \in A$ .

For each  $z \in \mathbb{T}$  there is a unitary  $U_z : \mathcal{F}_E \rightarrow \mathcal{F}_E$  satisfying  $U_z(\xi) = z^n \xi$  for  $\xi \in E^{\otimes n}$ . Writing  $\text{Ad} U_z \in \text{Aut}(\text{End}_A(\mathcal{F}_E))$  for conjugation by  $U_z$  it is routine to check that  $\text{Ad} U_z$  restricts to an automorphism  $\gamma_z$  of  $\mathcal{T}_E$ , and that the map  $z \mapsto \gamma_z$  is a strongly continuous action of  $\mathbb{T}$  on  $\mathcal{T}_E$ , called the *gauge action*. This action satisfies  $\gamma_z(T_e) = zT_e$  and  $\gamma_z(T_a) = T_a$  for  $e \in E$  and  $a \in A$ . Since  $\text{End}_A^0(\mathcal{F}_E)$  is  $\gamma$ -invariant,  $\gamma$  descends to an action, also denoted  $\gamma$  and called the gauge action, of  $\mathbb{T}$  on  $\mathcal{O}_E$ . Writing  $S_e = q(T_e)$  and  $S_a = q(T_a)$  we have

$$\gamma_z(S_e) = zS_e \quad \text{and} \quad \gamma_z(S_a) = S_a \quad \text{for all } e \in E \text{ and } a \in A. \quad (2.4)$$

## 2.2 Pimsner's six-term sequences

Some important  $KK$ -classes arise in the study of Cuntz–Pimsner algebras, and we summarise them now. The first is the class of the Morita-equivalence module  $\mathcal{F}_E$  given by

$$[\mathcal{F}_E] = [\mathrm{End}_A^0(\mathcal{F}_E), (\mathcal{F}_E)_A, 0] \in KK(\mathrm{End}_A^0(\mathcal{F}_E), A).$$

Since this  $KK$ -class is given by a Morita-equivalence bimodule, it is invertible in  $KK$  with inverse given by the class of the conjugate module  $[\overline{\mathcal{F}_E}] = [A, \overline{(\mathcal{F}_E)_{\mathrm{End}_A^0(\mathcal{F}_E)}}, 0]$ .

The next two  $KK$ -classes of particular importance to us arise from the inclusions  $\iota_{A,\mathcal{T}} : A \hookrightarrow \mathcal{T}_E$  and  $\iota_{E,\mathcal{T}} : \mathrm{End}_A^0(\mathcal{F}_E) \hookrightarrow \mathcal{T}_E$ . These homomorphisms define classes

$$[\iota_{A,\mathcal{T}}] = [A, (\mathcal{T}_E)_{\mathcal{T}_E}, 0] \quad \text{and} \quad [\iota_{E,\mathcal{T}}] = [\mathrm{End}_A^0(\mathcal{F}_E), (\mathcal{T}_E)_{\mathcal{T}_E}, 0]$$

in  $KK(A, \mathcal{T}_E)$  and  $KK(\mathrm{End}_A^0(\mathcal{F}_E), \mathcal{T}_E)$  respectively.

The fourth key  $KK$ -class was introduced by Pimsner in [30]. Let  $P$  denote the projection onto  $\mathcal{F}_E \ominus A := \bigoplus_{n=1}^{\infty} E^{\otimes n} \subseteq \mathcal{F}_E$ . Let  $\pi_0$  be the inclusion  $\mathcal{T}_E \hookrightarrow \mathrm{End}_A(\mathcal{F}_E)$ . There is a representation  $(\psi, \rho)$  of  $E$  on  $\mathrm{End}_A(\mathcal{F}_E)$  such that  $\psi(e) = T_e P$  and  $\rho(a) = T_a P$ . So the universal property of  $\mathcal{T}_E$  gives a homomorphism  $\pi_1 : \mathcal{T}_E \rightarrow \mathrm{End}_A(\mathcal{F}_E)$  such that  $\pi_1(T_a) = \rho(a)$  and  $\pi_1(T_e) = \psi(e)$  for all  $a \in A$  and  $e \in E$ . To describe  $\pi_1$  explicitly, observe that  $\mathcal{F}_E \ominus A$  is isomorphic to  $\mathcal{F}_E \otimes_A E$ . Under this isomorphism,  $\pi_1$  is identified with  $\pi_0 \otimes 1_E$ . That is

$$\pi_1(b)(e_1 \otimes \cdots \otimes e_n) = \pi_0(b)(e_1 \otimes \cdots \otimes e_{n-1}) \otimes e_n \quad \text{for } b \in \mathcal{T}_E \text{ and } e_i \in e. \quad (2.5)$$

In particular, the essential subspace of  $\pi_1$  is  $\mathcal{F}_E \ominus A \subseteq \mathcal{F}_E$ .

Pimsner defines a class  $[P] \in KK(\mathcal{T}_E, A)$  as the  $KK$ -class  $\left[ \left( \mathcal{T}_E, \pi_0 \oplus \pi_1(\mathcal{F}_E \oplus \mathcal{F}_E), \begin{pmatrix} 0 & \mathrm{Id} \\ \mathrm{Id} & 0 \end{pmatrix} \right) \right]$ . We obtain an equivalent Kasparov module by restricting to the essential submodule for  $\pi_0 \oplus \pi_1$  (see for instance [18, Lemma 8.3.8]). Explicitly, let  $\iota_{\mathcal{F}} : \mathcal{F}_E \ominus A \rightarrow \mathcal{F}_E$  be the inclusion map. Regard  $\pi_0 \oplus \pi_1$  as an adjointable left action of  $\mathcal{T}_E$  on  $\mathcal{F}_E \oplus \mathcal{F}_E \ominus A$ . Then  $[P]$  is represented by the nondegenerate Kasparov module

$$\left( \mathcal{T}_E, \begin{pmatrix} \mathcal{F}_E \\ \mathcal{F}_E \ominus A \end{pmatrix}_A, \begin{pmatrix} 0 & \iota_{\mathcal{F}} \\ P & 0 \end{pmatrix} \right). \quad (2.6)$$

It turns out (see Theorem 4.4 of [30]) that  $[P]$  and  $[\iota_{A,\mathcal{T}}]$  are mutually inverse  $KK$ -equivalences:

$$[\iota_{A,\mathcal{T}}] \otimes_{\mathcal{T}_E} [P] = \mathrm{Id}_{KK(A,A)}, \quad \text{and} \quad [P] \otimes_A [\iota_{A,\mathcal{T}}] = \mathrm{Id}_{KK(\mathcal{T}_E, \mathcal{T}_E)}.$$

In particular,  $[P] \otimes_A \cdot : K^*(A) \rightarrow K^*(\mathcal{T}_E)$  is an isomorphism in  $K$ -homology.

The fifth and final  $KK$ -class we will need later arises from the module  $E_A$  itself. Recalling that we assume that the left action of  $A$  on  $E$  is by compact endomorphisms, we obtain a class

$$[E] := [A, E_A, 0] \in KK(A, A).$$

Applying this construction with  $E = {}_A A_A$ , yields the Kasparov module  $(A, A_A, 0)$ , which is the identity element in the ring  $KK(A, A)$ . We will denote this variously by  $[A]$ ,  $\mathrm{Id}_{KK(A,A)}$ ,  $\mathrm{Id}_A$  or even just [1].

Pimsner [30, Theorem 4.9] combines the  $KK$ -equivalences  $[\mathcal{F}_E]$  and  $[\iota_{A,\mathcal{T}}]$  with the six-term exact sequence in  $KK$  (in the first variable) for the defining extension  $0 \rightarrow \mathrm{End}_A^0(\mathcal{F}_E) \xrightarrow{q} \mathcal{T}_E \xrightarrow{q} \mathcal{O}_E \rightarrow 0$

to show that for any separable  $C^*$ -algebra  $B$  we obtain the exact sequence

$$\begin{array}{ccccc}
KK^0(\mathcal{O}_E, B) & \xrightarrow{(\iota_{A,\mathcal{O}})^*} & KK^0(A, B) & \xrightarrow{(\text{Id}_{KK(A,A)} - [E]) \otimes_A \cdot} & KK^0(A, B) \\
\uparrow \partial & & & & \downarrow \partial \\
KK^1(A, B) & \xleftarrow{(\text{Id}_{KK(A,A)} - [E]) \otimes_A \cdot} & KK^1(A, B) & \xleftarrow{(\iota_{A,\mathcal{O}})^*} & KK^1(\mathcal{O}_E, B)
\end{array} \tag{2.7}$$

in which the boundary maps  $\partial$  are given by the taking the Kasparov product with the class of the defining extension  $0 \rightarrow \text{End}_A^0(\mathcal{F}_E) \rightarrow \mathcal{T}_E \xrightarrow{q} \mathcal{O}_E \rightarrow 0$ . More explicitly, the extension  $0 \rightarrow \text{End}_A^0(\mathcal{F}_E) \rightarrow \mathcal{T}_E \xrightarrow{q} \mathcal{O}_E \rightarrow 0$  yields a class  $\varepsilon \in KK^1(\mathcal{O}_E, \text{End}_A^0(\mathcal{F}_E))$ . Taking the Kasparov product with the Morita-equivalence bimodule  $\mathcal{F}_E$ , yields a class

$$[\text{ext}] := \varepsilon \otimes_{\text{End}} [\mathcal{F}_E] \in KK^1(\mathcal{O}_E, A). \tag{2.8}$$

The boundary maps  $\partial : K^i(A, B) \rightarrow KK^{1-i}(\mathcal{O}_E, B)$  are then given by  $\partial(\Theta) = [\text{ext}] \otimes_A \Theta$ . Explicit representatives of the class  $[\text{ext}] \in KK^1(\mathcal{O}_E, A)$  appear in [36, 17]. We use them to produce concrete representatives of various classes discussed in this paper.

We conclude this section by using the exact sequence (2.7) to see that for non-trivial dynamics the class  $[P]$  of (2.6) does not arise from a class over the Cuntz-Pimsner algebra. Specifically, we obtain the following non-lifting result.

**Proposition 2.2.** *Let  $E$  be a  $C^*$ -correspondence over  $A$  with  $A$  nuclear. Suppose that  $[E] \neq \text{Id}_{KK(A,A)}$  in  $KK(A, A)$ . Then there is no class  $x \in KK^0(\mathcal{O}_E, A)$  such that  $q^*x = [P]$ . If  $(A, E_A, 0) = (A, A_A, 0)$  then  $\mathcal{O}_E \cong A \otimes C(S^1)$  and, writing  $\text{ev} : C(S^1) \rightarrow \mathbb{C}$  for evaluation at 1, the class  $x = [(\mathcal{O}_E, \text{ev}A_A, 0)]$  satisfies  $q^*x = [P]$ .*

*Proof.* We prove the contrapositive. Suppose that  $x \in KK^0(\mathcal{O}_E, A)$  satisfies  $q^*x = [P]$ . Using the relations  $(\iota_{A,\mathcal{O}})^* = (\iota_{A,\mathcal{T}})^* \circ q^*$  and  $[\iota_{A,\mathcal{T}}] = [P]^{-1}$  we see that

$$(\iota_{A,\mathcal{O}})^*x = (\iota_{A,\mathcal{T}})^*q^*x = (\iota_{A,\mathcal{T}})^*[P] = \text{Id}_{KK(A,A)}.$$

So exactness of Pimsner's  $K$ -homology exact sequence

$$\dots \rightarrow KK(\mathcal{O}_E, A) \xrightarrow{(\iota_{A,\mathcal{O}})^*} KK(A, A) \xrightarrow{[A] - [E]} \dots$$

implies that  $([A] - [E]) \otimes_A \text{Id}_{KK(A,A)} = 0$ , forcing  $[A] - [E] = 0$ . Now suppose that  $(A, E_A, 0) = (A, A_A, 0)$ . Then the class  $x$  of the cycle  $(\mathcal{O}_E, \text{ev}A, 0)$  satisfies  $(\iota_{A,\mathcal{O}})^*x = (A, A, 0) = \text{Id}_{KK(A,A)}$ .  $\square$

### 2.3 Bi-Hilbertian bimodules and their Cuntz–Pimsner algebras

Some of our results require bimodules that admit  $A$ – $A$ -correspondence structures for both the left and right actions. Exploiting this bi-Hilbertian structure allows us to associate multiple Cuntz–Pimsner algebras to each such module. Here we clarify the relationships between these Cuntz–Pimsner algebras.

**Definition 2.3.** Let  $A$  be a separable  $C^*$ -algebra. Following [20], a *bi-Hilbertian  $A$ -bimodule* is a countably generated full right  $C^*$ - $A$ -module with inner product  $(\cdot | \cdot)_A$  which is also a countably generated full left  $C^*$ - $A$ -module with inner product  ${}_A(\cdot | \cdot)$  such that the left  $A$ -action is adjointable for  $(\cdot | \cdot)_A$ , and the right  $A$ -action is adjointable for  ${}_A(\cdot | \cdot)$ .

If  $E$  is a bi-Hilbertian  $A$ -bimodule, then it is complete in the norms induced by both inner products, and hence those norms are equivalent (see [36, Lemma 2.2]).

When the coefficient algebra  $A$  is unital, our main results apply to finitely generated modules. For nonunital  $A$ , our results apply to modules  $E$  arising as the restriction to  $A$  of a finitely generated module over some unitisation of  $A$ . As detailed in [37], this condition is closely related to the finiteness of the right Watatani index of  $E$ , defined in [20] as follows.

**Definition 2.4** (Kajiwara–Pinzari–Watatani). A countably generated bi-Hilbertian  $A$ – $B$  bimodule has *finite right numerical index* if there is a constant  $\lambda$  such that  $\|\sum_j {}_A(e_j | e_j)\| \leq \lambda \|\sum_j \Theta_{e_j, e_j}\|$  for all finite subsets  $\{e_j\}$  of  $E$ . It has *finite right Watatani index* if it has finite right numerical index and there is a frame  $(e_j)$  for  $E_B$  such that

$$\sum_j {}_A(e_j | e_j)$$

converges strictly in the multiplier algebra  $\text{Mult}(A)$ .

*Remark 2.5.* By [20, Corollaries 2.24 and 2.28], the bi-Hilbertian bimodule  ${}_A E_B$  has finite right Watatani index if and only if the left action of  $A$  is by compacts, and then the strict limit

$$e^\beta := \sum_j {}_A(e_j | e_j)$$

is independent of the choice of frame  $(e_j)$ , and is a positive central element of  $\text{Mult}(A)$ . We call this element the right Watatani index of  $E$ . When the left action is injective,  $e^\beta$  is also invertible (justifying our notation). For unital algebras, we have  $\text{Mult}(A) = A$  and the strict topology is the norm topology. In this case, bi-Hilbertian  $A$ -bimodules with finite right Watatani index are finitely generated and projective as right modules, [20, Corollary 2.25]. Left Watatani index is defined analogously.

*Remark 2.6.* Each left-Hilbert  $A$ -module  $E$  determines a right-Hilbert  $A^{\text{op}}$ -module  $E^{\text{op}}$ , and vice-versa. More precisely, let  $E^{\text{op}}$  be a copy of the Banach space  $E$  (we write  $e^{\text{op}} \in E^{\text{op}}$  for the element corresponding to  $e \in E$ ). There is a right action of  $A^{\text{op}}$  on  $E^{\text{op}}$  given by  $e^{\text{op}} \cdot a^{\text{op}} = (a \cdot e)^{\text{op}}$ , and there is an  $A^{\text{op}}$ -valued inner-product on  $E^{\text{op}}$  (see [28, page 1625]) given by

$$(e^{\text{op}} | f^{\text{op}})_{A^{\text{op}}} = {}_A(e | f)^{*{\text{op}}} = {}_A(f | e)^{\text{op}}.$$

So given a right Hilbert  $A$ -module  $E_A$ , the conjugate module  $\overline{E}$  of Section 2 determines a right-Hilbert  $A^{\text{op}}$  module  $\overline{E}^{\text{op}}$ .

**Lemma 2.7.** *Given  $A$ – $A$ -correspondences  $E_1, \dots, E_n$ , there is an isometric conjugate-linear map from  $E_1 \otimes_A \dots \otimes_A E_n$  to  $\overline{E}_1^{\text{op}} \otimes_{A^{\text{op}}} \dots \otimes_{A^{\text{op}}} \overline{E}_n^{\text{op}}$  such that*

$$e_1 \otimes e_2 \otimes \dots \otimes e_n \mapsto \overline{e}_1^{\text{op}} \otimes \overline{e}_2^{\text{op}} \otimes \dots \otimes \overline{e}_n^{\text{op}} \tag{2.9}$$

for all  $e_i \in E_i$ . If the modules  $E_i$  are bi-Hilbertian  $A$ -bimodules, then

$$e_1 \otimes e_2 \otimes \dots \otimes e_n \mapsto e_n^{\text{op}} \otimes \dots \otimes e_1^{\text{op}} \tag{2.10}$$

is a linear isomorphism from the left inner product  $A$ -module  $({}_A E_1)_A \otimes \dots \otimes_A ({}_A E_n)$  to the right inner product  $A^{\text{op}}$ -module  $\overline{E}_1^{\text{op}} \otimes_{A^{\text{op}}} \dots \otimes_{A^{\text{op}}} \overline{E}_n^{\text{op}}$ .



*Proof.* The result follows by an induction argument from the case  $n = 2$ . For  $e_1, e_2 \in E_1$  and  $f_1, f_2 \in E_2$ , we have

$$\begin{aligned} (\overline{e_1}^{\text{op}} \otimes \overline{f_1}^{\text{op}} \mid \overline{e_2}^{\text{op}} \otimes \overline{f_2}^{\text{op}})_{A^{\text{op}}} &= (\overline{f_1}^{\text{op}} \mid (\overline{e_1}^{\text{op}} \mid \overline{e_2}^{\text{op}})_{A^{\text{op}}} \cdot \overline{f_2}^{\text{op}})_{A^{\text{op}}} = (\overline{f_1}^{\text{op}} \mid ((e_1 \mid e_2)_A^*)^{\text{op}} \cdot \overline{f_2}^{\text{op}})_{A^{\text{op}}} \\ &= (\overline{f_1}^{\text{op}} \mid \overline{(e_1 \mid e_2)_A \cdot f_2}^{\text{op}})_{A^{\text{op}}} = ((f_1 \mid (e_1 \mid e_2)_A \cdot f_2)_A^*)^{\text{op}} \\ &= ((e_1 \otimes f_1 \mid e_2 \otimes f_2)_A^*)^{\text{op}}. \end{aligned}$$

So for any finite collection of elementary tensors  $e_{1,1} \otimes \cdots \otimes e_{n,1}, \dots, e_{1,J} \otimes \cdots \otimes e_{n,J}$ , we have  $\|\sum_{j=1}^J e_{1,j} \otimes \cdots \otimes e_{n,j}\|^2 = \|\sum_{j=1}^J \overline{e_{1,j}}^{\text{op}} \otimes \cdots \otimes \overline{e_{n,j}}^{\text{op}}\|^2$ . Hence the formula (2.9) determines a well-defined isometric linear map from  $E^{\otimes n}$  to  $(\overline{E}^{\text{op}})^{\otimes n}$ .

For the second statement we again consider the case  $n = 2$ . We have

$$\begin{aligned} (e_2^{\text{op}} \otimes e_1^{\text{op}} \mid f_2^{\text{op}} \otimes f_1^{\text{op}})_{A^{\text{op}}} &= (e_1^{\text{op}} \mid (e_2^{\text{op}} \mid f_2^{\text{op}})_{A^{\text{op}}} f_1^{\text{op}})_{A^{\text{op}}} = (e_1^{\text{op}} \mid A(f_2 \mid e_2)^{\text{op}} f_1^{\text{op}})_{A^{\text{op}}} \\ &= (e_1^{\text{op}} \mid (f_{1A}(f_2 \mid e_2))^{\text{op}})_{A^{\text{op}}} = A((f_{1A}(f_2 \mid e_2) \mid e_1)^{\text{op}}) \\ &= (A(e_1 \otimes e_2 \mid f_1 \otimes f_2))^{\text{op}*}. \end{aligned}$$

The remainder of the argument is as before.  $\square$

In general, given a bi-Hilbertian  $A$ -bimodule, the Cuntz–Pimsner algebra of  $E^{\text{op}}$  can be quite different from that of  $E$  (see Example 2.11). By contrast, the Cuntz–Pimsner algebras of  $E$  and  $\overline{E}^{\text{op}}$  are anti-isomorphic.

**Lemma 2.8.** *Let  $E$  be a countably generated right  $A$ -module with an adjointable and non-degenerate left action of  $A$ . Then the conjugate module  $\overline{E}^{\text{op}}$  is a correspondence over  $A^{\text{op}}$  and there is an isomorphism  $\mathcal{O}_E^{\text{op}} \cong \mathcal{O}_{\overline{E}^{\text{op}}}$  that carries  $S_e^{\text{op}}$  to  $S_{\overline{e}}^{\text{op}*}$  for all  $e \in E$ .*

*Proof.* Define  $\psi : E \rightarrow \mathcal{O}_{\overline{E}^{\text{op}}}^{\text{op}}$  by  $\psi(e) = S_{\overline{e}}^{\text{op}*}$ , and  $\pi : A \rightarrow \mathcal{O}_{\overline{E}^{\text{op}}}^{\text{op}}$  by  $\pi(a) = \iota_{A^{\text{op}}, \mathcal{O}_{\overline{E}^{\text{op}}}^{\text{op}}}(a^{\text{op}})^{\text{op}}$ . It is routine to check that  $(\psi, \pi)$  is a covariant representation of  $E$ , and so determines a homomorphism  $\mathcal{O}_E \rightarrow \mathcal{O}_{\overline{E}^{\text{op}}}^{\text{op}}$ . The image of this representation clearly contains all the generators of  $\mathcal{O}_{\overline{E}^{\text{op}}}^{\text{op}}$ , so it is surjective, and since it intertwines the two gauge actions, it is injective by the gauge-invariant uniqueness theorem [26, Section 6]. Taking opposite algebras now gives the result.  $\square$

## 2.4 Invertible $A$ – $A$ bimodules

An important special case of bi-Hilbertian bimodules is the class of  $C^*$ - $A$ – $A$ -correspondences that are invertible in the category whose objects are  $C^*$ -algebras and whose morphisms are isomorphism classes of  $C^*$ -correspondences [13, Lemma 2.4]. We will call these *invertible  $A$ – $A$  bimodules*, or just *invertible bimodules*; they are commonly called imprimitivity  $A$ – $A$ -bimodules or self-Morita-equivalence bimodules. The morphisms in the  $KK$ -category defined by invertible bimodules are also invertible.

**Definition 2.9.** Let  $A$  be a  $C^*$ -algebra. An *invertible bimodule* over  $A$  is a bi-Hilbertian  $A$ -bimodule  $E$  whose inner products are both full and satisfy the imprimitivity condition

$$A(e \mid f)g = e(f \mid g)_A, \quad \text{for all } e, f, g \in E.$$

These include the modules of sections of complex line bundles over locally compact spaces, and modules arising from automorphisms of  $C^*$ -algebras. Hence, the class of Cuntz–Pimsner algebras of invertible bimodules includes all crossed products by  $\mathbb{Z}$  (see [36, Section 2.1] for examples).

Pimsner identified  $\mathcal{O}_E$  with  $\mathcal{O}_{E \otimes_A \mathcal{O}_E^\gamma}$ , where  $\mathcal{O}_E^\gamma$  is the fixed point algebra for the gauge action (2.4). In [1] it was shown that  $E \otimes_A \mathcal{O}_E^\gamma$  is an invertible bimodule over  $\mathcal{O}_E^\gamma$ . Unfortunately, the relationship between  $A$  and  $\mathcal{O}_E^\gamma$  is typically fairly complicated. For example, in the standard realisation of the Cuntz algebra  $\mathcal{O}_n$  as the Cuntz–Pimsner algebra of  $\mathbb{C}^n$ , the coefficient algebra  $A$  is  $\mathbb{C}$ , while  $\mathcal{O}_E^\gamma$  is the UHF algebra  $M_{n^\infty}$ . So the isomorphism  $\mathcal{O}_E \cong \mathcal{O}_{E \otimes_A \mathcal{O}_E^\gamma}$  does not help us relate  $\mathcal{O}_E$  to  $A$  except when  $E$  is already an invertible bimodule, which happens if and only if  $\mathcal{O}_E^\gamma \cong A$ , [26].

If  $E$  is an invertible bimodule, the conjugate module  $\bar{E}$  satisfies  $E \otimes_A \bar{E} \cong A \cong \bar{E} \otimes_A E$ . In particular, writing  $E^{-\otimes n} := \bar{E}^{\otimes n}$  for  $n \geq 1$ , we have  $E^{\otimes m} \otimes_A E^{\otimes n} \cong E^{\otimes(m+n)}$  for all  $m, n \in \mathbb{Z}$ . We define the integer-graded Fock space of an invertible bimodule over  $A$  to be the module  $\mathcal{F}_{E, \mathbb{Z}} = \bigoplus_{n \in \mathbb{Z}} E^{\otimes n}$ .

**Lemma 2.10.** *Suppose the  $A$ -bimodule  $E$  is an invertible bimodule. Let  $(\psi, \pi)$  and  $(\psi^{\text{op}}, \pi^{\text{op}})$  denote the universal representations of  $E$  and  $E^{\text{op}}$  in their Cuntz–Pimsner algebras. There is a left action  $L$  of  $\mathcal{O}_E$  on  $\mathcal{F}_{E, \mathbb{Z}}$  by adjointable operators for the right  $A$ -module inner-product such that  $L(\pi(a))\xi = a \cdot \xi$  for all  $\xi$  and*

$$L(\psi(e))(\xi) = \begin{cases} e \otimes_A \xi & \text{if } \xi \in \bigcup_{n=1}^{\infty} E^{\otimes n} \\ e \cdot a \in E & \text{if } \xi = a \in A = E^{\otimes 0} \\ {}_A(e \mid e_1) \cdot \bar{e}_2 \otimes_A \cdots \otimes_A \bar{e}_n & \text{if } \xi = \bar{e}_1 \otimes_A \cdots \otimes_A \bar{e}_n \in \bigcup_{n=1}^{\infty} \bar{E}^{\otimes n}. \end{cases}$$

There is a right action of  $\mathcal{O}_{E^{\text{op}}}$  on  $\mathcal{F}_{E, \mathbb{Z}}$  by adjointable operators for the left  $A$ -module inner-product such that  $R(\pi^{\text{op}}(a^{\text{op}}))\xi = \xi \cdot a$  and

$$R(\psi^{\text{op}}(e^{\text{op}}))(\xi) = \begin{cases} \xi \otimes_A e & \text{if } \xi \in \bigcup_{n=1}^{\infty} E^{\otimes n} \\ a \cdot e & \text{if } \xi = a \in A = E^{\otimes 0} \\ \bar{e}_1 \otimes_A \cdots \otimes_A \bar{e}_{n-1} \cdot (e_n \mid e)_A & \text{if } \xi = \bar{e}_1 \otimes_A \cdots \otimes_A \bar{e}_n \in \bigcup_{n=1}^{\infty} \bar{E}^{\otimes n}. \end{cases}$$

Moreover, these actions commute.

*Proof.* The formulas in the lemma define left- and right-creation operators  $L_e, R_{e^{\text{op}}}$  for each  $e \in E$ . Routine calculations show that  $L_e$  is adjointable for  $(\cdot \mid \cdot)_A$  with adjoint given by

$$\begin{aligned} L_e^*(a) &= \bar{e} \cdot a, & L_e^*(e_1 \otimes \cdots \otimes e_n) &= (e \mid e_1)_A e_2 \otimes \cdots \otimes e_n, \\ L_e^*(\bar{e}_1 \otimes \cdots \otimes \bar{e}_k) &= \bar{e} \otimes \bar{e}_1 \otimes \bar{e}_2 \otimes \cdots \otimes \bar{e}_k, & \text{and } L_e^*(f) &= (e \mid f)_A. \end{aligned}$$

Similarly each  $R_{e^{\text{op}}}$  is adjointable for  ${}_A(\cdot \mid \cdot)$  with

$$\begin{aligned} R_{e^{\text{op}}}^*(a) &= a \bar{e}, & R_{e^{\text{op}}}^*(e_1 \otimes \cdots \otimes e_n) &= e_1 \otimes \cdots \otimes e_{n-1} (e_n \mid e)_A, \\ R_{e^{\text{op}}}^*(\bar{e}_1 \otimes \cdots \otimes \bar{e}_k) &= \bar{e}_1 \otimes \cdots \otimes \bar{e}_k \otimes \bar{e}, & \text{and } R_{e^{\text{op}}}^*(f) &= {}_A(f \mid e). \end{aligned}$$

Straightforward calculations using the imprimitivity condition and the formulas for  $L_e$  and  $L_f^*$  above show that  $L_e L_f^* = L(\pi({}_A(e \mid f)))$  for all  $e, f \in E$ . Since, for an invertible bimodule, the map  $\Theta_{e, f} \mapsto {}_A(e \mid f)$  is an isomorphism of  $\text{End}_A^0(E_A)$  onto  $A$ , it follows that  $L$  is Cuntz–Pimsner covariant. Similarly,  $R$  is Cuntz–Pimsner covariant.

To see that  $(L(x)\xi)R(y) = L(x)(\xi R(y))$ , it suffices to consider  $\xi$  an elementary tensor in some  $E^{\otimes n}$  and (by symmetry) to consider  $x$  of the form  $S_e$  and  $y$  either of the form  $\psi^{\text{op}}(f^{\text{op}})$  or of the form  $\psi^{\text{op}}(f^{\text{op}})^*$ . This is trivial for  $|n| \geq 2$ , and also when  $y = \psi^{\text{op}}(f^{\text{op}})$  and  $n \neq -1$  and when  $y = \psi^{\text{op}}(f^{\text{op}})^*$  and  $n \neq 0$ . When  $y = \psi^{\text{op}}(f^{\text{op}})$  and  $n = -1$ , commutation is exactly the imprimitivity condition. The last case is  $y = \psi^{\text{op}}(f^{\text{op}})^*$  and  $n = 0$ , so  $\xi = a \in A$ , with

$$R(\psi^{\text{op}}(f^{\text{op}})^*)(L(\psi(e))a) = R(\psi^{\text{op}}(f^{\text{op}})^*)(e \cdot a) = {}_A(e \cdot a | f) = {}_A(e | f \cdot a^*),$$

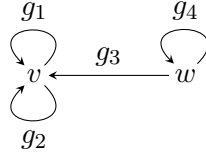
and

$$L(\psi(e))(R(\psi^{\text{op}}(f^{\text{op}})^*)a) = L(\psi(e))(a \cdot \bar{f}) = L(\psi(e))(\overline{f \cdot a^*}) = {}_A(e | f \cdot a^*). \quad \square$$

## 2.5 Relations between the two potential dual algebras

The Poincaré duality results of [21, 32] suggest that one might hope to prove that  $\mathcal{O}_E$  and  $\mathcal{O}_{E^{\text{op}}}$  are Poincaré dual for suitable modules  $E$ . On the other hand, Connes' picture of Poincaré self-duality, for the rotation algebras for example [9], suggests that we should aim to decide whether  $\mathcal{O}_E$  and  $\mathcal{O}_E^{\text{op}}$  are Poincaré dual. While we always have an isomorphism  $\mathcal{O}_E^{\text{op}} \cong \mathcal{O}_{\bar{E}^{\text{op}}}$ , our next example shows that  $\mathcal{O}_{E^{\text{op}}}$  can be quite different.

**Example 2.11.** Let  $G$  be the directed graph with two vertices  $G^0 = \{v, w\}$ , four edges  $G^1 = \{g_1, \dots, g_4\}$ , and range and source maps given by  $r(g_1) = s(g_1) = r(g_2) = s(g_2) = v$  and  $s(g_3) = r(g_4) = s(g_4) = w$ :



It is routine to check that the graph module (see Proposition 3.8)  $E = E_G = C(G^1)$  over  $C(G^0)$  has opposite module  $E^{\text{op}} = E_{G^{\text{op}}}$  where  $G^{\text{op}}$  is the graph obtained by reversing the edges in  $G$ . We use the conventions for graph algebras of [34]; so  $s_e^* s_e = p_{s(e)}$ . We have  $\mathcal{O}_E^{\text{op}} \cong C^*(G)^{\text{op}}$  and  $\mathcal{O}_{E^{\text{op}}} \cong C^*(G^{\text{op}})$ . To see that these are not isomorphic, first observe that  $C^*(G)$  can be realised as the  $C^*$ -algebra  $C^*(H_G)$  of a groupoid. In particular, the map  $\theta : C_c(H_G) \rightarrow C_c(H_G)$  given by  $\theta(f)(\gamma) = f(\gamma^{-1})$  extends to an isomorphism  $C^*(G) \cong C^*(G)^{\text{op}}$ , [5, Theorem 2.1]. So  $\mathcal{O}_E^{\text{op}} \cong C^*(G)$ . So it suffices to show that  $C^*(G^{\text{op}}) \not\cong C^*(G)$ .

By the universal property of  $C^*(G^{\text{op}})$ , there is a 1-dimensional representation of  $C^*(G^{\text{op}})$  given by  $\pi(p_w) = \pi(S_{g_4}) = 1 \in \mathbb{C}$  and  $\pi(p_v) = \pi(S_{g_i}) = 0$  for  $1 \leq i \leq 3$ . On the other hand, if  $\pi$  is any nonzero representation of  $C^*(G)$ , then  $\pi(p_v)$  is nonzero: since  $S_{g_i} = p_v S_{g_i}$  for  $i \leq 3$ , and since  $p_w = S_{g_3}^* S_{g_3}$  and  $S_{g_4} = p_w S_{g_4}$ , we see that  $p_v$  generates  $C^*(G)$  as an ideal. So any nonzero representation  $\pi$  of  $C^*(G)$  restricts to a nonzero representation of  $p_v C^*(G) p_v \cong \mathcal{O}_2$ , which cannot be 1-dimensional.

For invertible bimodules, the ambiguity between potential dual algebras disappears.

**Proposition 2.12.** *Let  $E$  be an invertible bimodule over  $A$ . Then  $\mathcal{O}_{E^{\text{op}}} \cong \mathcal{O}_E^{\text{op}}$ .*

*Proof.* The left inner-product gives an isomorphism  $\text{End}_A^0(E) \cong A$  satisfying  $\Theta_{e,f} \mapsto {}_A(e | f)$ . So by Lemma 2.10 and an application of the gauge-invariant uniqueness theorem [26, Section 6], we

have  $\mathcal{O}_E \cong C^*(L_e : e \in E) \subset \text{End}_A(\mathcal{F}_{E,\mathbb{Z}})$ . We claim that  $\psi : e^{\text{op}} \mapsto L_{e^{\text{op}}}^{\text{op}}, a \mapsto \pi(a)^{\text{op}}$  is a covariant Toeplitz representation of  $E^{\text{op}}$  in  $\text{End}_A(\mathcal{F}_{E,\mathbb{Z}})^{\text{op}}$ . To see this, fix  $e^{\text{op}}, f^{\text{op}} \in E^{\text{op}}$ . Then

$$\psi((e^{\text{op}} | f^{\text{op}})_{A^{\text{op}}}) = \psi(A(f | e)^{\text{op}}) = \pi(A(f | e))^{\text{op}} = (L_{f^{\text{op}}} L_{e^{\text{op}}}^*)^{\text{op}} = L_{e^{\text{op}}}^{*\text{op}} L_{f^{\text{op}}}^{\text{op}} = \psi(e)^* \psi(f),$$

and for covariance, we calculate that

$$\psi^{(1)}(\Theta_{e,f}) = L_e^{\text{op}} L_f^{*\text{op}} = (L_f^* L_e)^{\text{op}} = \pi((f | e)_A)^{\text{op}} = \psi((f | e)_A^{\text{op}}) = \psi(A^{\text{op}}(e | f)).$$

The universal property of  $\mathcal{O}_{E^{\text{op}}}$  gives a homomorphism  $\psi \times \pi : \mathcal{O}_E^{\text{op}} \rightarrow C^*(\{L_e^{\text{op}} : e \in E\})$ , and therefore induces a homomorphism  $\tilde{\psi} : \mathcal{O}_{E^{\text{op}}} \rightarrow \mathcal{O}_E^{\text{op}}$ . This  $\tilde{\psi}$  is surjective because the elements  $\tilde{\psi}(S_{e^{\text{op}}})$  generate  $\mathcal{O}_E^{\text{op}}$ . Injectivity follows from the gauge-invariant uniqueness theorem since  $\tilde{\psi} : A^{\text{op}} \rightarrow \mathcal{O}_E^{\text{op}}$  is injective and the gauge action on  $\mathcal{O}_E$  induces a gauge action on  $\mathcal{O}_E^{\text{op}}$ .  $\square$

### 3 Basic criteria for Poincaré duality of Cuntz–Pimsner algebras

In this section we derive conditions under which Poincaré self-duality of a  $C^*$ -algebra  $A$  induces Poincaré self-duality for the Cuntz–Pimsner algebra  $\mathcal{O}_E$  of a  $C^*$ -correspondence  $E$  over  $A$ . When  $E$  is a bi-Hilbertian bimodule, we also investigate when Poincaré self-duality for  $A$  induces Poincaré duality between  $\mathcal{O}_E$  and  $\mathcal{O}_{E^{\text{op}}}$ .

Following [24], we say that  $C^*$ -algebras  $A$  and  $B$  are *Poincaré dual* if there exist classes  $\mu \in KK^d(A \otimes B, \mathbb{C})$  (the Dirac class) and  $\beta \in KK^d(\mathbb{C}, A \otimes B)$  (the Bott or dual-Dirac class) such that

$$\beta \otimes_A \mu = (-1)^d \text{Id}_{KK(B,B)} \quad \text{and} \quad \beta \otimes_B \mu = \text{Id}_{KK(A,A)}. \quad (3.1)$$

The classes  $\mu$  and  $\beta$  implement isomorphisms

$$\cdot \otimes_B \mu : K_*(B) \xrightarrow{\cong} K^{*+d}(A), \quad \cdot \otimes_A \mu : K_*(A) \xrightarrow{\cong} K^{*+d}(B), \quad \text{and} \quad (3.2)$$

$$\beta \otimes_B \cdot : K^*(B) \xrightarrow{\cong} K_{*+d}(A), \quad \beta \otimes_A \cdot : K^*(A) \xrightarrow{\cong} K_{*+d}(B). \quad (3.3)$$

We call a class  $\mu \in KK(A \otimes B, \mathbb{C})$  implementing isomorphisms as in (3.2) a *K-homology fundamental class*, even if there is no corresponding class  $\beta$ . Similarly, we call a class  $\beta$  implementing isomorphisms as in (3.3) a *K-theory fundamental class*. If  $A$  and  $A^{\text{op}}$  are Poincaré dual, then we say that  $A$  is *Poincaré self-dual*.

Strictly speaking, the formulation here is appropriate only for unital algebras. For non-unital algebras Poincaré duality should be formulated using an appropriate analogue of compactly supported  $K$ -homology in Equations (3.2) and (3.3), though  $\beta$  and  $\mu$  need not be and usually are not compactly supported. For commutative algebras and  $C(X)$ -algebras, a suitable compactly supported theory is provided by  $RKK$ , defined by Kasparov in [25]. A version for some non-commutative algebras is presented in [35]. We will restrict our discussion to the formulation of Poincaré duality above, but include some additional non-unital results for future use.

Suppose that  $A$  is Poincaré self-dual, and consider a bi-Hilbertian  $A$ -bimodule  $E$ . Recall that  $[P]$  denotes the Kasparov class of the Kasparov module described in (2.6). The  $KK$ -equivalences between  $A$  and  $\mathcal{T}_E$  and between  $A$  and  $\text{End}_A^0(\mathcal{F}_E)$  described in Section 2.2, and the corresponding

equivalences between  $A^{\text{op}}$  and  $\mathcal{T}_E^{\text{op}}$  and between  $A^{\text{op}}$  and  $\text{End}_A^0(\mathcal{F}_E)^{\text{op}}$  lift  $(\mu, \beta)$  to Poincaré self-dualities  $(\mu_{\mathcal{T}}, \beta_{\mathcal{T}})$  for  $\mathcal{T}_E$  and  $(\mu_E, \beta_E)$  for  $\text{End}_A^0(\mathcal{F}_E)$  as follows:

$$\begin{aligned}\mu_{\mathcal{T}} &= ([P] \otimes [P^{\text{op}}]) \otimes_{A \otimes A^{\text{op}}} \mu_A, & \beta_{\mathcal{T}} &= \beta_A \otimes_{A \otimes A^{\text{op}}} (\iota_{A, \mathcal{T}} \otimes \iota_{A^{\text{op}}, \mathcal{T}^{\text{op}}}) \\ \mu_E &= ([\mathcal{F}_E] \otimes [\mathcal{F}_E^{\text{op}}]) \otimes_{A \otimes A^{\text{op}}} \mu_A, & \beta_E &= \beta_A \otimes_{A \otimes A^{\text{op}}} ([\overline{\mathcal{F}_E}] \otimes [\overline{\mathcal{F}_E^{\text{op}}}]).\end{aligned}$$

We cannot expect simple formulae of the same sort to describe classes implementing Poincaré self-duality for  $\mathcal{O}_E$ . One reason for this is Proposition 2.2. Another is that we expect a shift in parity: the algebra  $\mathcal{O}_E$  in important respects like a crossed product of  $A$  by  $\mathbb{Z}$ , so the passage from  $A$  to  $\mathcal{O}_E$  should add a noncommutative dimension, leading us to expect that the fundamental class for  $\mathcal{O}_E$  has parity  $d + 1$  if  $\mu$  has parity  $d \in \mathbb{Z}/2\mathbb{Z}$ .

### 3.1 Lifting Poincaré duality from the coefficient algebra

In this subsection, assuming Poincaré self-duality for the coefficient algebra  $A$ , we produce sufficient conditions for the existence of fundamental classes implementing a Poincaré duality

$$\delta \in KK^{d+1}(\mathbb{C}, \mathcal{O}_E \otimes \mathcal{O}_{E^{\text{op}}}) \quad \text{and} \quad \Delta \in KK^{d+1}(\mathcal{O}_E \otimes \mathcal{O}_{E^{\text{op}}}, \mathbb{C}),$$

and also for the existence of fundamental classes implementing a Poincaré duality

$$\bar{\delta} \in KK^{d+1}(\mathbb{C}, \mathcal{O}_E \otimes \mathcal{O}_E^{\text{op}}) \quad \text{and} \quad \bar{\Delta} \in KK^{d+1}(\mathcal{O}_E \otimes \mathcal{O}_E^{\text{op}}, \mathbb{C}).$$

Our sufficient conditions involve the dynamics encoded by  $E$ , the modules  $E^{\text{op}}$  and  $\overline{E}^{\text{op}}$ , and the existence of suitably  $E$ -invariant Poincaré self-duality classes for  $A$ . First, starting from the extension

$$0 \rightarrow \text{End}_A^0(\mathcal{F}_E) \xrightarrow{\iota_{E, \mathcal{T}}} \mathcal{T}_E \xrightarrow{q} \mathcal{O}_E \rightarrow 0 \tag{3.4}$$

and the analogous extensions for  $E^{\text{op}}$  and  $\overline{E}^{\text{op}}$ , we describe sufficient conditions under which classes  $\bar{\Delta} \in KK^{d+1}(\mathcal{O}_E \otimes \mathcal{O}_E^{\text{op}}, \mathbb{C})$  and  $\bar{\delta} \in KK^{d+1}(\mathbb{C}, \mathcal{O}_E \otimes \mathcal{O}_E^{\text{op}})$  yield isomorphisms

$$\begin{aligned}\cdot \otimes_{\mathcal{O}_E} \bar{\Delta} : K_*(\mathcal{O}_E) &\rightarrow K^{*+d+1}(\mathcal{O}_E^{\text{op}}), & \cdot \otimes_{\mathcal{O}_{E^{\text{op}}}} \bar{\Delta} : K_*(\mathcal{O}_E^{\text{op}}) &\rightarrow K^{*+d+1}(\mathcal{O}_E) \\ \bar{\delta} \otimes_{\mathcal{O}_E} \cdot : K^*(\mathcal{O}_E) &\rightarrow K_{*+d+1}(\mathcal{O}_E^{\text{op}}), & \bar{\delta} \otimes_{\mathcal{O}_{E^{\text{op}}}} \cdot : K^*(\mathcal{O}_E^{\text{op}}) &\rightarrow K_{*+d+1}(\mathcal{O}_E).\end{aligned} \tag{3.5}$$

We then give sufficient conditions for classes  $\Delta \in KK^{d+1}(\mathcal{O}_E \otimes \mathcal{O}_E^{\text{op}}, \mathbb{C})$  and  $\delta \in KK^{d+1}(\mathbb{C}, \mathcal{O}_E \otimes \mathcal{O}_E^{\text{op}})$  to yield isomorphisms

$$\begin{aligned}\cdot \otimes_{\mathcal{O}_E} \Delta : K_*(\mathcal{O}_E) &\rightarrow K^{*+d+1}(\mathcal{O}_{E^{\text{op}}}), & \cdot \otimes_{\mathcal{O}_{E^{\text{op}}}} \Delta : K_*(\mathcal{O}_{E^{\text{op}}}) &\rightarrow K^{*+d+1}(\mathcal{O}_E) \\ \delta \otimes_{\mathcal{O}_E} \cdot : K^*(\mathcal{O}_E) &\rightarrow K_{*+d+1}(\mathcal{O}_{E^{\text{op}}}), & \delta \otimes_{\mathcal{O}_{E^{\text{op}}}} \cdot : K^*(\mathcal{O}_{E^{\text{op}}}) &\rightarrow K_{*+d+1}(\mathcal{O}_E).\end{aligned} \tag{3.6}$$

Of course, the maps (3.5) must be isomorphisms if  $\bar{\delta}$  and  $\bar{\Delta}$  implement a Poincaré self-duality. The converse is false, but we show that the existence of classes  $(\bar{\delta}, \bar{\Delta})$  satisfying our sufficient conditions for (3.5) implies that  $\mathcal{O}_E$  and  $\mathcal{O}_{\overline{E}^{\text{op}}}$  are Poincaré dual, via a suitably modified pair of  $KK$ -classes. A related result, that isomorphisms  $KK(A \otimes B, C) \rightarrow KK(B, A^{\text{op}} \otimes C)$  for all  $B, C$  ensures  $A$  and  $A^{\text{op}}$  are Poincaré dual appears in [12].

Similarly, while not every pair  $(\delta, \Delta)$  satisfying our sufficient condition guaranteeing (3.6) implements a Poincaré duality between  $\mathcal{O}_E$  and  $\mathcal{O}_{E^{\text{op}}}$ , the existence of such a pair does guarantee Poincaré duality of  $\mathcal{O}_E$  and  $\mathcal{O}_{E^{\text{op}}}$ , again via a modified pair of  $KK$ -classes.

We begin by combining the  $K$ -theory exact sequence arising from the exact sequence (3.4) for  $\overline{E}^{\text{op}}$  with the  $K$ -homology exact sequence arising from (3.4) using the Poincaré self-duality of  $A$ .

**Theorem 3.1.** *Let  $E$  be a correspondence over  $A$  with compact and non-degenerate left action of  $A$ . Suppose that  $\mu \in KK^d(A \otimes A^{\text{op}}, \mathbb{C})$  and  $\beta \in KK^d(\mathbb{C}, A \otimes A^{\text{op}})$  implement a Poincaré self-duality for  $A$ . Let  $\iota_{A, \mathcal{O}} : A \rightarrow \mathcal{O}_E$  be the canonical inclusion.*

1. *Suppose that  $[E] \otimes_A \mu = [\overline{E}^{\text{op}}] \otimes_{A^{\text{op}}} \mu \in KK(A \otimes A^{\text{op}}, \mathbb{C})$ . Suppose that  $\overline{\Delta} \in KK^1(\mathcal{O}_E \otimes \mathcal{O}_E^{\text{op}}, \mathbb{C})$  satisfies*

$$\iota_{A, \mathcal{O}} \otimes_{\mathcal{O}_E} \overline{\Delta} = -[\overline{\text{ext}}^{\text{op}}] \otimes_{A^{\text{op}}} \mu \quad \text{and} \quad \iota_{A^{\text{op}}, \mathcal{O}^{\text{op}}} \otimes_{\mathcal{O}_E^{\text{op}}} \overline{\Delta} = [\text{ext}] \otimes_A \mu. \quad (3.7)$$

*Then the maps defined by  $\overline{\Delta}$  in (3.5) are isomorphisms, so  $\overline{\Delta}$  is a  $K$ -homology fundamental class.*

2. *Suppose that  $\beta \otimes_A [E] = \beta \otimes_{A^{\text{op}}} [\overline{E}^{\text{op}}] \in KK(\mathbb{C}, A \otimes A^{\text{op}})$ . Suppose that  $\overline{\delta} \in KK^{d+1}(\mathbb{C}, \mathcal{O}_E \otimes \mathcal{O}_E^{\text{op}})$  satisfies*

$$\beta \otimes_A \iota_{A, \mathcal{O}} = -\overline{\delta} \otimes_{\mathcal{O}_E^{\text{op}}} [\overline{\text{ext}}^{\text{op}}] \quad \text{and} \quad \beta \otimes_{A^{\text{op}}} \iota_{A^{\text{op}}, \mathcal{O}^{\text{op}}} = -\overline{\delta} \otimes_{\mathcal{O}_E} [\text{ext}]. \quad (3.8)$$

*Then the maps defined by  $\overline{\delta}$  in (3.5) are isomorphisms, so  $\overline{\delta}$  is a  $K$ -theory fundamental class.*

3. *Suppose that there exist classes  $\overline{\Delta}$  and  $\overline{\delta}$  satisfying the conditions in (1) and (2). Then  $\mathcal{O}_E$  is Poincaré self-dual.*

To prove the theorem, we first need two lemmas.

**Lemma 3.2.** *Let  $E$  be a correspondence over  $A$  with compact and non-degenerate left action of  $A$ . Suppose that  $\mu \in KK^d(A \otimes A^{\text{op}}, \mathbb{C})$  and  $\beta \in KK^d(\mathbb{C}, A \otimes A^{\text{op}})$  implement a Poincaré self-duality for  $A$ . Resume the notation of Section 2.2, and write  $\partial$  for all boundary maps in Pimsner's six-term  $K$ -theory and  $K$ -homology sequences. If  $[E] \otimes_A \mu = [\overline{E}^{\text{op}}] \otimes_{A^{\text{op}}} \mu$  then all subdiagrams consisting of solid arrows in the diagram*

$$\begin{array}{ccccc}
 & & K_0(A^{\text{op}}) & & \\
 & & \downarrow \otimes_{A^{\text{op}}} \mu & & \\
 & & K^d(A) & & \\
 \nearrow \partial^{\text{op}} & & \swarrow \cdot \otimes_{A^{\text{op}}} (\text{Id}_{A^{\text{op}}} - [\overline{E}^{\text{op}}]) & & \\
 & & K^d(A) & \longleftarrow & K_0(A^{\text{op}}) \\
 & \nearrow \iota_{A, \mathcal{O}}^* & & \swarrow \otimes_{A^{\text{op}}} \mu & \downarrow \iota_{A^{\text{op}}, \mathcal{O}^{\text{op}}} \\
 K_1(\mathcal{O}_E^{\text{op}}) - \frac{\theta}{\triangleright} & & K^d(\mathcal{O}_E) & & K^d(A) \\
 \uparrow \iota_{A^{\text{op}}, \mathcal{O}^{\text{op}}} & & \uparrow \partial & & \downarrow \partial \\
 K_1(A^{\text{op}}) & \xrightarrow{\otimes_{A^{\text{op}}} \mu} & K^{d+1}(A) & & K^{d+1}(\mathcal{O}_E) \xleftarrow{\frac{\theta}{\triangleleft}} K_0(\mathcal{O}_E^{\text{op}}) \\
 & \swarrow (\text{Id}_A - [E]) \otimes_A & & \swarrow \iota_{A, \mathcal{O}}^* & \downarrow \partial^{\text{op}} \\
 & & K^{d+1}(A) & & \\
 \nwarrow \cdot \otimes_{A^{\text{op}}} (\text{Id}_{A^{\text{op}}} - [\overline{E}^{\text{op}}]) & & \uparrow \otimes_{A^{\text{op}}} \mu & & \\
 & & K_1(A^{\text{op}}) & & 
 \end{array} \quad (3.9)$$

*commute. Any homomorphism  $\theta : K_*(\mathcal{O}_E^{\text{op}}) \rightarrow K^{*+1+d}(\mathcal{O}_E)$  making the whole diagram commute is an isomorphism.*

*Proof.* The completely defined squares in the upper right and lower left commute by the hypothesis on  $\mu$ . The remaining solid rectangles trivially commute because exactness of the two hexagonal sequences shows that the long sides of these rectangles are zero maps. The final assertion follows from the five lemma.  $\square$

**Lemma 3.3.** *Suppose that  $\overline{\Delta}$  and  $\overline{\delta}$  are as in Theorem 3.1(1) and (2) respectively. Write  $\sigma_{12} : \mathcal{O}_E^{\text{op}} \otimes \mathcal{O}_E \rightarrow \mathcal{O}_E \otimes \mathcal{O}_E^{\text{op}}$  for the flip isomorphism, and define  $v \in KK(\mathcal{O}_E, \mathcal{O}_E)$  by*

$$v = (\overline{\delta} \hat{\otimes}_{\mathbb{C}} \text{Id}_{KK(\mathcal{O}_E, \mathcal{O}_E)}) \hat{\otimes}_{\mathcal{O}_E \otimes \mathcal{O}_E^{\text{op}} \otimes \mathcal{O}_E} (\text{Id}_{KK(\mathcal{O}_E, \mathcal{O}_E)} \hat{\otimes}_{\mathbb{C}} \sigma_{12}^* \overline{\Delta}). \quad (3.10)$$

Then

$$\iota_{A, \mathcal{O}_E} \otimes_{\mathcal{O}_E} v = \iota_{A, \mathcal{O}_E} \quad \text{and} \quad v \otimes_{\mathcal{O}_E} [\text{ext}] = [\text{ext}].$$

*Proof.* We calculate:

$$\begin{aligned} \iota_{A, \mathcal{O}_E} \hat{\otimes}_{\mathcal{O}_E} v &= (\overline{\delta} \hat{\otimes}_{\mathbb{C}} \iota_{A, \mathcal{O}_E}) \hat{\otimes}_{\mathcal{O}_E \otimes \mathcal{O}_E^{\text{op}} \otimes \mathcal{O}_E} (\text{Id}_{KK(\mathcal{O}_E, \mathcal{O}_E)} \hat{\otimes}_{\mathbb{C}} \sigma_{12}^* \overline{\Delta}) \\ &= (\overline{\delta} \hat{\otimes}_{\mathcal{O}_E \otimes \mathcal{O}_E^{\text{op}}} (\text{Id}_{KK(\mathcal{O}_E, \mathcal{O}_E)} \hat{\otimes}_{\mathbb{C}} \text{Id}_{KK(\mathcal{O}_E^{\text{op}}, \mathcal{O}_E^{\text{op}})} \hat{\otimes}_{\mathbb{C}} \iota_{A, \mathcal{O}_E})) \hat{\otimes}_{\mathcal{O}_E \otimes \mathcal{O}_E^{\text{op}} \otimes \mathcal{O}_E} (\text{Id}_{KK(\mathcal{O}_E, \mathcal{O}_E)} \hat{\otimes}_{\mathbb{C}} \sigma_{12}^* \overline{\Delta}) \\ &= (\overline{\delta} \hat{\otimes}_{\mathcal{O}_E \otimes \mathcal{O}_E^{\text{op}}} (\text{Id}_{KK(\mathcal{O}_E, \mathcal{O}_E)} \hat{\otimes}_{\mathbb{C}} \iota_{A, \mathcal{O}_E} \hat{\otimes}_{\mathbb{C}} \text{Id}_{KK(\mathcal{O}_E^{\text{op}}, \mathcal{O}_E^{\text{op}})})) \hat{\otimes}_{\mathcal{O}_E \otimes \mathcal{O}_E^{\text{op}} \otimes \mathcal{O}_E} (\text{Id}_{KK(\mathcal{O}_E, \mathcal{O}_E)} \hat{\otimes}_{\mathbb{C}} \overline{\Delta}) \\ &= -\overline{\delta} \hat{\otimes}_{\mathcal{O}_E \otimes \mathcal{O}_E^{\text{op}}} \text{Id}_{KK(\mathcal{O}_E, \mathcal{O}_E)} \hat{\otimes}_{\mathbb{C}} [\overline{\text{ext}}^{\text{op}}] \hat{\otimes}_{A^{\text{op}}} \mu \\ &= \beta \otimes_A \iota_{A, \mathcal{O}_E} \otimes_{A^{\text{op}}} \mu \\ &= \iota_{A, \mathcal{O}_E}. \end{aligned}$$

The second equality is proved similarly, remembering the anti-symmetry of the external product to see that

$$\begin{aligned} v \otimes_{\mathcal{O}_E} [\text{ext}] &= \overline{\delta} \otimes_{\mathcal{O}_E^{\text{op}}} \overline{\Delta} \otimes_{\mathcal{O}_E} [\text{ext}] = (\overline{\delta} \otimes_{\mathcal{O}_E} [\text{ext}]) \otimes_{\mathcal{O}_E^{\text{op}}} \overline{\Delta} (-1)^{d+1} \\ &= -\beta \otimes_{A^{\text{op}}} \iota_{A^{\text{op}}, \mathcal{O}^{\text{op}}} \otimes_{\mathcal{O}_E^{\text{op}}} \overline{\Delta} (-1)^{d+1} = -\beta \otimes_{A^{\text{op}}} ([\text{ext}] \otimes_A \mu) (-1)^{d+1} \\ &= [\text{ext}] \otimes_A (\beta \otimes_{A^{\text{op}}} \mu) = [\text{ext}]. \quad \square \end{aligned}$$

*Proof of Theorem 3.1.* (1) By [24, Section 7], the boundary maps  $\partial$  and  $\partial^{\text{op}}$  are implemented by Kasparov products with  $[\text{ext}]$  and  $[\overline{\text{ext}}^{\text{op}}]$  respectively. So the conditions in (3.7) are equivalent to commutation of the diagram (3.9).

(2) There is a diagram dual to (3.9) in which  $\cdot \otimes_A \mu$  and  $\cdot \otimes_{A^{\text{op}}} \mu$  are replaced by maps  $\beta \otimes_A \cdot : K^*(A) \rightarrow K_{*+d}(A^{\text{op}})$  and  $\beta \otimes_{A^{\text{op}}} \cdot : K^*(A^{\text{op}}) \rightarrow K_{*+d}(A)$  from the inner exact hexagon to the outer one (and the arrows  $\theta$  are reversed). The proof of (2) is the same as that of (1) but applied to this dual diagram and the conditions in (3.8).

(3) Let  $v \in KK(\mathcal{O}_E, \mathcal{O}_E)$  denote the product  $v := \overline{\delta} \otimes_{\mathcal{O}_E^{\text{op}}} \overline{\Delta}$  (see (3.10)). Define  $\overline{\delta}' := \overline{\delta} \otimes_{\mathcal{O}_E} v^{-1}$ . We will show that  $\mathcal{O}_E$  is Poincaré self-dual with duality implemented by the pair  $\overline{\delta}', \overline{\Delta}$ . We calculate:

$$\begin{aligned} \overline{\delta}' \otimes_{\mathcal{O}_E^{\text{op}}} \overline{\Delta} &:= (\overline{\delta} \hat{\otimes}_{\mathcal{O}_E \otimes \mathcal{O}_E^{\text{op}}} (v^{-1} \hat{\otimes}_{\mathbb{C}} \text{Id}_{KK(\mathcal{O}_E^{\text{op}}, \mathcal{O}_E^{\text{op}})})) \hat{\otimes}_{\mathbb{C}} \text{Id}_{KK(\mathcal{O}_E, \mathcal{O}_E)} \hat{\otimes}_{\mathcal{O}_E \otimes \mathcal{O}_E^{\text{op}} \otimes \mathcal{O}_E} (\text{Id}_{KK(\mathcal{O}_E, \mathcal{O}_E)} \hat{\otimes}_{\mathbb{C}} \sigma_{12}^* \overline{\Delta}) \\ &= (\overline{\delta} \hat{\otimes}_{\mathbb{C}} \text{Id}_{KK(\mathcal{O}_E, \mathcal{O}_E)}) \hat{\otimes}_{\mathcal{O}_E \otimes \mathcal{O}_E^{\text{op}} \otimes \mathcal{O}_E} \\ &\quad (v^{-1} \hat{\otimes}_{\mathbb{C}} \text{Id}_{KK(\mathcal{O}_E^{\text{op}}, \mathcal{O}_E^{\text{op}})} \hat{\otimes}_{\mathbb{C}} \text{Id}_{KK(\mathcal{O}_E, \mathcal{O}_E)}) \hat{\otimes}_{\mathcal{O}_E \otimes \mathcal{O}_E^{\text{op}} \otimes \mathcal{O}_E} (\text{Id}_{KK(\mathcal{O}_E, \mathcal{O}_E)} \hat{\otimes}_{\mathbb{C}} \sigma_{12}^* \overline{\Delta}) \\ &= (\overline{\delta} \hat{\otimes}_{\mathbb{C}} \text{Id}_{KK(\mathcal{O}_E, \mathcal{O}_E)}) \hat{\otimes}_{\mathcal{O}_E \otimes \mathcal{O}_E^{\text{op}} \otimes \mathcal{O}_E} (v^{-1} \hat{\otimes}_{\mathbb{C}} \sigma_{12}^* \overline{\Delta}) \\ &= (\overline{\delta} \hat{\otimes}_{\mathbb{C}} \text{Id}_{KK(\mathcal{O}_E, \mathcal{O}_E)}) \hat{\otimes}_{\mathcal{O}_E \otimes \mathcal{O}_E^{\text{op}} \otimes \mathcal{O}_E} (\text{Id}_{KK(\mathcal{O}_E, \mathcal{O}_E)} \hat{\otimes}_{\mathbb{C}} \sigma_{12}^* \overline{\Delta}) \hat{\otimes}_{\mathcal{O}_E} v^{-1} \\ &= v \hat{\otimes}_{\mathcal{O}_E} v^{-1} = \text{Id}_{KK(\mathcal{O}_E, \mathcal{O}_E)}. \end{aligned}$$

Combining  $\bar{\delta}' \otimes_{\mathcal{O}_E^{\text{op}}} \bar{\Delta} = \text{Id}_{KK(\mathcal{O}_E, \mathcal{O}_E)}$  with the anti-symmetry of the external product, we see that

$$\begin{aligned} \bar{\delta}' \otimes_{\mathcal{O}_E} \bar{\Delta} &= \bar{\delta}' \otimes_{\mathcal{O}_E} (\bar{\delta}' \otimes_{\mathcal{O}_E^{\text{op}}} \bar{\Delta}) \otimes_{\mathcal{O}_E} \bar{\Delta} \\ &= ((-1)^{d+1} \bar{\delta}' \otimes_{\mathcal{O}_E^{\text{op}}} (\bar{\delta}' \otimes_{\mathcal{O}_E} \bar{\Delta})) \otimes_{\mathcal{O}_E} \bar{\Delta} = (-1)^{d+1} (\bar{\delta}' \otimes_{\mathcal{O}_E} \bar{\Delta}) \otimes_{\mathcal{O}_E^{\text{op}}} (\bar{\delta}' \otimes_{\mathcal{O}_E} \bar{\Delta}). \end{aligned}$$

Multiplying through by  $(-1)^{d+1}$  gives  $((-1)^{d+1} \bar{\delta}' \otimes_{\mathcal{O}_E} \bar{\Delta})^2 = (-1)^{d+1} \bar{\delta}' \otimes_{\mathcal{O}_E} \bar{\Delta}$ . So  $(-1)^{d+1} \bar{\delta}' \otimes_{\mathcal{O}_E} \bar{\Delta}$  is an idempotent in the group of units of the ring  $KK(\mathcal{O}_E^{\text{op}}, \mathcal{O}_E^{\text{op}})$ , and therefore equal to  $\text{Id}_{KK(\mathcal{O}_E^{\text{op}}, \mathcal{O}_E^{\text{op}})}$ . Hence  $\bar{\delta}' \otimes_{\mathcal{O}_E} \bar{\Delta} = (-1)^{d+1} \text{Id}_{KK(\mathcal{O}_E^{\text{op}}, \mathcal{O}_E^{\text{op}})}$ .  $\square$

If  $E$  is a bi-Hilbertian  $A$ -bimodule with left action by compacts, we can repeat the discussion above using the module  $E^{\text{op}}$ , replacing  $\overline{E^{\text{op}}}$  by  $E^{\text{op}}$ ,  $\mathcal{O}_E^{\text{op}}$  by  $\mathcal{O}_{E^{\text{op}}}$ , and  $[\overline{\text{ext}^{\text{op}}}]$  by  $[\text{ext}^{\text{op}}]$ . We obtain the following result.

**Theorem 3.4.** *Let  $E$  be a bi-Hilbertian bimodule over  $A$  with compact and non-degenerate left action of  $A$ . Suppose that  $\mu \in KK^d(A \otimes A^{\text{op}}, \mathbb{C})$  and  $\beta \in KK^d(\mathbb{C}, A \otimes A^{\text{op}})$  implement a Poincaré self-duality for  $A$ . Let  $\iota_{A, \mathcal{O}} : A \rightarrow \mathcal{O}_E$  be the canonical inclusion.*

1. *Suppose that  $[E] \otimes_A \mu = [E^{\text{op}}] \otimes_{A^{\text{op}}} \mu$ . Suppose that  $\Delta \in KK^1(\mathcal{O}_E \otimes \mathcal{O}_{E^{\text{op}}}, \mathbb{C})$  satisfies*

$$\iota_{A, \mathcal{O}_E} \otimes_{\mathcal{O}_E} \Delta = [\text{ext}^{\text{op}}] \otimes_{A^{\text{op}}} \mu \quad \text{and} \quad \iota_{A^{\text{op}}, \mathcal{O}_{E^{\text{op}}}} \otimes_{\mathcal{O}_{E^{\text{op}}}} \Delta = [\text{ext}] \otimes_A \mu.$$

*Then the maps defined by  $\Delta$  in (3.6) are isomorphisms, so  $\Delta$  is a  $K$ -homology fundamental class.*

2. *Suppose that  $\beta \otimes_A [E] = \beta \otimes_{A^{\text{op}}} [E^{\text{op}}]$ . Suppose that  $\delta \in KK^{d+1}(\mathbb{C}, \mathcal{O}_E \otimes \mathcal{O}_{E^{\text{op}}})$  satisfies*

$$\beta \otimes_A \iota_{A, \mathcal{O}_E} = \delta \otimes_{\mathcal{O}_{E^{\text{op}}}} [\text{ext}^{\text{op}}] \quad \text{and} \quad -\beta \otimes_{A^{\text{op}}} \iota_{A^{\text{op}}, \mathcal{O}_{E^{\text{op}}}} = \delta \otimes_{\mathcal{O}_E} [\text{ext}].$$

*Then the maps defined by  $\delta$  in (3.6) are isomorphisms, so  $\delta$  is a  $K$ -theory fundamental class.*

3. *Suppose that there exist classes  $\delta$  and  $\Delta$  satisfying the conditions in parts (1) and (2). Then  $\mathcal{O}_E$  and  $\mathcal{O}_{E^{\text{op}}}$  are Poincaré dual.*

## 3.2 Examples of Poincaré duality classes for the coefficient algebra

The restrictions on the classes  $\beta$  and  $\mu$  in the preceding section are fairly stringent, so we discuss two key examples where they are satisfied: Cuntz–Krieger algebras and crossed products by  $\mathbb{Z}$  in geometric settings. We will carry these examples through the remainder of the paper. We describe the Poincaré duality classes, and the conditions for commutation of the diagram.

### 3.2.1 Finite dimensional coefficient algebras and Cuntz–Krieger algebras

Our first example is  $A = \mathbb{C}$ . Clearly  $\mathbb{C} = \mathbb{C}^{\text{op}} = \mathbb{C} \otimes \mathbb{C}^{\text{op}}$ , but we avoid these identifications in the first instance to clarify how the components of our discussion relate to the general setting. The  $K$ -homology fundamental class is  $\mu = [\mathbb{C} \otimes \mathbb{C}^{\text{op}}, \mathbb{C}, 0]$ , where the left action is  $(w_1 \otimes w_2^{\text{op}}) \cdot z = w_1 w_2 z$ . The  $K$ -theory fundamental class is  $\beta = [\mathbb{C}, \mathbb{C} \otimes \mathbb{C}^{\text{op}}, 0]$  where the right action is  $z \cdot (w_1 \otimes w_2^{\text{op}}) = z w_1 w_2$  and the inner-product is  $(w \mid z) = (\bar{w} \otimes z^{\text{op}})$ . These classes trivially constitute a Poincaré self-duality for  $\mathbb{C}$ .



Tensoring with the explicit Morita equivalence cycle  $(M_n(\mathbb{C}), \mathbb{C}_\mathbb{C}^n, 0)$  and its inverse  $(\mathbb{C}, \mathbb{C}_{M_n(\mathbb{C})}^n, 0)$  yields a Poincaré self-duality for  $M_n(\mathbb{C})$ . We also obtain Poincaré duality classes for the compact operators, on  $\ell^2(\mathbb{N})$  say, but this takes us out of the finite-index setting needed later: see [37].

To extend the preceding paragraph from  $A = \mathbb{C}$  to  $A = \mathbb{C}^r$ , we record the following easy lemma.

**Lemma 3.5.** *Let  $A$  be a  $C^*$ -algebra, and suppose that*

$$\mu \in KK(A \otimes A^{\text{op}}, \mathbb{C}), \quad \text{and} \quad \beta \in KK(\mathbb{C}, A \otimes A^{\text{op}})$$

*implement a Poincaré self-duality for  $A$ . Then for each  $r \geq 1$ , the algebra  $A^r := \bigoplus_{i=1}^r A$  is Poincaré self-dual with respect to the classes*

$$\tilde{\mu} = \bigoplus_{i=1}^r \mu \in KK(A^r \otimes (A^{\text{op}})^r, \mathbb{C}), \quad \text{and} \quad \tilde{\beta} = \bigoplus_{i=1}^r \beta \in KK(\mathbb{C}, A^r \otimes (A^{\text{op}})^r).$$

*Proof.* We compute

$$\tilde{\beta} \otimes_{A^r} \tilde{\mu} = \bigoplus_{i,j=1}^r \beta \otimes_A \mu = \bigoplus_{i=1}^r \text{Id}_{KK(A^{\text{op}}, A^{\text{op}})} = \text{Id}_{KK((A^{\text{op}})^r, (A^{\text{op}})^r)},$$

and similarly for  $\tilde{\beta} \otimes_{(A^{\text{op}})^r} \tilde{\mu}$ . □

*Remark 3.6.* Lemma 3.5 shows that for any finite set  $X$ , the algebra  $C(X) := \mathbb{C}^X$  is Poincaré self-dual with respect to the classes

$$\begin{aligned} \mu &= [(C(X) \otimes C(X), C(X)_\mathbb{C}, 0)] \in KK(C(X) \otimes C(X), \mathbb{C}), \quad \text{and} \\ \beta &= [(\mathbb{C}, C(X)_{C(X) \otimes C(X)}, 0)] \in KK(\mathbb{C}, C(X) \otimes C(X)). \end{aligned} \tag{3.11}$$

The inner product is given on basis vectors of  $C(X)$  by

$$(\delta_x | \delta_y)_{C(X) \otimes C(X)} = \begin{cases} \delta_x \otimes \delta_x & x = y \\ 0 & \text{otherwise} \end{cases},$$

so the action of  $C(X)$  is diagonal. In the language of projections,

$$\beta = \sum_{x \in X} [\delta_x] \otimes [\delta_x^{\text{op}}]. \tag{3.12}$$

*Remark 3.7.* As discussed above for the 1-summand case, the Poincaré self-dualities in Remark 3.6 determine Poincaré self-dualities for algebras of the form  $\bigoplus_{x \in X} \mathcal{K}(\ell^2(I_x))$  for any collection of finite index sets  $I_x$  just by tensoring with the  $KK$ -classes of the invertible bimodules

$$\bigoplus_x \mathcal{K}(\ell^2(I_x)) \left( \bigoplus_x \ell^2(I_x) \right)_{C(X)}.$$

The following result shows that the hypotheses on  $\beta$  and  $\mu$  in Theorem 3.4 hold for finite graph algebras where  $A = C(G^0)$ , whether we use  $E^{\text{op}}$  or  $\overline{E}^{\text{op}}$ . We build on this to produce Poincaré duality classes for Cuntz–Krieger algebras and appropriate graph algebras in Sections 4.4.4 and 5.2.1.

**Proposition 3.8.** *Let  $G = (G^0, G^1, r, s)$  be a finite directed graph with no sources. Let  $\mu, \beta$  be as in Remark 3.6 for  $X = G^0$ . Let  $E$  be the edge module  $E = C(G^1)$  with*

$$(a \cdot e \cdot b)(g) = a(r(g))e(g)b(s(g)) \quad \text{for all } a, b \in A, \text{ all } e \in E \text{ and all } g \in G^1$$

and

$$(e | f)_A(v) = \sum_{s(g)=v} \overline{e(g)}f(g) \quad \text{and} \quad {}_A(e | f)(v) = \sum_{r(g)=v} e(g)\overline{f(g)}.$$

Then  $\beta \otimes_A [E] = \beta \otimes_{A^{\text{op}}} [\overline{E}^{\text{op}}]$  and  $[E] \otimes_A \mu = [\overline{E}^{\text{op}}] \otimes_{A^{\text{op}}} \mu$ . If  $G$  has no sinks then we also have  $\beta \otimes_A [E] = \beta \otimes_{A^{\text{op}}} [E^{\text{op}}]$  and  $[E] \otimes_A \mu = [E^{\text{op}}] \otimes_{A^{\text{op}}} \mu$ .

*Remark 3.9.* The hypothesis of ‘no sources’ is necessary for the injectivity of the left action of  $A$  on  $E$ . It also implies the injectivity of the left action of  $A^{\text{op}}$  on  $\overline{E}^{\text{op}}$ , and the right action of  $A^{\text{op}}$  on  $E^{\text{op}}$ . The hypothesis of ‘no sinks’ gives injectivity of the left action of  $A^{\text{op}}$  on  $E^{\text{op}}$ .

*Proof.* We first compute  $\beta \otimes_A [E] := \beta \otimes_{A \otimes A^{\text{op}}} ([E] \otimes [A^{\text{op}}])$ . Using the diagonality of the action and the inner product for the class  $\beta$ , one checks that  $C(G^0) \otimes_{C(G^0) \otimes C(G^0)} (C(G^1) \otimes C(G^0))_{C(G^0) \otimes C(G^0)}$  is isomorphic to  $C(G^1)$  as a linear space. The product module has action and inner product

$$(h \cdot (f_1 \otimes f_2))(g) = h(g)f_1(s(g))f_2(r(g)), \quad (h_1 | h_2)_A(v, w) = \sum_{s(g)=v, r(g)=w} \overline{h_1(g)}h_2(g) \quad (3.13)$$

for all  $h, h_1, h_2 \in C(G^1)$ , all  $f_1, f_2 \in C(G^0)$  and all  $g \in G^1$ . Hence

$$\beta \otimes_A [E] = [(C, C(G^1))_{sC(G^0) \otimes_r C(G^0)}, 0], \quad (3.14)$$

where we have labelled the actions by range and source to indicate that the right action is defined as in Equation (3.13). A similar line of reasoning shows that

$$\beta \otimes_{A^{\text{op}}} [E^{\text{op}}] := \beta \otimes_{A \otimes A^{\text{op}}} ([A] \otimes [E^{\text{op}}]) = [(C, C(G^1))_{sC(G^0) \otimes_r C(G^0)}, 0], \quad (3.15)$$

where the product module has action and inner product

$$(h \cdot (f_1 \otimes f_2))(g) = h(g)f_1(s(g))f_2(r(g)), \quad \text{and} \quad (h_1 | h_2)_A(v, w) = \sum_{r(g)=w, s(g)=v} \overline{h_1(g)}h_2(g)$$

for all  $h, h_1, h_2 \in C(G^1)$ , all  $f_1, f_2 \in C(G^0)$ , and all  $g \in G^1$ . Hence  $\beta \otimes_A E = \beta \otimes_{A^{\text{op}}} E^{\text{op}}$ .

The analogous identification  $V : C(G^0) \otimes_{C(G^0) \otimes C(G^0)} C(G^0) \otimes \overline{C(G^1)}^{\text{op}}_{C(G^0) \otimes C(G^0)} \rightarrow C(G^1)$  is given by  $V(a \otimes b \otimes \overline{h})(g) = a(s(g))b(s(g))\overline{h(g)}$ . This linear identification of modules shows that the class  $\beta \otimes_{A^{\text{op}}} [\overline{E}^{\text{op}}] := \beta \otimes_{A \otimes A^{\text{op}}} ([A] \otimes [\overline{E}^{\text{op}}])$  is represented by

$$\beta \otimes_{A^{\text{op}}} [\overline{E}^{\text{op}}] = [(C, C(G^1))_{sC(G^0) \otimes_r C(G^0)}, 0], \quad (3.16)$$

where

$$(h \cdot (f_1 \otimes f_2))(g) = h(g)f_1(s(g))f_2(r(g)) \quad \text{and} \quad (h_1 | h_2)_A(v, w) = \sum_{r(g)=w, s(g)=v} \overline{h_1(g)}h_2(g).$$

So  $\beta \otimes_{A^{\text{op}}} [\overline{E}^{\text{op}}] = \beta \otimes_A [E]$ . For the  $K$ -homology statements, we first check, using the same ideas as above, that

$$[E] \otimes_A \mu = ([E] \otimes [A^{\text{op}}]) \otimes_{A \otimes A^{\text{op}}} \mu = [(A \otimes A^{\text{op}}, {}_{r \otimes s} \ell^2(G^1), 0)], \quad (3.17)$$

where

$$\langle \xi_1, \xi_2 \rangle = \sum_{v \in G^0} \sum_{s(g)=v} \overline{\xi_1(g)}\xi_2(g) = \sum_{g \in G^1} \overline{\xi_1(g)}\xi_2(g) \quad (3.18)$$

is the standard  $\ell^2$  inner-product, and the actions are given by

$$((a \otimes b) \cdot \xi)(g) = a(r(g))b(s(g))\xi(g). \quad (3.19)$$

On the other hand,

$$[E^{\text{op}}] \otimes_{A^{\text{op}}} \mu = ([A] \otimes [E^{\text{op}}]) \otimes_{A \otimes A^{\text{op}}} \mu = (A \otimes A^{\text{op}}, r \otimes s \ell^2(G^1), 0),$$

where the actions are given by exactly the same formula as for  $[E] \otimes_A \mu$ , but the inner product is given by

$$\langle \xi_1, \xi_2 \rangle = \sum_{v \in G^0} \sum_{r(g)=v} \overline{\xi_1(g)} \xi_2(g) = \sum_{g \in G^1} \overline{\xi_1(g)} \xi_2(g),$$

which is (3.18), giving  $[E] \otimes_A \mu = [E^{\text{op}}] \otimes_{A^{\text{op}}} \mu$ .

For  $[\overline{E}^{\text{op}}] \otimes_{A^{\text{op}}} \mu$  there is a unitary  $U : C(G^0) \otimes \overline{C(G^1)^{\text{op}}} \otimes_{A \otimes A^{\text{op}}} \ell^2(G^0) \rightarrow \ell^2(G^1)$  such that  $U(f \otimes \overline{h} \otimes \xi)(g) = f(r(g))\overline{h(g)}\xi(r(g))$  for  $g \in G^1$ . The resulting unitary equivalence of Kasparov modules and Equation 3.17 yield

$$[A] \otimes [\overline{E}^{\text{op}}] \otimes_{A \otimes A^{\text{op}}} \mu = [(A \otimes A^{\text{op}}, r \otimes s \ell^2(G^1), 0)] = [E] \otimes_A \mu. \quad \square$$

### 3.2.2 Crossed products

Here we investigate the conditions of Theorem 3.1 and Theorem 3.4 for modules constructed from automorphisms of  $C^*$ -algebras. That is, we fix a  $C^*$ -algebra  $A$  satisfying Poincaré self-duality with respect to the classes  $\mu$  and  $\beta$ , and we consider an automorphism  $\alpha \in \text{Aut}(A)$ . We put  $E := {}_{\alpha}A_A$ , which has right action given by multiplication, inner-product  $(a | b)_A = a^*b$ , and left action given by  $a \cdot b = \alpha(a)b$ . A left inner product making  $E$  bi-Hilbertian is  ${}_A(a | b) = \alpha^{-1}(ab^*)$ . These definitions make  $E$  an invertible bimodule over  $A$ , so it yields an invertible Kasparov class

$$[E] = [(A, {}_{\alpha}A_A, 0)].$$

The opposite module  $E^{\text{op}}$  has a similar description: we have  $E^{\text{op}} = \text{id}A_{\alpha^{\text{op}}}$ , where  $\alpha^{\text{op}}$  is the automorphism of  $A^{\text{op}}$  implemented by  $\alpha$ ; that is  $\alpha^{\text{op}}(a^{\text{op}}) = \alpha(a)^{\text{op}}$ . So the left action of  $A^{\text{op}}$  on  $E^{\text{op}}$  is left multiplication (in  $A^{\text{op}}$ ), the right action of  $a^{\text{op}} \in A^{\text{op}}$  is by right-multiplication by  $\alpha^{\text{op}}(a)$ , and the inner-product is  $(e^{\text{op}} | f^{\text{op}})_{A^{\text{op}}} = (\alpha^{\text{op}})^{-1}(e^{*\text{op}}f^{\text{op}}) = \alpha^{-1}(fe^*)^{\text{op}}$ . We study the commutation of the diagram (3.9) for  $E^{\text{op}}$ .

**Lemma 3.10.** *Let  $A$  be a  $C^*$ -algebra, and take  $\alpha \in \text{Aut}(A)$ . Suppose that  $A$  satisfies Poincaré self-duality with respect to the classes  $\beta$  and  $\mu$ . Let  $E$  and  $E^{\text{op}}$  be as described above.*

1. *Suppose that  $\beta$  has representative  $\beta = [\mathbb{C}, X_{A \otimes A^{\text{op}}}, T]$  for which there is an invertible  $\mathbb{C}$ -linear map  $V : X \rightarrow X$  such that  $V^{-1}(VT - TV)$  is a compact adjointable endomorphism, and such that for all  $x, y \in X$ ,  $a \in A$  and  $b^{\text{op}} \in A^{\text{op}}$  we have*

$$V(x \cdot (a \otimes b^{\text{op}})) = V(x)(\alpha \otimes \alpha^{\text{op}})(a \otimes b^{\text{op}}), \quad \text{and} \quad (V(x) | V(y))_{A \otimes A^{\text{op}}} = (\alpha \otimes \alpha^{\text{op}})(x | y)_{A \otimes A^{\text{op}}}.$$

*Then  $\beta \otimes_A [E] = \beta \otimes_{A^{\text{op}}} [E^{\text{op}}]$ .*

2. Suppose that  $\mathcal{H}$  is a Hilbert space and  $\mathcal{A} \subseteq A$  is an  $\alpha$ -invariant dense  $*$ -subalgebra. Suppose that  $\phi : A \rightarrow \mathcal{B}(\mathcal{H})$  and  $\psi : \mathcal{A}^{\text{op}} \rightarrow \mathcal{B}(\mathcal{H})$  are  $*$ -homomorphisms that determine a spectral triple  $(A \otimes \mathcal{A}^{\text{op}}, \phi \otimes \psi, \mathcal{H}, \mathcal{D})$  representing  $\mu = [(A \otimes \mathcal{A}^{\text{op}}, \phi \otimes \psi, \mathcal{H}, \mathcal{D})]$ . Suppose that  $W_\alpha \in \mathcal{B}(\mathcal{H})$  is unitary and satisfies  $W_\alpha^* \phi(a) W_\alpha = \phi(\alpha(a))$  and  $W_\alpha^* \psi(a^{\text{op}}) W_\alpha = \psi(\alpha^{\text{op}}(a^{\text{op}}))$  for all  $a \in A$ . If  $[\mathcal{D}, W_\alpha]$  is bounded, then  $[E] \otimes_A \mu = [E^{\text{op}}] \otimes_{A^{\text{op}}} \mu$ .

*Proof.* (1) The map  $a^{\text{op}} \mapsto \alpha^{-1}(a)^{\text{op}}$  from  $\text{Id} A_{\alpha^{\text{op}}}^{\text{op}}$  to  ${}_{\alpha^{\text{op}-1}} A_{A^{\text{op}}}^{\text{op}}$  is an isomorphism of invertible bimodules, and so determines a unitary isomorphism of Kasparov modules

$$(A^{\text{op}}, \text{Id} A_{\alpha, A^{\text{op}}}^{\text{op}}, 0) \rightarrow (A^{\text{op}}, ({}_{\alpha^{\text{op}-1}} A_{A^{\text{op}}}^{\text{op}}), 0), \quad a^{\text{op}} \mapsto \alpha^{-1}(a)^{\text{op}}.$$

Hence

$$[E] \otimes [A^{\text{op}}] = (A \otimes A^{\text{op}}, {}_{\alpha \otimes \text{Id}} A \otimes A^{\text{op}}, 0), \quad \text{and} \quad [A] \otimes [E^{\text{op}}] = (A \otimes A^{\text{op}}, \text{Id} \otimes {}_{\alpha^{\text{op}-1}} A \otimes A^{\text{op}}, 0),$$

where the right actions are by multiplication and the inner products are the standard ones  $(a | b) = a^* b$ . We claim that there is a linear map  $\tilde{V} : X \otimes_{A \otimes A^{\text{op}}} (E \otimes A^{\text{op}}) \rightarrow X \otimes_{A \otimes A^{\text{op}}} (A \otimes E^{\text{op}})$  such that

$$\tilde{V}(x \otimes e \otimes b^{\text{op}}) = V(x) \otimes e \otimes b^{\text{op}} \quad \text{for all } x \in X, e \in E \text{ and } b \in A.$$

To see this, fix  $y_1, y_2 \in X$ ,  $e_1, e_2 \in E$  and  $a_1, a_2 \in A$  and calculate:

$$\begin{aligned} & (V(y_1) \otimes e_1 \otimes a_1^{\text{op}} | V(y_2) \otimes e_2 \otimes a_2^{\text{op}})_{A \otimes A^{\text{op}}} \\ &= (e_1 \otimes a_1^{\text{op}} | (V(y_1) | V(y_2))_{A \otimes A^{\text{op}}} \cdot e_2 \otimes a_2^{\text{op}})_{A \otimes A^{\text{op}}} \\ &= (e_1 \otimes a_1^{\text{op}} | (1 \otimes (\alpha^{\text{op}})^{-1})(V(y_1) | V(y_2))_{A \otimes A^{\text{op}}} e_2 \otimes a_2^{\text{op}})_{A \otimes A^{\text{op}}} \\ &= (e_1 \otimes a_1^{\text{op}} | (\alpha \otimes 1)(y_1 | y_2)_{A \otimes A^{\text{op}}} e_2 \otimes a_2^{\text{op}})_{A \otimes A^{\text{op}}} \\ &= (y_1 \otimes e_1 \otimes a_1^{\text{op}} | y_2 \otimes e_2 \otimes a_2^{\text{op}})_{A \otimes A^{\text{op}}}. \end{aligned}$$

Consequently, given  $y_i \in X$ ,  $e_i \in E$  and  $a_i \in A$ , we have

$$\begin{aligned} \left\| \sum_i V(y_i) \otimes e_i \otimes a_i \right\|^2 &= \sum_{i,j} (V(y_i) \otimes e_i \otimes a_i | V(y_j) \otimes e_j \otimes a_j)_{A \otimes A^{\text{op}}} \\ &= \sum_{i,j} (y_i \otimes e_i \otimes a_i | y_j \otimes e_j \otimes a_j)_{A \otimes A^{\text{op}}} \\ &= \left\| \sum_i y_i \otimes e_i \otimes a_i \right\|^2. \end{aligned}$$

Thus there is an isometric linear operator on  $\text{span}\{x \otimes e \otimes b^{\text{op}} : x \in X, e \in E, b \in A\}$  carrying each  $x \otimes e \otimes b^{\text{op}}$  to  $V(x) \otimes e \otimes b^{\text{op}}$ , and this extends to an isometric linear operator  $\tilde{V}$  on  $X \otimes_{A \otimes A^{\text{op}}} (E \otimes A^{\text{op}})$ .

Since  $V^{-1}(VT - TV)$  is a compact adjointable endomorphism, it is now straightforward to check that  $\tilde{V}^{-1}(\tilde{V}(T \otimes 1) - (T \otimes 1)\tilde{V})$  is also. Hence  $\tilde{V}(T \otimes 1)\tilde{V}^{-1}$  is homotopic to  $T \otimes 1$  via the straight line path. Thus  $(\mathbb{C}, X \otimes E \otimes A^{\text{op}}, T \otimes 1)$  is unitarily equivalent modulo compact perturbation to  $(\mathbb{C}, X \otimes A \otimes E^{\text{op}}, T \otimes 1)$ , completing the proof of the first statement.

(2) Let  $\mathcal{E} \subset E$  and  $\mathcal{E}^{\text{op}} \subset E^{\text{op}}$  be the submodules  $\mathcal{A}$  and  $\mathcal{A}^{\text{op}}$ . Then,

$$[E] \otimes_A \mu = [(\mathcal{E} \otimes \mathcal{A}^{\text{op}}) \otimes_{A \otimes A^{\text{op}}} (\mathcal{A} \otimes \mathcal{A}^{\text{op}}, \phi \otimes \psi, \mathcal{H}, \mathcal{D})] = [\mathcal{A} \otimes \mathcal{A}^{\text{op}}, \phi \circ \alpha \otimes \psi, \mathcal{H}, \mathcal{D}],$$

and

$$[E^{\text{op}}] \otimes_{A^{\text{op}}} \mu = [(\mathcal{A} \otimes \mathcal{E}^{\text{op}}) \otimes_{\mathcal{A} \otimes \mathcal{A}^{\text{op}}} (\mathcal{A} \otimes \mathcal{A}^{\text{op}}, \phi_{\otimes \psi} \mathcal{H}, \mathcal{D})] = [\mathcal{A} \otimes \mathcal{A}^{\text{op}}, \phi_{\otimes \psi \circ \alpha^{\text{op}-1}} \mathcal{H}, \mathcal{D}].$$

Using that  $\alpha$  is implemented by  $W_\alpha$  and that  $[W_\alpha, \mathcal{D}]$  is bounded, we see that

$$[(\mathcal{A} \otimes \mathcal{A}^{\text{op}}, \phi_{\circ \alpha \otimes \psi} \mathcal{H}, \mathcal{D})] = [(\mathcal{A} \otimes \mathcal{A}^{\text{op}}, \phi_{\otimes \psi} \mathcal{H}, W_\alpha^* \mathcal{D} W_\alpha)] = [(\mathcal{A} \otimes \mathcal{A}^{\text{op}}, \phi_{\otimes \psi} \mathcal{H}, \mathcal{D} + W_\alpha^* [\mathcal{D}, W_\alpha])].$$

Hence

$$[E] \otimes_A \mu = [(\mathcal{A} \otimes \mathcal{A}^{\text{op}}, \phi_{\otimes \psi} \mathcal{H}, \mathcal{D})].$$

A similar computation using that  $\alpha^{\text{op}}$  is also implemented by  $W_\alpha$  shows that

$$[E^{\text{op}}] \otimes_{A^{\text{op}}} \mu = [\mathcal{A} \otimes \mathcal{A}^{\text{op}}, \phi_{\otimes \psi} \mathcal{H}, \mathcal{D} + W_\alpha [\mathcal{D}, W_\alpha^*]] = [(\mathcal{A} \otimes \mathcal{A}^{\text{op}}, \phi_{\otimes \psi} \mathcal{H}, \mathcal{D})] = [E] \otimes_A \mu. \quad \square$$

*Remark 3.11.* If the operator  $T$  in the representative  $(\mathbb{C}, X, T)$  of  $\beta$  in Lemma 3.10(1) is unbounded and  $V^{-1}TV - T$  bounded, we can replace compact perturbation by bounded perturbation, as we did in the  $K$ -homology case. And vice versa.

We will show next that the criterion appearing in Lemma 3.10(1) holds for modules of the form  ${}_\alpha C(M)$  where  $M$  is a compact Riemannian  $\text{spin}^c$ -manifold and  $\alpha$  is an automorphism induced by a  $\text{spin}^c$ -structure-preserving isometry on  $M$ .

### 3.2.3 $\text{Spin}^c$ manifolds

The classical examples of  $C^*$ -algebras satisfying Poincaré self-duality are algebras of the form  $C_0(M)$  where  $(M, g)$  is a complete Riemannian  $\text{spin}^c$  manifold of dimension  $d$ . Given such a manifold  $(M, g)$  with a fixed  $\text{spin}^c$  structure, there is a spectral triple

$$(C_0^\infty(M) \otimes C_0^\infty(M), L^2(S, g), \mathcal{D}),$$

where  $S$  is the spinor bundle of the  $\text{spin}^c$  structure and  $\mathcal{D}$  the Dirac operator. This spectral triple represents the Dirac class  $\mu \in KK^d(C_0(M) \otimes C_0(M), \mathbb{C})$  in a Poincaré self-duality for  $C_0(M)$ . When  $M$  is non-compact, the product with the Dirac class gives an isomorphism  $K_*(C_0(M)) \xrightarrow{\cong} K_c^*(C_0(M))$ , where  $K_c^*$  is compactly supported  $K$ -homology, [25, 35].

Likewise the dual Bott class, described below, is well-defined for complete  $\text{spin}^c$  manifolds, giving a class in  $KK^d(\mathbb{C}, C_0(M) \otimes C_0(M))$ . By [25, Theorem 4.9], together with the Morita equivalence [31] between  $C_0(M)$  and  $\text{Cliff}_0(M)$ , the Bott and Dirac classes provide a Poincaré duality pair for  $C_0(M)$ , provided one uses compactly supported  $K$ -homology when  $M$  is non-compact, [35, Corollary 31]. For the non- $\text{spin}^c$  case, see Appendix A.

For the  $K$ -theory fundamental class, we recall the key elements of Kasparov's Bott class from [25]. Let  $U \subset M \times M$  be a neighbourhood of the diagonal such that for each  $(x, y) \in U$  there is a unique geodesic from  $x$  to  $y$ . Let  $\vec{xy}$  denote the tangent vector to this geodesic at  $x$ .

Let  $p_2 : U \rightarrow M$  be the projection on the second factor. Set  $X = \Gamma_0(p_2^* S)$ , a (non-full) right  $C^*$ -module over  $C_0(M) \otimes C_0(M)$ . There is a choice of numerical function  $\rho(x, y)$  of the distance  $d(x, y)$  such that the self-adjoint operator  $T \in \text{End}_{C_0(M) \otimes C_0(M)}(X)$  defined by

$$(T\sigma)(x, y) = \rho(x, y)\gamma(\vec{xy})\sigma(x, y), \quad \sigma \in L^2(S)$$

has the property that  $T^2 - 1$  is a compact endomorphism of  $X$ . Then the Bott class  $\beta \in KK^d(\mathbb{C}, C_0(M) \otimes C_0(M))$  is represented by the Kasparov module  $(\mathbb{C}, X, T)$ .

We consider a module  $E = \Gamma_0(M, Z)$  of continuous sections vanishing at infinity of a locally trivial vector bundle  $Z$  over  $M$  vanishing at infinity. To give  $E$  the structure of a bi-Hilbertian  $C_0(M)$ - $C_0(M)$ -bimodule we fix a diffeomorphism  $\phi$  of  $M$  defining an automorphism  $\alpha \in \text{Aut}(C_0(M))$  via  $\alpha(f)(x) = f(\phi^{-1}(x))$ . We define  ${}_{C_0(M)}(\cdot | \cdot)$  by

$${}_{C_0(M)}(e | f)(x) = \mathbb{C}(e(\phi(x)) | f(\phi(x))) = \alpha^{-1}((f | e)_{C_0(M)})(x)$$

and we define left and right actions by  $(a \cdot e \cdot b)(x) = a(\phi^{-1}(x))e(x)b(x) = \alpha(a)(x)e(x)b(x)$ . These definitions yield  $[E] \in KK(C_0(M), C_0(M))$ .

**Proposition 3.12.** *Let  $(M, g)$  be a complete Riemannian  $\text{spin}^c$  manifold,  $\mu$  and  $\beta$  the fundamental classes described above,  $Z \rightarrow M$  a vector bundle and  $\phi : M \rightarrow M$  a diffeomorphism with dual automorphism  $\alpha : C_0(M) \rightarrow C_0(M)$ . If  $\phi$  is  $\text{spin}^c$ -structure preserving, then  $[E] \otimes_{C_0(M)} \mu = [E^{\text{op}}] \otimes_{C_0(M)} \mu$ . If  $\phi$  is also an isometry then  $\beta \otimes_{C_0(M)} [E] = \beta \otimes_{C_0(M)} [E^{\text{op}}]$ .*

*Proof.* If  $\phi$  is  $\text{spin}^c$ -structure preserving, there exists  $V : \Gamma_0(S) \rightarrow \Gamma_0(S)$  such that for  $f \in C_0(M)$  acting by multiplication we have  $VfV^{-1}\sigma(x) = f(\phi^{-1}(x))\sigma(x)$ . For  $v \in T^*M$ ,  $\gamma(v \cdot d\phi^{-1}) = V \cdot \gamma(v) \cdot V^{-1}$ , where  $\gamma$  denotes the Clifford action of forms on spinors. (The scalar ambiguity in this characterisation of  $V$  is resolved precisely by the choice of  $\text{spin}^c$  structure.)

We can then compute the commutator

$$[\mathcal{D}, VfV^{-1}] = V[\mathcal{D}, f]V^{-1} + [[\mathcal{D}, V]V^{-1}, VfV^{-1}] = \gamma(df \cdot d\phi^{-1}) + [[\mathcal{D}, V]V^{-1}, f \circ \phi^{-1}].$$

Since  $V$  is a smooth map,  $[\mathcal{D}, V]V^{-1}$  is at most a first order differential operator, so  $[\mathcal{D}, VfV^{-1}]$  is bounded, and the conditions of Lemma 3.10(2) are satisfied and so  $[E] \otimes_{C_0(M)} \mu = [E^{\text{op}}] \otimes_{C_0(M)} \mu$ .

We now consider the  $K$ -theory fundamental class. Given a  $\text{spin}^c$ -structure-preserving diffeomorphism  $\phi$  of  $M$ , we obtain a lift  $V : X \rightarrow X$  of  $\phi$  satisfying  $(VfV^{-1}\sigma)(x, y) = f \circ \phi^{-1}(x)\sigma(x, y)$  and  $V(\sigma(f \otimes g))(x, y) = (V\sigma)(x, y)f \circ \phi^{-1}(x)g \circ \phi^{-1}(y)$ .

If  $\phi$  is an isometry, so that the distance and hence also  $\rho$  is invariant, then we can compute the commutator of  $T$  and  $V$  as follows: for each section  $\sigma$ , we have

$$\begin{aligned} (VT\sigma)(x, y) &= \rho(\phi^{-1}(x), \phi^{-1}(y))\gamma(\overrightarrow{x\hat{y}} \cdot d\phi^{-1})(V\sigma)(x, y) \\ &= \rho(x, y)\gamma(\overrightarrow{\phi^{-1}(x)\phi^{-1}(y)})V\sigma(x, y) = (TV\sigma)(x, y). \end{aligned}$$

So the conditions of Lemma 3.10(1) are satisfied and so  $\beta \otimes_{C_0(M)} [E] = \beta \otimes_{C_0(M)} [E^{\text{op}}]$ .  $\square$

## 4 The $K$ -theory fundamental class

In this section we start with a  $C^*$ -algebra  $A$  satisfying Poincaré self-duality with fundamental classes  $\beta, \mu$  of parity  $d$ .

We identify hypotheses on a bi-Hilbertian  $A$ -bimodule  $E$  that allow us to apply Theorem 3.4 to construct fundamental classes  $\delta \in KK^{d+1}(\mathbb{C}, \mathcal{O}_E \otimes \mathcal{O}_{\overline{E}^{\text{op}}})$  and  $\bar{\delta} \in KK^{d+1}(\mathbb{C}, \mathcal{O}_E \otimes \mathcal{O}_E^{\text{op}})$ . In what follows, if  $E_b$  is a full Hilbert module over a unitisation  $A_b$  of  $A$ , then the *restriction* of  $E_b$  to

$A$  is defined as  $E := E_b \otimes_{A_b} A$ . This condition arises as the most useful notion of (sections of) non-commutative vector bundles for non-unital algebras, [37].

To obtain the class  $\delta$ , we assume that:

1.  $E$  is a bi-Hilbertian  $A$ -bimodule which is the restriction of a module  $E_b$  over a unitisation  $A_b$  which is finitely generated as a left and right module over  $A_b$ ; and
2.  $\beta \otimes_A [E] = \beta \otimes_{A^{\text{op}}} [E^{\text{op}}]$ .

To obtain the class  $\bar{\delta}$ , we assume that:

1.  $E$  is an  $A$ - $A$  correspondence, and is the restriction of a module  $E_b$  over a unitisation  $A_b$  that is finitely generated as a right module over  $A_b$  with injective left action of  $A_b$ ; and
2.  $\beta \otimes_A [E] = \beta \otimes_{A^{\text{op}}} [\bar{E}^{\text{op}}]$ .

#### 4.1 The $K$ -theory fundamental class for $\mathcal{O}_{E^{\text{op}}}$

To construct the  $K$ -theory fundamental class, we first need to recall the mapping cone exact sequence in our setting, and some constructions from [1].

The inclusion  $\iota_{A, \mathcal{O}_E} : A \hookrightarrow \mathcal{O}_E$  gives rise to a mapping cone algebra

$$M(A, \mathcal{O}_E) = \{f \in C_0([0, \infty), \mathcal{O}_E) : f(0) \in \iota_{A, \mathcal{O}_E}(A)\}.$$

Write  $\mathcal{S}\mathcal{O}_E$  for the suspension  $C_0((0, \infty)) \otimes \mathcal{O}_E$ , and  $j : \mathcal{S}\mathcal{O}_E \rightarrow M(A, \mathcal{O}_E)$  for the inclusion. The evaluation map  $\text{ev} : M(A, \mathcal{O}_E) \rightarrow A$ , given by  $\text{ev}(f) = f(0)$ , induces a short exact sequence

$$0 \rightarrow \mathcal{S}\mathcal{O}_E \xrightarrow{j} M(A, \mathcal{O}_E) \xrightarrow{\text{ev}} A \rightarrow 0, \quad (4.1)$$

where  $j$  is the inclusion of the suspension  $\mathcal{S}\mathcal{O}_E$  into the mapping cone. Thus we obtain a  $K$ -theory exact sequence (see [1] and [7, Section 3])

$$\dots \rightarrow K_1(A) \xrightarrow{-\iota_{A, \mathcal{O}_E}^*} K_1(\mathcal{O}_E) \xrightarrow{j_*} K_0(M(A, \mathcal{O}_E)) \xrightarrow{\text{ev}_*} K_0(A) \xrightarrow{-\iota_{A, \mathcal{O}_E}^*} K_0(\mathcal{O}_E) \rightarrow \dots$$

Elements of  $K_0(M(A, \mathcal{O}_E))$  can be described as homotopy classes of partial isometries  $v$  over  $\tilde{\mathcal{O}}_E$  whose range and source projections  $v^*v$  and  $vv^*$  are projections over  $\tilde{A}$ , [33]. In this language,  $\text{ev}_*([v]) = [v^*v] - [vv^*]$ .

If  $E = E_b \otimes_{A_b} A$  is the restriction of a module over the minimal unitisation  $A_b$  of  $A$ , then [1, Section 6.2] describes an explicit Kasparov module representing a class  $[W] \in KK(A, M(A, \mathcal{O}_E))$ . To describe this representative and its key properties, we fix a frame  $(x_j)_{j=1}^k \subset E_b$  (or just  $E$  when  $A$  is unital), and define  $w \in M_k(\mathcal{O}_{E_b})$  by

$$w = \begin{pmatrix} S_{x_1}^* & 0 & \cdots & 0 \\ S_{x_2}^* & 0 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ S_{x_k}^* & 0 & \cdots & 0 \end{pmatrix}. \quad (4.2)$$

We have  $w^*w = \text{Id}_{\mathcal{O}_{\tilde{E}}} \oplus 0_{k-1} = \iota_{A \sim, \mathcal{O}_{\tilde{E}}}(\text{Id}_{A \sim}) \oplus 0_{k-1}$  and

$$ww^* = \begin{pmatrix} (x_1|x_1)_{A_b} & (x_1|x_2)_{A_b} & \cdots & (x_1|x_k)_{A_b} \\ (x_2|x_1)_{A_b} & (x_2|x_2)_{A_b} & \cdots & (x_2|x_k)_{A_b} \\ \vdots & & \ddots & \vdots \\ (x_k|x_1)_{A_b} & (x_k|x_2)_{A_b} & \cdots & (x_k|x_k)_{A_b} \end{pmatrix} = (x_i|x_j)_{i,j \geq 1} =: q \in M_k(A_b).$$

Then  $E_b \cong qA_b^k$ , with isomorphism given by  $e \mapsto ((x_1|e)_{A_b}, \dots, (x_k|e)_{A_b})^T$ . We can explicitly realise  $[w]$  as a difference of classes of projections over the minimal unitisation  $M(A_b, \mathcal{O}_{E_b})^\sim$  of the mapping cone  $M(A_b, \mathcal{O}_{E_b})$ .<sup>1</sup> Using [33], we have an identification of classes  $[w] = [e_w] - [1_k]$ , where

$$e_w(t) = \begin{pmatrix} 1_k - \frac{1}{1+t^2}q & \frac{-it}{1+t^2}w \\ \frac{it}{1+t^2}w^* & \frac{1}{1+t^2}\text{Id}_{\mathcal{O}_{E_b}} \end{pmatrix} = \begin{pmatrix} \frac{1}{1+t^2}(1_k - q) + \frac{t^2}{1+t^2}1_k & \frac{-it}{1+t^2}w \\ \frac{it}{1+t^2}w^* & \frac{1}{1+t^2}\text{Id}_{\mathcal{O}_{E_b}} \end{pmatrix}. \quad (4.3)$$

Then

$$\varphi(a) := ((x_i | \phi(a)x_j)_{A_b})_{i,j}$$

defines a left action of  $A_b$  on  $q(A_b)^k$ . Since  $w(0_{k-1} \oplus \phi(a))w^* = \varphi(a)$  and  $w^*\varphi(a)w = 0_{k-1} \oplus \phi(a)$ , it is straightforward to check that for all  $t \in [0, \infty)$

$$e_w(t) \begin{pmatrix} ((x_i | \phi(a)x_j)_{A_b})_{i,j} & 0 \\ 0 & \phi(a) \end{pmatrix} = \begin{pmatrix} ((x_i | \phi(a)x_j)_{A_b})_{i,j} & 0 \\ 0 & \phi(a) \end{pmatrix} e_w(t)$$

as operators on  $\mathcal{O}_{\tilde{E}}^{2k}$  (or  $(A_b)^{2k}$  for  $t = 0$ ). The last ingredient is the unitisation of  $M(A, \mathcal{O}_E)$  consisting of functions  $f : [0, \infty) \rightarrow \mathcal{O}_E + A_b$  which have a limit at infinity lying in  $A_b$ , so

$$M(A, \mathcal{O}_E)_b := \{f : [0, \infty) \rightarrow \mathcal{O}_E + A_b : f \text{ continuous, } f(0) \in A_b, \lim_{t \rightarrow \infty} f(t) \in A_b\}. \quad (4.4)$$

Then, using the injective and nondegenerate left action of  $A_b$  and [1, Lemma 6.1, Lemma 6.2], gives a Kasparov class

$$[W] = \left[ \left( A, \begin{pmatrix} e_w(M(A, \mathcal{O}_{E_A})_b)^{2k} \\ (M(A, \mathcal{O}_{E_A})_b)^k \end{pmatrix}, 0 \right) \right] \in KK(A, M(A, \mathcal{O}_{E_A})).$$

An important ingredient in the following arguments is a class  $[\widehat{\text{ext}}] \in KK(M(A, \mathcal{O}_E), A)$  which is  $KK$ -inverse to the class  $W$ , when  $A$  belongs to the bootstrap class. To describe  $[\widehat{\text{ext}}]$ , start from the mapping cone exact sequence (4.1) to obtain the exact sequence

$$\dots \xrightarrow{\text{ev}^*} KK^0(M(A, \mathcal{O}_E), A) \xrightarrow{j^*} KK^0(\mathcal{S}\mathcal{O}_E, A) \xrightarrow{\partial} KK^1(A, A) \rightarrow \dots$$

In this exact sequence, the boundary map  $\partial$  is given (up to sign and Bott periodicity) by the inclusion  $\iota_{A, \mathcal{O}} : A \hookrightarrow \mathcal{O}_E$ , [7, Lemma 3.1]. Restricting the extension class  $[\text{ext}]$  to  $A \subset \mathcal{O}_E$  gives the zero class in  $KK(A, A)$ , because the class of the extension  $[\text{ext}]$  implements the boundary map in the Pimsner exact sequence in  $K$ -theory. Thus the boundary map  $\partial$  in the mapping cone exact sequence applied to  $[\text{ext}]$  gives zero. This implies the existence of a class  $[\widehat{\text{ext}}] \in KK^0(M(A, \mathcal{O}_E), A)$  such that  $j^*[\widehat{\text{ext}}] = j \otimes_M [\widehat{\text{ext}}] = [\text{ext}]$ .

We now recall the key relation between  $[W]$  and the  $KK$ -class  $[\widehat{\text{ext}}]$  described in [1].

<sup>1</sup> Since  $e_w(\infty) = 1_k$ , we obtain a class in the  $KK$  group for  $M(A, \mathcal{O}_E)$ . See [24, Corollary 1, Section 7]



**Lemma 4.1.** (*[1, Lemma 6.1]*) Let  $[\text{ext}] \in KK^1(\mathcal{O}_E, A) = KK(\mathcal{S}\mathcal{O}_E, A)$  be the class of the defining extension for  $\mathcal{O}_E$  and let  $[W] \in KK(A, M(A, \mathcal{O}_E))$  be as above. Let  $[\widehat{\text{ext}}] \in KK(M(A, \mathcal{O}_E), A)$  be a class such that  $j^*[\widehat{\text{ext}}] = [\text{ext}]$  as above. Then

$$[W] \otimes_M [\widehat{\text{ext}}] = -\text{Id}_{KK(A, A)}.$$

Let  $\mathbb{M} := M(A \otimes A^{\text{op}}, \mathcal{O}_E \otimes \mathcal{O}_{E^{\text{op}}})$  be the mapping cone algebra for the inclusion  $A \otimes A^{\text{op}} \hookrightarrow \mathcal{O}_E \otimes \mathcal{O}_{E^{\text{op}}}$ . Using the canonical identification  $\mathcal{S}(\mathcal{O}_E \otimes \mathcal{O}_{E^{\text{op}}}) \cong \mathcal{S}\mathcal{O}_E \otimes \mathcal{O}_{E^{\text{op}}}$ , we have an exact sequence

$$0 \rightarrow \mathcal{S}\mathcal{O}_E \otimes \mathcal{O}_{E^{\text{op}}} \xrightarrow{\mathbf{j}} \mathbb{M} \xrightarrow{\text{ev}} A \otimes A^{\text{op}} \rightarrow 0. \quad (4.5)$$

By reasoning similar to that used to define  $W$ , we obtain a Kasparov module

$$\mathbb{W} = \left( A \otimes A^{\text{op}}, \left( \begin{array}{c} (e_{w \otimes \text{Id}_{A^{\text{op}}}}) \mathbb{M}_b^{2k} \\ \mathbb{M}_b^k \end{array} \right)_{\mathbb{M}}, 0 \right),$$

where the mapping  $\mathbb{M}$  has been unitised by considering functions  $f : [0, \infty) \rightarrow \mathcal{O}_E \otimes \mathcal{O}_{E^{\text{op}}} + A_b \otimes A_b^{\text{op}}$  as in Equation (4.4).

If  $E$  is the restriction of a bimodule over  $A_b$  that is finitely generated on both sides, then  $E^{\text{op}}$  is similarly a restriction of a bimodule over  $A_b^{\text{op}}$ . So we may apply the discussion above to the module  $E^{\text{op}}$  over  $A^{\text{op}}$  to obtain a partial isometry  $w^{\text{op}}$  and a class  $[W^{\text{op}}]$ , and [1, Lemma 6.1] gives  $[W] \otimes_{M^{\text{op}}} [\widehat{\text{ext}}]^{\text{op}} = -\text{Id}_{KK(A^{\text{op}}, A^{\text{op}})}$ . So we obtain a class

$$\mathbb{W}^{\text{op}} = \left( A \otimes A^{\text{op}}, \left( \begin{array}{c} (e_{\text{Id}_A \otimes w^{\text{op}}}) \mathbb{M}_b^{2k} \\ \mathbb{M}_b^k \end{array} \right)_{\mathbb{M}}, 0 \right).$$

**Definition 4.2.** Suppose that  $E$  is the restriction of a finitely generated  $A_b$ -bimodule, and let  $\beta$  be a  $K$ -theory fundamental class in  $KK^d(\mathbb{C}, A \otimes A^{\text{op}})$ . We define

$$\hat{\delta}_{E, \beta} := \beta \otimes_{A \otimes A^{\text{op}}} \mathbb{W} - \beta \otimes_{A \otimes A^{\text{op}}} \mathbb{W}^{\text{op}} \in KK^d(\mathbb{C}, \mathbb{M}).$$

We will generally suppress the subscripts  $E, \beta$  and just denote this class by  $\hat{\delta}$ .

**Lemma 4.3.** *Suppose that  $E$  is the restriction of a finitely generated  $A_b$ -bimodule. Given a  $K$ -theory fundamental class  $\beta \in KK^d(\mathbb{C}, A \otimes A^{\text{op}})$  satisfying  $\beta \otimes_A [E] = \beta \otimes_{A^{\text{op}}} [E^{\text{op}}]$ , the class  $\hat{\delta}$  satisfies*

$$\hat{\delta} \otimes_{\mathbb{M}} \text{ev} = 0.$$

*There exists a class  $\delta \in K_d(\mathcal{S}\mathcal{O}_E \otimes \mathcal{O}_{E^{\text{op}}})$  such that  $\hat{\delta} = \delta \otimes_{\mathcal{S}\mathcal{O}_E \otimes \mathcal{O}_{E^{\text{op}}}} \mathbf{j}$  where  $\mathbf{j} \in KK(\mathcal{S}\mathcal{O}_E \otimes \mathcal{O}_{E^{\text{op}}}, \mathbb{M})$  is the class of the inclusion.*

*Proof.* The second statement will follow from the first by exactness of the  $K$ -theory exact sequence. So we compute

$$\begin{aligned} \hat{\delta} \otimes_{\mathbb{M}} \text{ev} &= \beta \otimes_{A \otimes A^{\text{op}}} (([A] - [E]) \otimes [A^{\text{op}}]) - \beta \otimes_{A \otimes A^{\text{op}}} ([A] \otimes ([A^{\text{op}}] - [E^{\text{op}}])) \\ &= \beta \otimes_{A^{\text{op}}} [E^{\text{op}}] - \beta \otimes_A [E] = 0 \end{aligned}$$

by assumption on the class  $\beta$ . □

The preceding lemma provides us with the tools we need to check that the product of the class  $\delta \in KK^d(\mathbb{C}, \mathcal{S}\mathcal{O}_E \otimes \mathcal{O}_{E^{\text{op}}}) = KK^{d+1}(\mathbb{C}, \mathcal{O}_E \otimes \mathcal{O}_{E^{\text{op}}})$  with the extension class satisfies the condition appearing in Theorem 3.1(2).

To do this we consider the mapping cone exact sequence (4.5) and apply the same reasoning that we did for the ‘one-variable’ mapping cone sequence (4.1). This shows that restricting the class  $[\text{ext}] \otimes \text{Id}_{KK(\mathcal{O}_{E^{\text{op}}}, \mathcal{O}_{E^{\text{op}}})}$  to  $A \otimes A^{\text{op}}$  gives the zero class. Hence there is a lift of  $[\text{ext}] \otimes \text{Id}_{KK(\mathcal{O}_{E^{\text{op}}}, \mathcal{O}_{E^{\text{op}}})}$  in  $KK(\mathbb{M}, A \otimes \mathcal{O}_{E^{\text{op}}})$ . We claim that we can choose a lift  $\widehat{[\text{ext}]}$  such that

$$\mathbf{j} \otimes_{\mathbb{M}} \widehat{[\text{ext}]} = j \otimes_M \widehat{[\text{ext}]} \otimes \text{Id}_{KK(\mathcal{O}_{E^{\text{op}}}, \mathcal{O}_{E^{\text{op}}})}. \quad (4.6)$$

To see this, pick any representative  $(M(A, \mathcal{O}_E), Y_A, S)$  of the class  $\widehat{[\text{ext}]}$  such that the action of  $M(A, \mathcal{O}_E)$  on  $Y_A$  is non-degenerate. We compute the right hand side of (4.6) to find

$$\begin{aligned} j \otimes_M \widehat{[\text{ext}]} \otimes \text{Id}_{KK(\mathcal{O}_{E^{\text{op}}}, \mathcal{O}_{E^{\text{op}}})} &= (\mathcal{S}\mathcal{O}_E, M_M, 0) \otimes_M (M, Y_A, S) \otimes (\mathcal{O}_{E^{\text{op}}}, (\mathcal{O}_{E^{\text{op}}})_{\mathcal{O}_{E^{\text{op}}}}, 0) \\ &= (\mathcal{S}\mathcal{O}_E \otimes \mathcal{O}_{E^{\text{op}}}, Y_A \otimes (\mathcal{O}_{E^{\text{op}}})_{\mathcal{O}_{E^{\text{op}}}}, S \otimes 1) \\ &= (\mathcal{S}\mathcal{O}_E \otimes \mathcal{O}_{E^{\text{op}}}, \mathbb{M}_{\mathbb{M}}, 0) \otimes_{\mathbb{M}} (\mathbb{M}, Y_A \otimes (\mathcal{O}_{E^{\text{op}}})_{\mathcal{O}_{E^{\text{op}}}}, S \otimes 1). \end{aligned}$$

Let  $\widehat{[\text{ext}]}$  be the class of the Kasparov module  $(\mathbb{M}, Y_A \otimes (\mathcal{O}_{E^{\text{op}}})_{\mathcal{O}_{E^{\text{op}}}}, S \otimes 1)$ . Then

$$j \otimes_M \widehat{[\text{ext}]} \otimes \text{Id}_{KK(\mathcal{O}_{E^{\text{op}}}, \mathcal{O}_{E^{\text{op}}})} = \mathbf{j} \otimes_{\mathbb{M}} \widehat{[\text{ext}]},$$

and

$$(\mathcal{S}\mathcal{O}_E \otimes \mathcal{O}_{E^{\text{op}}}, \mathbb{M}_{\mathbb{M}}, 0) \otimes_{\mathbb{M}} (\mathbb{M}, Y_A \otimes (\mathcal{O}_{E^{\text{op}}})_{\mathcal{O}_{E^{\text{op}}}}, S \otimes 1) = (\mathcal{S}\mathcal{O}_E \otimes \mathcal{O}_{E^{\text{op}}}, Y_A \otimes (\mathcal{O}_{E^{\text{op}}}), S \otimes 1),$$

which represents  $[\text{ext}] \otimes \text{Id}_{KK(\mathcal{O}_{E^{\text{op}}}, \mathcal{O}_{E^{\text{op}}})}$ . So any such  $\widehat{[\text{ext}]}$  provides the desired lift.

*Remark 4.4.* The class  $\delta$  of Lemma 4.3 need not be unique. With some additional regularity properties on  $E$  and  $[\text{ext}]$ , we can explicitly construct a concrete representative of such a lift: see [1, 7].

**Theorem 4.5.** *Suppose that  $E$  is the restriction of a finitely generated  $A_b$ -bimodule for some unitisation  $A_b$  of  $A$ . Suppose that  $\beta$  is a  $K$ -theory fundamental class in  $KK^d(\mathbb{C}, A \otimes A^{\text{op}})$ , and suppose that  $\beta \otimes_A [E] = \beta \otimes_{A^{\text{op}}} [E^{\text{op}}]$ . Let  $\delta \in K_d(\mathcal{S}\mathcal{O}_E \otimes \mathcal{O}_{E^{\text{op}}})$  be the class obtained from Lemma 4.3. Then*

$$\delta \otimes_{\mathcal{O}_E} [\text{ext}] = -\beta \otimes_{A^{\text{op}}} \iota_{A^{\text{op}}, \mathcal{O}_{E^{\text{op}}}} \quad \text{and} \quad \delta \otimes_{\mathcal{O}_{E^{\text{op}}}} [\text{ext}^{\text{op}}] = \beta \otimes_A \iota_{A, \mathcal{O}_E}.$$

*In particular,  $\delta$  defines isomorphisms  $K^*(\mathcal{O}_E) \rightarrow K_{*+d+1}(\mathcal{O}_{E^{\text{op}}})$  and  $K^*(\mathcal{O}_{E^{\text{op}}}) \rightarrow K_{*+d+1}(\mathcal{O}_E)$ .*

*Proof.* We have

$$\begin{aligned} \delta \otimes_{\mathcal{S}\mathcal{O}_E \otimes \mathcal{O}_{E^{\text{op}}}} [\text{ext}] &:= \delta \otimes_{\mathcal{S}\mathcal{O}_E \otimes \mathcal{O}_{E^{\text{op}}}} ([\text{ext}] \otimes \text{Id}_{\mathcal{O}_{E^{\text{op}}}}) = \delta \otimes_{\mathcal{S}\mathcal{O}_E \otimes \mathcal{O}_{E^{\text{op}}}} ((j \otimes_M \widehat{[\text{ext}]}) \otimes \text{Id}_{\mathcal{O}_{E^{\text{op}}}}) \\ &= \delta \otimes_{\mathcal{S}\mathcal{O}_E \otimes \mathcal{O}_{E^{\text{op}}}} \mathbf{j} \otimes_{\mathbb{M}} \widehat{[\text{ext}]} = \hat{\delta} \otimes_{\mathbb{M}} \widehat{[\text{ext}]}. \end{aligned}$$

Let  $X$  be any Stinespring dilation module for the Fock module of  $E$ , and let  $P : X \rightarrow \mathcal{F}_E$  denote the projection onto the Fock space. Let  $w \in M_k(\mathcal{O}_E)$  be as in (4.2) above. Define

$$\tilde{w} := (P \otimes \text{Id}_{\mathcal{O}_{E^{\text{op}}}} \otimes 1_k)(w \otimes \text{Id}_{\mathcal{O}_{E^{\text{op}}}})(P \otimes \text{Id}_{\mathcal{O}_{E^{\text{op}}}} \otimes 1_k) : w^* w X^k \otimes \mathcal{O}_{E^{\text{op}}} \rightarrow w w^* X^k \otimes \mathcal{O}_{E^{\text{op}}}.$$

Regard  $\text{Index}(\tilde{w}) = [\ker(\tilde{w})] - [\ker(\tilde{w}^*)]$  as an element of  $KK(A \otimes A^{\text{op}}, A \otimes \mathcal{O}_{E^{\text{op}}})$  as in Lemma 4.1. Then [6, Theorem 2.11] gives

$$\mathbb{W} \otimes_{\mathbb{M}} \widehat{[\text{ext}]} = -\text{Index}(\tilde{w}),$$

So, just as in Lemma 4.1, we have

$$\mathbb{W} \otimes_{\mathbb{M}} \widehat{[\text{ext}]} = -(A \otimes A^{\text{op}}, (A \otimes \mathcal{O}_{E^{\text{op}}})_{A \otimes \mathcal{O}_{E^{\text{op}}}}, 0) = -\text{Id}_{KK(A, A)} \otimes \iota_{A^{\text{op}}, \mathcal{O}_{E^{\text{op}}}}.$$

The product  $\mathbb{W}^{\text{op}} \otimes_{\mathbb{M}} \widehat{[\text{ext}]}$  is zero, because the restriction of  $\widehat{[\text{ext}]}$  to  $A \otimes \mathcal{O}_{E^{\text{op}}}$  is zero. Thus

$$\delta \otimes_{\mathcal{O}_E} [\text{ext}] = -\beta \otimes_{A^{\text{op}}} \iota_{A^{\text{op}}, \mathcal{O}_{E^{\text{op}}}}.$$

An analogous argument gives  $\delta \otimes_{\mathcal{O}_{E^{\text{op}}}} [\text{ext}^{\text{op}}] = \beta \otimes_A \iota_{A, \mathcal{O}_E}$ .  $\square$

## 4.2 The $K$ -theory fundamental class for $\mathcal{O}_E^{\text{op}}$ .

Suppose that  $E$  is the restriction to  $A$  of a finitely generated right  $A_b$ -module  $E_b$  with injective and nondegenerate left action by  $A_b$ , and  $\beta \otimes_A [E] = \beta \otimes_{A^{\text{op}}} [\overline{E}^{\text{op}}]$ . Then  $\overline{E}^{\text{op}}$  is the restriction of a finitely generated right  $A^{\text{op}}$  module, and so we can produce a class  $\bar{\delta}$  analogous to  $\delta$ . The difference between this construction and the one in the preceding section is illustrated by the following three lemmas. The first is standard: we include it for completeness since we need an explicit description of the isomorphism.

**Lemma 4.6.** *For any  $C^*$ -algebra  $B$  there is an isomorphism  $K_0(B) \cong K_0(B^{\text{op}})$ . When  $B$  is unital and  $p^{\text{op}} \in M_n(B^{\text{op}})$  is a projection, the isomorphism sends  $[p^{\text{op}}]$  to  $[p^T]$ , where we now regard the entries of  $p$  as elements of  $B$ .*

*Proof.* It suffices to prove the lemma for unital algebras. Given a finitely generated and projective right  $B^{\text{op}}$  module  $p^{\text{op}}(B^{\text{op}})^n$  (thought of as columns) we obtain a finitely generated and projective left  $B^{\text{op}}$  module  $(B^{\text{op}})^n(p^{\text{op}})^T$ , thought of as rows.

A finitely generated and projective left  $B^{\text{op}}$  module is the same thing as a finitely generated and projective right  $B$  module, namely  $p^T B^n$ .

As this construction is plainly symmetric in  $B$  and  $B^{\text{op}}$ , we obtain the stated isomorphism.  $\square$

**Lemma 4.7.** *Use Lemma 2.8 to identify  $\mathcal{O}_E^{\text{op}}$  and  $\mathcal{O}_{\overline{E}^{\text{op}}}$ . The isomorphism*

$$K_0(M(A^{\text{op}}, \mathcal{O}_E^{\text{op}})) = K_0(M(A, \mathcal{O}_E)^{\text{op}}) \cong K_0(M(A, \mathcal{O}_E))$$

*of Lemma 4.6 carries the class of*

$$w_{\overline{E}^{\text{op}}} = \begin{pmatrix} S_{x_1}^* & 0 & \cdots & 0 \\ S_{x_2}^* & 0 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ S_{x_k}^* & 0 & \cdots & 0 \end{pmatrix}$$

*to the class of*

$$w_E^* = \begin{pmatrix} S_{x_1} & S_{x_2} & \cdots & S_{x_k} \\ 0 & 0 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

*Proof.* The isomorphism  $\mathcal{O}_{\overline{E}^{\text{op}}} \rightarrow \mathcal{O}_E^{\text{op}}$  of Lemma 2.8 is given on generators by  $S_{\overline{x}} \mapsto S_x^{*\text{op}}$ . Thus

$$w_{\overline{E}^{\text{op}}} = \begin{pmatrix} S_{x_1}^{*\text{op}} & 0 & \cdots & 0 \\ S_{x_2}^{*\text{op}} & 0 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ S_{x_k}^{*\text{op}} & 0 & \cdots & 0 \end{pmatrix} \mapsto \begin{pmatrix} S_{x_1}^{\text{op}} & 0 & \cdots & 0 \\ S_{x_2}^{\text{op}} & 0 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ S_{x_k}^{\text{op}} & 0 & \cdots & 0 \end{pmatrix} \mapsto \begin{pmatrix} S_{x_1} & S_{x_2} & \cdots & S_{x_k} \\ 0 & 0 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} = w_E^*,$$

where the second isomorphism is the isomorphism of Lemma 4.6.  $\square$

We can now define analogues of the classes  $W^{\text{op}}$ ,  $\mathbb{W}^{\text{op}}$  and so forth using  $\overline{E}^{\text{op}}$  in place of  $E^{\text{op}}$ . We denote the resulting classes and their representatives by  $\overline{W}^{\text{op}}$ ,  $\overline{\mathbb{W}}^{\text{op}}$  and so on.

An argument similar to Lemma 4.3, but using Lemma 4.7, proves the following lemma.

**Lemma 4.8.** *Suppose that  $E$  is the restriction of a finitely generated right  $A_b$ -module  $E_b$  to  $A$ . Let  $\mathbf{j} \in KK(\mathcal{S}\mathcal{O}_E \otimes \mathcal{O}_E^{\text{op}}, \mathbb{M})$  be the class of the inclusion. Given a  $K$ -theory fundamental class  $\beta \in KK^d(\mathbb{C}, A \otimes A^{\text{op}})$  satisfying  $\beta \otimes_A [E] = \beta \otimes_{A^{\text{op}}} [\overline{E}^{\text{op}}]$ , the class  $\hat{\delta}$  defined by*

$$\hat{\delta} = \beta_{A \otimes A^{\text{op}}} \mathbb{W} + \beta \otimes_{A \otimes A^{\text{op}}} \overline{\mathbb{W}}^{\text{op}}$$

satisfies

$$\hat{\delta} \otimes_{\mathbb{M}} \text{ev} = 0.$$

Hence there exists a class  $\overline{\delta} \in K_d(\mathcal{S}\mathcal{O}_E \otimes \mathcal{O}_E^{\text{op}})$  such that  $\hat{\delta} = \overline{\delta} \otimes_{\mathcal{S}\mathcal{O}_E \otimes \mathcal{O}_E^{\text{op}}} \mathbf{j}$ .

So, just as before, we obtain a  $K$ -theory fundamental class.

**Theorem 4.9.** *Suppose that  $E$  is the restriction of a finitely generated right  $A_b$ -module  $E_b$  to  $A$ . Given a  $K$ -theory fundamental class  $\beta \in KK^d(\mathbb{C}, A \otimes A^{\text{op}})$  satisfying  $\beta \otimes_A [E] = \beta \otimes_{A^{\text{op}}} [\overline{E}^{\text{op}}]$ ,*

$$\overline{\delta} \otimes_{\mathcal{O}_E} [\text{ext}] = -\beta \otimes_{A^{\text{op}}} \iota_{A^{\text{op}}, \mathcal{O}_E^{\text{op}}} \quad \text{and} \quad \overline{\delta} \otimes_{\mathcal{O}_E^{\text{op}}} [\overline{\text{ext}}^{\text{op}}] = -\beta \otimes_A \iota_{A, \mathcal{O}_E}.$$

Consequently  $\overline{\delta}$  defines isomorphisms  $K^*(\mathcal{O}_E) \rightarrow K_{*+d+1}(\mathcal{O}_E^{\text{op}})$  and  $K^*(\mathcal{O}_E^{\text{op}}) \rightarrow K_{*+d+1}(\mathcal{O}_E)$ .

### 4.3 The $K$ -theory classes for an invertible bimodule

Comparing Lemma 4.3 with Lemma 4.8, we see a discrepancy of sign between the definitions of  $\hat{\delta}$  and  $\overline{\delta}$ . The following proposition reconciles this difference in the context of our Poincaré duality pairings in the situation of invertible bimodules  $E$ .

In this section, we write  $\mathcal{O}_E$  for the dense  $*$ -subalgebra of  $\mathcal{O}_E$  generated by  $A$  and the elements  $\{S_e : e \in E\}$ , so

$$\mathcal{O}_E = \text{span} \{S_\eta S_\zeta^* : \eta, \zeta \in \bigcup_{n \geq 0} E^{\otimes n}\}. \quad (4.7)$$

**Proposition 4.10** ([36]). *Suppose that  $E$  is an invertible bimodule. Let  $N$  be the densely-defined number operator on  $(\mathcal{F}_{E, \mathbb{Z}})_A$  such that  $N\rho := n\rho$  and  $N\overline{\rho} = -n\overline{\rho}$  for  $n \geq 0$  and  $\rho \in E^{\otimes n}$ . Then  $[\text{ext}] \in KK^1(\mathcal{O}_E, A)$  has representative*

$$(\mathcal{O}_E, (\mathcal{F}_{E, \mathbb{Z}})_A, N).$$

Similarly, the class  $[\text{ext}^{\text{op}}] \otimes_{\text{End}^{\text{op}}} [\mathcal{F}_{E^{\text{op}}}] \in KK^1(\mathcal{O}_{E^{\text{op}}}, A^{\text{op}})$  is represented by

$$(\mathcal{O}_{E^{\text{op}}}, (F_{E, \mathbb{Z}})_{A^{\text{op}}}, N) = (\mathcal{O}_E^{\text{op}}, (\mathcal{F}_{E, \mathbb{Z}})_{A^{\text{op}}}, N).$$

The class  $[\overline{\text{ext}}^{\text{op}}] \in KK^1(\mathcal{O}_E^{\text{op}}, A^{\text{op}})$  has representative

$$(\mathcal{O}_{\overline{E}^{\text{op}}}, (\mathcal{F}_{\overline{E}^{\text{op}}, \mathbb{Z}})_{A^{\text{op}}}, N_{\overline{E}}) = (\mathcal{O}_E^{\text{op}}, (\mathcal{F}_{E, \mathbb{Z}})_{A^{\text{op}}}, -N)$$

where we identify algebras by  $\mathcal{O}_{\overline{E}^{\text{op}}} \ni S_{\overline{e}} \mapsto S_e^{*\text{op}} \in \mathcal{O}_E^{\text{op}}$ , and we do nothing else except notice that  $(\mathcal{F}_{\overline{E}, \mathbb{Z}})_{A^{\text{op}}} = (\mathcal{F}_{E, \mathbb{Z}})_{A^{\text{op}}}$  and that  $N_{\overline{E}}$  acts on  $(\mathcal{F}_{E, \mathbb{Z}})_{A^{\text{op}}}$  as  $-N$ .

When  $E$  is an invertible bimodule, we can identify  $\mathcal{O}_{E^{\text{op}}} \cong \mathcal{O}_E^{\text{op}}$ . So in this situation we can compare the different extension classes, and so obtain a relationship between  $\delta$  and  $\overline{\delta}$ :

**Corollary 4.11.** *Let  $E$  be an invertible bimodule which is the restriction to  $A$  of an (invertible)  $A_b$ -bimodule  $E_b$  for some unitisation  $A_b$  of  $A$ . Identifying  $\mathcal{O}_{E^{\text{op}}}$  with  $\mathcal{O}_E^{\text{op}}$  via the isomorphism of Proposition 2.12, we have*

$$[\text{ext}^{\text{op}}] = -[\overline{\text{ext}}^{\text{op}}] \in KK^1(\mathcal{O}_E^{\text{op}}, A^{\text{op}})$$

and

$$(\delta - \overline{\delta}) \otimes_{\mathcal{O}_E^{\text{op}}} [\text{ext}^{\text{op}}] = (\delta - \overline{\delta}) \otimes_{\mathcal{O}_E^{\text{op}}} [\overline{\text{ext}}^{\text{op}}] = (\delta - \overline{\delta}) \otimes_{\mathcal{O}_E} [\text{ext}] = 0.$$

*Proof.* Proposition 4.10 gives the first statement. We know that

$$\mathbb{W}^{\text{op}} \otimes_{\mathcal{O}_E^{\text{op}}} [\text{ext}^{\text{op}}] = -\text{Id}_{KK(A^{\text{op}}, A^{\text{op}})} \quad \text{and} \quad \overline{\mathbb{W}}^{\text{op}} \otimes_{\mathcal{O}_E^{\text{op}}} [\overline{\text{ext}}^{\text{op}}] = -\text{Id}_{KK(A^{\text{op}}, A^{\text{op}})}.$$

Using Proposition 4.10 again yields

$$\mathbb{W}^{\text{op}} \otimes_{\mathcal{O}_E^{\text{op}}} [\overline{\text{ext}}^{\text{op}}] = \text{Id}_{KK(A^{\text{op}}, A^{\text{op}})} \quad \text{and} \quad \overline{\mathbb{W}}^{\text{op}} \otimes_{\mathcal{O}_E^{\text{op}}} [\text{ext}^{\text{op}}] = \text{Id}_{KK(A^{\text{op}}, A^{\text{op}})}.$$

Since

$$\delta - \overline{\delta} = -\beta \otimes_{A \otimes A^{\text{op}}} (\mathbb{W}^{\text{op}} + \overline{\mathbb{W}}^{\text{op}})$$

the result follows from associativity of the Kasparov product.  $\square$

## 4.4 Examples

In the following examples, we construct explicit representatives of the  $K$ -theory fundamental class.

### 4.4.1 The circle

The simplest example is when  $A = E = \mathbb{C}$  as algebra and bimodule. Here clearly  $\beta = [1_{\mathbb{C}} \otimes 1_{\mathbb{C}}]$  is  $E$ -invariant. So we recover the fact that  $C(S^1) = \mathcal{O}_E$  satisfies Poincaré duality in the  $K$ -theory sense. Since  $E$  is an invertible bimodule, we can realise  $\mathcal{O}_E$  as shift operators on  $F_{E, \mathbb{Z}} = \ell^2(\mathbb{Z}) \cong L^2(S^1)$ . Since  $E$  is singly generated by  $1_{\mathbb{C}}$ , the partial isometry  $w$  is the unitary given by the bilateral shift (i.e. multiplication by  $z$ ). Following the recipe for constructing  $\delta$  we obtain

$$\delta_{C(S^1)} = [z \otimes 1_{C(S^1)}] - [1_{C(S^1)} \otimes z] = [z] \otimes [\iota_{\mathbb{C}, C(S^1)}] - [\iota_{\mathbb{C}, C(S^1)}] \otimes [z] \in KK^1(\mathbb{C}, C(S^1) \otimes C(S^1)).$$

#### 4.4.2 The rotation algebras

Let  $A = C(S^1)$  and let  $E$  be the bimodule implementing the automorphism of rotation by an angle  $\theta$ , defined as in Subsection 3.2.2. That is, define  $\alpha : C(S^1) \rightarrow C(S^1)$  by  $\alpha(a)(e^{i\phi}) = a(e^{i(\phi+\theta)})$ , and then define

$$E = A, \quad a \cdot e \cdot b := \alpha(a)eb, \quad \text{and} \quad a, b \in A, \quad e \in E.$$

Then  $\mathcal{O}_E$  is isomorphic to the rotation algebra  $A_\theta$ . Since  $\beta := [z \otimes 1_{C(S^1)}] - [1_{C(S^1)} \otimes z] \in KK^1(C(S^1) \otimes C(S^1), \mathbb{C})$  satisfies  $\beta \otimes_{C(S^1)} [E] = \beta \otimes_{C(S^1)} [E^{\text{op}}] = \beta$  (the class of the unitary  $z$  is invariant under rotations), we obtain a  $K$ -theory fundamental class  $\delta$  for  $A_\theta$  as follows.

Since  $E$  is an invertible bimodule,  $\mathcal{O}_E$  can be realised as the algebra generated by the shift operators on  $F_{E, \mathbb{Z}} = \ell^2(\mathbb{Z}) \otimes C(S^1)$ . Specifically, writing  $\{e_n : n \in \mathbb{Z}\}$  for the standard orthonormal basis of  $\ell^2(\mathbb{Z})$ ,

$$a \cdot (e_n \otimes b) = e_n \otimes \alpha^n(a)b \quad \text{for all } a, b \in C(S^1).$$

As  $E$  is generated by  $1_{C(S^1)}$ , the partial isometry  $w$  is the unitary given by the bilateral shift (which we continue to denote by  $w$ ). This  $w$  does not commute with the left action of  $z \in C(S^1)$ : we have  $wz = e^{-i\theta}zw$ .

To determine  $\delta$ , we follow the recipe of Section 5 (remembering the antisymmetry of the external product) and obtain

$$\begin{aligned} \delta_{A_\theta} &= ([z \otimes 1_{C(S^1)^{\text{op}}}] - [1_{C(S^1)} \otimes z^{\text{op}}]) \otimes_{C(S^1) \otimes C(S^1)^{\text{op}}} [\mathbb{W}] \\ &\quad - ([z \otimes 1_{C(S^1)^{\text{op}}}] - [1_{C(S^1)} \otimes z^{\text{op}}]) \otimes_{C(S^1) \otimes C(S^1)^{\text{op}}} [\mathbb{W}^{\text{op}}] \\ &= [z] \otimes_{C(S^1)} [W] \otimes_{\mathbb{C}} [\iota_{\mathbb{C}, A_\theta^{\text{op}}}] + [w] \otimes_{\mathbb{C}} [z^{\text{op}}] - [z] \otimes_{\mathbb{C}} [w^{\text{op}}] - [\iota_{\mathbb{C}, A_\theta}] \otimes_{\mathbb{C}} [z^{\text{op}}] \otimes_{C(S^1)^{\text{op}}} [W^{\text{op}}]. \end{aligned}$$

Up to sign, this expression agrees with the Bott element identified by Connes in [9], and identifies the class of the Powers-Rieffel projector with  $[z] \otimes_{C(S^1)} [W] \in K_0(A_\theta)$ .

#### 4.4.3 Automorphisms

More generally, given an algebra  $A$  with Bott class  $\beta$  that is invariant under an automorphism  $\alpha \in \text{Aut}(A)$  in the sense that  $\beta \otimes_A \alpha A = \beta \otimes_{A^{\text{op}}} (\alpha A)^{\text{op}}$ , we obtain a Bott class  $\delta$  for  $A \rtimes_\alpha \mathbb{Z}$ . This applies in particular to isometric actions of  $\mathbb{Z}$  on compact  $\text{spin}^c$  manifolds, and more generally when we have a Bott class satisfying the conditions of Lemma 3.10(1).

Let  $U \in A \rtimes_\alpha \mathbb{Z}$  be the unitary implementing  $\alpha$ . The projection  $e_w$  of Equation (4.3) is given by

$$e_w(t) = \frac{1}{1+t^2} \begin{pmatrix} t^2 & -itU \\ itU^* & 1 \end{pmatrix}.$$

So we obtain an explicit representative of  $\delta$  from an explicit representative of  $\beta$ .

#### 4.4.4 Cuntz–Krieger algebras and graph algebras

Consider a finite directed graph  $G$ . Suppose that there is at most one (directed) edge between any pair of vertices in  $G$ . This is a constraint on the graphs we consider, but not the algebras: replacing a graph with its dual graph does not change the graph  $C^*$ -algebra, and the dual graph has at most one edge between any two vertices, [34].

Let  $E$  be the graph bimodule of  $G$  as in Section 3.2.1. We proved there that the diagram 3.9 for  $E^{\text{op}}$  commutes. Here we compute  $\delta$  by taking the products of  $\beta = (\mathbb{C}, C_0(G^0), 0)$  with the classes  $\mathbb{W}$  and  $\mathbb{W}^{\text{op}}$  described in Definition 4.2 and Lemma 4.3.

Enumerate  $G^1 = g_1, \dots, g_{|G^1|}$ . For  $i \leq |G^1|$ , write  $e_i \in C(G^1)$  for the point-mass function.

Let  $e_{w\otimes 1}$  be the projection over the mapping cone for  $C(G^0) \otimes C(G^0) \hookrightarrow C^*(G) \otimes C^*(G)$  determined by Equation (4.3) from the partial isometry  $w$  over  $C^*(G)$  in Equation (4.2). Using the Cuntz-Krieger relations it is not hard to show that  $(p_v \otimes p_v^{\text{op}})e_{w\otimes 1}$  is the projection  $e_V$  associated as in Equation (4.3) to the partial isometry

$$V = \sum_{s(e_i)=v} E_{i1} S_{g_i}^* \otimes p_v^{\text{op}},$$

where the  $E_{i1}$  are matrix units. With this observation and  $\beta = \sum_{v \in G^0} p_v \otimes p_v^{\text{op}}$  one checks that

$$\beta \otimes_{A \otimes A^{\text{op}}} \mathbb{W} = \sum_{v \in G^0} \sum_{s(g_i)=v} [E_{i1} S_{e_i}^* \otimes p_v^{\text{op}}] \in KK(\mathbb{C}, M(A \otimes A^{\text{op}}, \mathcal{O}_E \otimes \mathcal{O}_{E^{\text{op}}}).$$

Similarly<sup>2</sup>, writing  $f_j^{\text{op}}$  for the point mass function  $\delta_j \in C(G^1)$  regarded as an element of  $E^{\text{op}}$ , we compute

$$\beta \otimes_{A \otimes A^{\text{op}}} \mathbb{W}^{\text{op}} = \sum_{v \in G^0} \sum_{r(g_i)=v} [p_v \otimes E_{i1} S_{f_i^{\text{op}}}^*] \in KK(\mathbb{C}, M(A \otimes A^{\text{op}}, \mathcal{O}_E \otimes \mathcal{O}_{E^{\text{op}}}).$$

We have

$$\left( \sum_{s(g_j)=w} E_{1j} S_{e_j} \otimes p_w^{\text{op}} \right) \left( \sum_{s(g_i)=v} E_{i1} S_{e_i}^* \otimes p_v^{\text{op}} \right) = p_v \otimes p_v^{\text{op}}$$

and

$$\left( \sum_{r(g_j)=w} p_w \otimes E_{1j} S_{f_j^{\text{op}}} \right) \left( \sum_{r(g_i)=v} p_v \otimes E_{i1} S_{f_i^{\text{op}}}^* \right) = p_v \otimes p_v^{\text{op}}.$$

So we can use [7, Lemma 3.3] to compute that

$$\begin{aligned} \beta \otimes_{A \otimes A^{\text{op}}} \mathbb{W} - \beta \otimes_{A \otimes A^{\text{op}}} \mathbb{W}^{\text{op}} &= \sum_{v, w \in G^0} \left( \sum_{s(g_i)=w} [E_{i1} S_{e_i}^* \otimes p_w^{\text{op}}] - \sum_{r(g_j)=v} [p_v \otimes E_{j1} S_{f_j^{\text{op}}}^*] \right) \\ &= \sum_{v, w \in G^0} \sum_{s(g_i)=w} \sum_{r(g_j)=v} [(E_{i1} S_{e_i}^* \otimes p_w^{\text{op}})(p_v \otimes E_{1j} S_{f_j^{\text{op}}})] \\ &= \sum_{v, w \in G^0} \sum_{s(g_i)=w} \sum_{r(g_j)=v} [E_{ij} S_{e_i}^* \otimes S_{f_j^{\text{op}}}] \delta_{v, r(e_i)} \delta_{w, s(f_j)} \\ &= \sum_{g_j \in G^1} [E_{jj} S_{e_j}^* \otimes S_{f_j^{\text{op}}}] \quad \text{as each } |vG^1w| \leq 1 \\ &= \sum_{g_j \in G^1} [S_{e_j}^* \otimes S_{f_j^{\text{op}}}] \end{aligned}$$

Identifying  $E^{\text{op}}$  with the edge module of the opposite graph, we deduce that the class constructed in Lemma 4.3 recovers the  $K$ -theory Poincaré duality class of Kaminker and Putnam [21], but for any finite graph with at most one edge between any two vertices.

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<sup>2</sup>This identification of left and right frames does not hold in general, and relies heavily on the orthonormality of the frames in this example

In the non-unital case, Kajiwara, Pinzari and Watatani [20, Section 6.1] showed that a left inner product can be defined on  $C_0(G^1)$  making  $C_0(G^1)$  bi-Hilbertian precisely when the in- and out-valences of the graph are uniformly bounded. The requirement that  $C_0(G^1)$  is the restriction of a finitely generated module over  $C_b(G^1)$  is proved in [1]. The construction of the classes  $W$  and so  $\delta$  extend to this generality, but the construction of the  $K$ -homology fundamental class discussed in the next section does not immediately extend.

## 5 Examples of $K$ -homology fundamental classes

We have been unable to identify a general procedure for lifting  $K$ -homology fundamental classes from  $A$  to  $\mathcal{O}_E$ . For the special cases of crossed products by  $\mathbb{Z}$  and graph algebras, we *can* produce the required  $K$ -homology class; but the procedure in each case is ad hoc.

### 5.1 Crossed products by $\mathbb{Z}$

For this subsection, we suppose that  $\alpha : A \rightarrow A$  is an automorphism, and that  $\mu$  is a  $K$ -homology fundamental class for  $A$  such that  $\mu$  and  $\alpha$  satisfy the conditions of Lemma 3.10(2). Thus  $\mu$  is represented by a spectral triple  $(A \otimes A^{\text{op}}, \pi\mathcal{H}, \mathcal{D})$  with  $\overline{A\mathcal{H}} = \overline{A^{\text{op}}\mathcal{H}} = \mathcal{H}$ , both  $\alpha$  and  $\alpha^{\text{op}}$  preserve the subalgebras  $A$  and  $A^{\text{op}}$ , and  $\alpha$  is implemented on  $\mathcal{H}$  by a unitary  $W_\alpha$  such that  $[\mathcal{D}, W_\alpha]$  is bounded. The main constructions of this section do not require  $A$  to be unital.

**Theorem 5.1.** *Take  $A$ ,  $\alpha$ , and  $\mu = [(A \otimes A^{\text{op}}, \pi\mathcal{H}, \mathcal{D})] \in KK^d(A \otimes A^{\text{op}}, \mathbb{C})$  as above. Let  $E := {}_\alpha A_A$ . Write  $S_1 \in \mathcal{O}_E$  for the generator corresponding to  $1_A$  regarded as an element of  $E$ . Let  $\mathcal{O}_E \subseteq \mathcal{O}_E$  be the subalgebra described at (4.7), and similarly for  $\mathcal{O}_{E^{\text{op}}}$ . Let  $U \in \mathcal{B}(\bigoplus_{n \in \mathbb{Z}} \mathcal{H})$  be the shift,  $(U\xi)_n = \xi_{n+1}$ . There is a representation  $\tilde{\pi}$  of  $\mathcal{O}_E \otimes \mathcal{O}_E^{\text{op}}$  on  $\bigoplus_{n \in \mathbb{Z}} \mathcal{H}$  such that for all  $a, b \in A$  and  $\xi \in \bigoplus_{n \in \mathbb{Z}} \mathcal{H}$ , we have*

$$\begin{aligned} (\tilde{\pi}(a \otimes b^{\text{op}})\xi)_n &= \pi(\alpha^n(a) \otimes b^{\text{op}})\xi, & (\tilde{\pi}(S_1 \otimes 1)\xi)_n &= (U\xi)_n = \xi_{n+1}, \text{ and} \\ (\tilde{\pi}(1 \otimes S_{1^{\text{op}}})\xi)_n &= (U^{-1}W_\alpha\xi)_n = (W_\alpha U^{-1}\xi)_n = W_\alpha\xi_{n-1}. \end{aligned} \tag{5.1}$$

Let  $N : \bigoplus_{n \in \mathbb{Z}} \mathcal{H} \rightarrow \bigoplus_{n \in \mathbb{Z}} \mathcal{H}$  be the densely defined number operator  $(N\xi)_n = n\xi_n$ . If  $d$  is even, write  $\mathcal{H}_+$  and  $\mathcal{H}_-$  for the even and odd subspaces of  $\mathcal{H}$  so that  $\mathcal{H} = \begin{pmatrix} \mathcal{H}_+ \\ \mathcal{H}_- \end{pmatrix}$  and  $\mathcal{D} = \begin{pmatrix} 0 & \mathcal{D}_- \\ \mathcal{D}_+ & 0 \end{pmatrix}$ .

Define

$$\Delta_0 := \begin{cases} \left( \mathcal{O}_E \otimes \mathcal{O}_E^{\text{op}}, \bigoplus_{n \in \mathbb{Z}} (\mathcal{H} \otimes \mathbb{C}^2), \begin{pmatrix} 0 & N - i\mathcal{D} \\ N + i\mathcal{D} & 0 \end{pmatrix} \right) & \text{if } d \text{ is odd} \\ \left( \mathcal{O}_E \otimes \mathcal{O}_E^{\text{op}}, \bigoplus_{n \in \mathbb{Z}} \begin{pmatrix} \mathcal{H}_+ \\ \mathcal{H}_- \end{pmatrix}, \begin{pmatrix} N & \mathcal{D}_- \\ \mathcal{D}_+ & -N \end{pmatrix} \right), & \text{if } d \text{ is even.} \end{cases}$$

If both  $[\mathcal{D}, \alpha^n(a)]$  and  $[\mathcal{D}, \alpha^n(a)^{\text{op}}]$  are uniformly norm-bounded in  $n$ , then  $\Delta_0$  is an unbounded Kasparov module. If in addition the operators  $W_\alpha^n[\mathcal{D}, W_\alpha^{-n}](\mathcal{D} \pm i)^{-1}$  are uniformly bounded in  $n$ , the class  $\Delta \in KK^{d+1}(\mathcal{O}_E \otimes \mathcal{O}_E^{\text{op}}, \mathbb{C})$  that  $\Delta_0$  defines is a  $K$ -homology fundamental class.

*Proof.* The universal properties of  $\mathcal{O}_E$  and  $\mathcal{O}_E^{\text{op}}$ , together with that of the tensor product, show that there is a representation of  $\mathcal{O}_E \otimes \mathcal{O}_E^{\text{op}}$  whose restriction to  $\mathcal{O}_E \otimes \mathcal{O}_E^{\text{op}}$  satisfies the desired formulas.



We will first construct the product of the extension class and the  $K$ -homology fundamental class for  $A$ . We only present the argument for  $d$  odd, as the case for  $d$  even is similar. The extension class  $[\text{ext}]$  is represented by  $(\mathcal{O}_E, \bigoplus_{n \in \mathbb{Z}} A, N)$  [36, Theorem 3.1], as described in Proposition 4.10.

The internal product of  $\bigoplus_{n \in \mathbb{Z}} A$  with  $\mathcal{H}$  is just  $\bigoplus_{n \in \mathbb{Z}} \mathcal{H}$  because the action is non-degenerate. Since both  $[\text{ext}]$  and  $\mu$  are odd, we need to double both of them to even classes using the Clifford algebra  $\text{Cliff}_1$ . We omit the details but refer to [3, Appendix] for the mechanics and determination of signs.

Abusing notation slightly, we write  $N$  for  $N \otimes 1$  on  $\bigoplus_{n \in \mathbb{Z}} A \otimes_A \mathcal{H}$  and we write  $\mathcal{D}$  for the operator  $\bigoplus_{n \in \mathbb{Z}} \mathcal{D}$  on the same space. We will show that

$$\left( \mathcal{O}_E \otimes \mathcal{A}^{\text{op}}, \bigoplus_{n \in \mathbb{Z}} (\mathcal{H} \otimes \mathbb{C}^2), \begin{pmatrix} 0 & N - i\mathcal{D} \\ N + i\mathcal{D} & 0 \end{pmatrix} \right) \quad (5.2)$$

is a spectral triple representing the Kasparov product  $[\text{ext}] \otimes_A \mu$ . To see this, first note that the operator

$$N \# \mathcal{D} := \begin{pmatrix} 0 & N - i\mathcal{D} \\ N + i\mathcal{D} & 0 \end{pmatrix}$$

is self-adjoint by [29, Proposition 3.12, Theorem 3.18 and Lemma 4.2], and has locally compact resolvent by [19, Theorem 6.7].

By assumption we have uniform boundedness of  $[\mathcal{D}, \alpha^n(a)]$  and  $[\mathcal{D}, \alpha^n(a)^{\text{op}}]$ . For commutators with the other generators of  $\mathcal{O}_E \otimes \mathcal{O}_E^{\text{op}}$  we just need to recall that  $[\mathcal{D}, W_\alpha]$  is assumed to be bounded, and observe that

$$[\mathcal{D}, U] = 0, \quad [N, U] = U, \quad \text{and} \quad [N, W_\alpha] = 0.$$

Hence  $N \# \mathcal{D}$  has bounded commutators not only with elements of  $\mathcal{O}_E \otimes \mathcal{A}^{\text{op}}$ , but also with all of  $\mathcal{O}_E \otimes \mathcal{O}_E^{\text{op}}$ . (Similar but more general conclusions were reached in [2, Lemma 3.4, Proposition 3.5].)

We deduce that  $[N \# \mathcal{D}, \mathcal{O}_E \otimes \mathcal{A}^{\text{op}}] \subseteq \mathcal{B}(\bigoplus_{n \in \mathbb{Z}} \mathcal{H})$ , and so (5.2) is indeed a spectral triple. Theorem 4.4 of [29] shows that this triple represents  $[\text{ext}] \otimes_A \mu$ . Using the fact that we have bounded commutators with all of  $\mathcal{O}_E \otimes \mathcal{O}_E^{\text{op}}$ , we obtain a spectral triple  $\Delta_0$  for  $\mathcal{O}_E \otimes \mathcal{O}_E^{\text{op}}$ , whose class we denote by  $\Delta$ . Plainly  $\iota_{\mathcal{A}^{\text{op}}, \mathcal{O}_E^{\text{op}}} \otimes_{\mathcal{O}_E^{\text{op}}} \Delta$  coincides with  $[\text{ext}] \otimes_A \mu$ .

For the final statement set  $V = \bigoplus_{n \in \mathbb{Z}} W_\alpha^{-n}$ , and assume that  $V[\mathcal{D}, V^*](\mathcal{D} \pm i)^{-1}$  is bounded. We need to show that  $\iota_{\mathcal{A}, \mathcal{O}_E} \otimes_{\mathcal{O}_E} \Delta = [\text{ext}^{\text{op}}] \otimes_{\mathcal{A}^{\text{op}}} \mu$ . Since  $V$  is unitary,  $\Delta$  is represented by the spectral triple

$$\left( \mathcal{O}_E \otimes \mathcal{O}_{E^{\text{op}}, V\tilde{\pi}(\cdot)V^*}(\bigoplus_n \mathcal{H} \otimes \mathbb{C}^2), \begin{pmatrix} 0 & N - iV\mathcal{D}V^* \\ N + iV\mathcal{D}V^* & 0 \end{pmatrix} \right)$$

obtained from  $\Delta_0$  by unitary equivalence and by identifying  $\mathcal{O}_E^{\text{op}}$  with  $\mathcal{O}_{E^{\text{op}}}$  using Proposition 2.12.

We have  $[V, N] = 0$ , and  $V[\mathcal{D}, V^*](\mathcal{D} \pm i)^{-1}$  is bounded by assumption. A simple calculation shows that the straight-line path  $\mathcal{D}_t := \mathcal{D} + tV[\mathcal{D}, V^*]$  between  $\mathcal{D}$  and  $V\mathcal{D}V^*$  is graph norm differentiable [8, Definition 6], and so  $\mathcal{D}_t(1 + \mathcal{D}_t^2)^{-1/2}$  is an operator homotopy by [8, Theorem 20]. We deduce that

$$\Delta = \left[ \left( \mathcal{O}_E \otimes \mathcal{O}_{E^{\text{op}}, V\tilde{\pi}(\cdot)V^*}(\bigoplus_n \mathcal{H} \otimes \mathbb{C}^2), \begin{pmatrix} 0 & N - i\mathcal{D} \\ N + i\mathcal{D} & 0 \end{pmatrix} \right) \right].$$

For  $a, b \in \mathcal{A}$  and  $\xi \in \bigoplus_{n \in \mathbb{Z}} \mathcal{H}$ , the representation  $V\tilde{\pi}(\cdot)V^*$  satisfies

$$\begin{aligned} (V\tilde{\pi}(a \otimes b^{\text{op}})V^*\xi)_n &= \pi(a \otimes \alpha^{-n}(b)^{\text{op}})\xi, & (\tilde{\pi}(1 \otimes S_{1^{\text{op}}})\xi)_n &= (U^{-1}\xi)_n = \xi_{n-1}, \text{ and} \\ (\tilde{\pi}(S_1 \otimes 1)\xi)_n &= (UW_\alpha\xi)_n = (W_\alpha U\xi)_n = W_\alpha\xi_{n+1}. \end{aligned}$$

Lemma 3.10 gives  $E^{\text{op}} = {}_{\alpha^{\text{op}-1}}A_{A^{\text{op}}}^{\text{op}}$ , and so, just as above, we obtain

$$\iota_{A, \theta_E} \otimes_{\theta_E} \Delta = [\text{ext}^{\text{op}}] \otimes_{A^{\text{op}}} \mu.$$

This is equal to  $-\overline{[\text{ext}^{\text{op}}]} \otimes_{A^{\text{op}}} \mu$  by Proposition 4.10.  $\square$

Given a Riemannian manifold  $M$ , by an *almost-isometry* on  $M$  we mean a diffeomorphism  $\phi : M \rightarrow M$  such that for every  $f \in C_c^\infty(M)$ , we have the differentials of  $f \circ \phi^k$  uniformly bounded, so  $\sup_k \|d(f \circ \phi^k)\| < \infty$ . Given an almost-isometry  $\phi$  of a manifold  $M$ , we can construct a  $K$ -homology fundamental class for the crossed product  $C_0(M) \rtimes_\alpha \mathbb{Z}$  of  $C_0(M)$  by the automorphism  $\alpha$  dual to  $\phi$ .

**Corollary 5.2.** *Let  $(M, g)$  be a complete Riemannian  $\text{spin}^c$  manifold. Suppose that  $\phi : M \rightarrow M$  is a  $\text{spin}^c$ -structure-preserving almost-isometry. Define  $\alpha : C_0^\infty(M) \rightarrow C_0^\infty(M)$  by  $\alpha(f) = f \circ \phi$ . Then there exists a class  $\Delta \in KK^{d+1}(C_0(M) \rtimes_\alpha \mathbb{Z} \otimes (C_0(M) \rtimes_\alpha \mathbb{Z})^{\text{op}}, \mathbb{C})$  satisfying part 1. of Theorem 3.4.*

*If  $M$  is compact, then  $\Delta$  is a  $K$ -homology fundamental class for  $C(M) \rtimes_\alpha \mathbb{Z}$  represented by the spectral triple  $\Delta_0$ . In particular,*

$$\begin{aligned} K_*(C(M) \rtimes_\alpha \mathbb{Z}) &\cong K^{*+\dim(M)+1}((C(M) \rtimes_\alpha \mathbb{Z})^{\text{op}}) \text{ and} \\ K_*((C(M) \rtimes_\alpha \mathbb{Z})^{\text{op}}) &\cong K^{*+\dim(M)+1}(C(M) \rtimes_\alpha \mathbb{Z}). \end{aligned}$$

*If  $\phi$  is an isometry and  $M$  is compact then  $C(M) \rtimes_\alpha \mathbb{Z}$  is Poincaré self-dual.*

*Proof.* The first statement is a direct consequence of Theorem 5.1 and the uniform boundedness of the differentials  $d(\phi^k)$ . If  $\phi$  is an isometry and  $M$  is compact then the discussion of the example of subsection 4.4.3 combined with Proposition 3.12 gives a  $K$ -theory fundamental class, and then Theorem 3.1(3) gives a Poincaré self-duality.  $\square$

**Corollary 5.3.** *Let  $(M_\theta, g)$  be a  $\theta$ -deformation of a complete Riemannian  $\text{spin}^c$  manifold. Suppose that  $\alpha : C_0^\infty(M_\theta) \rightarrow C_0^\infty(M_\theta)$  is an automorphism unitarily implemented on  $L^2(S_\theta)$  and commuting with the Dirac operator. Then there exists a class  $\Delta \in KK^{d+1}(C_0(M_\theta) \rtimes_\alpha \mathbb{Z} \otimes (C_0(M_\theta) \rtimes_\alpha \mathbb{Z})^{\text{op}}, \mathbb{C})$  satisfying part 1. of Theorem 3.4.*

*If  $M$  is compact, then  $\Delta$  is a  $K$ -homology fundamental class for  $C_0(M_\theta) \rtimes_\alpha \mathbb{Z}$ . In particular,*

$$\begin{aligned} K_*(C(M_\theta) \rtimes_\alpha \mathbb{Z}) &\cong K^{*+\dim(M_\theta)+1}((C(M_\theta) \rtimes_\alpha \mathbb{Z})^{\text{op}}) \text{ and} \\ K_*((C(M_\theta) \rtimes_\alpha \mathbb{Z})^{\text{op}}) &\cong K^{*+\dim(M_\theta)+1}(C(M_\theta) \rtimes_\alpha \mathbb{Z}). \end{aligned}$$

*If  $\alpha$  is unitarily implemented and  $M$  is compact then  $C(M_\theta) \rtimes_\alpha \mathbb{Z}$  is Poincaré self-dual.*

*Proof.* The  $KK$ -equivalence  $C(M_\theta) \sim_{KK} C(M)$ , [39], gives us fundamental classes for  $C(M_\theta)$ .  $\square$

*Remark 5.4.* An analogous construction can be given when  $(M, g)$  is oriented but not necessarily  $\text{spin}^c$ . This construction, which uses Kasparov's fundamental class [25], starts from a Poincaré duality between  $C(M)$  and  $\text{Cliff}(M)$  and produces a Poincaré duality between  $C(M) \rtimes_\alpha \mathbb{Z}$  and  $\text{Cliff}(M) \rtimes_\alpha \mathbb{Z}$ . See Appendix A.

*Remark 5.5.* The important point in the above constructions is that we have an explicit representative  $\Delta_0$  of the fundamental class. Likewise representatives of the dual  $K$ -theory class can be obtained from the Bott class and the isometry.

## 5.2 Cuntz–Krieger algebras and graph algebras

For unital Cuntz–Krieger algebras we have fairly complete information. Let  $E$  be the edge module for a finite graph with no sources nor sinks. When the graph algebra has associated shift space a Cantor set, Kaminker and Putnam provided an extension representing a  $K$ -homology fundamental class  $\Delta$  relating  $\mathcal{O}_E$  and  $\mathcal{O}_{E^{\text{op}}}$ , [21]. Goffeng and Mesland provided a Kasparov module representing this extension in [16].

We show how Goffeng and Mesland’s construction fits our framework. Given further assumptions we produce a  $K$ -homology fundamental class relating  $\mathcal{O}_E$  and  $\mathcal{O}_E^{\text{op}}$ , and so deduce a  $KK$ -equivalence between  $\mathcal{O}_{E^{\text{op}}}$  and  $\mathcal{O}_E^{\text{op}}$ .

### 5.2.1 The $K$ -homology fundamental class for $C^*(G^{\text{op}})$

Fix a finite directed graph  $G = (G^0, G^1, r, s)$  with no sources and no sinks, let  $A = C(G^0)$ , and write  $E$  for the Cuntz–Krieger module  $C(G^1)$  of  $G$ . We regard  $E$  as a bi-Hilbertian bimodule with

$$(e \mid f)_A(v) = \sum_{r(g)=v} \overline{e(g)}f(g) \quad \text{and} \quad {}_A(e \mid f)(v) = \sum_{s(g)=v} e(g)\overline{f(g)}.$$

We write  $\phi : A \rightarrow \mathbb{C}$  for the functional  $\phi(a) = \sum_{v \in G^0} a_v$ . Observe that

$$\mu = [(A \otimes A^{\text{op}}, L^2(A, \phi), 0)] \tag{5.3}$$

is a  $K$ -homology fundamental class for  $A$  by Remark 3.6. The functional  $\phi$  is invariant for  $E$  in the sense that  $\phi((e \mid f)_A) = \phi({}_A(e \mid f))$  for all  $e, f \in E$ , cf [36, Section 4].

Since  $\phi$  is faithful, we can form the Hilbert space  $L^2(\mathcal{F}_E, \phi)$  with inner product  $\langle \xi, \eta \rangle = \phi((\xi \mid \eta)_A)$ . On this Hilbert space we can define operators  $L_e, R_e$  for  $e \in E$  via the formulae

$$L_e \xi = e \otimes_A \xi, \quad R_e \xi = \xi \otimes_A e.$$

We have  $[L_e, R_f] = 0$  for all  $e, f \in E$ . For  $b \in L^2(A, \phi)$  we have  $[L_e, R_f^*]b = -{}_A(eb \mid f)P_0$ , where  $P_0$  is the projection onto the degree zero part  $A \subseteq \mathcal{F}_E$  of the Fock space. Since  $A$  has finite linear dimension, we obtain commuting representations of  $\mathcal{O}_E$  and  $\mathcal{O}_{E^{\text{op}}}$  modulo compacts, and hence an extension

$$0 \rightarrow \mathcal{K}(L^2(\mathcal{F}_E, \phi)) \rightarrow C^*(L, R) \rightarrow \mathcal{O}_E \otimes \mathcal{O}_{E^{\text{op}}} \rightarrow 0. \tag{5.4}$$

This extension is almost the extension used by Kaminker and Putnam to define the  $K$ -homology fundamental class: the difference is that Kaminker and Putnam’s extension replaces  $L^2(A, \phi)$  in degree zero by  $\mathbb{C}$ . (For us,  $L^2(A, \phi) = \ell^2(G^0) = C(G^0)$ .)

A Fredholm module representing Kaminker and Putnam’s extension appears in [16]. Our class is based directly on their idea. In the situation of Cuntz–Krieger algebras, Goffeng and Mesland’s Fredholm module is constructed from the Hilbert space  $L^2(\mathcal{O}_E) \otimes L^2(\mathcal{O}_{E^{\text{op}}})$ —where the GNS  $L^2$ -spaces are defined with respect to the unique KMS-states [15] for the gauge actions—and the natural left action. There is an isometric embedding of  $L^2(\mathcal{F}_E)$  into  $L^2(\mathcal{O}_E) \otimes L^2(\mathcal{O}_{E^{\text{op}}})$ . Writing  $P$  for the projection onto the image of this embedding, the Fredholm operator appearing in Goffeng and Mesland’s Fredholm module is  $2P - 1$ .

To adapt Goffeng and Mesland’s approach to our setting, we require two pieces of information from [36]. One is the construction of representatives of the classes [ext] and [ext<sup>op</sup>]. The other is an

expectation  $\mathcal{O}_E \rightarrow A$  which, when composed with the  $E$ -invariant functional  $\phi : A \rightarrow \mathbb{C}$ , yields a KMS functional on  $\mathcal{O}_E$ .

In order to obtain these ingredients, we make an assumption on the asymptotic behaviour of the Watatani indices of the tensor powers of  $E$ . When  $E$  is the module associated to an irreducible matrix that is not a permutation matrix, our assumption is automatically satisfied, [36, Example 3.8].

Recall that if  $(a_n)$  and  $(b_n)$  are sequences of positive real numbers, we say that  $(a_n) \in O(b_n)$  if there exist  $N \in \mathbb{N}$  and  $C \geq 0$  such that  $a_n \leq Cb_n$  for all  $n \geq N$ .

**Assumption 1.** *For each  $k \geq 0$ , let  $e^{\beta k}$  denote the right Watatani index of  $E^{\otimes k}$ . We assume that for every  $k \in \mathbb{N}$ , there is a  $\delta > 0$  such that for each  $\nu \in E^{\otimes k}$  there exists  $\tilde{\nu} \in E^{\otimes k}$  satisfying*

$$\left( \|e^{-\beta n} \nu e^{\beta n - k} - \tilde{\nu}\| \right)_{n=1}^{\infty} \in O(n^{-\delta}).$$

**Definition 5.6.** Given Assumption 1, [36, Proposition 3.5] describes an expectation  $\Phi_{\infty} : \mathcal{O}_E \rightarrow A$ . We define  $\Xi_E$  to be the completion of  $\mathcal{O}_E$  in the norm determined by the inner product  $(a | b)_A := \Phi_{\infty}(a^*b)$ . When Assumption 1 holds for  $E^{\text{op}}$ , we obtain the analogous module  $\Xi_{E^{\text{op}}}$ .

The delta functions  $\delta_{\alpha}$  on paths  $\alpha \in G^k$  of length  $k$  constitute a frame for  $E^{\otimes k}$ . For  $\alpha \in G^k$ , as a notational convenience, we will write  $S_{\alpha}$  for the element  $S_{\delta_{\alpha}} \in \mathcal{O}_E$ . We have  $\mathcal{O}_E = \overline{\text{span}}\{S_{\alpha}S_{\beta}^* : s(\alpha) = s(\beta)\}$ . We denote the image of  $S_{\alpha}S_{\beta}^*$  in  $\Xi_E$  by  $W_{\alpha,\beta}$ . As a notational convenience, we interchangeably write  $W_{\alpha,1}$  or  $W_{\alpha,s(\alpha)}$  for the image of  $S_{\alpha}$  and  $W_{1,\alpha}$  or  $W_{s(\alpha),\alpha}$  for the image of  $S_{\alpha}^*$ . We also denote the set of finite paths in the graph  $G$  by  $G^* = \bigcup_k G^k$ .

The opposite module  $E^{\text{op}}$  coincides with the Cuntz–Krieger module of the opposite graph  $G^{\text{op}}$ , which has vertex set  $(G^{\text{op}})^0 = G^0$ , edges  $(G^{\text{op}})^1 = \{g^{\text{op}} : g \in G^1\}$  and range and source maps  $r(g^{\text{op}}) = s(g)$  and  $s(g^{\text{op}}) = r(g)$ . If  $\alpha = \alpha_1 \cdots \alpha_k \in G^k$  then  $\alpha^{\text{op}} = \alpha_k^{\text{op}} \cdots \alpha_1^{\text{op}} \in (G^{\text{op}})^k$ , so we obtain elements  $\{W_{\alpha^{\text{op}},\beta^{\text{op}}} : \alpha, \beta \in E^*, r(\alpha) = r(\beta)\} \in \Xi_{E^{\text{op}}}$ .

By [36, Theorem 3.14], the subspace  $\overline{\text{span}}\{W_{\mu,1} : \mu \in G^*\}$  is isometrically isomorphic to  $\mathcal{F}_E$  and is complemented in  $\Xi_E$ . We write  $P_{\mathcal{F}_E}$  for the projection on this subspace. We have

$$[\text{ext}] = [(\mathcal{O}_E, \Xi_E, 2P_{\mathcal{F}_E} - 1)] \in KK^1(\mathcal{O}_E, A). \quad (5.5)$$

This is the representative of  $[\text{ext}]$  that we require.

By [36, Proposition 4.7],  $\phi \circ \Phi_{\infty} : \mathcal{O}_E \rightarrow \mathbb{C}$  is a  $\text{KMS}_1$  functional for the dynamics  $S_e \mapsto e^{i\beta t} S_e$ ,  $e \in E$ ,  $t \in \mathbb{R}$ .

*Notation 5.7.* The following notation will prove very helpful throughout our calculations below. Let  $G$  be a directed graph. Given  $\lambda = \lambda_1 \cdots \lambda_m \in G^m$ , and  $0 \leq j \leq m$ , we define

$$\underline{\lambda}_j := \begin{cases} \lambda_1 \cdots \lambda_j & \text{if } j \geq 1 \\ r(\lambda) & \text{if } j = 0 \end{cases} \quad \text{and} \quad \overline{\lambda}^{\text{op}(m-j)} := \begin{cases} \lambda_m^{\text{op}} \cdots \lambda_{j+1}^{\text{op}} & \text{if } j < m \\ s(\lambda)^{\text{op}} & \text{if } j = m. \end{cases}$$

Equivalently, if  $\lambda = \alpha\beta$  is the unique factorisation with  $|\alpha| = j$  and  $|\beta| = |\lambda| - j$ , then  $\underline{\lambda}_j = \alpha$  and  $\overline{\lambda}^{\text{op}(m-j)} = \beta^{\text{op}}$ .

**Lemma 5.8** (see also [16, Section 2.3]). *Let  $G$  be a finite directed graph with no sources and no sinks, let  $A = C(G^0)$  and let  $E$  be the associated edge module, regarded as a finitely generated bi-Hilbertian  $A$ -bimodule. Assume that  $E$  satisfies Assumption 1. There is an isometry*

$$\mathcal{V} : L^2(\mathcal{F}_E, \phi) \rightarrow \Xi_E \otimes \Xi_{E^{\text{op}}} \otimes_{A \otimes A^{\text{op}}} L^2(G^0, \phi)$$

such that, for  $\lambda = \lambda_1 \cdots \lambda_m \in G^m$ , we have

$$\mathcal{V}\delta_\lambda = \sum_{j=0}^m \frac{1}{\sqrt{m+1}} W_{\underline{\lambda}_j,1} \otimes W_{\overline{\lambda}^{\text{op}(m-j)},1} \otimes_{A \otimes A^{\text{op}}} \delta_{s(\lambda_j)}.$$

For  $\alpha, \beta, \mu, \nu \in G^*$  with  $s(\alpha) = s(\beta)$  and  $r(\mu) = r(\nu)$  and  $x \in S$ , we have

$$\begin{aligned} & \mathcal{V}^*(W_{\alpha,\beta} \otimes W_{\mu^{\text{op}},\nu^{\text{op}}} \otimes x) \\ &= \begin{cases} \delta_{r(\beta),s(\nu)} \frac{x_{r(\beta)}}{\sqrt{|\alpha|-|\beta|+|\mu|-|\nu|+1}} \delta_{\underline{\alpha}} \cdot \Phi_\infty(S_\beta S_\beta^*) \Phi_\infty^{\text{op}}(S_{\nu^{\text{op}}} S_{\nu^{\text{op}}}^*) \cdot \delta_{\overline{\mu}} & \text{if } \alpha = \underline{\alpha}\beta \text{ and } \mu = \nu\overline{\mu} \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

where  $\Phi_\infty$  is as in Definition 5.6.

*Proof.* Fix  $\lambda = \lambda_1 \cdots \lambda_m$  and  $\mu = \mu_1 \cdots \mu_n \in G^*$  and calculate

$$\begin{aligned} & \sum_{j,k} \frac{1}{\sqrt{m+1}} \frac{1}{\sqrt{n+1}} \langle W_{\underline{\lambda}_j,1} \otimes W_{\overline{\lambda}^{\text{op}(m-j)},1} \otimes \delta_{s(\lambda_j)}, W_{\underline{\mu}_k,1} \otimes W_{\overline{\mu}^{\text{op}(n-k)},1} \otimes \delta_{s(\mu_k)} \rangle \\ &= \delta_{m,n} \sum_{j,k} \frac{1}{m+1} \langle \delta_{s(\lambda_j)}, ((\underline{\lambda}_j | \underline{\mu}_k)_A \otimes (\overline{\lambda}^{\text{op}(m-j)} | \overline{\mu}^{\text{op}(n-k)})_{A^{\text{op}}}) \cdot \delta_{s(\mu_k)} \rangle \\ &= \delta_{m,n} \sum_j \frac{1}{m+1} \phi((\delta_{s(\lambda_j)} | (\underline{\lambda}_j | \underline{\mu}_j)_A \delta_{s(\mu_j)_A} (\overline{\mu}^{(m-j)} | \overline{\lambda}^{(m-j)}))_A) \\ &= \begin{cases} \sum_{j=0}^m \frac{1}{m+1} \phi(\delta_{s(\lambda_j)} | \delta_{s(\lambda_j)}) & \text{if } \lambda = \mu \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} 1 & \text{if } \lambda = \mu \\ 0 & \text{otherwise} \end{cases} \\ &= \langle \delta_\lambda, \delta_\mu \rangle_{L^2(\mathcal{F}_E)}. \end{aligned}$$

So there is an isometry  $\mathcal{V}$  satisfying the desired formula. Now fix  $\lambda, \alpha, \beta, \mu, \nu \in G^*$  with  $s(\alpha) = s(\beta)$  and  $s(\mu) = s(\nu)$  and  $x \in S$ . Put  $l = |\lambda|$ , and calculate

$$\begin{aligned} & \langle \mathcal{V}\delta_\lambda, W_{\alpha,\beta} \otimes W_{\mu^{\text{op}},\nu^{\text{op}}} \otimes_{A \otimes A^{\text{op}}} x \rangle \\ &= \sum_{j=0}^l \frac{1}{\sqrt{l+1}} \langle W_{\underline{\lambda}_j,1} \otimes W_{\overline{\lambda}^{\text{op}(l-j)},1} \otimes \delta_{s(\lambda_j)}, W_{\alpha,\beta} \otimes W_{\mu^{\text{op}},\nu^{\text{op}}} \otimes_{A \otimes A^{\text{op}}} x \rangle \\ &= \sum_{j=0}^l \frac{1}{\sqrt{l+1}} \phi\left(\left(\delta_{s(\lambda_j)} \left| \left( (W_{\underline{\lambda}_j,1} | W_{\alpha,\beta})_A \otimes (W_{\overline{\lambda}^{\text{op}(l-j)},1} | W_{\mu^{\text{op}},\nu^{\text{op}}})_{A^{\text{op}}} \right) \cdot x \right)\right) \\ &= \sum_{j=0}^l \frac{1}{\sqrt{l+1}} \phi\left(\left(\delta_{s(\lambda_j)} \left| \Phi_\infty(S_{\underline{\lambda}_j}^* S_\alpha S_\beta^*) \cdot x \cdot \Phi_\infty^{\text{op}}(S_{\overline{\lambda}^{\text{op}(l-j)}}^* S_{\mu^{\text{op}}} S_{\nu^{\text{op}}}^*) \right)\right)\right) \\ &= \sum_{j=0}^l \frac{1}{\sqrt{l+1}} \Phi_\infty(S_{\underline{\lambda}_j}^* S_\alpha S_\beta^*)(s(\lambda_j)) \phi\left(\left(\delta_{s(\lambda_j)} \left| x \right)\right) \Phi_\infty^{\text{op}}(S_{\overline{\lambda}^{\text{op}(l-j)}}^* S_{\mu^{\text{op}}} S_{\nu^{\text{op}}}^*)(s(\lambda_j)). \end{aligned}$$

This is nonzero only if  $\alpha = \underline{\alpha}\beta$  and  $\mu = \nu\bar{\mu}$  and  $\lambda = \underline{\alpha}\bar{\mu}$ , in which case  $r(\beta) = s(\underline{\alpha}) = r(\bar{\mu}) = s(\nu)$ , and the final line of the preceding calculation collapses to

$$\frac{1}{\sqrt{l+1}}\Phi_\infty(S_\beta S_\beta^*)(r(\beta))x_{r(\beta)}\Phi_\infty^{\text{op}}(S_{\nu^{\text{op}}} S_{\nu^{\text{op}}}^*)(r(\beta)).$$

This is precisely the inner-product of  $\delta_\lambda$  with the right-hand side of the expression given for  $\mathcal{V}^*(W_{\alpha,\beta} \otimes W_{\mu^{\text{op}},\nu^{\text{op}}} \otimes x)$ .  $\square$

**Proposition 5.9.** *Let  $G$  be a finite directed graph with no sources and no sinks, let  $A = C(G^0)$  and let  $E$  be the associated edge module which we assume satisfies Assumption 1, regarded as a finitely generated bi-Hilbertian  $A$ -bimodule. Let  $W$  be as in Lemma 5.8, and let  $P := \mathcal{V}\mathcal{V}^*$ . Then*

$$(\mathcal{O}_E \otimes \mathcal{O}_{E^{\text{op}}}, \Xi_E \otimes \Xi_{E^{\text{op}}}, 2P - 1)$$

defines a Kasparov module, whose class  $\Delta$  represents the extension (5.4).

*Proof.* It suffices to show that  $[P, a]$  is compact for every  $a \in \mathcal{O}_E \otimes \mathcal{O}_{E^{\text{op}}}$ : since  $P = P^*$  and  $(2P - 1)^2 = 1$ , the result will then follow from standard Busby-invariant arguments.

We compute for generators  $S_e \otimes 1$  of  $\mathcal{O}_E$ , where  $e \in G^1$ . Together with the Leibniz rule, an identical calculation for  $1 \otimes S_{f^{\text{op}}}$ , and routine approximation, this will suffice to show that  $P$  has compact commutators with  $\mathcal{O}_E \otimes \mathcal{O}_{E^{\text{op}}}$ .

We have

$$\mathcal{V}\mathcal{V}^*(S_e \otimes 1) - (S_e \otimes 1)\mathcal{V}\mathcal{V}^* = (\mathcal{V}\mathcal{V}^*(S_e \otimes 1)\mathcal{V}\mathcal{V}^* - (S_e \otimes 1)\mathcal{V}\mathcal{V}^*) + \mathcal{V}\mathcal{V}^*(S_e \otimes 1)(1 - \mathcal{V}\mathcal{V}^*),$$

so it suffices to show that each of  $(\mathcal{V}\mathcal{V}^*(S_e \otimes 1)\mathcal{V}\mathcal{V}^* - (S_e \otimes 1)\mathcal{V}\mathcal{V}^*)$  and  $\mathcal{V}\mathcal{V}^*(S_e \otimes 1)(1 - \mathcal{V}\mathcal{V}^*)$  is compact. For the former, we fix  $\lambda \in G^*$ , say  $|\lambda| = l$ , so that  $\mathcal{V}\delta_\lambda$  is a typical spanning element of  $\mathcal{V}\mathcal{V}^*\Xi$ . If  $s(e) \neq r(\lambda)$ , then  $(S_e \otimes 1)\mathcal{V}\delta_\lambda = 0$ , and so  $(\mathcal{V}\mathcal{V}^*(S_e \otimes 1)\mathcal{V}\mathcal{V}^* - (S_e \otimes 1)\mathcal{V}\mathcal{V}^*)\mathcal{V}\delta_\lambda = 0$ . If  $s(e) = r(\lambda)$ , then

$$\begin{aligned} (\mathcal{V}\mathcal{V}^*(S_e \otimes 1)\mathcal{V}\mathcal{V}^* - (S_e \otimes 1)\mathcal{V}\mathcal{V}^*)\mathcal{V}\delta_\lambda &= ((\mathcal{V}\mathcal{V}^* - 1)(S_e \otimes 1)) \sum_{j=0}^l \frac{1}{\sqrt{l+1}} W_{\underline{\lambda}_j,1} \otimes W_{\bar{\lambda}^{\text{op}(l-j)},1} \otimes \delta_{s(\underline{\lambda}_j)} \\ &= (\mathcal{V}\mathcal{V}^* - 1) \sum_{j=0}^l \frac{1}{\sqrt{l+1}} W_{e\underline{\lambda}_j,1} \otimes W_{\bar{\lambda}^{\text{op}(l-j)},1} \otimes \delta_{s(\underline{\lambda}_j)} \\ &= (\mathcal{V}\mathcal{V}^* - 1) \frac{1}{\sqrt{l+1}} (\sqrt{l+2}\mathcal{V}\delta_{e\lambda} - W_{r(e),1} \otimes W_{(e\lambda)^{\text{op}},1} \otimes \delta_{r(e)}) \\ &= \frac{-(\mathcal{V}\mathcal{V}^* - 1)}{\sqrt{l+1}} (W_{r(e),1} \otimes W_{(e\lambda)^{\text{op}},1} \otimes \delta_{r(e)}) = \frac{-1}{\sqrt{l+1}} (\mathcal{V} \frac{1}{l+2} \delta_{e\lambda} - W_{r(e),1} \otimes W_{(e\lambda)^{\text{op}},1} \otimes \delta_{r(e)}) \\ &= \frac{-1}{\sqrt{l+1}} \left( \left( \frac{1}{l+2} \sum_{j=0}^{l+1} W_{e\underline{\lambda}_j,1} \otimes W_{\bar{\lambda}^{\text{op}(l+1-j)},1} \otimes \delta_{s(\underline{\lambda}_j)} \right) - W_{r(e),1} \otimes W_{(e\lambda)^{\text{op}},1} \otimes \delta_{r(e)} \right) \\ &= \frac{1}{\sqrt{l+1}} W_{r(e),1} \otimes W_{(e\lambda)^{\text{op}},1} \otimes \delta_{r(e)} - \frac{1}{\sqrt{l+1}(l+2)} \sum_{j=0}^{l+1} W_{e\underline{\lambda}_j,1} \otimes W_{\bar{\lambda}^{\text{op}(l+1-j)},1} \otimes \delta_{s(\underline{\lambda}_j)} \\ &= \frac{\sqrt{l+1}}{l+2} W_{r(e),1} \otimes W_{(e\lambda)^{\text{op}},1} \otimes \delta_{r(e)} - \frac{1}{\sqrt{l+1}(l+2)} \sum_{j=1}^{l+1} W_{e\underline{\lambda}_j,1} \otimes W_{\bar{\lambda}^{\text{op}(l+1-j)},1} \otimes \delta_{s(\underline{\lambda}_j)}. \end{aligned}$$

The terms in this sum are mutually orthogonal, and  $\|W_{e\lambda_j,1} \otimes W_{e\lambda^{\text{op}(l+1-j)},1} \otimes \delta_{s(e\lambda_j)}\| = 1$  for each  $j$ . So we have

$$\|(\mathcal{V}\mathcal{V}^*(S_e \otimes 1) - (S_e \otimes 1)\mathcal{V}\mathcal{V}^*)\mathcal{V}\delta_\lambda\| \sim \sqrt{\frac{l+1}{(l+2)^2} + \frac{l^2}{(l+1)(l+2)^2}} \sim \sqrt{\frac{2}{l}}.$$

Moreover, if  $|\lambda'| = |\lambda|$ , then the terms in the above sum for  $\lambda$  are all orthogonal to all of the terms in the corresponding sum for  $\lambda'$ . Thus  $\|(\mathcal{V}\mathcal{V}^*(S_e \otimes 1) - (S_e \otimes 1)\mathcal{V}\mathcal{V}^*)|_{\text{span}\{\mathcal{V}\delta_\lambda : |\lambda|=l\}}\| \sim \sqrt{\frac{2}{l}}$ . The subspaces  $Y_l := \text{span}\{\mathcal{V}\delta_\lambda : |\lambda|=l\}$  are all finite-dimensional. Writing  $P_l$  for the projection onto  $Y_l$ , we see that

$$\|(\mathcal{V}\mathcal{V}^*(S_e \otimes 1)\mathcal{V}\mathcal{V}^* - (S_e \otimes 1)\mathcal{V}\mathcal{V}^*)(1 - P_l)\| \sim \sqrt{\frac{2}{l}} \rightarrow 0.$$

So  $(\mathcal{V}\mathcal{V}^*(S_e \otimes 1)\mathcal{V}\mathcal{V}^* - (S_e \otimes 1)\mathcal{V}\mathcal{V}^*) = \lim_l (\mathcal{V}\mathcal{V}^*(S_e \otimes 1)\mathcal{V}\mathcal{V}^* - (S_e \otimes 1)\mathcal{V}\mathcal{V}^*)P_l$  is compact.

Now we must compute  $\mathcal{V}\mathcal{V}^*(S_e \otimes 1)(1 - \mathcal{V}\mathcal{V}^*)$ .

Fix paths  $\alpha, \beta, \mu, \nu \in G^*$  with  $s(\alpha) = s(\beta)$  and  $r(\mu) = r(\nu)$  and  $x \in C(G^0)$ . We compute  $\mathcal{V}\mathcal{V}^*(S_e \otimes 1)(1 - \mathcal{V}\mathcal{V}^*)(W_{\alpha,\beta} \otimes W_{\mu^{\text{op}},\nu^{\text{op}}} \otimes x)$ . If  $s(e) \neq r(\alpha)$  then both  $\mathcal{V}\mathcal{V}^*(S_e \otimes 1)(W_{\alpha,\beta} \otimes W_{\mu^{\text{op}},\nu^{\text{op}}} \otimes x)$  and  $(S_e \otimes 1)\mathcal{V}\mathcal{V}^*(W_{\alpha,\beta} \otimes W_{\mu^{\text{op}},\nu^{\text{op}}} \otimes x)$  are zero, so we can assume that  $r(\alpha) = s(e)$ .

Similarly, both terms are zero unless  $(e\alpha) = \zeta\beta$  for some  $\zeta$  and  $\mu = \nu\tau$ , and  $s(\zeta) = r(\tau)$ . So we suppose that this is also the case. Two cases remain: either  $\alpha = \eta\beta$  for some  $\eta$ , or  $e\alpha = \beta$ .

First suppose that  $\alpha = \eta\beta$ . Let  $l := |\alpha| - |\beta| + |\mu| - |\nu|$ , and let  $\lambda := \eta\tau$ . Then

$$\begin{aligned} & (\mathcal{V}\mathcal{V}^*(S_e \otimes 1))(W_{\alpha,\beta} \otimes W_{\mu^{\text{op}},\nu^{\text{op}}} \otimes x) \\ &= \mathcal{V}\mathcal{V}^*(W_{e\alpha,\beta} \otimes W_{\mu^{\text{op}},\nu^{\text{op}}} \otimes x) \\ &= \frac{x_{r(\beta)}(\Phi_\infty(S_\beta S_\beta^*)\Phi_\infty^{\text{op}}(S_{\nu^{\text{op}}} S_{\nu^{\text{op}}}^*))(r(\beta))}{\sqrt{|e\eta| + |\tau| + 1}} \mathcal{V}\delta_{e\eta\tau} \\ &= \frac{x_{r(\beta)}(\Phi_\infty(S_\beta S_\beta^*)\Phi_\infty^{\text{op}}(S_{\nu^{\text{op}}} S_{\nu^{\text{op}}}^*))(r(\beta))}{l+2} \sum_{j=0}^{l+1} W_{e\lambda_j,1} \otimes W_{e\lambda^{\text{op}(l-j)},1} \otimes \delta_{s(e\lambda_j)}. \end{aligned} \quad (5.6)$$

On the other hand,

$$\begin{aligned} & (\mathcal{V}\mathcal{V}^*(S_e \otimes 1)\mathcal{V}\mathcal{V}^*)(W_{\alpha,\beta} \otimes W_{\mu^{\text{op}},\nu^{\text{op}}} \otimes x) \\ &= \mathcal{V}\mathcal{V}^*(S_e \otimes 1) \frac{x_{r(\beta)}(\Phi_\infty(S_\beta S_\beta^*)\Phi_\infty^{\text{op}}(S_{\nu^{\text{op}}} S_{\nu^{\text{op}}}^*))(r(\beta))}{\sqrt{|\eta| + |\tau| + 1}} \mathcal{V}\delta_{\eta\tau} \\ &= \mathcal{V}\mathcal{V}^*(S_e \otimes 1) \frac{x_{r(\beta)}(\Phi_\infty(S_\beta S_\beta^*)\Phi_\infty^{\text{op}}(S_{\nu^{\text{op}}} S_{\nu^{\text{op}}}^*))(r(\beta))}{\sqrt{l+1}} \frac{1}{\sqrt{l+1}} \sum_{j=0}^l W_{\lambda_j,1} \otimes W_{\lambda^{\text{op}(l-j)},1} \otimes \delta_{s(\lambda_j)} \\ &= \frac{x_{r(\beta)}(\Phi_\infty(S_\beta S_\beta^*)\Phi_\infty^{\text{op}}(S_{\nu^{\text{op}}} S_{\nu^{\text{op}}}^*))(r(\beta))}{l+1} \sum_{j=0}^l \mathcal{V}\mathcal{V}^*(S_e \otimes 1)(W_{\lambda_j,1} \otimes W_{\lambda^{\text{op}(l-j)},1} \otimes \delta_{s(\lambda_j)}). \end{aligned}$$

Our calculation (5.6) of  $(\mathcal{VV}^*(S_e \otimes 1))(W_{\alpha,\beta} \otimes W_{\mu^{\text{op}},\nu^{\text{op}}} \otimes x)$  applies to each term in this sum, and we obtain

$$\begin{aligned}
& (\mathcal{VV}^*(S_e \otimes 1)\mathcal{VV}^*)(W_{\alpha,\beta} \otimes W_{\mu^{\text{op}},\nu^{\text{op}}} \otimes x) \\
&= \frac{x_{r(\beta)}(\Phi_\infty(S_\beta S_\beta^*)\Phi_\infty^{\text{op}}(S_{\nu^{\text{op}}} S_{\nu^{\text{op}}}^*))(r(\beta))}{l+1} \sum_{j=0}^l \frac{1}{l+2} \sum_{k=0}^{l+1} (W_{\underline{e}\lambda_k,1} \otimes W_{\underline{e}\lambda^{\text{op}}(l-k),1} \otimes \delta_{s(\underline{e}\lambda_k)}) \\
&= \frac{x_{r(\beta)}(\Phi_\infty(S_\beta S_\beta^*)\Phi_\infty^{\text{op}}(S_{\nu^{\text{op}}} S_{\nu^{\text{op}}}^*))(r(\beta))}{l+2} \sum_{k=0}^{l+1} (W_{\underline{e}\lambda_k,1} \otimes W_{\underline{e}\lambda^{\text{op}}(l-k),1} \otimes \delta_{s(\underline{e}\lambda_k)}) \\
&= (\mathcal{VV}^*(S_e \otimes 1))(W_{\alpha,\beta} \otimes W_{\mu^{\text{op}},\nu^{\text{op}}} \otimes x).
\end{aligned}$$

Hence

$$\mathcal{VV}^*(S_e \otimes 1)(1 - \mathcal{VV}^*)(W_{\alpha,\beta} \otimes W_{\mu^{\text{op}},\nu^{\text{op}}} \otimes x) = 0.$$

Now suppose that  $e\alpha = \beta$  and  $\mu = \nu\tau$ . Then  $(\mathcal{VV}^*(S_e \otimes 1)\mathcal{VV}^*)(W_{\alpha,\beta} \otimes W_{\mu^{\text{op}},\nu^{\text{op}}} \otimes x) = 0$ , whereas

$$\begin{aligned}
& (\mathcal{VV}^*(S_e \otimes 1))(W_{\alpha,\beta} \otimes W_{\mu^{\text{op}},\nu^{\text{op}}} \otimes x) \\
&= \mathcal{VV}^*(W_{e\alpha,\beta} \otimes W_{\mu^{\text{op}},\nu^{\text{op}}} \otimes x) \\
&= \frac{x_{r(e)}(\Phi_\infty(S_{e\alpha} S_{e\alpha}^*)\Phi_\infty^{\text{op}}(S_{\nu^{\text{op}}} S_{\nu^{\text{op}}}^*))(r(e))}{\sqrt{|\tau|+1}} \mathcal{V}\delta_\tau \\
&= \frac{x_{r(e)}(\Phi_\infty(S_{e\alpha} S_{e\alpha}^*)\Phi_\infty^{\text{op}}(S_{\nu^{\text{op}}} S_{\nu^{\text{op}}}^*))(r(e))}{|\tau|+1} \sum_{j=0}^{|\tau|} W_{\underline{e}\tau_j,1} \otimes W_{\overline{e}\tau^{\text{op}}(|\tau|-j),1} \otimes \delta_{s(\underline{e}\tau_j)} \\
&= \frac{x_{r(e)}(\Phi_\infty(S_\beta S_\beta^*)\Phi_\infty^{\text{op}}(S_{\nu^{\text{op}}} S_{\nu^{\text{op}}}^*))(r(e))}{\sqrt{|\tau|+1}} \mathcal{V}\delta_\tau.
\end{aligned}$$

For  $\lambda \in G^*$ , let  $P_\lambda$  be the projection onto the 1-dimensional subspace  $\mathbb{C}\mathcal{V}\delta_\lambda$  of  $\Xi \otimes \Xi^{\text{op}} \otimes \ell^2(S, \phi)$ . We will prove that  $\mathcal{VV}^*(S_e \otimes 1)(1 - \mathcal{VV}^*) = \lim_{l \rightarrow \infty} \sum_{\lambda \in G^l} P_\lambda \mathcal{VV}^*(S_e \otimes 1)(1 - \mathcal{VV}^*)$ . Since each  $G^l$  is finite, this will complete the proof that the commutator is a compact operator.

To prove the result, first observe that we always have  $W_{\alpha,\beta} \otimes W_{\mu^{\text{op}},\nu^{\text{op}}} \otimes x = x(r(\beta))W_{\alpha,\beta} \otimes W_{\mu^{\text{op}},\nu^{\text{op}}} \otimes 1$ , where  $1 \in \mathbb{C}G^0$  is the vector  $1_v = 1$  for all  $v$ . Thus we can always assume that  $x = 1$ .

Next, if  $\mathcal{VV}^*(S_e \otimes 1)(1 - \mathcal{VV}^*)(W_{\alpha_1,\beta_1} \otimes W_{\mu_1^{\text{op}},\nu_1^{\text{op}}} \otimes 1) \in \mathbb{C}\mathcal{V}\delta_{\lambda_1}$  and  $\mathcal{VV}^*(S_e \otimes 1)(1 - \mathcal{VV}^*)(W_{\alpha_2,\beta_2} \otimes W_{\mu_2^{\text{op}},\nu_2^{\text{op}}} \otimes 1) \in \mathbb{C}\mathcal{V}\delta_{\lambda_2}$  with  $\lambda_1 \neq \lambda_2$ , and both are nonzero, then  $e\alpha_i = \zeta_i\beta_i$  and  $\mu_i = \nu_i\tau_i$  and we have  $\zeta_i\tau_i = \lambda_i$ . So either  $\zeta_1 \neq \zeta_2$  or  $\tau_1 \neq \tau_2$ , and we deduce that  $W_{\alpha_1,\beta_1} \otimes W_{\mu_1^{\text{op}},\nu_1^{\text{op}}} \otimes 1 \perp W_{\alpha_2,\beta_2} \otimes W_{\mu_2^{\text{op}},\nu_2^{\text{op}}} \otimes 1$ .

It follows that for any finite  $F \subseteq G^*$ , we have

$$\left\| \left( 1 - \sum_{\lambda \in F} P_\lambda \right) \mathcal{VV}^*(S_e \otimes 1)(1 - \mathcal{VV}^*) \right\| = \sup_{\lambda \notin F} \|P_\lambda \mathcal{VV}^*(S_e \otimes 1)(1 - \mathcal{VV}^*)\|.$$

So it suffices to show that for any  $\varepsilon > 0$  there is a finite set  $F \subseteq G^*$  such that

$$\|P_\lambda \mathcal{VV}^*(S_e \otimes 1)(1 - \mathcal{VV}^*)\| < \varepsilon \quad \text{for all } \lambda \notin F. \quad (5.7)$$

For this, first note from our calculations above that  $\mathcal{VV}^*(S_e \otimes 1)(1 - \mathcal{VV}^*)(W_{\alpha,\beta} \otimes W_{\mu^{\text{op}},\nu^{\text{op}}} \otimes 1)$  is either zero, or perpendicular to  $\mathbb{C}\mathcal{V}\delta_\lambda$  unless  $e\alpha = \beta$  and  $\mu = \nu\lambda$ . So, defining

$$H_{e,\lambda} := \overline{\text{span}}\{W_{\alpha,e\alpha} \otimes W_{(\nu\lambda)^{\text{op}},\nu^{\text{op}}} \otimes 1 : r(\alpha) = s(e), s(\nu) = r(\lambda) = r(e)\},$$



we have

$$\|P_\lambda \mathcal{V}\mathcal{V}^*(S_e \otimes 1)(1 - \mathcal{V}\mathcal{V}^*)\| = \|\mathcal{V}\mathcal{V}^*(S_e \otimes 1)(1 - \mathcal{V}\mathcal{V}^*)|_{H_{e,\lambda}}\|.$$

The Cuntz–Krieger relation gives  $W_{\alpha,e\alpha} = \sum_{\alpha' \in s(\alpha)E^k} W_{\alpha\alpha',e\alpha\alpha'}$  for any  $\alpha \in s(e)G^*$  and any  $k$ . Similarly, each  $W_{(\nu\lambda)^{\text{op}},\nu^{\text{op}}} = \sum_{\nu' \in G^k r(\nu)} W_{(\nu'\nu\lambda)^{\text{op}},(\nu'\nu)^{\text{op}}}$  for any  $k$ . So putting

$$H_{e,\lambda,k} = \text{span}\{W_{\alpha,e\alpha} \otimes W_{(\nu\lambda)^{\text{op}},\nu^{\text{op}}} \otimes 1 : \alpha \in s(e)G^k, \nu \in G^k r(\lambda)\}, \quad (5.8)$$

we have a filtration  $H_{e,\lambda} = \bigcup_k H_{e,\lambda,k}$ . This implies that

$$\|\mathcal{V}\mathcal{V}^*(S_e \otimes 1)(1 - \mathcal{V}\mathcal{V}^*)|_{H_{e,\lambda}}\| = \sup_k \|\mathcal{V}\mathcal{V}^*(S_e \otimes 1)(1 - \mathcal{V}\mathcal{V}^*)|_{H_{e,\lambda,k}}\|.$$

The spanning elements (5.8) are mutually orthogonal and span  $H_{e,\lambda,k}$ . Let  $l := |\lambda|$ . Using the calculations of  $(\mathcal{V}\mathcal{V}^*(S_e \otimes 1)(1 - \mathcal{V}\mathcal{V}^*))(W_{\alpha,\beta} \otimes W_{\mu^{\text{op}},\nu^{\text{op}}} \otimes x)$  above, we see that for a spanning element  $W_{\alpha,e\alpha} \otimes W_{(\nu\lambda)^{\text{op}},\nu^{\text{op}}} \otimes 1$  of  $H_{e,\lambda,k}$ , we have

$$(\mathcal{V}\mathcal{V}^*(S_e \otimes 1)(1 - \mathcal{V}\mathcal{V}^*))(W_{\alpha,e\alpha} \otimes W_{(\nu\lambda)^{\text{op}},\nu^{\text{op}}} \otimes 1) = \frac{(\Phi_\infty(S_{e\alpha}S_{e\alpha}^*)\Phi_\infty^{\text{op}}(S_{\nu^{\text{op}}}S_{\nu^{\text{op}}}^*))(r(e))}{\sqrt{l+1}} \mathcal{V}\delta_\lambda.$$

We have

$$\begin{aligned} \|W_{\alpha,e\alpha} \otimes W_{(\nu\lambda)^{\text{op}},\nu^{\text{op}}} \otimes 1\|^2 &= \phi((1 | (W_{\alpha,e\alpha} \otimes W_{(\nu\lambda)^{\text{op}}} | W_{\alpha,e\alpha} \otimes W_{(\nu\lambda)^{\text{op}}})_{A \otimes A^{\text{op}}})) \\ &= \phi(\Phi_\infty(S_{e\alpha}S_{e\alpha}^*) \cdot 1 \Phi_\infty^{\text{op}}(S_{\nu^{\text{op}}}S_{\nu^{\text{op}}}^*) \Phi_\infty(S_{(\nu\lambda)^{\text{op}}}S_{(\nu\lambda)^{\text{op}}}^*)) \\ &= \sum_{v \in G^0} \Phi_\infty(S_{e\alpha}S_{e\alpha}^*)(v) \Phi_\infty^{\text{op}}(S_{\nu^{\text{op}}}S_{\nu^{\text{op}}}^*)(v) \\ &= (\Phi_\infty(S_{e\alpha}S_{e\alpha}^*)\Phi_\infty^{\text{op}}(S_{\nu^{\text{op}}}S_{\nu^{\text{op}}}^*))(r(e)). \end{aligned}$$

For  $\alpha \in s(e)G^k$  and  $\nu \in G^k r(\lambda)$ , let

$$\kappa_{\alpha,\nu} := \sqrt{(\Phi_\infty(S_{e\alpha}S_{e\alpha}^*)\Phi_\infty^{\text{op}}(S_{\nu^{\text{op}}}S_{\nu^{\text{op}}}^*))(r(e))}$$

and

$$h_{\alpha,\nu} := \kappa_{\alpha,\nu}^{-1} W_{\alpha,e\alpha} \otimes W_{(\nu\lambda)^{\text{op}},\nu^{\text{op}}} \otimes 1.$$

Then  $\{h_{\alpha,\nu} : \alpha \in s(e)G^k, \nu \in G^k r(\lambda)\}$  is an orthonormal basis for  $H_{e,\lambda,k}$ , and

$$(\mathcal{V}\mathcal{V}^*(S_e \otimes 1)(1 - \mathcal{V}\mathcal{V}^*))h_{\alpha,\nu} = \frac{\kappa_{\alpha,\nu}}{\sqrt{l+1}} \mathcal{V}\delta_\lambda$$

for each  $\alpha, \nu$ . The Cuntz–Krieger relation in each of  $C^*(G)$  and  $C^*(G^{\text{op}})$  shows that

$$\sum_{\alpha,\nu} \kappa_{\alpha,\nu}^2 = \sum_{\alpha,\nu} (\Phi_\infty(S_{e\alpha}S_{e\alpha}^*)\Phi_\infty^{\text{op}}(S_{\nu^{\text{op}}}S_{\nu^{\text{op}}}^*))(r(e)) = (\Phi_\infty(S_e S_e^*)\Phi_\infty^{\text{op}}(P_{r(e)}))(r(e)) = \Phi_\infty(S_e S_e^*)(r(e)),$$

where  $P_{r(e)} \in \mathcal{O}_E$  is the projection corresponding to the function  $\delta_{r(e)} \in C(G^0)$ . Hence

$$\|(\mathcal{V}\mathcal{V}^*(S_e \otimes 1)(1 - \mathcal{V}\mathcal{V}^*))|_{H_{e,\lambda,k}}\| = \left( \sum_{\alpha,\nu} \left( \frac{\kappa_{\alpha,\nu}}{\sqrt{l+1}} \right)^2 \right)^{1/2} = \left( \frac{\Phi_\infty(S_e S_e^*)(r(\lambda))}{l+1} \right)^{1/2}.$$

Now fix  $\varepsilon > 0$  and choose  $l$  large enough so that  $\frac{1}{\sqrt{l+1}} < \varepsilon$ . Let  $F := \bigcup_{k \leq l} G^k$ . Then  $F$  is finite, and the calculations above show that

$$\sup_{\lambda \notin F} \|P_\lambda \mathcal{V} \mathcal{V}^* (S_e \otimes 1) (1 - \mathcal{V} \mathcal{V}^*)\| \leq \left( \frac{(\Phi_\infty(S_e S_e^*) \Phi_\infty^{\text{op}}(s_{r(\lambda)}))(r(\lambda))}{l+1} \right)^{1/2} < \varepsilon,$$

which is (5.7).  $\square$

**Theorem 5.10.** *Let  $G$  be a finite graph with no sources and no sinks and let  $E$  be the associated edge module. Assume that the Watatani indices of  $E^{\otimes m}$  and  $E^{\text{op} \otimes m}$  satisfy Assumption 1. Then the class  $\Delta$  is a  $K$ -homology fundamental class.*

*Proof.* Let  $\mu = (\mathbb{C}^k \otimes \mathbb{C}^k, \mathbb{C}^k, 0)$  be the  $K$ -homology fundamental class for  $\mathbb{C}^k$ . Proposition 3.8 shows that this class satisfies the hypothesis of Theorem 3.4(1). So by that theorem, it suffices to check that

$$\iota_{A, \mathcal{O}_E} \otimes_{\mathcal{O}_E} \Delta = [\text{ext}^{\text{op}}] \otimes_{A^{\text{op}}} \mu \quad \text{and} \quad \iota_{A^{\text{op}}, \mathcal{O}_{E^{\text{op}}}} \otimes_{\mathcal{O}_{E^{\text{op}}}} \Delta = [\text{ext}] \otimes_A \mu.$$

First we compute  $[\text{ext}] \otimes_A \mu$ . As described at (5.3),  $\mu$  is represented by  $(\mathbb{C}^k \otimes \mathbb{C}^k, L^2(\mathbb{C}^k, \phi), 0)$  where  $\phi : \mathbb{C}^k \rightarrow \mathbb{C}$  is given by  $\phi(a) = \sum_j a_j$ . Since  $\mathbb{C}^k \otimes \mathbb{C}^k$  acts diagonally on  $\mathbb{C}^k$ , writing  $\Phi_\infty : \mathcal{O}_E \rightarrow A$  for the expectation of [36, Proposition 3.5], we have

$$(\Xi_{E,A} \otimes A^{\text{op}}) \otimes_{A \otimes A^{\text{op}}} \mathbb{C}^k \cong L^2(\mathcal{O}_E, \phi \circ \Phi_\infty).$$

By [36, Theorem 3.14], there is an isometric inclusion

$$Y : L^2(\mathcal{F}_E, \phi) \rightarrow L^2(\mathcal{O}_E, \phi)$$

satisfying  $Y(\mu) := [S_\mu]$ . We write  $Q := Y Y^* : L^2(\mathcal{O}_E, \phi) \rightarrow Y(L^2(\mathcal{F}_E, \phi))$ . Then  $[\text{ext}] \otimes_A \mu$  is represented by the Kasparov module  $(\mathcal{O}_E \otimes A^{\text{op}}, L^2(\mathcal{O}_E, \phi \circ \Phi_\infty), 2Q - 1)$ .

For  $t \in [0, \infty]$ , define

$$\mathbb{P}_t := \frac{1}{1+t^2} \begin{pmatrix} \mathcal{V} \mathcal{V}^* & t \mathcal{V}(Y^* \otimes 1) \\ t(Y \otimes 1) \mathcal{V}^* & t^2 Q \end{pmatrix}.$$

A direct computation shows that each  $\mathbb{P}_t$  is a projection, and that

$$F_t := \left( \mathcal{O}_E \otimes A^{\text{op}}, \begin{pmatrix} L^2(\mathcal{O}_E) \otimes L^2(\mathcal{O}_{E^{\text{op}}}) \\ L^2(\mathcal{O}_E) \end{pmatrix}, 2\mathbb{P}_t - 1 \right)$$

is a Fredholm module. At  $t = 0$ , this Fredholm module represents  $\iota_{A^{\text{op}}, \mathcal{O}_{E^{\text{op}}}} \otimes_{\mathcal{O}_{E^{\text{op}}}} \Delta$ . At  $t = \infty$ , we have

$$\left( \mathcal{O}_E \otimes A^{\text{op}}, \begin{pmatrix} L^2(\mathcal{O}_E) \otimes L^2(\mathcal{O}_{E^{\text{op}}}) \\ L^2(\mathcal{O}_E) \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 2Q - 1 \end{pmatrix} \right).$$

Since  $(\mathcal{O}_E \otimes A^{\text{op}}, L^2(\mathcal{O}_E) \otimes L^2(\mathcal{O}_{E^{\text{op}}}), -1)$  is a degenerate Kasparov module, we deduce that

$$[F_\infty] = [\mathcal{O}_E \otimes A^{\text{op}}, L^2(\mathcal{O}_E), 2Q - 1]$$

in  $KK(\mathcal{O}_E \otimes A^{\text{op}}, \mathbb{C})$ . Since  $A \otimes A^{\text{op}} = \mathbb{C}^k \otimes \mathbb{C}^k$  acts diagonally on  $L^2(A, \phi) = \mathbb{C}^k$ , we deduce from [10, Theorem A.5] that

$$\begin{aligned} [(\mathcal{O}_E \otimes A^{\text{op}}, \Xi_E \otimes A^{\text{op}}, (2Q - 1) \otimes 1_{A^{\text{op}}})] \otimes_{A \otimes A^{\text{op}}} [(A \otimes A^{\text{op}}, A_{\mathbb{C}}, 0)] \\ = [(\mathcal{O}_E \otimes A^{\text{op}}, L^2(\mathcal{O}_E), 2Q - 1)] = [F_\infty], \end{aligned}$$

so  $F_\infty$  represents  $[\text{ext}] \otimes_A \mu$ . Since the  $F_t$  constitute a homotopy, this shows that  $\iota_{A^{\text{op}}, \mathcal{O}_{E^{\text{op}}}} \otimes_{\mathcal{O}_{E^{\text{op}}}} \Delta = [\text{ext}] \otimes_A \mu$ .

The same argument applied to the opposite graph shows that  $\iota_{A, \mathcal{O}_E} \otimes_{\mathcal{O}_E} \Delta = [\text{ext}^{\text{op}}] \otimes_{A^{\text{op}}} \mu$ , and this completes the proof.  $\square$

**Example 5.11.** Verifying Assumption 1 in Theorem 5.10 in general seems complicated, but it is checkable in concrete instances. For example,  $SU_q(2)$ , realised as a graph  $C^*$ -algebra as in [36, Example 3.10], is easily seen to satisfy Assumption 1. Likewise, by [36, Lemma 3.7 and Example 3.8], Cuntz–Krieger modules associated to primitive non-negative matrices all satisfy Assumption 1.

**Corollary 5.12.** *Let  $G$  be a finite directed graph with no sinks and no sources. Suppose that the edge modules of  $G$  and  $G^{\text{op}}$  both satisfy Assumption 1. Then  $C^*(G)$  is Poincaré dual to the graph algebra of the opposite graph  $C^*(G^{\text{op}})$ . Thus there are isomorphisms*

$$K_*(C^*(G)) \rightarrow K^{**+1}(C^*(G^{\text{op}})), \quad K_*(C^*(G^{\text{op}})) \rightarrow K^{**+1}(C^*(G)).$$

We require no assumptions on the associated shift space here. From the characterisation of the  $K$ -theory of graph algebras [34, Theorem 7.18] we obtain

**Corollary 5.13.** *Suppose the finite directed graphs  $G = (G^0, G^1, r, s)$  and  $G^{\text{op}}$  have no sinks and no sources, and that their edge modules satisfy Assumption 1. Then the even  $K$ -homology groups of  $C^*(G)$  and  $C^*(G^{\text{op}})$  are torsion-free and have the same rank as the corresponding odd  $K$ -theory groups.*

### 5.2.2 The $K$ -homology fundamental class for $C^*(G)^{\text{op}}$

It turns out that proving that  $C^*(G)^{\text{op}}$  is a Poincaré dual for  $C^*(G)$  requires more assumptions than for  $C^*(G^{\text{op}})$ .

We let  $A = C(G^0)$  and  $E = C(G^1)$  be the algebra and module for a finite directed graph  $G$  with no sources. We let  $\phi : A \rightarrow \mathbb{C}$  be the state given by  $\phi(a) = \sum_{v \in G^0} a(v)$ .

Suppose that the bi-Hilbertian  $A$ -bimodule  $E$  satisfies Assumption 1, and additionally that for all  $\nu \in E^{\otimes k}$  there is  $c_{|\nu|} \in A$  (necessarily central) which is invertible and such that

$$\mathbf{q}(\nu) := \lim_{n \rightarrow \infty} e^{-\beta n} \nu e^{\beta n - k} = c_{|\nu|} \nu. \quad (5.9)$$

This assumption is a special case of a key assumption in [17]; see [17] for more detail. This condition is not satisfied for all graph algebras. For example, the bimodule of the graph with vertices  $v, w$  and edges  $e, f, g$  satisfying  $s(e) = r(e) = s(f) = v$  and  $r(f) = s(g) = r(g) = w$ , whose  $C^*$ -algebra is  $SU_q(2)$ , does not satisfy this hypothesis. On the other hand, the graph-modules associated to Cuntz–Krieger algebras of primitive 0–1 matrices all do, [36, Example 3.8].

We can now produce a fundamental  $K$ -homology class

$$\overline{\Delta} \in KK^1(\mathcal{O}_E \otimes_{\mathcal{O}_E^{\text{op}}}, \mathbb{C}) = KK^1(\mathcal{O}_E \otimes_{\mathcal{O}_{E^{\text{op}}}}, \mathbb{C}).$$

The process is largely the same as for  $\mathcal{O}_{E^{\text{op}}}$ , so we just describe it briefly. We first claim that there is an isometry

$$\overline{\mathbf{v}} : L^2(\mathcal{F}_E, \phi) \hookrightarrow \Xi_E \otimes_{\Xi_{E^{\text{op}}}} \otimes_{A \otimes A^{\text{op}}} L^2(A, \phi)$$

such that for every elementary tensor  $\lambda \in E^{\otimes |\lambda|}$ , we have

$$\overline{\mathcal{V}}(\lambda) = \sum_{j=0}^{|\lambda|} \frac{1}{\sqrt{|\lambda|+1}} W_{\lambda_j, 1} \otimes W_{1, c_{|\lambda|}^{-1/2} \lambda^{\text{op}(|\lambda|-j)}} \otimes \delta_{s(\lambda_j)}.$$

Here  $\overline{\lambda}^{\text{op}}$  refers to the element of the conjugate module  $\overline{E}^{\text{op}}$  corresponding to  $\lambda$ . We write  $\overline{\lambda}^{\text{op}(|\lambda|-j)}$  for the element of the conjugate module corresponding to the tail of  $\lambda$  of length  $|\lambda| - j$ . By Lemma 2.7 we see that here we do *not* reverse the order of elements in a tensor product when we factor a simple tensor: that is,  $\delta_{\overline{\lambda}^{\text{op}}} = \delta_{\overline{\lambda}_j^{\text{op}}} \otimes \delta_{\overline{\lambda}^{\text{op}(|\lambda|-j)}}$  under the identification of  $(\overline{E}^{\text{op}})^{\otimes |\lambda|}$  with  $(\overline{E}^{\text{op}})^{\otimes j} \otimes (\overline{E}^{\text{op}})^{\otimes |\lambda|-j}$ . The need for the additional assumption (5.9) becomes clear in the following computation, which uses [36, Lemmas 3.2 and 3.3]:

$$\begin{aligned} (W_{1, c_{|\lambda|}^{-1/2} \lambda}^{\text{op}} | W_{1, c_{|\rho|}^{-1/2} \rho}^{\text{op}})_{A^{\text{op}}} &= {}_{A^{\text{op}}}(\overline{c_{|\lambda|}^{-1/2} \lambda}^{\text{op}} | \mathbf{q}(\overline{c_{|\rho|}^{-1/2} \rho}^{\text{op}})) = {}_A(\mathbf{q}(c_{|\rho|}^{-1/2} \rho) | c_{|\lambda|}^{-1/2} \lambda)^{\text{op}} \\ &= \delta_{|\rho|, |\lambda|} {}_A(\mathbf{q}(c_{|\lambda|}^{-1/2} c_{|\rho|}^{-1/2} \rho) | \lambda)^{\text{op}} = {}_A(\rho | \lambda)^{\text{op}} = (\lambda^{\text{op}} | \rho^{\text{op}})_{A^{\text{op}}}. \end{aligned}$$

With this calculation in place, the remainder of the computations used to prove that  $\overline{\mathcal{V}}$  is an isometry are identical to those for the  $E^{\text{op}}$  case. The formula for the adjoint is slightly modified, becoming

$$\overline{\mathcal{V}}^*(W_{\alpha, \beta} \otimes W_{\overline{\mu}^{\text{op}}, \overline{\nu}^{\text{op}}} \otimes x) = x_{r(\beta)} \Phi_{\infty}(S_{\overline{\alpha}} S_{\beta}^*)(r(\beta)) \Phi_{\infty}(S_{\overline{\nu}} S_{\mu}^*)(r(\beta)) \delta_{\underline{\alpha}, \underline{\alpha}} c_{|\underline{\alpha}|}^{1/2}.$$

The additional factor of  $c_{|\underline{\alpha}|}^{1/2}$  does not alter any subsequent computations because it is central and invertible. For computing commutators of  $\mathcal{O}_E \otimes \mathcal{O}_{\overline{E}^{\text{op}}}$  with  $\overline{\mathcal{V}} \overline{\mathcal{V}}^*$ , the computations are mostly as for the case of  $E^{\text{op}}$ , with only minor differences. We summarise the results as follows:

**Theorem 5.14.** *Let  $G$  be a finite directed graph with no sources,  $A = C(G^0)$  and  $E = C(G^1)$ . Assume that  $E$  satisfies Assumption 1 and Equation (5.9). Then there is a  $K$ -homology fundamental class*

$$\overline{\Delta} \in KK^1(\mathcal{O}_E \otimes \mathcal{O}_{\overline{E}^{\text{op}}}, \mathbb{C}).$$

Hence there are isomorphisms

$$\cdot \otimes_{\mathcal{O}_E} \overline{\Delta} : K_*(\mathcal{O}_E) \xrightarrow{\cong} K^{*+1}(\mathcal{O}_{\overline{E}^{\text{op}}}) \quad \text{and} \quad \cdot \otimes_{\mathcal{O}_{\overline{E}^{\text{op}}}} \overline{\Delta} : K_*(\mathcal{O}_{\overline{E}^{\text{op}}}) \xrightarrow{\cong} K^{*+1}(\mathcal{O}_E).$$

### 5.2.3 A comparison of the dual algebras

We now discuss what happens when we can construct  $\delta \in KK^{d+1}(\mathbb{C}, \mathcal{O}_E \otimes \mathcal{O}_{\overline{E}^{\text{op}}})$  and  $\overline{\Delta} \in KK^{d+1}(\mathcal{O}_E \otimes \mathcal{O}_{\overline{E}^{\text{op}}}, \mathbb{C})$  (or  $\overline{\delta}$  and  $\overline{\Delta}$ ). Our methods guarantee that the Kasparov product with these classes gives us isomorphisms

$$K^*(\mathcal{O}_{\overline{E}^{\text{op}}}) \xrightarrow{\delta \otimes \cdot} K_{*+d+1}(\mathcal{O}_E) \xrightarrow{\cdot \otimes \overline{\Delta}} K^*(\mathcal{O}_{\overline{E}^{\text{op}}})$$

and

$$K_*(\mathcal{O}_{\overline{E}^{\text{op}}}) \xrightarrow{\cdot \otimes \overline{\Delta}} K_{*+d+1}(\mathcal{O}_E) \xrightarrow{\delta \otimes \cdot} K_*(\mathcal{O}_{\overline{E}^{\text{op}}}).$$

Hence when  $A$ , and so  $A^{\text{op}}$ ,  $\mathcal{O}_{\overline{E}^{\text{op}}}$  and  $\mathcal{O}_E^{\text{op}}$  are in the bootstrap class, we find that  $\mathcal{O}_{\overline{E}^{\text{op}}}$  and  $\mathcal{O}_E^{\text{op}}$  are  $KK$ -equivalent. Thus the two potential dual algebras are indistinguishable at the level of  $KK$ -theory. Indeed if, for example,  $\mathcal{O}_E^{\text{op}}$  and  $\mathcal{O}_{\overline{E}^{\text{op}}}$  are UCT Kirchberg algebras, then they coincide.

**Theorem 5.15.** *Let  $A$  be unital and Poincaré self-dual with fundamental classes  $\beta$  and  $\mu$ . If  $E$  is a bi-Hilbertian bimodule with finite left and right Watatani indices such that  $\beta \otimes_A [E] = \beta \otimes_{A^{\text{op}}} [E^{\text{op}}]$  and  $[E] \otimes_A \mu = [\overline{E}^{\text{op}}] \otimes_{A^{\text{op}}} \mu$  then  $\mathcal{O}_{E^{\text{op}}}$  and  $\mathcal{O}_E^{\text{op}}$  are  $KK$ -equivalent. Hence  $\mathcal{O}_E$  is Poincaré dual to both  $\mathcal{O}_{E^{\text{op}}}$  and  $\mathcal{O}_E^{\text{op}}$ .*

**Corollary 5.16.** *Suppose that the finite graph  $G$  has no sinks and no sources, and that the edge modules  $E$  and  $E^{\text{op}}$  satisfy Assumption 1 and Equation (5.9). Then  $C^*(G^{\text{op}})$  and  $C^*(G)^{\text{op}}$  are  $KK$ -equivalent and both are Poincaré dual to  $C^*(G)$ .*

## A General Poincaré duality

We briefly comment on a more general situation than that of lifting a Poincaré self-duality for  $A$  to one for  $\mathcal{O}_A$ . We thank Magnus Goffeng for pointing out the applications of our approach to this setting. Let  $A, B$  be Poincaré dual unital  $C^*$ -algebras, with the duality realised by classes  $\mu \in KK^d(A \otimes B, \mathbb{C})$  and  $\beta \in KK^d(A \otimes B, \mathbb{C})$ . Suppose that  $E$  is an  $A$ - $A$ -correspondence and  $F$  is a  $B$ - $B$ -correspondence.

By considering diagrams like that in Lemma 3.2 with  $B$  in place of  $A^{\text{op}}$ , and applying arguments like those in Section 3, we arrive at the following statement.

**Theorem A.1.** *With  $A, B, E$  and  $F$  as above, suppose that  $[E] \otimes_A \mu = [F] \otimes_B \mu \in KK(A \otimes B, \mathbb{C})$  and  $\beta \otimes_A [E] = \beta \otimes_B [F] \in KK(\mathbb{C}, A \otimes B)$ . Then classes  $\Delta \in KK^{d+1}(\mathcal{O}_E \otimes \mathcal{O}_F, \mathbb{C})$  and  $\delta \in KK^{d+1}(\mathbb{C}, \mathcal{O}_E \otimes \mathcal{O}_F)$  define isomorphisms as in Lemma 3.2 if*

$$\iota_{A, \mathcal{O}_E} \otimes_{\mathcal{O}_E} \Delta = \text{ext}_B \otimes_B \mu \quad \text{and} \quad \iota_{B, \mathcal{O}_F} \otimes_{\mathcal{O}_F} \Delta = \text{ext}_A \otimes_A \mu$$

and

$$-\delta \otimes_{\mathcal{O}_E} \text{ext}_A = \beta \otimes_B \iota_{B, \mathcal{O}_F} \quad \text{and} \quad \delta \otimes_{\mathcal{O}_F} \text{ext}_B = \beta \otimes_A \iota_{A, \mathcal{O}_E}.$$

The same arguments as in Theorem 4.5 allow us to obtain a  $K$ -theory fundamental class for  $\mathcal{O}_E \otimes \mathcal{O}_F$ .

**Theorem A.2.** *Let  $A$  and  $B$  be unital  $C^*$ -algebras. Suppose that  $A$  and  $B$  are Poincaré dual with invariant classes  $\beta, \mu$  as in Theorem A.1, and assume that  $E$  and  $F$  are both finitely generated as right modules. Construct  $\mathbb{W}_A$  from  $E_A$  and  $\mathbb{W}_B$  from  $F_B$  as in Definition 4.2 and Lemma 4.3. Then*

$$\delta := \beta \otimes_{A \otimes B} \mathbb{W}_A - \beta \otimes_{A \otimes B} \mathbb{W}_B \in KK(\mathbb{C}, A \otimes B)$$

*is a  $K$ -theory fundamental class for  $\mathcal{O}_E \otimes \mathcal{O}_F$ .*

Obtaining  $K$ -homology fundamental classes is harder, but we mention one important case.

**Example A.3.** Let  $(M, g)$  be a compact oriented Riemannian manifold of dimension  $d$  and  $\alpha$  the automorphism of  $C^\infty(M)$  dual to an orientation-preserving isometry.

Insisting on an actual isometry, as opposed to an almost-isometry, ensures that we obtain an automorphism  $\tilde{\alpha}$  of the bundle of Clifford algebras  $\text{Cliff}(M, g)$ . Thus we obtain correspondences  $E = {}_\alpha C(M)_{C(M)}$  and  $F = \tilde{\alpha} \text{Cliff}(M, g)_{\text{Cliff}(M, g)}$ .

The orientable version of Kasparov's Bott class for  $M$  is as we described in the proof of Proposition 3.12, except the spinor bundle is replaced by the Clifford bundle. Thus  $\beta$  has a representative

$(\mathbb{C}, X_{C(M) \otimes \text{Cliff}(M, g)}, T)$  with  $\beta \in KK^d(\mathbb{C}, C(M) \otimes \text{Cliff}(M, g))$ . Just as in the proof of Proposition 3.12, we can implement  $\alpha$  and  $\tilde{\alpha}$  via a  $\mathbb{C}$ -linear map  $V : X \rightarrow X$  satisfying conditions analogous to those of part 1 of Lemma 3.10. In particular, we can show that  $\beta \otimes_{C(M)} [E] = \beta \otimes_{\text{Cliff}(M, g)} [F]$ , and so we can obtain a  $K$ -theory fundamental class  $\delta \in KK^{d+1}(\mathbb{C}, C(M) \otimes \text{Cliff}(M, g))$ . As in the  $\text{spin}^c$  case, we restrict to compact manifolds to obtain the  $K$ -theory class.

We can also produce the  $K$ -homology fundamental class, and this does not require compactness, though our formulation of Poincaré duality does require compactness. Kasparov's fundamental class [25, 28] for the oriented manifold  $(M, g)$  is

$$\lambda = \left[ \left( C^\infty(M) \otimes \text{Cliff}(M, g), \pi L^2(\Lambda_+ T^* M \oplus \Lambda_- T^* M), \begin{pmatrix} 0 & (d + d^*)_- \\ (d + d^*)_+ & 0 \end{pmatrix} \right) \right],$$

where  $\pi$  is the representation defined for  $f \in C(M)$  and  $v \in \Gamma(T^* M)$  by

$$\pi(f \otimes v)\omega(x) = f(x)(v(x) \wedge \omega(x) + v(x) \lrcorner \omega(x)), \quad \omega \in L^2(\Lambda^* T^* M).$$

Just as for the Bott class, we can implement  $\alpha$  and  $\tilde{\alpha}$  on  $L^2(\Lambda^* T^* M)$  via a  $\mathbb{C}$ -linear map  $W$ . One checks that  $W$  and  $\lambda$  satisfy analogues of the conditions of part 2 of Lemma 3.10. In particular, we can show that  $[E] \otimes_{C(M)} \lambda = [F] \otimes_{\text{Cliff}(M, g)} \lambda$ . So mildly modifying the constructions of Section 5.1, we obtain a fundamental class implementing duality between the crossed product of the functions and the crossed product of the Clifford algebra,

$$\Delta = \left[ \left( (C^\infty(M) \rtimes_\alpha \mathbb{Z}) \otimes (\text{Cliff}(M, g) \rtimes_\alpha \mathbb{Z}), \tilde{\pi} \ell^2(\mathbb{Z}) \otimes L^2(\Lambda T^* M), \begin{pmatrix} N & (d + d^*)_- \\ (d + d^*)_+ & -N \end{pmatrix} \right) \right],$$

where the representation  $\tilde{\pi}$  is defined analogously to Equation (5.1). See [25] and [28] for more information about Kasparov's fundamental class.

## B Relationships between the extension classes

If  $E$  is not an invertible bimodule, then constructing a representative of the extension class is more complicated than for an invertible module [36, Theorem 3.14]. As a result, the relationship between the extension class for  $\mathcal{O}_E$  and that for  $\mathcal{O}_E^{\text{op}}$  is also more complicated than in Section 4.3, as we now explain. Throughout this section we assume that  $A$  is unital, and that  $E$  is finitely generated and bi-Hilbertian and satisfies Assumption 1 of subsection 5.2.1.

Just as for  $K$ -theory where we compared  $K_*(A)$  and  $K_*(A^{\text{op}})$  in Lemma 4.6, we can also produce an explicit isomorphism for  $KK$ -groups of algebras and their opposites. This will allow us to compare the classes  $[\text{ext}]$  and  $[\overline{\text{ext}}^{\text{op}}]$ . (When  $E$  is not an invertible bimodule, there can be no corresponding comparison with  $[\text{ext}^{\text{op}}]$ .)

**Proposition B.1.** *For any  $C^*$ -algebras  $A, B$  there is an isomorphism*

$$\text{OP} : KK(A, B) \xrightarrow{\cong} KK(A^{\text{op}}, B^{\text{op}})$$

given on cycles by the map

$$(A, X_B, T) \mapsto (A^{\text{op}}, \overline{X}_{B^{\text{op}}}, \overline{T}).$$

Here  $\overline{X}_{B^{\text{op}}}$  is the (left  $B$ -)conjugate module considered as a right  $B^{\text{op}}$  module, and for  $a \in A, x \in X$  we define  $a^{\text{op}} \overline{x} = \overline{a^* x}$  and  $\overline{T} \overline{x} = \overline{T x}$ .

*Proof.* Given a Kasparov module  $(A, \phi X_B, T)$  (bounded or unbounded), the data  $(A^{\text{op}}, \overline{X}_{B^{\text{op}}}, \overline{T})$  defines a Kasparov  $A^{\text{op}}-B^{\text{op}}$ -module. To see that  $(A, \phi X_B, T) \mapsto (A^{\text{op}}, \overline{X}_{B^{\text{op}}}, \overline{T})$  descends to a well-defined map from  $KK(A, B)$  to  $KK(A^{\text{op}}, B^{\text{op}})$ , observe that if  $(A, Y_{B \otimes C([0,1])}, \mathbf{T})$  is a homotopy of Kasparov  $A-B$ -modules, then  $(A^{\text{op}}, \overline{Y}_{(B \otimes C([0,1]))^{\text{op}}}, \overline{\mathbf{T}})$  is a homotopy of the corresponding Kasparov  $A^{\text{op}}-B^{\text{op}}$ -modules. Since this entire discussion is symmetric in  $A-B$  and  $A^{\text{op}}-B^{\text{op}}$ , we are done.  $\square$

**Proposition B.2.** *Let  $A$  be unital, and  $E$  a finitely generated bi-Hilbertian  $A$  bimodule satisfying Assumption 1. Let  $(\mathcal{O}_E, \Xi_{E,A}, 2P-1)$  be the representative of  $[\text{ext}]$  described in Equation 5.5, let  $(\mathcal{O}_E^{\text{op}}, \overline{\Xi}_{E, A_{A^{\text{op}}}}, 2\overline{P}-1)$  be the class provided by Proposition B.1, and let  $(\mathcal{O}_E^{\text{op}}, \Xi_{\overline{E}^{\text{op}}, A^{\text{op}}}, 2P^{\text{op}}-1)$  be the representative of  $[\overline{\text{ext}}^{\text{op}}]$ . Then the map*

$$S_{\overline{E}^{\text{op}}} S_{\overline{D}^{\text{op}}}^* \mapsto \overline{S}_{\mu} \overline{S}_{\nu}^{\text{op}}$$

*extends to a unitary isomorphism of Kasparov modules*

$$(\mathcal{O}_E^{\text{op}}, \Xi_{\overline{E}^{\text{op}}, A^{\text{op}}}, 2P^{\text{op}}-1) \cong (\mathcal{O}_E^{\text{op}}, \overline{\Xi}_{E, A_{A^{\text{op}}}}, 2\overline{P}-1).$$

*Hence under the isomorphism  $KK(\mathcal{O}_E, A) \rightarrow KK(\mathcal{O}_E^{\text{op}}, A^{\text{op}})$  of Proposition B.1, the class  $[\text{ext}]$  is mapped to the class  $[\overline{\text{ext}}^{\text{op}}]$ .*

*Proof.* Fix elementary tensors  $\mu, \nu, \rho, \sigma \in \mathcal{F}_E$ . Assumption 1 provides a positive adjointable (for both inner products) map  $q : \mathcal{F}_E \rightarrow \mathcal{F}_E$  given by  $q(\mu) = \lim_{n \rightarrow \infty} e^{-\beta_n} \nu e^{\beta_n - |\mu|}$ .

We write  $\mu = \mu_{in} \otimes \mu_f$  where  $\mu_{in}$  is an initial tensor factor of  $\mu$  whose length will be clear from context, and  $\mu_f$  the corresponding final tensor factor. Write  $W_{\overline{E}^{\text{op}}, \overline{D}^{\text{op}}}$  for the image of  $S_{\overline{E}^{\text{op}}} S_{\overline{D}^{\text{op}}}^*$  in the completion  $\Xi_{\overline{E}^{\text{op}}, A^{\text{op}}}$  and  $W_{\mu, \nu}$  for the image of  $S_{\mu, \nu}$  in  $\Xi_{E, A}$ . We have

$$\begin{aligned} (W_{\overline{E}^{\text{op}}, \overline{D}^{\text{op}}} | W_{\overline{E}^{\text{op}}, \overline{\sigma}^{\text{op}}})_{A^{\text{op}}} &= \Phi_{\infty}(S_{\overline{E}^{\text{op}}} S_{\overline{D}^{\text{op}}}^* S_{\overline{\sigma}^{\text{op}}} S_{\overline{\sigma}^{\text{op}}}^*) \\ &= \begin{cases} A^{\text{op}}(\overline{D}^{\text{op}}(\overline{\mu}^{\text{op}} | \overline{\rho_{in}}^{\text{op}})_{A^{\text{op}}} \overline{\rho_f}^{\text{op}} | \overline{q}^{\text{op}}(\overline{\sigma}^{\text{op}})) & |\rho| \geq |\mu| \\ A^{\text{op}}(\overline{D}^{\text{op}} | \overline{q}^{\text{op}}(\overline{\sigma}^{\text{op}}(\overline{\rho}^{\text{op}} | \overline{\mu_{in}}^{\text{op}})_{A^{\text{op}}}) \overline{\mu_f}^{\text{op}}) & |\rho| \leq |\mu| \end{cases} \\ &= \begin{cases} A^{\text{op}}(\overline{D}^{\text{op}}_A(\overline{\rho_{in}} | \overline{\mu})^{\text{op}} \overline{\rho_f}^{\text{op}} | \overline{q}^{\text{op}}(\overline{\sigma}^{\text{op}})) & |\rho| \geq |\mu| \\ A^{\text{op}}(\overline{D}^{\text{op}} | \overline{q}^{\text{op}}(\overline{\sigma}^{\text{op}}_A(\overline{\mu_{in}} | \overline{\rho})^{\text{op}}) \overline{\mu_f}^{\text{op}}) & |\rho| \leq |\mu| \end{cases} \\ &= \begin{cases} A^{\text{op}}(\overline{\nu}(\overline{\mu} | \overline{\rho_{in}})_A \overline{\rho_f}^{\text{op}} | \overline{q}(\overline{\sigma})^{\text{op}}) & |\rho| \geq |\mu| \\ A^{\text{op}}(\overline{D}^{\text{op}} | \overline{q}(\overline{\sigma}(\overline{\rho} | \overline{\mu_{in}})_A \overline{\mu_f})^{\text{op}}) & |\rho| \leq |\mu| \end{cases} \\ &= \begin{cases} A(q(\sigma) | \nu(\mu | \rho_{in})_A \rho_f)^{\text{op}} & |\rho| \geq |\mu| \\ A(q(\sigma(\rho | \mu_{in})_A \mu_f) | \nu)^{\text{op}} & |\rho| \leq |\mu|. \end{cases} \end{aligned}$$

Likewise in the conjugate module  $\overline{\Xi}_{E, A_{A^{\text{op}}}}$  we have

$$\begin{aligned} (\overline{W}_{\mu, \nu}^{\text{op}} | \overline{W}_{\rho, \sigma}^{\text{op}})_{A^{\text{op}}} &= A(\overline{W}_{\rho, \sigma} | \overline{W}_{\mu, \nu})^{\text{op}} = (W_{\rho, \sigma} | W_{\mu, \nu})_A^{\text{op}} = \Phi_{\infty}(S_{\sigma} S_{\rho}^* S_{\mu} S_{\nu}^*) \\ &= \begin{cases} A(\sigma | q(\nu(\mu | \rho_{in})_A \rho_f))^{\text{op}} & \text{if } |\rho| \geq |\mu| \\ A(\sigma(\rho | \mu_{in})_A \mu_f | q(\nu))^{\text{op}} & \text{if } |\rho| \leq |\mu|. \end{cases} \end{aligned}$$

The self-adjointness of  $q$  and sesquilinearity of the inner product show that there is an isometry

$$U : \Xi_{\overline{E}^{\text{op}}, A^{\text{op}}} \rightarrow \overline{\Xi}_{E, A_{A^{\text{op}}}} \quad \text{such that} \quad UW_{\overline{E}^{\text{op}}, \overline{D}^{\text{op}}} = \overline{W}_{\mu, \nu}^{\text{op}}.$$

This  $U$  carries the embedded image  $\overline{\text{span}}\{W_{\bar{\mu},1}^{\text{op}} : \mu \in \mathcal{F}_E\}$  of  $\mathcal{F}_{\bar{E}}^{\text{op}}$  to the embedded image  $\overline{\text{span}}\{\overline{W}_{\mu,1} : \mu \in \mathcal{F}_E\}$  of  $\overline{\mathcal{F}}_E^{\text{op}}$ . Hence

$$UP^{\text{op}}U^* = \overline{P}.$$

Likewise,

$$US_{\bar{e}^{\text{op}}}W_{\bar{\rho}^{\text{op}},\bar{\sigma}^{\text{op}}} = UW_{\bar{e}\bar{\rho}^{\text{op}},\bar{\sigma}^{\text{op}}} = \overline{W_{e\rho,\sigma}^{\text{op}}} = S_e^{*\text{op}}UW_{\bar{\rho}^{\text{op}},\bar{\sigma}^{\text{op}}}.$$

and we deduce that the actions are also intertwined.  $\square$

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