HIGHER DIMENSIONAL GENERALIZATIONS OF THE THOMPSON GROUPS VIA HIGHER RANK GRAPHS

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Abstract. We construct a family of groups from suitable higher rank graphs which are analogues of the finite symmetric groups. We introduce homological invariants showing that many of our groups are, for example, not isomorphic to nV, when $n \geq 2$.

1. Introduction

The goal of this paper is to show how to construct a family of groups from suitable higher rank graphs. By a theorem of Matui [22, Theorem 3.10], these groups are complete invariants of the usual groupoids associated with such higher rank graphs.

To understand a group properly usually requires the group to be acting on a suitable geometric structure. In the case of the classical groups, the groups arise as groups of units of matrix rings and therefore act on geometric structures constructed from vector spaces. In this paper, we generalize [17] and show how to construct a group from a suitable higher rank graph. Our approach is closely related to the one adopted in [2]. As part of this construction, we show that the group arises as a group of units of a Boolean inverse monoid. Such inverse monoids have ring-theoretic characteristics, without themselves being rings [29]. In addition, their sets of idempotents form Boolean algebras on which the group acts. Thus showing that a group is a group of units of a Boolean inverse monoid brings with it geometric information. We now state the main theorem we prove in this paper. All undefined terms will be defined in due course.

Main Theorem. *Let C be a higher rank graph having a finite number of identities and no sources and being row finite. If C is aperiodic and cofinal then we may associate with C a countable, simple Boolean inverse $\wedge$-monoid $\mathsf{B}(C)$. There are two possibilities:*

(1) *The Boolean inverse monoid $\mathsf{B}(C)$ is finite and isomorphic to a finite symmetric inverse monoid. Its group of units is a finite symmetric group.*
(2) *The Boolean inverse monoid $\mathsf{B}(C)$ is countably infinite. Its group of units is then isomorphic to a full subgroup of the group of self-homeomorphisms of the Cantor space which acts minimally and in which each element has clopen support.*

Remark 1.1. In the case where $\mathsf{B}(C)$ is countably infinite, we are interested in the situation where its commutator subgroup is simple. We conjecture that when the higher rank graph satisfies, in addition, the condition of [9, Proposition 4.9] then the commutator subgroup should be simple.

Under non-commutative Stone duality, our Boolean inverse monoids are related to Hausdorff étale topological groupoids which are effective, minimal and have a space of identities homeomorphic to the Cantor space. By [22, Theorem 3.10], our group is therefore a complete invariant for the étale groupoid. This suggests that both the homology of the groupoid, and the K-groups of its C^*-algebra should be interesting group-theoretically. This theme is developed in Sections 7 and 8. Section 7 develops the invariants of the groups depending on the 1-skeleton of the k-graph. Section 8 deals with explicit constructions of infinite families of pair-wise non-isomorphic k-graphs, for every $k \geq 2$. Using the invariants of the Section 8 we show that the corresponding groups are non-isomorphic as well. In particular, they are not isomorphic to the known examples of groups nV for $n \geq 2$.

We begin this paper by considering more general categories than the higher rank graphs, such as those in [27, 28].

Notation

- $I(X, \tau)$ the inverse semigroup of all partial homeomorphisms between the open subsets of the topological space (X, τ). When X is endowed with the discrete topology, we simply denote this inverse semigroup by $I(X)$, the symmetric inverse monoid on the set X. Section 2.1.
- S^e the inverse semigroup of all elements $s \in S$ where both $s^{-1}s$ and ss^{-1} are essential idempotents. Section 2.1.
- $\mathsf{RI}(C)$ the inverse semigroup of all bijective morphisms between the right ideals of the category C. Section 2.2.
- $\mathsf{R}(C)$ the inverse semigroup of all bijective morphisms between the finitely generated right ideals of the finitely aligned category C. Section 2.2.
- $\mathcal{G}(C) = \mathsf{R}(C)^e/\sigma$ the group constructed from the category C which is finitely aligned and has only a finite number of identities. Section 2.2.
- $\Sigma(C)$ the inverse hull of the cancellative category C. Section 2.3.
- $\mathsf{P}(C)$ the inverse monoid of all bijective morphisms between finitely generated right ideals generated by codes when C is a strongly finitely aligned conical cancellative category with a finite number of identities. Section 2.4.
- $C_{\mathbf{m}}$ the set of all elements of the higher rank k-graph with degree equal to $\mathbf{m}$ where $\mathbf{m} \in \mathbb{N}^k$. Section 3.
- If X is a subset of a distributive inverse semigroup, then $X^\vee$ denotes the set of all binary joins of compatible elements of X. Beginning of Section 4.
- $a \leq_e b$ the element a is essential in the element b. Section 4.1.
- $\mathsf{B}(C) = \mathsf{R}(C)/\equiv$, a Boolean inverse monoid with group of units isomorphic to $\mathcal{G}(C)$ when C is a strongly finitely aligned higher rank graph with a finite number of identities with no sources and is row finite. Section 4.2.
- C^∞ the set of all k-tilings of the k-graph C. Section 6.
- $\mathcal{G}(C)$ the groupoid associated with the k-graph C. Section 6.

2. Constructing groups from suitable categories

In this section, we shall show how to construct a group from a category under certain assumptions on the category. Later in this paper, we shall specialize the results of this section to those categories that arise as higher rank graphs.

2.1. **Background on inverse semigroups.** We shall construct our groups using partial bijections and so our work is rooted in inverse semigroup theory. We refer the reader to [10] for background on such semigroups but we recall some important definitions here. An *inverse semigroup* is a semigroup in which for each element a there is a unique element, denoted by a^{-1}, such that $a = aa^{-1}a$ and $a^{-1} = a^{-1}aa^{-1}$. The partial isometries of a C^*-algebra *almost* form an inverse semigroup (see [10, Section 4.2]) and, perhaps for this reason, inverse semigroups have come to play an important role in the theory of C^*-algebras. The set of idempotents in S is denoted by $\mathsf{E}(S)$. It is called the *semilattice of idempotents* of S. Observe that if e is any idempotent and a is any element then aea^{-1} is an idempotent. Thus the 'conjugate of an idempotent is an idempotent'. If S is an inverse *monoid* its group of units is denoted by $\mathsf{U}(S)$. Define $\mathbf{d}(a) = a^{-1}a$ and $\mathbf{r}(a) = aa^{-1}$. Define the *natural partial order* on S by $a \leq b$ if and only if $a = ba^{-1}a$. It can be proved that with respect to this order, an inverse semigroup is partially ordered but observe that $a \leq b$ implies that $a^{-1} \leq b^{-1}$. An inverse semigroup is called *E-unitary* if $e \leq a$, where e is an idempotent, implies that a is an idempotent. Define the *compatibility relation* $a \sim b$ precisely when $a^{-1}b$ and ab^{-1} are both idempotents; this relation is reflexive and symmetric, but not transitive. If $a \sim b$ we say that a and b are *compatible.* A non-empty subset X of an inverse semigroup is said to be *compatible* if each pair of elements of X is compatible. Observe that if $a, b \leq c$ then $a \sim b$. It follows that $a \sim b$ is a necessary condition for a and b to have a join $a \vee b$ with respect to the natural partial order. The compatibility relation plays an important role in this paper. The following is [10, Lemma 1.4.11] and [10, Lemma 1.4.12].

Lemma 2.1. *In an inverse semigroup, we have the following:*

(1) *$a \sim b$ if and only if all of the following hold: $a \wedge b$ exists, $\mathbf{d}(a \wedge b) = \mathbf{d}(a)\mathbf{d}(b)$, and $\mathbf{r}(a \wedge b) = \mathbf{r}(a)\mathbf{r}(b)$.*
(2) *If $a \sim b$ then $a \wedge b = ab^{-1}b = bb^{-1}a = ba^{-1}a = aa^{-1}b$.*

Observe that the meet $a \wedge b$ may exist without a and b being compatible. An inverse semigroup is called a *$\wedge$-semigroup* if each pair of elements has a meet with respect to the natural partial order. Inverse $\wedge$-semigroups were first studied in [18] and will play an important role in this paper.

Let ρ be a congruence on a semigroup S; it is said to be *idempotent-pure* if $a \,\rho\, e$, where e is an idempotent, implies that a is an idempotent; it is said to be *0-restricted* if if $a \,\rho\, 0$ implies that $a = 0$. Let $\theta \colon S \to T$ be a homomorphism between semigroups; it is said to be *0-restricted* if $\theta(a)$ is zero if and only if a is zero; it is said to be *idempotent-pure* if $\theta(a)$ is an idempotent if and only if a is an idempotent. The proofs of the following are straightforward from the definitions.

Lemma 2.2. *Let S be an inverse semigroup:*

(1) *The congruence ρ is idempotent-pure if and only if $\rho \subseteq \sim$.*
(2) *The homomorphism $\theta \colon S \to T$ is idempotent-pure if and only if $\theta(a) \sim \theta(b)$ implies that $a \sim b$.*

If (X, τ) is a topological space, then the set $I(X, \tau)$ of all homeomorphisms between the open subsets of X is an inverse monoid. The elements of $I(X, \tau)$ are called *partial homeomorphisms.* If τ is the discrete topology we just write $I(X)$ instead of $I(X, \tau)$ and call it the *symmetric inverse monoid* on X. If A is a subset of X then the identity function defined on A is denoted by 1_A.

One way, of course, to construct a group from an inverse monoid is to consider its group of units. We now consider an alternative approach. Let S be an inverse semigroup. There is a congruence σ defined on S such that S/σ is a group and if ρ is any congruence on S such that S/ρ is a group then $\sigma \subseteq \rho$. Thus σ is the *minimum group congruence.* In fact, $s \,\sigma\, t$ if and only if there exists $z \leq s, t$. See [10, Section 2.4]. The following was proved as [10, Theorem 2.4.6].

Proposition 2.3. *Let S be an inverse semigroup. Then S is E-unitary if and only if $\sigma = \sim$.*

The following is well-known and easy to check.

Lemma 2.4. *Let S be an E-unitary inverse semigroup. Then for $a, b \in S$ we have $a \sim b$ if and only if $ab^{-1}b = ba^{-1}a$.*

Intuitively, the above lemma says that the partial bijections a and b are identified precisely when they agree on the intersection of their domains of definition.

Groups often arise as groups of symmetries but, sometimes, how they arise is more elusive. For example, the *abstract commensurator* of a group G is the set of all isomorphisms between subgroups of finite index factored out by the equivalence that identifies two such isomorphisms if they agree on a subgroup of finite index. This forms a group $\mathrm{Comm}(G)$, called the *abstract commensurator* of G [23]. In fact, this group is best understood using inverse semigroup theory. The set, $\Omega(G)$, of all isomorphisms between subgroups of finite index is an inverse semigroup. The group $\mathrm{Comm}(G)$ is then $\Omega(G)/\sigma$ where σ is the minimum group congruence on $\Omega(G)$. The elements of $\mathrm{Comm}(G)$ are 'hidden symmetries' to use the terminology of Farb and Weinberger [4]. The elements of $\Omega(G)$ are, in some sense, 'large'.

We now describe an analogous procedure to the one described above for constructing a group from an inverse semigroup (of partial bijections). Let S be an inverse semigroup (of partial isomorphisms, for example). Let $S' \subseteq S$ be an inverse subsemigroup whose elements are, in some sense, large; whatever this might mean, we require that S' does not contain a zero. Then we obtain a group S'/σ. We regard the elements of S'/σ as hidden symmetries of the structure that gives rise to S. *We now define what 'large' means in the context of this paper.* A non-zero idempotent e of an inverse semigroup S is said to be *essential* if $ef \neq 0$ for all non-zero idempotents f of S. An element s is said to be *essential* if both $s^{-1}s$ and ss^{-1} are essential. Denote by S^e the set of all essential elements of S. It follows by [11, Lemma 4.2], that S^e is an inverse semigroup (without zero). We therefore expect the group S^e/σ to be interesting. This will be the basis of our construction of a group from an inverse semigroup:

$$\text{inverse semigroup } S \Rightarrow \text{ inverse semigroup of essential elements } S^e \Rightarrow \text{ group } S^e/\sigma.$$

Let S be an inverse semigroup with zero. We write $a \perp b$, and say that a and b are *orthogonal*, if $\mathbf{d}(a)\mathbf{d}(b) = 0$ and $\mathbf{r}(a)\mathbf{r}(b) = 0$. Observe that if e and f are idempotents then $e \perp f$ means precisely that $ef = 0$. If $a \perp b$ and $a \vee b$ exists we speak of an *orthogonal join*.

We need a little notation from the theory of posets. Let $(X, \leq)$ be a poset. If $A \subseteq X$ then $A^{\uparrow}$ is the set of all elements of X above some element of A and $A^{\downarrow}$ is the set of all elements of X below some element of A. If $A = \{a\}$ we write $a^{\uparrow}$ instead of $\{a\}^{\uparrow}$ and $a^{\downarrow}$ instead of $\{a\}^{\downarrow}$. If $A = A^{\downarrow}$ we say that A is an *order ideal*.

2.2. **Constructing a group from a suitable category.** In this section, we show how to construct a group from any finitely aligned category with a finite number of identities. We shall follow the procedure outlined in the previous section by constructing an inverse semigroup $\mathsf{R}(C)$ of bijective morphisms between the finitely generated right ideals of the category C and then building the group from $\mathsf{R}(C)$.

We regard a category as a generalized monoid or 'monoid with many identities'. Thus, the set of identities of the category C, denoted by C_o, is a subset of C and there are two maps $\mathbf{d}, \mathbf{r} \colon C \to C_o$, called, respectively, *domain* and *codomain*. The elements of the category C are *arrows* such that $\mathbf{r}(a) \xleftarrow{a} \mathbf{d}(a)$. If $a, b \in C$ then the product ab is defined if $\mathbf{d}(a) = \mathbf{r}(b)$; in this case, we often write $\exists ab$. A category C is said to be *cancellative* if $ab = ac$ implies that $b = c$, and $ba = ca$ implies that $b = c$. An arrow x is *invertible* if there is an arrow y such that xy and yx are identities. Clearly, every identity is invertible. A category in which the identities are the only invertible arrows is said to be *conical*.

Since categories generalize monoids, we can generalize monoid-theoretic definitions to a category-theoretic setting. If C is a category and $a \in X$ then we can consider the set $aC = \{ax \colon x \in C\}$. We call this the *principal right ideal* generated by a. Observe that $aC \subseteq \mathbf{r}(a)C$. More generally, if $X \subseteq C$ then XC is the *right ideal* of C generated by X; we allow the set X to be empty and so the empty set is counted as a right ideal. We say that a right ideal A is *finitely generated* if there is a finite set $X \subseteq C$ such that $A = XC$.

Lemma 2.5. *Let C be a category. Then C is finitely generated as a right ideal if and only if it has a finite number of identities.*

Proof. Suppose that $C = XC$ where X is a finite set. Let $e \in C_o$. Then $e \in XC$. It follows that $e = xy$ for some $x \in X$ and $y \in Y$. Thus $e = \mathbf{r}(x)$. We have proved that every identity of C is the range of an element of X. But X is a finite set. Thus the number of identities is finite. Conversely, suppose that the number of identities is finite. Then $C = C_oC$ and so C is finitely generated as a right ideal. □

The intersection of two right ideals is always a right ideal but we need something stronger. We say that C is *finitely aligned* if $aC \cap bC$ is always finitely generated (we include the possibility that it is empty).

Lemma 2.6. *Let C be a category. Then the intersection of any two finitely generated right ideals is finitely generated if and only if C is finitely aligned.*

Proof. It is clear that being finitely aligned is a necessary condition, we now show that it is sufficient. Let XC and YC be two finitely generated right ideals. Then $XC \cap YC = \bigcup_{x \in X, y \in Y} (xC \cap yC)$. If C is finitely aligned then each $xC \cap yC$ is finitely generated and so too is $XC \cap YC$. □

A function $\theta \colon R_1 \to R_2$ between two right ideals of a category C is called a *morphism* if $\theta(rs) = \theta(r)s$ for all $r \in R_1$ and $s \in C$. As usual, if α is a bijective morphism then α^{-1} is also a morphism.

Lemma 2.7. *Let $\theta \colon R_1 \to R_2$ be a morphism between two right ideals. Let $XC \subseteq R_1$ be a right ideal. Then $\theta(XC)$ is a right ideal contained in R_2. If XC is finitely generated then $\theta(XC)$ is finitely generated. In particular, if $\theta \colon XC \to YC$ is a bijective morphism, we can always assume that $\theta(X) = Y$.*

Proof. We claim that $\theta(XC)$ is a right ideal. Let $c \in C$ be arbitrary and let $\theta(xa) \in \theta(XC)$. Then $\theta(xa)c = \theta(x(ac))$, since θ is a morphism. We claim that $\theta(XC) = \theta(X)C$. Clearly, $\theta(XC) \subseteq \theta(X)C$ since $\theta(xc) = \theta(x)c$. Conversely, $\theta(x)c = \theta(xc)$, since θ is a morphism. It follows from this that if XC is finitely generated then $\theta(XC)$ is finitely generated. We now prove the last claim. Suppose that $\theta \colon XC \to YC$ is a bijective morphism. Then $\theta(X)C = YC$. Replace Y by $\theta(X)$. □

Definition. Denote by $\mathsf{RI}(C)$ the set of all bijective morphisms between the right ideals of C and denote by $\mathsf{R}(C)$ the set of all bijective morphisms between finitely generated right ideals of C.

Proposition 2.8. *Let C be a category. Then $\mathsf{RI}(C)$ is an inverse monoid. If C is finitely aligned and has a finite number of identities then $\mathsf{R}(C)$ is an inverse submonoid of $\mathsf{RI}(C)$.*

Proof. The whole category C is a right ideal and so the identity function on C is a bijective morphism, and is an identity for $\mathsf{RI}(C)$. The intersection of two right ideals is a right ideal. It follows by Lemma 2.7 that the composition of two bijective morphisms is a bijective morphism. It is now clear that $\mathsf{RI}(C)$ is an inverse monoid. Suppose now that C is finitely aligned and has a finite number of identities. Then by Lemma 2.5, the identity function on C is the identity of $\mathsf{R}(C)$. By Lemma 2.6, the intersection of any two finitely generated right ideals is a finitely generated right ideal. By Lemma 2.7, it is now easy to see that $\mathsf{R}(C)$ is an inverse submonoid of $\mathsf{RI}(C)$. □

Let C be a finitely aligned category with a finite number of identities. We are now interested in the structure of the inverse monoid $\mathsf{R}(C)^e$. We say that a right ideal XC of C is *essential* if it intersects every right ideal of C in a non-empty set. Thus the inverse monoid $\mathsf{R}(C)^e$ consists of bijective morphisms between the finitely generated essential right ideals of C. The following result tells us that if we want $\mathsf{R}(C)^e$ to be non-empty then we must assume that C has a finite number of identities.

Lemma 2.9. *Let C be a category. If XC is a finitely generated essential right ideal of C then for each $e \in C_o$, there exists an $x \in X$ such that $\exists ex$. It follows that the number of identities is finite.*

Proof. Let $e \in C_o$. Then $eC \cap XC$ must be non-empty. It follows that there exists $x \in X$ such that $eu = xv$ for some $u, v \in C$. Thus $e = \mathbf{r}(x)$. We have proved that each identity in C is the range of an element of X. But X is a finite set. It follows that the number of identities is finite. □

Lemma 2.10. *Let C be a finitely aligned category. Then 1_{XC} is an essential idempotent in $\mathsf{R}(C)^e$ if and only if for each element $a \in C$ we have that $aC \cap XC \neq \varnothing$.*

Proof. Suppose that 1_{XC} is an essential idempotent in $\mathsf{R}(C)^e$. Then $1_{aC}1_{XC} \neq \varnothing$. But this is simply the identity function on the set $aC \cap XC$. It follows that $aC \cap XC \neq \varnothing$. The proof of the converse is similar. □

We shall need the following definitions. Let C be a category and let $a, b \in C$. We say that a and b are *dependent* if $aC \cap bC \neq \varnothing$; otherwise, we say that they are *independent.*[1] If $aC \cap bC \neq \varnothing$ then we also say that *a is dependent on b* or that *b is dependent on a*. A subset $X \subseteq C$ is said to be *large in C* if each $a \in C$ is dependent on an element of X. A subset $X \subseteq aC$ is said to be *large in aC* if each $b \in aC$ is dependent on an element of X.

Remark 2.11. What we call a 'large' subset of aC is called 'exhaustive' in, say, [20]

Lemma 2.12. *Let C be a category. Then X is large if and only if XC is essential.*

Proof. Suppose that X is large. We prove that XC is essential. Consider a principal right ideal aC. Then $au = xv$ for some $x \in X$ and $u, v \in C$. Thus $aC \cap xC \neq \varnothing$. It follows that XC is essential. Suppose that XC is essential. Let $a \in C$ be arbitrary. Then $aC \cap XC \neq \varnothing$. Thus $au = xv$ for some $x \in X$ and $u, v \in C$. It follows that X is large. □

We can now define the group that we shall be interested in. Recall from just before Proposition 2.3 that the minimal group congruence on $\mathsf{R}(C)^e$ is the congruence such that $x \,\sigma\, y$ iff there exists $z \leq x, y$.

Definition. Let C be a finitely aligned category with a finite number of identities. Then

$$\mathcal{G}(C) = \mathsf{R}(C)^e/\sigma$$

is the *group associated with C*.

[1] The terms 'comparable' and 'incomparable' were used in [17]. Strictly speaking, we should say 'dependent on the right' and 'independent on the right' but we only work 'on the right' in this paper, anyway.

2.3. **The cancellative case.** We shall now revisit the construction of the previous section under the additional assumption that C is both cancellative and conical.

Lemma 2.13. *Let C be a category that is conical and cancellative. Then $aC = bC$ if and only if $a = b$.*

Proof. Suppose that $aC = bC$. Then $a = bx$ and $b = ay$ for some $x, y \in C$. Thus $a = ayx$ and $b = bxy$. By cancellation xy and yx are identities. This implies x and y are invertible. But C is conical. Thus x and y are identities. It follows that $a = b$. The converse is immediate. ☐

The above result tells us that when the category is conical and cancellative, we can identify principal right ideals by the unique elements that generate them.

We begin by constructing some special elements of the inverse monoid $\mathsf{R}(C)$ in the case where C is cancellative. Let $a \in C$. Define $\lambda_a : \mathbf{d}(a)C \to aC$ by $\mathbf{d}(a)C \ni x \mapsto ax$. It is easy to check, given C is cancellative, that λ_a is a bijection. We denote the inverse of this map by λ_a^{-1}. These maps are elements of the symmetric inverse monoid $I(C)$ and so generate an inverse subsemigroup $\Sigma(C)$ we call the *inverse hull* of C. A product of the form $\lambda_a\lambda_b^{-1}$ is the empty function unless $\mathbf{d}(a)(C) \cap \mathbf{d}(b)C \neq \varnothing$. But this only occurs if $\mathbf{d}(a) = \mathbf{d}(b)$. We therefore need $\mathbf{d}(a) = \mathbf{d}(b)$ in order for $\lambda_a\lambda_b^{-1} \colon bC \to aC$ to be an honest-to-goodness function. We describe the properties of this function. Suppose that $(\lambda_a\lambda_b^{-1})(x) = (\lambda_a\lambda_b^{-1})(y)$ where $x, y \in bC$. Then $x = be$ and $y = bf$ for some $e, f \in C$. Then $(\lambda_a\lambda_b^{-1})(x) = ae$ and $(\lambda_a\lambda_b^{-1})(y) = af$. By assumption, $ae = af$. Thus by left cancellation $e = f$ and so $be = bf$ giving $x = y$. It follows that $\lambda_a\lambda_b^{-1}$ is injective. Now, let $ax \in aC$ be arbitrary. Then $\mathbf{d}(a) = \mathbf{r}(x)$. But $\mathbf{d}(a) = \mathbf{d}(b)$. It follows that bx is defined. It is clear that $(\lambda_a\lambda_b^{-1})(bx) = ax$. We have therefore proved that $\lambda_a\lambda_b^{-1}$ is a bijection. Now, let $bx \in bC$. Then for any $s \in C$ we have that $(\lambda_a\lambda_b^{-1})(bxs) = axs$. Thus $(\lambda_a\lambda_b^{-1})((bx)s) = (\lambda_a\lambda_b^{-1})(bx)s$. We have proved explicitly that $\lambda_a\lambda_b^{-1}$ is a bijective morphism.

Definition. We work in a conical, cancellative category. Let $\mathbf{d}(a) = \mathbf{d}(b)$. Then ab^{-1} is the bijective morphism from bC to aC given by $bx \mapsto ax$. We call this a *basic morphism*. Observe that $\mathbf{d}(ab^{-1}) = bb^{-1}$ and $\mathbf{r}(ab^{-1}) = aa^{-1}$.

In general, the inverse hull of a cancellative category is hard to describe but under the assumption that C is finitely aligned the product of two basic morphisms can be explicitly computed. Note that, in the specific instance of higher-rank grpahs, our basic morphisms are closely related to the building blocks of the inverse semigroup constructed in [6].

Lemma 2.14. *Let C be a finitely aligned conical cancellative category with a finite number of identities. Suppose that $(\lambda_a\lambda_b^{-1})(\lambda_c\lambda_d^{-1})$ is non-empty and that $bC \cap cC = \{x_1, \ldots, x_m\}C$. Then*

$$(\lambda_a\lambda_b^{-1})(\lambda_c\lambda_d^{-1}) = \bigcup_{i=1}^{n} \lambda_{ap_i}\lambda_{dq_i}^{-1}$$

where $x_i = bp_i = cq_i$ where $1 \leq i \leq m$.

Proof. We have to calculate the product $aC \leftarrow bC \colon \lambda_a\lambda_b^{-1}$ with $cC \leftarrow dC \colon \lambda_c\lambda_d^{-1}$. Let $bC \cap cC = \{x_1, \ldots, x_m\}C$ and $e = \mathbf{r}(b) = \mathbf{r}(c)$. Then $\{x_1, \ldots, x_m\}C \subseteq eC$. Let $x_i = bp_i = cq_i$ where $1 \leq i \leq m$. Observe that $\mathbf{d}(a) = \mathbf{d}(b) = \mathbf{r}(p_i)$ and so ap_i is defined. Similarly, $\mathbf{d}(d) = \mathbf{d}(c) = \mathbf{r}(q_i)$ and so dq_i is defined. Observe that $\mathbf{d}(x_i) = \mathbf{d}(p_i) = \mathbf{d}(q_i)$. Thus the product $\lambda_{ap_i}\lambda_{dq_i}^{-1}$ is non-empty. Observe that $\lambda_{ap_i}\lambda_{dq_i}^{-1}$ and $\lambda_{ap_j}\lambda_{dq_j}^{-1}$ are compatible (when $i \neq j$); this is easily checked by calculating $(\lambda_{ap_i}\lambda_{dq_i}^{-1})^{-1}\lambda_{ap_j}\lambda_{dq_j}^{-1}$ and $\lambda_{ap_i}\lambda_{dq_i}^{-1}(\lambda_{ap_j}\lambda_{dq_j}^{-1})^{-1}$ and showing that both are idempotents. To prove that

$$(\lambda_a\lambda_b^{-1})(\lambda_c\lambda_d^{-1}) = \bigcup_{i=1}^{n} \lambda_{ap_i}\lambda_{dq_i}^{-1}$$

it is enough, by symmetry, to check that both left and right hand sides have the same domains and that the maps do the same thing, both of which are routine. □

In the case where C is a finitely aligned cancellative category, the inverse hull $\Sigma(C)$ is an inverse submonoid of $\mathsf{R}(C)$, but it is much easier to work with the latter than the former; we describe the mathematical relationship between them below. Lemma 2.15, Lemma 2.16 and Lemma 2.17 show the important role played by the basic morphisms in the inverse monoid $\mathsf{R}(C)$.

Lemma 2.15. *Let C be a finitely aligned conical cancellative category with a finite number of identities. Let $\theta\colon XC \to YC$ be a bijective morphism between two finitely generated right ideals of C. Then θ is a join of a finite number of basic morphisms.*

Proof. We may assume that θ induces a bijection between X and Y by Lemma 2.7. Let $x \in X$ and let $y_x = \theta(x)$. Observe that $\mathbf{d}(x) = \mathbf{d}(y_x)$ by using right identities. We may therefore form the basic morphism $y_x x^{-1}$. We claim that $\theta = \bigvee_{x\in X} y_x x^{-1}$. Let $xc \in XC$ Then $\theta(xc) = \theta(x)c = y_x c$. But $(y_x x^{-1})(xc) = y_x c$. □

Lemma 2.16. *Let C be a finitely aligned conical cancellative category with a finite number of identities.*

1. *$xy^{-1} \leq uv^{-1}$ if and only if $(x, y) = (us, vs)$ for some $s \in C$. It follows that if xy^{-1} is an idempotent so too is uv^{-1}.*
2. *$xx^{-1} \perp yy^{-1}$ if and only if x and y are incomparable in C.*
3. *Let $xC \cap yC = UC$ where U is a finite set. Then $xx^{-1}yy^{-1} = \bigvee_{u\in U} uu^{-1}$.*

Proof. (1) By the definition of the order on partial functions, we have that $yC \subseteq vC$ and $xC \subseteq uC$. In addition, xy^{-1} and uv^{-1} agree on elements of yC. We have that $y = va$ and $x = ub$. Now, $(xy^{-1})(y) = x$. But $(uv^{-1})(y) = ua$. It follows that $x = ua$ and so $ub = ua$ and so $a = b$. The result now follows with $s = a = b$. In order that xy^{-1} be an idempotent, we must have that $x = y$. It is therefore immediate that if xy^{-1} is an idempotent then so too is uv^{-1}.

(2) The idempotents xx^{-1} and yy^{-1} are orthogonal if and only if $xC \cap yC = \varnothing$. But this is equivalent to saying that x and y are incomparable.

(3) The product of xx^{-1} and yy^{-1} is the identity function on $xC \cap yC$ which is the identity function on UC. □

The following is a key property since it shows how the basic morphisms sit inside the inverse monoid $\mathsf{R}(C)$.

Lemma 2.17. *Let C be a finitely aligned conical cancellative category with a finite number of identities. Let $xy^{-1} \leq \bigvee_{j=1}^{n} u_j v_j^{-1}$ then $xy^{-1} \leq u_j v_j^{-1}$ for some j.*

Proof. Observe that $yC \subseteq \{v_1, \ldots, v_n\}C$. Thus $y = v_j p$ for some j and $p \in C$. Now $(xy^{-1})(y) = x$ whereas $(u_j v_j^{-1}) = u_j p$. It follows that $xy^{-1} \leq u_j v_j^{-1}$. □

An inverse semigroup is said to be *distributive* if each pair of compatible elements has a join and multiplication distributes over such joins.

Proposition 2.18. *Let C be a finitely aligned conical cancellative category with a finite number of identities. Then $\mathsf{R}(C)$ is a distributive inverse $\wedge$-monoid.*

Proof. By proposition 2.8, we know that $\mathsf{R}(C)$ is an inverse monoid. It is distributive because Lemma 2.6 the union of two finitely generated right ideals is a finitely generated right ideal. We now prove that $\mathsf{R}(C)$ has all binary meets. We use [18, Theorem 1.9]. Let $\theta \colon XC \to YC$ be a bijective morphism. We may assume that $\theta(X) = Y$ by Lemma 2.7. We are interested in the fixed-point set of θ. Then $\theta(xc) = xc$ if and only if $\theta(x)c = xc$. It follows by right cancellation that $\theta(x) = x$. Define $X' \subseteq X$ to be those elements $x \in X$ such that $\theta(x) = x$. Then the fixed-point set of θ is the finitely generated right ideal $X'C$. □

A *morphism* of distributive inverse semigroups is a homomorphism that preserves compatible joins. Let S be an inverse semigroup. We say that T, a distributive inverse semigroup, is the *distributive completion* of S, if there is a homomorphism $\iota \colon S \to T$ such that for any homomorphism $\alpha \colon S \to D$, to a distributive inverse semigroup D, there is a morphism $\beta \colon T \to D$ such that $\alpha = \beta\iota$. The exact mathematical relationship between $\Sigma(C)$ and $\mathsf{R}(C)$ can now be spelled out.

Proposition 2.19. *Let C be a finitely aligned conical cancellative category with a finite number of identities. Then $\mathsf{R}(C)$ is the distributive completion of $\Sigma(C)$.*

Proof. Let $\alpha \colon \Sigma(C) \to D$ be any homomorphism to a distributive inverse monoid D. Define $\beta(\bigvee_{j=1}^{n} u_j v_j^{-1}) = \bigvee_{j=1}^{n} \alpha(u_j v_j^{-1})$. We need to check that this is well-defined. Suppose that

$$\bigvee_{j=1}^{n} u_j v_j^{-1} = \bigvee_{i=1}^{m} a_i b_i^{-1}$$

in $\mathsf{R}(C)$. Then by Lemma 2.17, for each j there exists an i such that $u_j v_j^{-1} \leq a_i b_i^{-1}$. Thus $\alpha(u_j v_j^{-1}) \leq \alpha(a_i b_i^{-1})$. From this result, and symmetry, the well-definedness of β follows. That β is a homomorphism is immediate by Lemma 2.14 and the fact that α is a homomorphism. □

The following is important in the construction of the associated group.

Proposition 2.20. *Let C be a finitely aligned conical cancellative category with a finite number of identities. Then the inverse monoid $\mathsf{R}(C)^e$ is E-unitary.*

Proof. Let $\alpha \colon XC \to YC$ be a bijective morphism between two finitely generated essential right ideals. Suppose that α is the identity when restricted to the finitely generated essential right ideal ZC where $ZC \subseteq XC$. Let $xa \in XC$. Then, since ZC is an essential right ideal, we have that $xaC \cap ZC \neq \varnothing$. It follows that $xab = zc$ for some $b, c \in C$. But $\alpha(zc) = zc$ and so $\alpha(xab) = xab$. But α is a morphism and so $\alpha(xab) = \alpha(x)ab$. By cancellation, it follows that $\alpha(x) = x$. We have therefore proved that α is the identity on X and so α is also an idempotent. □

Suppose that C is a finitely aligned conical cancellative category with a finite number of identities. Then by Proposition 2.20, Proposition 2.3 and Lemma 2.4, we can say that two elements of $\mathsf{R}(C)^e$ are identified under σ if they agree on the intersection of their domains of definition. This process for constructing a group from an inverse semigroup of partial bijections is identical to the one used in [1] though our group is quite different from the one defined in [1].

The following result summarizes some important properties of the distributive inverse $\wedge$-monoid $\mathsf{R}(C)$. It uses Proposition 2.18, Lemma 2.17, Lemma 2.16.

Proposition 2.21. *Let C be a finitely aligned conical cancellative category with a finite number of identities. Put $\mathcal{B}$ equal to the set of basic morphisms in $\mathsf{R}(C)$. Then the following three properties hold:*

(1) *Each element of $\mathsf{R}(C)$ is a join of a finite number of elements of $\mathcal{B}$.*

(2) *If $e, a \in \mathcal{B}$, e is a non-zero idempotent and $e \leq a$ then a is an idempotent.*
(3) *If $a \leq \bigvee_{i=1}^{m} b_i$ where $a, b_i \in \mathcal{B}$ then $a \leq b_i$ for some i.*

Remark 2.22. We can see that the reason $\mathsf{R}(C)$ has all binary meets is due to property (2) above. To see why, we carry out a small calculation first. Let $a = \bigvee_{i=1}^{m} a_i$, where $a_i \in \mathcal{B}$, be an arbitrary element of $S = \mathsf{R}(C)$. Let $e = \bigvee_{j=1}^{n} e_j$, where $e_i \in \mathcal{B}$, be any idempotent such that $e \leq a$. Observe that for each j we have that $e_j \leq e$. It follows by property (3), that for each j there exists an i such that $e_j \leq a_i$. By property (2), it follows that a_i is an idempotent. Now, we may split the join $a = \bigvee_{i=1}^{m} a_i$ into two parts so that $a = (\bigvee_{k=1}^{p} b_k) \vee (\bigvee_{l=1}^{q} f_l)$ where $b_k, f_l \in \mathcal{B}$ and all the f_l are idempotents and none of the b_k is an idempotent. It follows by our calculations above, that $\bigvee_{l=1}^{q} f_l$ is the largest idempotent less than or equal to a. This proves that S is a $\wedge$-semigroup by [18].

2.4. **The group described in terms of maximal codes.** From a finitely aligned conical cancellative category C with a finite number of identities we have constructed a group $\mathcal{G}(C)$ but we have little idea about the structure of this group. To gain a better handle on this structure, we need to make further assumptions on the category C. First of all, we shall need to strengthen the notion of finite alignment.

A *code* is a finite subset $X \subseteq C$ such that any two distinct elements are independent. Observe that 'finiteness' is part of the definition of a code in this paper. A *maximal code* is a large code. Given an identify e of C, a *code in e* is a finite subset $X \subseteq eC$ such that any two distinct elements are independent. A *maximal code in e* is a code in e which is large in eC.

We now have the following refinement of the notion of a category's being finitely aligned.

Definition. We say that a category C is *strongly finitely aligned* if the set $xC \cap yC$, when non-empty, is finitely generated by independent elements; thus $xC \cap yC$ is generated by a code.

We now strengthen Lemma 2.13.

Lemma 2.23. *Let C be a conical cancellative category. Let U and V be codes such that $UC = VC$. Then $U = V$.*

Proof. Let $u \in U$. Then $u = va$ for some $v \in V$ and $a \in C$. Let $v = u'b$ for some $u' \in U$ and $b \in C$. Then $u = va = u'ba$. But U is a code and so $u = u'$. By cancellation, ba is an identity. Similarly, ab is an identity. But C is conical and so a and b are identities. We have therefore proved that $U \subseteq V$. By symmetry, $V \subseteq U$ and so $U = V$ as claimed. □

Let C be a strongly finitely aligned cancellative category. Let $x, y \in C$. Define $x \vee y$ to be the finite subset of C such that $xC \cap yC = (x \vee y)C$. We assume that $x \vee y$ is a code.

Lemma 2.24. *Let C be a strongly finitely aligned conical cancellative category with a finite number of identities. Let XC and YC be finitely generated right ideals both generated by codes. Then $XC \cap YC$ is either empty or a finitely generated right ideal generated by a code.*

Proof. We have that $XC \cap YC = \bigcup_{x \in X, y \in Y} (x \vee y)X$. Thus $XC \cap YC$ is certainly finitely generated. We prove that the set $Z = \bigcup_{x \in X, y \in Y} x \vee y$ is a code. Let $a \in x_i \vee y_j$ and $b \in x_k \vee y_l$ where $x_i, x_k \in X$ and $y_j, y_l \in Y$. Suppose that a and b are dependent. Then $z = au = bv$ for some $u, v \in C$. But $a \in x_iC \cap y_jC$ and $b \in x_kC \cap y_lC$. It follows that $z \in x_iC \cap y_jC \cap x_kC \cap y_lC$. Thus $x = x_i = x_k$, since

$x_i, x_k \in X$ and X is a code and $y = y_j = y_l$, since $y_j, y_l \in Y$ and Y is a code. It follows that $a, b \in x \vee y$. But $a, b \in x \vee y$ and $x \vee y$ is a code and so $a = b$. □

Definition. Let C be a strongly finitely aligned cancellative category with a finite number of identities. Define $\mathsf{P}(C)$ to be the set of all bijective morphisms between finitely generated right ideals generated by codes.

Lemma 2.25. *Let C be a strongly finitely aligned conical cancellative category with a finite number of identities. Then $\mathsf{P}(C)$ is an inverse subsemigroup of $\mathsf{R}(C)$.*

Proof. By Lemma 2.24, the intersection of any two finitely generated right ideals generated by codes is itself a finitely generated right ideal generated by a code. Let $\alpha \colon XC \to YC$ be a bijective morphism between two finitely generated right ideals. Let $ZC \subseteq XC$ be a finitely generated right ideal generated by a code. We prove that $\alpha(Z)$ is also a code. Suppose that $\alpha(z)$ and $\alpha(z')$ are dependent for some $z, z' \in Z$. Then $\alpha(z)u = \alpha(z')v$ for some $u, v \in C$. Then $\alpha(zu) = \alpha(z'v)$ since α is a morphism. Thus $zu = z'v$ since α is a bijection. But Z is a code and so $z = z'$. It follows that $\alpha(z) = \alpha(z')$. We have therefore proved that $\alpha(Z)$ is a code. It follows that α restricts to a bijective morphism $ZC \to \alpha(Z)C$. The fact that $\mathsf{P}(S)$ is an inverse subsemigroup of $\mathsf{R}(S)$ is now immediate, □

The elements of $\mathsf{P}(C)^e$ are (using Lemma 2.12) the bijective morphisms between the right ideals of C generated by maximal codes.

Lemma 2.26. *Let C be a strongly finitely aligned conical cancellative category with a finite number of identities. Then $\mathsf{P}(C)^e$ is an inverse submonoid of $\mathsf{R}(C)^e$.*

Proof. We prove first that $\mathsf{P}(C)^e$ is contained in $\mathsf{R}(C)^e$. Observe that an idempotent in $\mathsf{P}(C)^e$ is an identity function defined on a finitely generated right ideal of C generated by a code which intersects every finitely generated right ideal of C generated by a code. In particular, it intersects principal right ideals of C. It is now clear that $\mathsf{P}(C)^e$ is contained in $\mathsf{R}(C)^e$. The composition in $\mathsf{P}(C)^e$ is just the restriction of the composition in $\mathsf{R}(C)^e$. □

Condition (MC). Let C be a strongly finitely aligned conical cancellative category with a finite number of identities. We assume that if XC is any finitely generated essential right ideal then there is a $YC \subseteq XC$ where Y is a maximal code.

Lemma 2.27. *Let C be a strongly finitely aligned conical cancellative category with a finite number of identities satisfying condition (MC). Then each element of $\mathsf{R}(S)^e$ is above an element of $\mathsf{P}(S)^e$.*

Proof. The condition (MC) is essentially a condition on idempotents. Let $\alpha \colon XC \to YC$ be a bijective morphism between two finitely generated essential right ideals; thus $\alpha \in \mathsf{R}(C)^e$. We assume that $\alpha(X) = Y$ by Lemma 2.7. Let $ZC \subseteq XC$ be a finitely generated right ideal generated by a maximal code. We prove that $\alpha(Z)$ is also a maximal code. To do this, we need to show that every element of C is dependent on an element of $\alpha(Z)$. Let $a \in C$. Since YC is an essential right ideal we have that $au = yv$ for some $u, v \in C$. Now $y \in Y$ and so there is an $x \in X$ such that $\alpha(x) = y$. It follows that $au = \alpha(x)v$. Thus $au = \alpha(xv)$. Now, Z is a maximal code. It follows that $xvp = zs$ for $p, s \in C$. Thus $\alpha(x)vp = \alpha(z)s$. Hence, $aup = \alpha(x)vp\alpha(z)s$. We have proved that $a \in C$ is dependent on an element of $\alpha(Z)$. It follows that $\alpha(Z)$ is a maximal code. □

The following was proved as [17, Lemma 7.10].

Lemma 2.28. *Let S be an inverse subsemigroup of an inverse semigroup T. Suppose that each element of T lies above an element of S in the natural partial order. Then $S/\sigma \cong T/\sigma$.*

On the strength of Lemma 2.27 and Lemma 2.28, we have proved the following.

Theorem 2.29. *Let C be a strongly finitely aligned conical cancellative category with a finite number of identities satisfying condition (MC). Then*

$$\mathsf{P}(C)^e/\sigma \cong \mathsf{R}(C)^e/\sigma.$$

A typical element of $\mathsf{P}(C)^e$ is a bijective morphism $\theta\colon XC \to YC$ where X and Y are maximal codes. We may assume that $\theta(X) = Y$. Then $\theta = \bigvee_{x \in X} \theta(x)x^{-1}$ by Lemma 2.15. By Lemma 2.16, this is an orthogonal join.

3. The group associated with a higher rank graph

The goal of this section is to show how to construct the group $\mathcal{G}(C)$, described in Section 2, in the case where C is a higher rank graph (under some suitable assumptions on C). Higher-rank graphs were introduced in [9] as combinatorial models for the systems of matrices, and associated C^*-algebras, studied in [25]. The following definition comes from [9].

Definition. A countable category C is said to be a *higher rank graph* or a *k-graph* if there is a functor $d\colon C \to \mathbb{N}^k$, called the *degree map*, satisfying the *unique factorization property* (UFP): if $d(a) = \mathbf{m} + \mathbf{n}$ then there are unique elements a_1 and a_2 in C such that $a = a_1a_2$ where $d(a_1) = \mathbf{m}$ and $d(a_2) = \mathbf{n}$. We call $d(x)$ the *degree* of x. A *morphism* of k-graphs is a degree-preserving functor.

Repeated applications of the UFP show that if C is a k-graph and $a \in C$, and if $0 \le \mathbf{m} \le \mathbf{n} \le d(a)$, then there is a unique factorisation $a = a'a''a'''$ such that $d(a') = \mathbf{m}$, $d(a'') = \mathbf{n} - \mathbf{m}$ and $d(a''') = d(a) - \mathbf{n}$. We define $a[\mathbf{m}, \mathbf{n}] := a''$. Again, the UFP implies that for any $0 \le \mathbf{m}_1 \le \mathbf{m}_2 \le \cdots \le \mathbf{m}_l \le d(a)$, we have

$$a = a[0, \mathbf{m}_1]a[\mathbf{m}_1, \mathbf{m}_2] \cdots a[\mathbf{m}_{l-1}, \mathbf{m}_l]a[\mathbf{m}_l, d(a)]. \tag{1}$$

It follows, for example, that if $a = bc$ and $\mathbf{m} \le d(b)$, then $a[0, \mathbf{m}] = b[0, \mathbf{m}]$.

By [9, Remarks 1.2], we have the following. All are easy to prove directly.

Lemma 3.1. *Let C be a k-graph.*

1. *C is cancellative.*
2. *C is conical.*
3. *The elements of C of degree $\mathbf{0}$ are precisely the identities.*

The following two definitions are important.

Definition. We say that C has *no sources* if for each identity e of C and element $\mathbf{m} \in \mathbb{N}^k$ there exists an arrow $x \in C$ such that $\mathbf{r}(x) = e$ and $d(x) = \mathbf{m}$. We say that C is *row finite* if for each identity e of C, the number of elements of eC of degree $\mathbf{m}$ is finite.

We shall now derive some properties of higher rank graphs that will be important later. The following is proved by a simple application of the UFP. It generalizes [17, Lemma 3.10].

Lemma 3.2. *Let $d\colon C \to \mathbb{N}^k$ be a k-graph. Suppose that $xy = uv$ where $d(x) \ge d(u)$. Then there exists an element $t \in C$ such that $x = ut$ and $v = ty$. In particular, if $d(x) = d(u)$ then $x = u$.*

The above lemma proves the following.

Lemma 3.3. *Let $d\colon C \to \mathbb{N}^k$ be a k-graph. If $a \neq b$ and $d(a) = d(b)$ then a and b are independent.*

Proof. Suppose that $ax = by$ for some $x, y \in C$. Then by Lemma 3.2, we have that $a = b$ which contradicts our assumption. □

By the above lemma, any finite set of elements of C in which all elements have the same degree is independent and so forms a code.

Let $\mathbf{m} \in \mathbb{N}^k$. Define $C_{\mathbf{m}}$ to be the subset of C which consists of all elements of degree $\mathbf{m}$.

Lemma 3.4. *Let $d\colon C \to \mathbb{N}^k$ be a k-graph with no sources. Then $C_{\mathbf{m}}$ is a maximal code.*

Proof. By Lemma 3.3, $C_{\mathbf{m}}$ is a code. We prove that it is maximal. Let $a \in C$ be arbitrary. Since C is assumed to have no sources, there is an element x such that ax is defined and $d(x) = \mathbf{m}$. It follows that $d(ax) \geq \mathbf{m}$. By the UFP, we can write $ax = by$ where $d(b) = \mathbf{m}$. This proves the claim. □

Lemma 3.5. *Let C be a k-graph with no sources, let $\mathbf{m} \in \mathbb{N}^k$ and let e be an identity. Then the set $eC_{\mathbf{m}}$ of all elements of C with range e and degree $\mathbf{m}$ is a maximal code in eC.*

Proof. Fix $c \in eC$. By Lemma 3.4, there exists $a \in C_{\mathbf{m}}$ such that $a \vee c$ exists, say $aa' = a \vee c = cc'$. Hence $\mathbf{r}(a) = \mathbf{r}(aa') = \mathbf{r}(cc') = \mathbf{r}(c) = e$, so $a \in eC$. □

We can now show that condition (MC) holds for higher rank graphs.

Lemma 3.6. *Let $d\colon C \to \mathbb{N}^k$ be a k-graph with no sources. Let XC be an essential finitely generated right ideal. Put $\mathbf{m} = \bigvee_{x \in X} d(x)$. Then $C_{\mathbf{m}}C \subseteq XC$. Thus, every finitely generated essential right ideal contains a right ideal generated by a maximal code.*

Proof. Let $y \in C_{\mathbf{m}}$. We are assuming that XC is an essential right ideal and so X is a large subset by Lemma 2.12. It follows that $ya = xb$ for some $x \in X$ and $a, b \in C$. But $d(y) \geq d(x)$. Thus by Lemma 3.2, there is $t \in C$ such that $y = xt$. We have therefore proved that $C_{\mathbf{m}}C \subseteq XC$. □

The following result is well-known but we prove it for the sake of completeness.

Lemma 3.7. *Let $d\colon C \to \mathbb{N}^k$ be a k-graph and suppose that $a, b, c \in C$ satisfy $cC \subseteq aC, bC$. Then there is an element $e \in C$ such that $cC \subseteq eC \subseteq aC \cap bC$ where $d(e) = d(a) \vee d(b)$.*

Proof. By assumption, $c = au = bv$ for some $u, v \in C$. Then $d(c) = d(a) + d(u)$ and $d(c) = d(b) + d(v)$. It follows that $d(a), d(b) \leq d(c)$. Thus $d(a) \vee d(b) \leq d(c)$ (in the lattice-orderd group $\mathbb{Z}^k$). Hence, in the notation of Equation 1, $e := c[0, d(a) \vee d(b)] \in C_{d(a) \vee d(b)}$ and $f := c[d(a) \vee d(b), d(c)]$ satisfy $c = ef$. Hence $cC = efC \subseteq eC$. Since $au = c = ef = e[0, d(a)]e[d(a), d(e)]f$, the UFP forces $e[0, d(a)] = a$. Hence $eC = ae[d(a), d(e)]C \subseteq aC$. A symmetric argument gives $eC \subseteq bC$. □

Let $a, b \in C$. We define the notation $a \vee b$ (it will agree with the notation introduced in Section 2.4). If a and b are independent, define $a \vee b = \varnothing$. Otherwise, $a \vee b$ consists of all elements $e \in aC \cap bC$ such that $d(e) = d(a) \vee d(b)$.

Lemma 3.8. *Let $d\colon C \to \mathbb{N}^k$ be a k-graph. Then $aC \cap bC = \bigcup_{x \in a \vee b} xC$.*

Proof. Without loss of generality, we assume that $a \vee b$ is non-empty. Let $y \in aC \cap bC$. Then by Lemma 3.7, we have that $y = ex$ for some $e \in a \vee b$. It follows that the LHS is contained in the RHS. The reverse inclusion is immediate. □

The following is immediate from the definitions and Lemma 3.8.

Lemma 3.9. *A higher rank graph that is row finite is finitely aligned.*

The following is immediate by Lemma 3.3 and shows that the categories underlying higher rank graphs are in fact strongly finitely aligned.

Lemma 3.10. *Let $d \colon C \to \mathbb{N}^k$ be a k-graph. Let $a, b \in C$. If $a \vee b$ is non-empty and finite it is a code.*

Note that the set $a \vee b$ is precisely the set $\mathrm{MCE}(a, b)$ of [24].

Example 3.11. This example is well-known but it is included for completeness. Let G be a finite directed graph. Denote by G^* the free category generated by G. This consists of all finite allowable strings of elements of G with the identities being identified with the vertices of G. Let $x \in G^*$. Then $x = x_1 \dots x_n$ where $x_1, \dots, x_n$ are edges of G so that the edge x_i begins where the edge x_{i+1} ends. Define $\mathbf{d}(x)$ to be the identity at the source of x_n and define $\mathbf{r}(x)$ to be the identity at the target of x_1. Thus, in particular, G^* has a finite number of identities. Suppose that $xG^* \cap yG^*$ is non-empty. Then $\mathbf{r}(x) = \mathbf{r}(y)$. There are therefore three possibilities: $x = y$ or $x = yu$ for some $u \in G^*$ or $y = xu$ for some $u \in G^*$. In the first case, $xG^* = yG^*$, in the second case, $xG^* \subseteq yG^*$, and in the third case, $yG^* \subseteq xG^*$. It follows that $xG^* \cap yG^*$ is either empty or a principal right ideal. It follows that G^* is finitely aligned. See [8] for more on such categories.

Proposition 3.12. *The 1-graphs are precisely the countable free categories. There are a finite number of identities and row finiteness holds precisely when the free category is generated by a finite directed graph.*

Proof. Example 1.3 of [9] gives the first statement. If G is finite then, clearly, G^* has a finite number of identities and is row finite. Conversely if G^* is finite with finitely many identities, then the vertex set of G is the finite set G^*_o and the edge set of G is a finite union $\bigcup_{e \in G^*_o} eG^*_1$ of finite sets. So G is finite. □

We now show how to construct a group from a suitable higher rank graph. By Lemma 3.1, higher rank graphs are conical and cancellative. By Lemma 3.4 and Lemma 3.6, condition (MC) holds. By Lemma 3.10, if a higher rank graph is finitely aligned then it is strongly finitely aligned. If C is row finite then by Lemma 3.9, it is finitely aligned. We therefore assume that C is a row-finite higher rank graph with a finite number of identities and no sources. We may therefore define the group $\mathcal{G}(C)$ either in the fashion of Section 2.2 or in the fashion of Section 2.4. Theorem 2.29 guarantees that the two approaches yield isomorphic groups.

4. The group $\mathcal{G}(C)$ as a group of units of a Boolean inverse monoid

In this section, we shall prove that when C is a higher rank graph with a finite number of identities, no sources and is row finite, then the group $\mathcal{G}(C)$ is isomorphic to the group of units of a Boolean inverse $\wedge$-monoid. By non-commutative Stone duality [12, 13, 14, 16], this implies that $\mathcal{G}(C)$ is isomorphic to the topological full group of a second-countable Hausdorff étale topological groupoid whose space of identities is a compact, Hausdorff zero-dimensional space (that is, a *Boolean space*). Such a groupoid is itself said to be *Boolean*. A distributive inverse semigroup is said to be *Boolean* if its semilattice of idempotents is a generalized Boolean algebra. A distributive inverse monoid is said to be *Boolean* if its semilattice of idempotents is a Boolean algebra. The complement of an element e of a Boolean algebra is denoted by $\bar{e}$. If $X \subseteq S$ define $X^{\vee}$ to be the set of all joins of finite compatible subsets of X.

4.1. **Generalities.** In this section, we shall reprove some results from [17, Section 9] in a slightly more general setting. We recall first a definition due to Daniel Lenz [19].

Let S be an inverse semigroup with zero. Define the relation $\equiv$ on S as follows: $s \equiv t$ if and only if for each $0 < x \leq s$ we have that $x^{\downarrow} \cap t^{\downarrow} \neq 0$ and for each $0 < y \leq t$ we have that $y^{\downarrow} \cap s^{\downarrow} \neq 0$. Then $\equiv$ is a 0-restricted congruence on S. We denote the $\equiv$-class of a by $[a]$. We dub $\equiv$ the *Lenz congruence*.

Let $a \leq b$. We say that a is *essential in* b if $0 < x \leq b$ implies that $x^{\downarrow} \cap a^{\downarrow} \neq 0$. In this case, we write $a \leq_e b$. Clearly,

$$a \leq_e b \Leftrightarrow a \equiv b \text{ and } a \leq b.$$

Lemma 4.1. *In an inverse semigroup, suppose that $a \leq b$. Then $a \leq_e b$ if and only if $\mathbf{d}(a) \leq_e \mathbf{d}(b)$.*

Proof. Suppose that $a \leq_e b$. We prove that $\mathbf{d}(a) \leq_e \mathbf{d}(b)$. Let $0 < e \leq \mathbf{d}(a)$. Then $0 < be \leq b$. Now $a, be \leq b$ and so $a \sim be$. By assumption, $a \wedge be \neq 0$. Thus $\mathbf{d}(a)e \leq 0$. We have therefore proved that $\mathbf{d}(a) \leq_e \mathbf{d}(b)$. Conversely, suppose that $\mathbf{d}(a) \leq_e \mathbf{d}(b)$. Let $0 < x \leq b$. Then $0 < \mathbf{d}(x) \leq \mathbf{d}(b)$. By assumption, $\mathbf{d}(x)\mathbf{d}(a) \neq 0$. But $x, a \leq b$ implies that $x \sim a$. It follows that $x \wedge a$ exists and is non-zero. We have therefore proved that $a \leq_e b$. □

Let $\theta \colon S \to T$ be a homomorphism. We say that it is *essential* if $a \leq_e b$ implies that $\theta(a) = \theta(b)$. We say that a congruence ρ on a semigroup is *essential* if $a \leq_e b$ implies that $a \, \rho \, b$.

Lemma 4.2. *Let S be an inverse semigroup. If ρ is any 0-restricted, idempotent-pure essential congruence on S then $a \, \rho \, b$ implies that $a \wedge b$ is defined and $(a \wedge b) \leq_e a, b$.*

Proof. Suppose that $a \, \rho \, b$. Then $a \sim b$ by Lemma 2.2 and so $a \wedge b$ exists by Lemma 2.1. We prove that $a \wedge b \leq_e a$; the fact that $a \wedge b \leq_e b$ follows by symmetry. Since $a \sim b$ we have that $a \wedge b = ab^{-1}b$. Let $0 < x \leq b$. Then $x \sim a \wedge b$ since $x, a \wedge b \leq b$. Thus $(a \wedge b) \wedge x = (a \wedge b)x^{-1}x$. But $(ab^{-1}b)x^{-1}x \, \rho \, x$. We are given that $x \neq 0$ and so $(a \wedge b) \wedge x \neq 0$. The claim now follows. □

Lemma 4.3. *Let S be an inverse semigroup in which $\equiv$ is idempotent-pure. Then $a \equiv b$ if and only if there exists $c \leq_e a, b$.*

Proof. It is immediate that if there exists $c \leq_e a, b$ then $a \equiv b$. The converse follows by Lemma 4.2. □

Proposition 4.4. *Let S be an inverse semigroup on which $\equiv$ is idempotent-pure. Then $\equiv$ is the unique 0-restricted, idempotent-pure essential congruence on S.*

Proof. By definition $\equiv$ is 0-restricted, it is idempotent-pure by assumption and it is an essential congruence by virtue of its definition. Let ρ be any 0-restricted, idempotent-pure essential congruence on S. We shall prove that $\rho = \equiv$. Let $a \, \rho \, b$. By Lemma 4.2 there exists $x \leq_e a, b$, and so Lemma 4.3 gives $a \equiv b$. We have therefore shown that $\rho \subseteq \equiv$. We now prove the reverse inclusion. Let $a \equiv b$. Then Lemma 4.2 shows that $a \wedge b$ is defined and $a \wedge b \leq_e a, b$ and hence $a \, \rho \, b$ because ρ is an essential congruence. □

The following is now immediate by Lemma 4.3.

Lemma 4.5. *Let S be an inverse semigroup on which $\equiv$ is idempotent-pure. Let ρ any congruence on S such that if $a \leq_e b$ then $\rho(a) = \rho(b)$. Then $\equiv$ is contained in ρ.*

Proposition 4.6. *Let S be a distributive inverse semigroup. Let $\mathcal{B}$ be a subset of S having the following properties:*

(1) *Each element of S is a finite join of elements from $\mathcal{B}$.*
(2) *If $a \leq \bigvee_{i=1}^{m} a_i$ where $a, a_i \in \mathcal{B}$ then $a \leq a_i$ for some i.*
(3) *If $a \leq b$, where $a, b \in \mathcal{B}$ and a is a non-zero idempotent, then b is an idempotent.*

Then $\equiv$ is idempotent-pure on S.

Proof. Suppose that $a \equiv e$ where e is a non-zero idempotent. Then $(\bigvee_{i=1}^{m} a_i) \equiv \left(\bigvee_{j=1}^{n} e_j\right)$ where $a_i, e_j \in \mathcal{B}$. We assume all elements are non-zero. For each i, we have that $a_i \leq \bigvee_{i=1}^{m} a_i$. Thus, from the definition of $\equiv$, there is a non-zero element z in S such that $z \leq a_i$ and $z \leq \bigvee_{j=1}^{n} e_j$. Without loss of generality, we may assume that $z \in \mathcal{B}$. But z is an idempotent and so a_i is an idempotent. It follows that all of $a_1, \ldots, a_m$ are idempotents and so a is an idempotent, as required. □

The following was proved as [17, Lemma 9.12].

Lemma 4.7. *Let S be a distributive inverse semigroup. If ρ is idempotent-pure then S/ρ is a distributive inverse semigroup and the natural map from S to S/ρ is a morphism of distributive inverse semigroups. If, in addition, S is a $\wedge$-semigroup then S/ρ is a $\wedge$-semigroup and the morphism preserves meets.*

The following refines Lemma 4.5.

Lemma 4.8. *Let S be a distributive inverse semigroup. Let $\mathcal{B}$ be a subset of S having the following properties:*

(1) *Each element of S is a finite join of elements from $\mathcal{B}$.*
(2) *If $a \leq \bigvee_{i=1}^{m} a_i$ where $a, a_i \in \mathcal{B}$ then $a \leq a_i$ for some i.*
(3) *If $a \leq b$, where $a, b \in \mathcal{B}$ and a is a non-zero idempotent, then b is an idempotent.*

Let $\theta \colon S \to T$ be a morphism of distributive inverse semigroups such that $b \leq_e a$, where $a \in \mathcal{B}$, implies that $\theta(b) = \theta(a)$. Then $\equiv$ is contained in the kernel of θ.

Proof. Suppose that $a \leq_e b$ where $b = \bigvee_i b_i$ such that $b_i \in \mathcal{B}$. Then $a \wedge b_i \leq_e b_i$ for all i. Observe that $a \wedge b_i$ is algebraically defined since $a \sim b_i$ by Lemma 2.1. It follows that $\theta(b_i) \leq \theta(a)$ for all i giving $\theta(b) \leq \theta(a)$. But $\theta(a) \leq \theta(b)$ and so $\theta(a) = \theta(b)$. □

Lemma 4.9. *Let T be an inverse monoid. Then e is an essential idempotent if and only if $e \equiv 1$.*

Proof. Suppose that e is an essential idempotent. Then by definition for any non-zero idempotent f we have that $ef \neq 0$. This proves that $e \equiv 1$. The proof of the converse is immediate. □

The proof of the following can be deduced using the proof of [17, Lemma 9.7].

Lemma 4.10. *Let T be an inverse monoid in which $\equiv$ is idempotent-pure and suppose that T^e is E-unitary. Then the group of units of $T/\equiv$ is isomorphic to T^e/σ.*

Let S be an inverse semigroup and let $\{a_1, \ldots, a_m\} \subseteq a^{\downarrow}$. We say that this is a *tight cover* of a if for every $0 < z \leq a$ we have that $z \wedge a_i \neq 0$ for some i. The proof of the following is routine using [14, Lemma 2.5(4)].

Lemma 4.11. *Let S be a distributive inverse semigroup. Then $\{a_1, \ldots, a_m\}$ is a tight cover of a if and only if $\bigvee_{i=1}^{m} a_i \leq_e a$.*

Let S be an inverse semigroup. A subset $A \subseteq S$ is a *filter* if $A = A^{\uparrow}$ and if $a, b \in A$ there exists $c \in A$ such that $c \leq a, b$. It is *proper* if it does not contain 0. The proper filter A is *tight* if for every $a \in A$ and every tight cover $\{a_1, \ldots, a_m\}$ of a, there exists $i \leq m$ such that $a_i \in A$. A maximal proper filter is called an *ultrafilter*. If S is a distributive inverse semigroup, then a proper filter A is said to be *prime* if $a \vee b \in A$ implies that $a \in A$ or $b \in A$. By Zorn's lemma, every proper filter is contained in an ultrafilter.

Lemma 4.12. *Let S be a distributive inverse semigroup. Every tight filter is a prime filter.*

Proof. Let A be a tight filter and suppose that $a = \bigvee_{i=1}^{m} a_i \in A$. Observe that $\{a_1, \ldots, a_m\}$ is a tight cover of a using [14, Lemma 2.5(4)]. Thus $a_i \in A$, as claimed. □

By [16, Proposition 5.10] and Lemma 4.12, we have that: every ultrafilter is a tight filter, and every tight filter is a prime filter.

Lemma 4.13. *Let S be an inverse semigroup with zero in which $\equiv$ is idempotent-pure. Let X be a tight filter in S.*

1. *If $x \in X$ and $y \leq_e x$ then $y \in X$.*
2. *If $x \in X$ and $y \equiv x$ then $y \in X$.*

Proof. (1) By definition, $\{y\}$ is a tight cover of x. It follows that $y \in X$. (2) This follows by (1) and Lemma 4.3. □

To prove that a distributive inverse semigroup is Boolean, we have to prove, by [16, Lemma 3.20] that every prime filter is an ultrafilter. By Lemma 4.7, if S is distributive and $\equiv$ is idempotent-pure then $S/{\equiv}$ is distributive. The following theorem is now relevant.

Theorem 4.14. *Let S be a distributive inverse semigroup on which $\equiv$ is idempotent-pure. Then $S/{\equiv}$ is Boolean if and only if every tight filter in S is an ultrafilter.*

Proof. Put $T = S/{\equiv}$ and let $\theta \colon S \to T$ be the associated natural map. We shall prove that there is a bijection between the set of ultrafilters in S and the set of ultrafilters in T; we shall also prove that there is an order-isomorphism between the set of tight filters in S and the set of prime filters in T.

We describe first the relationship between filters in T and filters in S.

Let A be a proper filter in T and put $A' = \theta^{-1}(A)$. Since θ is 0-restricted, it follows that A' does not contain zero. Suppose that $a', b' \in A'$. Then $\theta(a'), \theta(b') \in A$. Since A is a filter there exists $c \in A$ such that $c \leq \theta(a'), \theta(b')$. It follows that $\theta(a')c^{-1}c = \theta(b')c^{-1}c$. The map θ is surjective and so there exists $e \in S$ such that $\theta(e) = c^{-1}c$. It follows that $\theta(a'e) = \theta(b'e)$. But then $a'e \sim b'e$ since we are assuming that $\equiv$ is idempotent-pure. Thus $a'e \wedge b'e$ exists. Put $d = a'e \wedge b'e$. Then $d \leq a', b'$ and $\theta(d) = c$. Thus $d \in A'$. Let $a' \in A$ and $a' \leq b'$. Then $\theta(a') \in A$ and $\theta(a') \leq \theta(b')$. Thus $\theta(b') \in A$. It follows that $b' \in A'$. We have therefore proved that A' is a proper filter.

We now go in the opposite direction. Let X be a proper filter in S. Since θ is 0-restricted, we know that $\theta(X)$ does not contain zero. Let $\theta(a'), \theta(b') \in \theta(X)$. Then $a' \wedge b' \in X$ and so there exists $c' \in X$ such that $c' \leq a', b'$. Thus $\theta(c') \leq \theta(a'), \theta(b')$. It follows that $\theta(X)^{\uparrow}$ is a proper filter in T. Write $\overline{X} = \theta(X)^{\uparrow}$.

It is easy to check that $A = \overline{A'}$ for each proper filter A in T.

Let X be a proper filter in S. Then $X \subseteq \overline{X}'$ always. Suppose, now, that X is a tight filter in S. Let $y \in \overline{X}'$. Then $\theta(y) \in \overline{X}$. By definition, $\theta(x) \leq \theta(y)$ for some $x \in X$. Observe that $yx^{-1}x \leq y$ and that $\theta(yx^{-1}x) = \theta(x)$. It follows that $x \equiv yx^{-1}x$. But $x \in X$ implies that $yx^{-1}x \in X$ and so $y \in X$ since X is a tight filter by Lemma 4.13. Thus $X = \overline{X}'$ when X is a tight filter in S.

We now prove that A is an ultrafilter in T if and only if A' is an ultrafilter in S. Suppose that A is an ultrafilter. Let $A' \subseteq Y$ where Y is an ultrafilter in S. Observe that since $A' \subseteq Y$ we have that $\overline{A'} \subseteq \overline{Y}$. But $A = \overline{A'}$ Thus $A \subseteq \overline{Y}$. But A is assumed to be an ultrafilter and so $A = \overline{Y}$. Thus $A = \theta(Y)^{\uparrow}$. It follows that $A' = \overline{Y}'$. So $Y \subseteq \overline{Y}' = A' \subseteq Y$ giving equality throughout. Hence A' is an ultrafilter.

Suppose, now, that A' is an ultrafilter. We prove that A is an ultrafilter. Suppose that $A \subseteq B$ where B is an ultrafilter. Then $A' = \theta^{-1}(A) \subseteq \theta^{-1}(B)$. But A' is an ultrafilter. Thus $\theta^{-1}(A) = \theta^{-1}(B)$. It follows that $A = \overline{B}'$. So $B \subseteq \overline{B}' = A \subseteq B$, and hence $A = B$. So A is an ultrafilter.

We prove that $X \mapsto \overline{X}$ is a bijection from the set of ultrafilters in S to the set of ultrafilters in T, with inverse $A \mapsto A'$. Let X be an ultrafilter in S. Suppose that $\overline{X} \subseteq A$, where A is a proper filter. Then $\overline{X}' \subseteq A'$. But then $X \subseteq A'$ and so $X = A'$, since X is an ultrafilter. Thus $\overline{X} = \overline{A'} = A$. We have proved that $\overline{X}$ is an ultrafilter. Suppose that X and Y are ultrafilters in S and that $\overline{X} = \overline{Y}$. Then $\overline{X}' = \overline{Y}'$. Hence $X = \overline{X}' = \overline{Y}' = Y$. Let A be an ultrafilter in T. Then A' is an ultrafilter in S and $A = \overline{A'}$. This establishes our bijection.

We now prove that A is a prime filter in T if and only if A' is a tight filter in S. Suppose first that A' is a tight filter. We prove that A is a prime filter. Suppose that $a \vee b \in A$. Let $\theta(a') = a$ and $\theta(b') = b$. Since $a \sim b$ and θ is idempotent-pure, we have that $a' \sim b'$. Thus $a' \vee b' \in A'$. Now $\{a', b'\} \subseteq (a' \vee b')^{\downarrow}$ is a tight cover. Thus, by assumption and without loss of generality, we have that $a' \in A'$ and so $a \in A$. It follows that A is a prime filter. Suppose now that A is a prime filter. We prove that A' is a tight filter. Let $a' \in A$ and let $b'_1, \ldots, b'_n$ be a tight cover of a'. Then $\bigvee_{i=1}^{n} b'_i \leq_e a'$ by Lemma 4.11. Thus $\theta(a') = \theta(b'_1) \vee \ldots, \theta(b'_n)$ since θ is essential. But A is a prime filter. It follows that $\theta(b'_i) \in A$ for some i and so $b'_i \in A'$ for some i, proving that A' is a tight filter.

If X is a tight filter in S then $\overline{X}$ is a prime filter in T. Let $\bigvee_{i=1}^{m} t_i \in \overline{X}$. Then $\theta(x) \leq \bigvee_{i=1}^{m} t_i$ for some $x \in X$. Let $\theta(y_i) = t_i$. Since $\{t_1, \ldots, t_m\}$ is a compatible set so too is $\{y_1, \ldots, y_m\}$ since θ is idempotent-pure. Put $y = \bigvee_{i=1}^{m} y_i$. Put $x' = yx^{-1}x$. Then $\theta(x') = \theta(x)$. Thus $x \equiv x'$ and $x \in X$ and so $x' \in X$ by Lemma 4.13. But every tight filter is a prime filter and so $y_i \in X$ for some i. It follows that $t_i \in \overline{X}$ for some i.

We prove that $X \mapsto \overline{X}$ is an order-isomorphism between the set of tight filters in S and the set of prime filters in T. Let X be a tight filter in S. Then $\overline{X}$ is a prime filter in T and $X = \overline{X}'$. Let A be a prime filter in T. Then A' is a tight filter in S and $A = \overline{A'}$. This establishes the bijection. The fact that it is an order-isomorphism is straightforward on the basis of what we have proved.

We now prove the theorem. Suppose first that every tight filter in S is an ultrafilter. We prove that every prime filter in T is an ultrafilter which proves that T is Boolean. Let P be a prime filter in T and let $P \subseteq Q$ where Q is a proper filter in T. Then $P' \subseteq Q'$. It follows that P' is a tight filter and so, by assumption, it is an ultrafilter. Thus $P' = Q'$. Now $\overline{P'} = \overline{Q'}$. It follows that $P = Q$ and we have shown that P is an ultrafilter.

Conversely, suppose that T is Boolean which is equivalent to saying that every prime filter in T is an ultrafilter. Let X be a tight filter in S. We prove that X

is an ultrafilter. Let $X \subseteq Y$ where Y is an ultrafilter. Then $\overline{X} \subseteq \overline{Y}$. Then $\overline{Y}$ is an ultrafilter and $\overline{X}$ is a prime filter. Thus $\overline{X} = \overline{Y}$, by assumption. It follows that $\overline{X}' = \overline{Y}'$ and so $X = Y$, as required. □

The following can easily be deduced from [12, 13]. Let A be a filter in the inverse semigroup S. Define $\mathbf{d}(A) = (A^{-1}A)^{\uparrow}$. Then $\mathbf{d}(A)$ is a filter in S which is also an inverse subsemigroup. Furthermore, $A = (a\mathbf{d}(A))^{\uparrow}$ for any $a \in A$. Clearly, $0 \in A$ if and only if $0 \in \mathbf{d}(A)$. Also, $\mathbf{d}(A) \cap \mathbf{E}(S)$ is a filter in $\mathsf{E}(S)$. The following result shows that to check whether every tight filter is an ultrafilter it is enough to restrict attention to the distributive lattice of idempotents.

Lemma 4.15. *Let A be a filter in an inverse semigroup S. Then $x \in \mathbf{d}(A)$ if and only if $a^{-1}a \leq x$ for some $a \in A$.*

Proof. Suppose that $x \in \mathbf{d}(A)$. Then $a^{-1}b \leq x$ for some $a, b \in A$. But A is a filter and so there exists $c \leq a, b$. It follows that $c^{-1}c \leq x$. Conversely, suppose that $a^{-1}a \leq x$ for some $a \in A$. Then it is immediate that $x \in \mathbf{d}(A)$. □

Proposition 4.16. *Let S be a distributive inverse semigroup. Then each tight filter in S is an ultrafilter in S if and only if each tight filter in $\mathsf{E}(S)$ is an ultrafilter in $\mathsf{E}(S)$.*

Proof. Observe first that we have the following: A is a prime filter (respectively, ultrafilter, tight filter) in S if and only if $\mathbf{d}(A)$ is a prime filter (respectively, ultrafilter, tight filter) in S if and only if $\mathsf{E}(\mathbf{d}(A))$ is a prime filter (respectively, ultrafilter, tight filter) in $\mathsf{E}(S)$.

We first prove the results relating A and $\mathbf{d}(A)$. Suppose that A is a prime filter. We prove that $\mathbf{d}(A)$ is a prime filter. Let $x \vee y \in \mathbf{d}(A)$. Then $a^{-1}a \leq x \vee y$ for some $a \in A$. We have that $a^{-1}a = xa^{-1}a \vee ya^{-1}a$. Whence $a = axa^{-1}a \vee aya^{-1}a$. But A is prime. Without loss of generality, we may assume that $axa^{-1}a \in A$. Thus $ax \in A$. Now, $a^{-1}ax \in \mathrm{d}(A)$. It follows that $x \in \mathbf{d}(A)$. We now prove the converse. Suppose that $\mathrm{d}(A)$ is prime. Let $x \vee y \in A$. Then $\mathbf{d}(x) \vee \mathbf{d}(y) \in \mathbf{d}(A)$. But $\mathbf{d}(A)$ is prime. Without loss of generality, we may assume that $\mathbf{d}(x) \in \mathbf{d}(A)$. It follows that $(x \vee y)\mathbf{d}(x) \in A$ and so $x \in A$. It is routine to check that A is an ultrafilter if and only if $\mathbf{d}(A)$ is an ultrafilter. The fact that A is a tight filter if and only if $\mathbf{d}(A)$ is a tight filter follows by [16, Lemma 5.9(1)]. By [16, Lemma 5.9(2)], we have that $\mathbf{d}(A)$ is a tight filter (respectively, ultrafilter) if and only if $\mathsf{E}(\mathbf{d}(A))$ is a tight filter (respectively, ultrafilter).

We can now prove the proposition. Suppose that each tight filter in $\mathsf{E}(S)$ is an ultrafilter in $\mathsf{E}(S)$. Let A be a tight filter in S. Then $\mathbf{d}(A)$ is a tight filter in S. Thus $\mathsf{E}(\mathbf{d}(A))$ is a tight filter in $\mathsf{E}(S)$. By assumption, $\mathsf{E}(\mathbf{d}(A))$ is an ultrafilter in $\mathsf{E}(S)$. Thus $\mathbf{d}(A)$ is an ultrafilter in S. Finally, we deduce that A is an ultrafilter in S. The proof of the converse is now straightforward. □

4.2. **Specifics.** We now apply the results of the previous section to the specific inverse monoids constructed in this paper. By Lemmas 2.15, 2.16 and Lemma 2.17, the inverse monoid $\mathsf{R}(C)$ satisfies the conditions of Proposition 4.6 and so $\equiv$ is idempotent-pure. By Proposition 2.18, $\mathsf{R}(C)$ is a distributive inverse $\wedge$-monoid and so by Lemma 4.7, it follows that $\mathsf{R}(C)/\equiv$ is a distributive inverse $\wedge$-monoid. By Proposition 2.20, $\mathsf{R}(C)^e$ is E-unitary. Thus by Lemma 4.10, the group of units of $\mathsf{R}(C)/\equiv$ is isomorphic to $\mathcal{G}(C)$. We have therefore proved the following.

Theorem 4.17. *Let C be a higher rank graph with a finite number of identities which has no sources and is row finite. Then the group $\mathcal{G}(C)$, defined at the end of Section 3, is isomorphic with the group of units of $\mathsf{R}(C)/\equiv$.*

Definition. Let C be a higher rank graph with a finite number of identities which has no sources and is row finite. Put $\mathsf{B}(C) = \mathsf{R}(C)/\equiv$.

To prove that $\mathsf{R}(C)/\equiv$ is Boolean, it is enough to prove that every tight filter in $\mathsf{R}(C)$ is an ultrafilter by Theorem 4.14. To do this, we shall relate filters in $\mathsf{R}(C)$ to appropriate subsets of C.

Let C be a category. Let $A \subseteq C$ be a non-empty subset. We say that it is a *filter* if it satisfies the following two conditions:

(1) If $x, y \in A$ then there exist $u, v \in C$ such that $xu = yv \in A$. In particular, this implies that the elements of A are pairwise comparable.
(2) If $x = yz$ and $x \in A$ then $y \in A$.

Lemma 4.18. *Let C be a finitely aligned category. Let A be a filter in C and let $a, b \in A$. Suppose that $aC \cap bC = \{c_1, \ldots, c_n\}C$. Then $c_i \in A$ for some i.*

Proof. Let $u, v \in C$ such that $z = au = bv \in A$. Then $z \in aC \cap bC$ and so $z = c_i p$ for some i and some $p \in C$. But $z \in A$ and A is a filter and so $c_i \in A$. ☐

Let $a \in C$ and $\{a_1, \ldots, a_m\}$ such that $a_i = ap_i$. We say that $\{a_1, \ldots, a_m\}$ is a *tight cover* of a if whenever $z = ap$ there exists an i and a u such that $u = zq = a_i r$; thus z is comparable with some element a_i. Let $A \subseteq C$ be a filter. We say that it is *tight* if the following condition holds: if $\{a_1, \ldots, a_m\}$ is a tight cover of a, where $a \in A$, then $a_i \in A$ for some i.

Lemma 4.19. *Let C be a finitely aligned cancellative category. Then*

$$\{a_1a_1^{-1}, \ldots, a_m a_m^{-1}\} \subseteq (aa^{-1})^{\downarrow}$$

is a tight cover in $\mathsf{R}(C)$ if and only if

$$\{a_1, \ldots, a_m\} \subseteq aC$$

is a large subset.

Proof. Suppose that $\{a_1, \ldots, a_m\} \subseteq aC$ is a large subset. Let $0 < bb^{-1} \leq aa^{-1}$. Then $b = ap$ for some p. Thus $b \in aC$. It follows that b is comparable with some element a_i. Thus $z = bu = a_i v$ for some $u, v \in C$. But then $zz^{-1} \leq bb^{-1}, a_i a_i^{-1}$. Now let $\bigvee_{j=1}^{n} b_j b_j^{-1} \leq aa^{-1}$. Then $0 < b_j b_j^{-1} \leq aa^{-1}$ for all j. Then $b_j = ap_j$ for some $p_j \in C$. In particular, $b_j \in aC$. It follows that b_j is comparable with some element a_i. Suppose that $\{a_1a_1^{-1}, \ldots, a_m a_m^{-1}\} \subseteq (aa^{-1})^{\downarrow}$ is a tight cover. Let $b \in aC$. Then $b = ap$. Thus $bb^{-1} \leq aa^{-1}$. It follows that there is $zz^{-1} \leq bb^{-1}, a_i a_i^{-1}$ for some i. Thus $z = bu = a_i v$ It follows that $\{a_1, \ldots, a_m\} \subseteq aC$ is a large subset. ☐

Proposition 4.20. *Let C be a strongly finitely aligned cancellative category with a finite number of identities. Given a filter $A \subseteq C$, define*

$$\mathsf{P}(A) = \{xx^{-1} \colon x \in A\}^{\uparrow} \cap \mathsf{E}(\mathsf{R}(S)).$$

Then P is a bijective correspondence between filters in C (resp. maximal filters) and prime filters in $\mathsf{E}(\mathsf{R}(C))$ (resp. maximal filters). Under this bijective correspondence, tight filters correspond to tight filters. The inverse of F is given by

$$\mathsf{F}(P) = \{x \in C \colon xx^{-1} \in P\}$$

for each prime filter P in $\mathsf{E}(\mathsf{R}(C))$.

Proof. Let $A \subseteq C$ be a filter in C. We claim that $\mathsf{P}(A)$ is a filter in $\mathsf{E}(\mathsf{R}(S))$. Because of Lemma 2.17, whenever we have a relation $xx^{-1} \leq \bigvee_{i=1}^{m} a_i a_i^{-1}$ then, in fact, $xx^{-1} \leq a_i a_i^{-1}$ for some i. Bearing this in mind, let $aa^{-1}, bb^{-1} \in \mathsf{P}(A)$. Then $xx^{-1} \leq aa^{-1}$ and $yy^{-1} \leq bb^{-1}$ for some $x, y \in A$. Thus $x = ap$ and $y = bq$ for some $p, q \in C$. But A is a filter and $a, b \in A$. By assumption, $aC \cap bC \cap A \neq \varnothing$. It follows that $au = bv \in A$ for some $u, v \in S$. Put $z = au = bv$. Then $zz^{-1} \in \mathsf{P}(A)$ and $zz^{-1} \leq aa^{-1}, bb^{-1}$. It is now clear that $\mathsf{P}(A)$ is a filter in $\mathsf{E}(\mathsf{R}(S))$; it is a proper filter by construction. We prove that $\mathsf{P}(A)$ is a prime filter. Suppose that $\bigvee_{i=1}^{m} x_i x_i^{-1} \in \mathsf{P}(A)$. Then $xx^{-1} \leq \bigvee_{i=1}^{m} x_i x_i^{-1}$ for some $x \in A$. Thus by Lemma 2.17, we have that $xx^{-1} \leq x_i x_i^{-1}$ for some i. Hence $x = x_i p$. Since $x \in A$ it follows that $x_i \in A$ and so $x_i x_i^{-1} \in \mathsf{P}(A)$.

Now let P be a prime filter in $\mathsf{E}(\mathsf{R}(C))$. Since P is a prime filter the set $\mathsf{F}(P)$ is non-empty, and it is then routine to check that $\mathsf{F}(P)$ is a filter in C.

It remains to show that the maps $A \mapsto \mathsf{P}(A)$ and $P \mapsto \mathsf{F}(P)$ are mutually inverse and order-preserving. Clearly, $A \subseteq \mathsf{F}(\mathsf{P}(A))$. Let $x \in \mathsf{F}(\mathsf{P}(A))$. Then $yy^{-1} \leq xx^{-1}$ where $y \in A$. Then $y = xp$. Thus $x \in A$. It follows that $A = \mathsf{F}(\mathsf{P}(A))$. Clearly, $\mathsf{P}(\mathsf{F}(P)) \subseteq P$. Let $\bigvee_{i=1}^{m} x_i x_i^{-1} \in P$. Then $x_i x_i^{-1} \in P$ for some i. Thus $x_i \in \mathsf{F}(P)$ and so $\bigvee_{i=1}^{m} x_i x_i^{-1} \in \mathsf{P}(\mathsf{F}(P))$. It follows that $\mathsf{P}(\mathsf{F}(P)) = P$.

Suppose that A and B are filters in C such that $A \subseteq B$. Then $\mathsf{P}(A) \subseteq \mathsf{P}(B)$. If P and Q are prime filters in $\mathsf{E}(\mathsf{R}(C))$ such that $P \subseteq Q$ then $\mathsf{F}(P) \subseteq \mathsf{F}(Q)$. It follows that P is an order-isomorphism between the poset of filters in C and the poset of prime filters in $\mathsf{E}(\mathsf{R}(C))$.

Since P is an order isomorphism, it restricts to a bijection between the ultrafilters of A and the maximal prime filters of $\mathsf{E}(\mathsf{R}(C))$. Since every ultrafilter of $\mathsf{E}(\mathsf{R}(C))$ is a prime filter, the maximal prime filters of $\mathsf{E}(\mathsf{R}(C))$ are the ultrafilters. So P restricts to a bijection between ultrafilters of A and ultrafilters of $\mathsf{E}(\mathsf{R}(C))$.

We conclude by showing that tight filters correspond to tight filters. Suppose now that A is a tight filter in C. We prove that $\mathsf{P}(A)$ is a tight filter in $\mathsf{E}(\mathsf{R}(C))$. Let $\{a_1 a_1^{-1}, \ldots, a_m a_m^{-1}\}$ be a tight cover of xx^{-1} where $xx^{-1} \in \mathsf{P}(A)$. Then $\{a_1, \ldots, a_m\}$ is a tight cover of x where $x \in A$. By assumption, $a_i \in A$ for some i. Thus $a_i a_i^{-1} \in \mathsf{P}(A)$ for some i. It follows that $\mathsf{P}(A)$ is a tight filter in $\mathsf{E}(\mathsf{R}(C))$. We now prove the converse. Let P be a tight filter in $\mathsf{E}(\mathsf{R}(C))$. We prove that $\mathsf{F}(P)$ is a tight filter in C. Let $x \in \mathsf{F}(P)$ and suppose that $\{a_1, \ldots, a_m\} \subseteq aC$ is large. Then by Lemma 4.19, we have that $\{a_1 a_1^{-1}, \ldots, a_m a_m^{-1}\}$ is a tight cover of aa^{-1}. But $aa^{-1} \in P$ and P is a tight filter and so $a_i a_i^{-1} \in P$ for some i. It follows that $a_i \in \mathsf{F}(P)$ for some i, as required. ☐

We define a subset $A \subseteq C$ to be *good* if it has the following two properties:

1. (1) Any two elements of A are comparable.
2. (2) For each $\mathbf{m} \in \mathbb{N}^k$ there exists $a \in A$ such that $d(a) = \mathbf{m}$.

Remark 4.21. If A is a good subset of C then there is an identity e in C such that $A \subseteq eC$. This follows from the fact that any two elements in A are comparable. In fact, $e \in A$ since e is the unique element of A such that $d(e) = \mathbf{0}$.

Observe that if $a, b \in C$ and $d(a) = d(b)$ then, in fact, $a = b$ since, being comparable, we have that $au = bv$ for some $u, v \in C$ and then we apply Lemma 3.2. Our rationale for defining good subsets is explained by the following result which is fundamental.

Lemma 4.22. *Let C be a finitely aligned k-graph. We prove that the following three classes of subsets are the same:*

1. (1) *Good subsets.*
2. (2) *Tight filters.*
3. (3) *Maximal filters.*

Proof. We prove first that good subsets are filters. There are two claims we have to prove.

Let A be a good subset of C. Let $x, y \in A$. We prove that $xC \cap yC \cap A \neq \varnothing$. By assumption, $z = xu = yv$ for some $u, v \in C$. Let $z' \in A$ such that $d(z') = d(z)$. Since $z', x \in A$ we have that $z's = xt$ for some $s, t \in C$. Now, $d(z') = d(z) = d(x) + d(u)$ thus $d(z') > d(x)$. It follows by Lemma 3.2 that $z' = xp$ for some $p \in C$. Likewise, $z' = yq$ for some $q \in C$. Thus $z' = xu = yq$. It follows that $z' \in xC \cap yC \cap A$.

Let $x = yz$ where $x \in A$. We want to prove that $y \in A$. Let $y' \in A$ be the unique element such that $d(y') = d(y)$. Since $x, y' \in A$ we have that $xu = y'v$ for some $u, v \in C$. But then $yzu = y'v$. However, $d(y) = d(y')$, thus by Lemma 3.2, it follows that $y = y'$ and so $y \in A$, as claimed.

We now prove that good subsets are in fact tight filters. Let A be a good subset. Let $a \in A$ and suppose that $\{a_1, \dots, a_m\}$ is a tight cover of a. We need to prove that $a_i \in A$ for some i. Observe that $d(a_i) \geq d(a)$ for all i since $a_i = ap_i$ for some $p_i \in C$. Put $\mathbf{m} = \bigvee_{i=1}^{m} d(a_i)$. By assumption, there is a unique $z \in A$ such that $d(z) = \mathbf{m}$. But $d(z) \geq d(a)$ and, since $a, z \in A$, they are comparable. It follows by Lemma 3.2 that $z = as$ for some $s \in C$. By assumption, z is comparable with some element a_i. But $d(z) \geq d(a_i)$. Thus $z = a_i t$ for some t by Lemma 3.2. But A is a filter, $z \in A$ and so $a_i \in A$, as required.

We now prove that every tight filter is a good subset. Let A be a tight filter. We prove that it is a good subset. Let $\mathbf{m} \in \mathbb{N}^k$ be arbitrary. We prove that there is an element $y \in A$ such that $d(y) = \mathbf{m}$. Let $a \in A$ be arbitrary. If $d(a) \geq \mathbf{m}$ then $a = ya'$ where $d(y) = \mathbf{m}$. But A is a filter and so $y \in A$ and we are done. It follows that, putting $\mathbf{n} = d(a) \vee \mathbf{m}$, we can assume in what follows that $\mathbf{n} > d(a)$. Let $p_1, \dots, p_s \in C$ be all the elements such that $d(ap_i) = \mathbf{n}$. (We assume that there are only finitely many which is fine since they all have the same degree $\mathbf{n} - \mathbf{m}$.) We shall prove below that $\{ap_1, \dots, ap_s\}$ is a tight cover of a. Assuming this result, then $ap_j \in A$ for some j since A is a tight filter. But $d(ap_j) = \mathbf{n} = \mathbf{m} + (\mathbf{n} - \mathbf{m})$. Thus A contains an element of degree $\mathbf{m}$ by the UFP and the properties of filters. It remains to prove that $\{ap_1, \dots, ap_s\}$ is a tight cover of a where $p_1, \dots, p_s \in C$ is the set of all the elements such that $d(ap_i) = \mathbf{n}$. Observe that all these elements have the same range, e say, and they are all the elements in eC with degree $\mathbf{n} - \mathbf{m}$. Let $z = ap$ for some p. We prove that z is comparable with some ap_k. Observe that $p \in eC$. Choose u such that $d(pu) \geq \mathbf{n} - \mathbf{m}$. Then $pu = p_k v$ for some $v \in C$ and k. Then $apu = ap_k v$ and so $zu = ap_k v$.

We now prove that good subsets are maximal filters. Let A be a good subset and suppose that $A \subseteq B$ where B is a filter. Let $b \in B$. Then, by assumption, there exists $a \in A$ such that $d(a) = d(b)$. But $a, b \in B$ and B is a filter and so a and b are comparable. It follows by Lemma 3.2 that $a = b$. Thus $b \in A$. It follows that $A = B$ and so A is maximal.

Finally, we prove that maximal filters are good subsets; in fact, we prove that maximal filters are tight filters. Let A be a maximal filter. Let x be an element of C which is comparable with every element of A. Let $x = yx'$. Then y is comparable with every element of A since x is. Put $X = \{y \in C \colon x = yx' \text{ for some } x' \in C\}$. Then $A \cup X$ is a filter. But A is maximal and so $X \subset A$; this simply means that $x \in A$. We prove that maximal filters are tight filters. Let $a \in A$ and suppose that $\{a_1, \dots, a_m\}$ is a tight cover of a. If $a_i \notin A$ then there is some $b_i \in a_i$ so that b_i and a_i are not comparable. Now $b_1, \dots, b_m \in A$. By repeated application of part (1) of the definition of a filter, we can find elements $u_1, \dots, u_m$ such that $b_1 u_1 = b_2 u_2 = \ldots = b_m u_m \in A$. In fact, we can assume that there is an element u such that $z = au = b_1 u_1 = b_2 u_2 = \ldots = b_m u_m \in A$. By assumption, $zs = a_i t$ for some s, t. It follows that $b_i u_i s = a_i t$. But this says that b_i is comparable to a_i which is a contradiction. It follows that some $a_j \in A$ and so A is a tight filter. $\square$

By Theorem 4.14, Lemma 4.22 and Proposition 4.16 we deduce that $\mathsf{R}(C)/\equiv$ is a Boolean inverse monoid. By Theorem 4.17 the group of units of $\mathsf{R}(C)/\equiv$ is isomorphic to $\mathsf{R}(C)^e/\sigma$ which is the group $\mathcal{G}(C)$. We have therefore proved the following.

Theorem 4.23. *Let C be a higher rank graph with a finite number of identities which has no sources and is row finite. Then $\mathsf{B}(C) = \mathsf{R}(C)/\equiv$ is a Boolean inverse monoid whose group of units is isomorphic to $\mathcal{G}(C)$.*

5. Properties of the Boolean inverse monoid $\mathsf{B}(C)$

In this section, we shall impose natural conditions on the k-graph C and then determine the properties acquired by the Boolean inverse monoid $\mathsf{B}(C)$ as a result.

5.1. **Aperiodicity.** Our first definition is taken from [20]. Let C be a k-graph. We say that C is *aperiodic* if for all $a, b \in C$ such that $a \neq b$ and $\mathbf{d}(a) = \mathbf{d}(b)$ there exists an element $u \in C$ such that au and bu are not comparable. To make use of this concept in an inverse semigroup setting we need some definitions. A non-zero element a of an inverse semigroup with zero is called an *infinitesimal* if $a^2 = 0$. Observe that a is an infinitesimal if and only if $\mathbf{d}(a) \perp \mathbf{r}(a)$ by [15, Lemma 2.5]. An inverse semigroup S is said to be *fundamental* if the only elements of S that commute with all the idempotents of S are themselves idempotents. The following is an easy consequence of [10, Theorem 5.2.9].

Lemma 5.1. *If a fundamental inverse semigroup has a finite number of idempotents then it is itself finite.*

Now, let S be a Boolean inverse monoid. Denote the group of units of S by $\mathsf{U}(S)$ and the Boolean algebra of idempotents of S by $\mathsf{E}(S)$. There is an action of $\mathsf{U}(S)$ on $\mathsf{E}(S)$ given by $e \mapsto geg^{-1}$. We call this the *natural action.* The following was proved as [15, Proposition 3.1].

Lemma 5.2. *Let S be a Boolean inverse monoid. Then the natural action is faithful if and only if S is fundamental.*

The following is a souped up version of [15, Lemma 3.2]. In the proof of the lemma below, we use the result that in a Boolean algebra $e \leq f$ if and only if $e\bar{f} = 0$.

Lemma 5.3. *Let S be a Boolean inverse monoid. Let e be an idempotent and a an element such that $ea \neq ae$. Then there is an idempotent $f \leq e$ such that $f \perp afa^{-1}$.*

Proof. There are two cases.

Case 1: Suppose that $e\overline{(aea^{-1})} \neq 0$. Put $f = e\overline{(aea^{-1})} \leq e$. Then $f(afa^{-1}) = e\overline{(aea^{-1})}a(e\overline{(aea^{-1})})a^{-1}$. Thus

$$f(afa^{-1}) = e\overline{(aea^{-1})}a(ea^{-1}a\overline{(aea^{-1})})a^{-1} = e\overline{(aea^{-1})}(aea^{-1})a\overline{(aea^{-1})}a^{-1} = 0,$$

and the result is proved.

Case 2: Now, suppose that $e\overline{(aea^{-1})} = 0$. Then $e \leq aea^{-1}$ and so $ea \leq ae$. For the sake of a contradiction, suppose that $aea^{-1}\bar{e} = 0$. Then $aea^{-1} \leq e$ and so $ae \leq ea$. It follows that $ea = ae$, which is a contradiction. Thus, in fact, $aea^{-1}\bar{e} \neq 0$. Put $h = aea^{-1}\bar{e}$ and $f = a^{-1}ha$. Suppose that $f = 0$. Then $aha^{-1} = 0$. But $aha^{-1} = h \neq 0$. It follows that $f \neq 0$. Also, $f = a^{-1}ha = a^{-1}aea^{-1}\bar{e}a = ea^{-1}\bar{e}a \leq e$. Finally, by direct computation, $afa^{-1}f = 0$. ☐

The following is now immediate by the above lemma when we put $b = af$.

Corollary 5.4. *Suppose that $ae \neq ea$. Then there is an infinitesimal b such that $b \leq a$ and $b^{-1}b \leq e$.*

Proposition 5.5. *Let S be a Boolean inverse monoid. Then the following are equivalent:*

(1) *S is fundamental*
(2) *Each non-idempotent element of S is above an infinitesimal.*

Proof. (1)⇒(2). Let a be a non-idempotent element. Then, since S is fundamental, there is an idempotent e such that $ae \neq ea$. Thus, by Corollary 5.4, there is an infinitesimal $b \leq a$.

(2)⇒(1). Let a be a non-idempotent element. Then $b \leq a$ where b is an infinitesimal. In particular, $b \neq 0$. Suppose that a commutes with $b^{-1}b$. Then $ab^{-1}b = b^{-1}ba$. But $ab^{-1}b = b$ because $b \leq a$, and so $b = b^{-1}ba$. It follows that $b^2 = ba$. Thus $ba = 0$ and so $b^{-1}ba = 0$. It follows that $b = 0$, which is a contradiction. We have shown that a cannot commute with $b^{-1}b$. □

Lemma 5.6. *Let C be a k-graph. Then C is aperiodic if and only if each non-idempotent basic morphism ab^{-1} of $\mathsf{R}(C)$ lies above an infinitesimal basic morphism.*

Proof. Suppose that C is aperiodic. Let ab^{-1} be a non-idempotent basic morphism. Thus $a \neq b$ and $\mathbf{d}(a) = \mathbf{d}(b)$. By assumption, there is an element $u \in C$ such that au and bu are incomparable. Observe that $(au)(bu)^{-1} \leq ab^{-1}$ by Lemma 2.16. To say that au and bu are incomparable means precisely that the idempotents $(au)(au)^{-1}$ and $(bu)(bu)^{-1}$ are orthogonal by Lemma 2.16. Thus $(au)(bu)^{-1}$ is an infinitesimal and is below ab^{-1}.

Let $a \neq b$ and $\mathbf{d}(a) = \mathbf{d}(b)$. Then ab^{-1} is a non-idempotent basic morphism. By assumption, there exists an infinitesimal cd^{-1} such that $cd^{-1} \leq ab^{-1}$. By Lemma 2.16, we have that $c = au$ and $d = bu$ for some $u \in C$. By assumption, $(au)(au)^{-1}$ and $(bu)(bu)^{-1}$ are orthogonal and so au and bu are incomparable. We have therefore proved that C is aperiodic. □

Lemma 5.7. *Let S be a Boolean inverse semigroup. Then $\bigvee_{j=1}^{n} a_j$ an infinitesimal implies that each a_j is an infinitesimal.*

Proof. By assumption, $\bigvee_{j=1}^{n} \bigvee_{i=1}^{n} a_j a_i = 0$. In particular, $a_j^2 = 0$ for each $1 \leq j \leq n$. □

The following theorem is key.

Theorem 5.8. *Let C be a strongly finitely aligned higher rank graph with a finite number of identities which has no sources and is row finite. Then the following are equivalent:*

(1) *$\mathsf{B}(C)$ is fundamental.*
(2) *C is aperiodic.*

Proof. (1)⇒(2). We shall use Lemma 5.6 to prove that C is aperiodic. Let ab^{-1} be a non-idempotent basic morphism of $\mathsf{R}(C)$. Then $[ab^{-1}]$ is not an idempotent since $\equiv$ is idempotent-pure. Thus, by Proposition 5.5, $[\bigvee_{j=1}^{n} c_j d_j^{-1}] \leq [ab^{-1}]$ where $[\bigvee_{j=1}^{n} c_j d_j^{-1}]$ is an infinitesimal. Thus by Lemma 5.7, each $[c_j d_j^{-1}]$ is an infinitesimal. But then each $c_j d_j^{-1}$ is an infinitesimal, since $\equiv$ is 0-restricted. Observe that $(ab^{-1})(\bigvee_{j=1}^{n} d_j d_j^{-1})$ maps to $[\bigvee_{j=1}^{n} c_j d_j^{-1}]$. Thus $(ab^{-1})(\bigvee_{j=1}^{n} d_j d_j^{-1})$ is an infinitesimal and lies below ab^{-1}. By relabelling if necessary, we can assume that $(ab^{-1})d_1 d_1^{-1}$ is non-zero and an infinitesimal in $\mathsf{R}(C)$ and lies below ab^{-1}. But $(ab^{-1})d_1 d_1^{-1}$ is a join of basic morphisms and each of these basic morphisms must be either zero or an infinitesimal. Pick a non-zero such basic morphism cd^{-1}. Then $cd^{-1} \leq ab^{-1}$ and is an infinitesimal. It follows by Lemma 5.6 that C is aperiodic.

(2)⇒(1). Let $[\bigvee_{i=1}^{m} a_i b_i^{-1}]$ be a non-idempotent element of $\mathsf{B}(C)$. Then the element $\bigvee_{i=1}^{m} a_i b_i^{-1}$ is a non-idempotent in $\mathsf{R}(C)$ since $\equiv$ is idempotent-pure. It follows that for some i we have that $a_i \neq b_i$. By Lemma 5.6, there is an infinitesimal basic morphism $cd^{-1} \leq a_i b_i^{-1}$. Thus $[cd^{-1}] \leq [\bigvee_{i=1}^{m} a_i b_i^{-1}]$. Because cd^{-1} is non-zero it follows that $[cd^{-1}]$ is non-zero. We have therefore proved that each non-idempotent element of $\mathsf{B}(C)$ is above an infinitesimal. It follows by Proposition 5.5, that $\mathsf{B}(C)$ is fundamental. □

5.2. **Cofinality.** Our second definition is also taken from [20]. Define C to be *cofinal* if for all $e, f \in C_o$ there exists a large subset $X \subseteq fC$ such that $eC\mathbf{d}(x) \neq \varnothing$ for all $x \in X$.

A semigroup ideal I of a Boolean inverse semigroup is said to be an *additive ideal* if $a, b \in I$ and $a \sim b$ implies that $a \vee b \in I$. A Boolean inverse semigroup is called 0-*simplifying* if it has no non-trivial additive ideals. A Boolean inverse semigroup that is both fundamental and 0-simplifying is called *simple*.

Lemma 5.9. *Let C be a k-graph. Then each non-trivial additive ideal of $\mathsf{B}(C)$ contains an idempotent of the form $[ee^{-1}]$ where e is an identity of C.*

Proof. We begin with some generalities on non-trivial additive ideals I in Boolean inverse semigroups S. Let $a \in I$. Then $aa^{-1} \in I$ since I is a semigroup ideal. On the other hand if $a^{-1}a \in I$ then $a \in I$ since $a = a(a^{-1}a)$. It follows that non-trivial additive ideals always contain idempotents. If $a \in I$ then $ae \in I$ for all $e \in \mathsf{E}(S)$. It follows that additive ideals are always order ideals. Let I be a non-trivial additive ideal in $\mathsf{B}(C)$. Then, by the above, we can assume that $[xx^{-1}] \in \mathsf{B}(C)$ for some $x \in C$. Let $e = \mathbf{d}(x)$. Then ex^{-1} is a well-defined basic morphism and so belongs to $\mathsf{R}(C)$. Observe that $\mathbf{d}(ex^{-1}) = xx^{-1}$ and so $[ex^{-1}] \in I$. It follows that $[ee^{-1}] \in I$. □

Let S be a Boolean inverse monoid. If $X \subseteq S$ define $\mathbf{d}(X) = \{\mathbf{d}(x) \colon x \in X\}$ and $\mathbf{r}(X) = \{\mathbf{r}(x) \colon x \in X\}$. Let e and f be non-zero idempotents in S. We write $e \preceq f$, and say there is a *pencil* from e to f, if there is a finite set X of S such that $e = \bigvee \mathbf{d}(X)$ and $\bigvee \mathbf{r}(X) \leq f$. By [14, Lemma 4.1], if I is an additive ideal of S and $f \preceq e \in I$ then $f \in I$. The following was first proved in [19] but we give the proof for completeness.

Lemma 5.10. *Let S be a Boolean inverse semigroup. Let e be an idempotent. Then $I = (SeS)^{\vee}$ is an additive ideal of S and $f \in I$ is an idempotent if and only if $f \preceq e$.*

Proof. We may write $f = \bigvee_{i=1}^{m} e_i$ where $e_1, \ldots, e_m \in SeS$. But it is easy to prove that $e_i = x_i^{-1}x_i$, where $x_i \in eSe_i$ Thus $x_i x_i^{-1} \leq e$. It follows that $X = \{x_1, \ldots, x_m\}$ is a pencil from f to e.

We now prove the converse. Suppose that $f \preceq e$. Then there is a pencil X from f to e. Now $\mathbf{r}(X) \leq e$ and so $\mathbf{r}(X) \in (SeS)^{\vee}$. Thus $\mathbf{r}(x) \in (SeS)^{\vee}$ and so $\mathbf{d}(x) \in (SeS)^{\vee}$ for all $x \in X$. Thus $f \in (SeS)^{\vee}$. □

Lemma 5.11. *Let C be a finitely aligned k-graph. The following are equivalent:*

(1) *C is cofinal.*
(2) *For all idempotents in $\mathsf{R}(C)$ of the form ee^{-1} and ff^{-1}, where $e, f \in C_o$, there exists a tight cover $\{x_1x_1^{-1}, \ldots, x_nx_n^{-1}\}$ of ff^{-1} and elements $a_1a_1^{-1}, \ldots, a_na_n^{-1} \leq ee^{-1}$ such that $a_1x_1^{-1}, \ldots, a_nx_n^{-1}$ are well-defined basic morphisms.*

We can now prove our main result about cofinality.

Theorem 5.12. *Let C be a strongly finitely aligned higher rank graph with a finite number of identities which has no sources and is row finite. Then the following are equivalent:*

(1) *C is cofinal.*
(2) *$\mathsf{B}(S)$ is 0-simplifying.*

Proof. (1)⇒(2). Let I be a non-trivial additive ideal of $\mathsf{B}(S)$. By Lemma 5.9, it contains an idempotent of the form $[ee^{-1}]$ for some $e \in C_o$. Let $f \in C_o$ be arbitrary. Then by Lemma 5.11, there exits a tight cover $\{x_1x_1^{-1}, \ldots, x_nx_n^{-1}\}$ of ff^{-1} and elements $a_1a_1^{-1}, \ldots, a_na_n^{-1} \leq ee^{-1}$ such that $a_1x_1^{-1}, \ldots, a_nx_n^{-1}$ are well-defined basic morphisms. Observe that $\bigvee \mathbf{d}[a_ix_i^{-1}] = [ff^{-1}]$ and that $\mathbf{r}[a_ix_i^{-1}] \leq [ee^{-1}]$. Thus $\{[a_1x_1^{-1}], \ldots [a_nx_n^{-1}]\}$ is a pencil from $[ff^{-1}]$ to $[ee^{-1}]$. But I is an additive ideal and so $[ff^{-1}] \in I$. Thus I contains every idempotent of the form $[ff^{-1}]$ where $f \in C_o$. Let $[ab^{-1}]$ be arbitrary. Then $\mathbf{d}(ab^{-1}) = bb^{-1} \leq \mathbf{r}(b)\mathbf{r}(b)^{-1}$ and $\mathbf{r}(ab^{-1}) = aa^{-1} \leq \mathbf{r}(a)\mathbf{r}(a)^{-1}$. But I being an additive ideal is also an order ideal. Thus $\mathbf{d}(ab^{-1}), \mathbf{r}(ab^{-1}) \in I$. But in an inverse semigroup, an ideal that contains $x^{-1}x$ must contain x. Thus $[ab^{-1}] \in I$. But I is additive and so I contains all elements of $\mathsf{B}(S)$. We have therefore shown that $I = \mathsf{B}(C)$.

(2)⇒(1). Let e and f be identities in the category C. Observe that $I = (\mathsf{B}(S)[ee^{-1}]\mathsf{B}(S))^{\vee}$ is an additive ideal of $\mathsf{B}(S)$ containing $[ee^{-1}]$. Then, by assumption, $I = \mathsf{B}(S)$. Thus $[ff^{-1}] \in I$. It follows by Lemma 5.10 that $[ff^{-1}] \preceq [ee^{-1}]$. Thus there is a pencil $X = \{\mathbf{x}_1, \ldots, \mathbf{x}_m\}$ in $\mathsf{B}(S)$ such that $[ff^{-1}] = \bigvee_{i=1}^{m} \mathbf{d}(\mathbf{x}_i)$ and $\mathbf{r}(\mathbf{x}_i) \leq [ee^{-1}]$. Without loss of generality, we can assume that $X = \{[a_1b_1^{-1}], \ldots, [a_mb_m^{-1}]\}$. We therefore have that $ff^{-1} \equiv \bigvee_{i=1}^{m} b_ib_i^{-1}$ and $a_ia_i^{-1} \equiv ee^{-1}a_ia_i^{-1}$. We carry out the multiplications $ee^{-1}a_ib_i^{-1}ff^{-1}$, remove any zero elements, write as joins of basic morphisms and then, without loss of generality, assume that our elements are all basic morphisms. We therefore obtain the following (relabelling where necessary): there is a set of elements $\{b_1, \ldots, b_m\} \subseteq fC$ which is large a set $\{a_1, \ldots, a_m\} \subseteq eC$ and the basic morphisms $a_1b_1^{-1}, \ldots, a_mb_m^{-1}$ are all defined. By Lemma 5.11, it follows that C is cofinal. □

By theorem 5.8 and Theorem 5.12, we have proved the following.

Theorem 5.13. *Let C be a finitely aligned higher rank graph with a finite number of identities which has no sources and is row finite. Then the following are equivalent:*

(1) *C is aperiodic and cofinal.*
(2) *The Boolean inverse monoid $\mathsf{B}(C)$ is simple.*

It is clear that the Boolean inverse monoids $\mathsf{B}(C)$ are countable. We now apply some results from [14]. We call the countable atomless Boolean algebra the *Tarski algebra* and use the term *Tarski monoid* to mean a countable, Boolean inverse $\wedge$-monoid whose semilattice of idempotents is the Tarski algebra. The following is [14, Proposition 4.4].

Proposition 5.14. *let S be a countable Boolean inverse $\wedge$-monoid. If S is 0-simplifying then either S is a Tarski monoid or the semilattice of idempotents of S is finite.*

By Lemma 5.1, a fundamental inverse semigroup in which the semilattice of idempotents is finite must itself be finite. The finite simple Boolean inverse monoids are precisely the finite symmetric inverse monoids [13, Theorem 4.18]; the groups of units of the finite symmetric inverse monoids are the finite symmetric inverse monoids. With the help of [14, Theorem 2.22], we have therefore proved the following which is our main theorem.

Theorem 5.15. *Let C be a finitely aligned higher rank graph with a finite number of identities which has no sources and is row finite. If C is aperiodic and cofinal then there are two possibilities:*

(1) *The Boolean inverse monoid* $\mathsf{B}(C)$ *is finite and isomorphic to a finite symmetric inverse monoid. Its group of units is a finite symmetric group.*
(2) *The Boolean inverse monoid* $\mathsf{B}(C)$ *is countably infinite. Its group of units is isomorphic to a full subgroup of the group of self-homeomorphisms of the Cantor space which acts minimally and in which each element has clopen support.*

6. The groupoid associated with $\mathsf{B}(C)$

The goal of this section is to prove that the étale groupoid associated with the Boolean inverse monoid $\mathsf{B}(C)$ under non-commutative Stone duality is the usual groupoid $\mathcal{G}(C)$ associated with the higher rank graph C.

Let C be a k-graph. Define Ω_k to be the category of all ordered pairs $(\mathbf{m}, \mathbf{n}) \in \mathbb{N}^k \times \mathbb{N}^k$ where $\mathbf{m} \leq \mathbf{n}$; see [9, Examples 1.7(ii)]. A *k-tiling* in C is a degree-preserving functor w from Ω_k to C. These are called *infinite paths* in C elsewhere in the literature (for example in [9]). Explicitly, w satisfies the following three conditions:

(1) $w(\mathbf{m}, \mathbf{m})$ is an identity.
(2) $w(\mathbf{m}, \mathbf{n})w(\mathbf{n}, \mathbf{p}) = w(\mathbf{m}, \mathbf{p})$
(3) $d(w(\mathbf{m}, \mathbf{n})) = \mathbf{n} - \mathbf{m}$.

Denote the set of all k-tilings of C by C^∞. For each $\mathbf{p}$ and $w \in C^\infty$ define $\sigma^{\mathbf{p}}(w) \in C^\infty$ by $\sigma^{\mathbf{p}}(w)(\mathbf{m}, \mathbf{n}) = w(\mathbf{p} + \mathbf{m}, \mathbf{p} + \mathbf{n})$. Put $e = w(\mathbf{0}, \mathbf{0})$, an identity. Denote by $C^\infty(e)$ all k-tilings w such that $e = w(\mathbf{0}, \mathbf{0})$.

Define $\mathcal{G}(C)$, the groupoid associated with C, as follows [9, 6]. Its elements are triples $(w_1, \mathbf{n}, w_w) \in C^\infty \times \mathbb{Z}^k \times C^\infty$ where there are elements $\mathbf{l}, \mathbf{m} \in \mathbb{N}^k$ such that $\mathbf{n} = \mathbf{l} - \mathbf{m}$ and $\sigma^{\mathbf{l}}(w_1) = \sigma^{\mathbf{m}}(w_2)$. Define $\mathbf{d}(w_1, \mathbf{n}, w_2) = (w_2, \mathbf{0}, w_2)$ and define $\mathbf{r}(w_1, \mathbf{n}, w_2) = (w_1, \mathbf{0}, w_1)$. Multiplication is defined by $(w_1, \mathbf{m}, w_2)(w_2, \mathbf{n}, w_3) = (w_1, \mathbf{m} + \mathbf{n}, w_3)$ and the inverse by $(w_2, \mathbf{n}, w_1)^{-1} = (w_1, -\mathbf{n}, w_2)$. A topology is defined on $\mathcal{G}(C)$ with basis elements of the form

$$Z(x, y) = \{(xw, d(x) - d(y), yw) \colon w \in C^\infty(\mathbf{d}(x))\}$$

where $\mathbf{d}(x) = \mathbf{d}(y)$.

The remainder of this section is devoted to proving that the groupoid $\mathcal{G}(C)$ is the étale groupoid associated with the Boolean inverse monoid $\mathsf{B}(C)$ when C is a k-graph.

Lemma 6.1. *Let C be a k-graph. For each k-tiling w, let*

$$\mathcal{C}_w := \{w(\mathbf{0}, \mathbf{m}) \colon \mathbf{m} \in \mathbb{N}^k\}.$$

Then the map $w \mapsto C_w$ is a bijection between k-tilings of C and good subsets of C.

Proof. Let $w \colon \Omega_k \to C$ be a k-tiling. Observe that $\mathbf{r}(w(\mathbf{0}, \mathbf{m})) = \mathbf{r}(w(\mathbf{0}, \mathbf{0})) = e$, say. Thus $\mathcal{C}_w \subseteq eC$. The elements of $\mathcal{C}_w$ are pairwise comparable and for each $\mathbf{n} \in \mathbb{N}^k$ there exists $x \in \mathcal{C}_w$ such that $d(x) = \mathbf{n}$. So $\mathcal{C}_w$ is a good subset.

Now let $A \subseteq C$ be a good subset. Then [9, Remarks 2.2] whose that there is a unique k-tiling w_A such that $\mathcal{C}_{w_A} = A$. □

Let A be a good subset of C. By Lemma 4.22, this is (precisely) a maximal filter in C. Define $\mathsf{F}(A) = \{[xx^{-1}] \colon x \in A\}^\uparrow$ in $\mathsf{B}(C)$. By Theorem 4.14 and Proposition 4.20, this is an ultrafilter in $\mathsf{B}(C)$. It is worth recalling the proof of this. If $x, y \in A$ then, since A is a good subset, we have that $xC \cap yC \cap A \neq \varnothing$ by Lemma 4.22. Thus $a = xu = yv$ for some $u, v \in C$ and $a \in A$. Observe that $aa^{-1} \leq xx^{-1}, yy^{-1}$. It follows that $[aa^{-1}] \leq [xx^{-1}], [yy^{-1}]$. It is a prime filter (and so an ultrafilter by Theorem 4.23 and [16, Lemma 3.20]) as a result of the following argument. Let $[xx^{-1}] \leq [yy^{-1}]$ where $x \in A$. Suppose that

$xC \cap yC = \{u_1, \ldots, u_m\}C$. Then $xx^{-1} \equiv \bigvee_{i=1}^{m} u_i u_i^{-1}$. Thus $\{u_1, \ldots, u_m\} \subseteq xC$ is a tight cover. It follows that $u_i \in A$, for some i, since good subsets are tight filters by Lemma 4.22. Now $u_i = yc$ for some $c \in C$. Thus $y \in A$.

Definition. Let C be a higher rank graph. A subset $A \subseteq C$ is called *expanding* if each pair of elements of A is comparable and for each $\mathbf{m} \in \mathbb{N}^k$ there exists $a \in A$ such that $d(a) \geq \mathbf{m}$.

Lemma 6.2. *Let C be a higher rank graph. Let A be an expanding subset of C. Define $\mathsf{Pref}(A)$ to be all elements $x \in C$ such that $a = xu$ for some $a \in A$ and $u \in C$. Then $\mathsf{Pref}(A)$ is a good subset.*

Proof. Let $x, y \in \mathsf{Pref}(A)$. Then $a = xu$ and $b = yv$ for some $a, b \in A$ and $u, v \in C$. The elements a and b are comparable. Thus $xuc = yvd$ for some $c, d \in C$. It follows that x and y are comparable. Now let $\mathbf{m} \in \mathbb{N}^k$ be arbitrary. Then there exists $a \in A$ such that $d(a) \geq \mathbf{m}$. Let $d(a) = \mathbf{m} + \mathbf{n}$. By the UFP, there exists $x, y \in C$ such that $a = xy$, $d(x) = \mathbf{m}$ and $d(y) = \mathbf{n}$. By definition, $x \in \mathsf{Pref}(A)$ and $d(x) = \mathbf{m}$. We have therefore proved that $\mathsf{Pref}(A)$ is a good subset. ☐

Lemma 6.3. *Let C be a k-graph. Suppose that A is an expanding subset such that $A \subseteq \mathbf{d}(x)C$. Then xA is an expanding subset.*

Proof. Let $xa, xb \in xA$. Then $a, b \in A$ and so are comparable. Thus $au = bv$ for some $u, v \in C$. It follows that $(xa)u = (xb)v$ and so xa and xb are comparable. Let $\mathbf{m} \in \mathbb{N}^k$. Then there exists $a \in A$ such that $d(a) \geq \mathbf{m}$. It follows that $d(xa) \geq \mathbf{m}$. We have therefore proved that xA is an expanding subset. ☐

Definition. Let C be a higher rank graph. Let A be a good subset and let $x \in A$. Define $x^{-1}A$ to be all elements a such that $xa \in A$.

Lemma 6.4. *Let C be a higher rank graph. Let A be a good subset and let $x \in A$. Then $x^{-1}A$ is an expanding set.*

Proof. The product $x\mathbf{d}(x)$ is defined. Thus $\mathbf{d}(x) \in x^{-1}A$. Let $u, v \in x^{-1}X$. Then $xu, xv \in A$. Thus $xub = xvc$ for some $b, c \in C$. By cancellation, $ub = vc$. Thus u and v are comparable. Let $\mathbf{m} \in \mathbb{N}^k$. Put $\mathbf{n} = \mathbf{m} + d(x)$. Then there exists $a \in A$ such that $\mathbf{n} = d(a)$. By the UFP, we can write $a = xy$ where $d(y) = \mathbf{m}$. Thus $y \in x^{-1}A$. ☐

The proof of the following is immediate.

Lemma 6.5. *Let C be a k-graph and let A be a good subset. Suppose that $\exists xy$ and $xy \in A$. Then $(xy)^{-1}A = y^{-1}(x^{-1}A)$.*

Let A be a good subset and let $\mathbf{m} \in \mathbb{N}^k$. Then there is a unique $x \in A$ such that $d(x) = \mathbf{m}$. Define

$$\sigma^{\mathbf{m}}(A) = x^{-1}A.$$

We shall now describe the groupoid $\mathcal{G}(C)$ using good subsets. We say that a triple $(A, \mathbf{n}, B)$ where A and B are good subsets and $\mathbf{n} \in \mathbb{Z}^k$ is *allowable* if there exists $x \in A$ and $y \in B$ such that $\mathbf{d}(x) = \mathbf{d}(y)$, $\mathbf{n} = d(x) - d(y)$, and $x^{-1}A = y^{-1}B$.

Lemma 6.6. *Let C be a k-graph. Let $(A, \mathbf{n}, B)$ be an allowable triple where A and B are good subsets, $\mathbf{n} \in \mathbb{Z}^k$, $x \in A$ and $y \in B$ are such that $\mathbf{d}(x) = \mathbf{d}(y)$, $\mathbf{n} = d(x) - d(y)$, and $x^{-1}A = y^{-1}B$. Let $\mathbf{m} \geq d(y)$. Then there exists $y_1 \in B$ such that $d(y_1) = \mathbf{m}$ and there is an $x_1 \in A$ such that $\mathbf{d}(x_1) = \mathbf{d}(y_1)$, $\mathbf{n} = d(x_1) - d(y_1)$, and $x_1^{-1}A = y_1^{-1}B$.*

Proof. Given any $\mathbf{m} \in \mathbb{N}^k$ there exists $y_1 \in B$ such that $d(y_1) = \mathbf{m}$ since B is a good subset. But y and y_1 are comparable, again because B is a good subset, and so by Lemma 3.2 there exists $t \in C$ such that $y_1 = yt$. Now $t \in y^{-1}B = x^{-1}A$. It follows that $xt \in A$. Define $x_1 = xt$. Then $d(x_1) - d(y_1) = \mathbf{m}$. The fact that $x_1^{-1}A = y_1^{-1}B$ follows by Lemma 6.5. □

Lemma 6.7. *Let C be a k-graph. The set of allowable triples forms a groupoid isomorphic to the groupoid $\mathcal{G}(C)$ defined above.*

Proof. Observe that for any good subset, $(B, \mathbf{0}, B)$ is a well-defined triple since B contains a unique identity. It follows that defining $\mathbf{d}(A, \mathbf{n}, B) = (B, \mathbf{0}, B)$ and $\mathbf{r}(A, \mathbf{n}, B) = (A, \mathbf{0}, A)$ makes sense. It is clear that if $(A, \mathbf{n}, B)$ is an allowable triple so, too, is $(B, -\mathbf{n}, A)$. Let $(A, \mathbf{m}, B)$ and $(B, \mathbf{n}, C)$ be allowable triples. The fact that $(A, \mathbf{m} + \mathbf{n}, C)$ is an allowable triple follows by Lemma 6.6. This proves that the allowable triples form a groupoid. To prove that the isomorphism holds we map $(w_2, \mathbf{m}, w_1)$ to $(\mathcal{C}_{w_2}, \mathbf{n}, \mathcal{C}_{w_1})$ using Lemma 6.1. This is a well-defined bijection and functor. □

The basis sets for the topology on the set of allowable triples has the following form. Let $x, y \in C$ such that $\mathbf{d}(x) = \mathbf{d}(y)$. Then

$$Z'(x, y) = \{(\mathsf{Pref}(xA), d(x) - d(y), \mathsf{Pref}(yA)) \colon A \subseteq \mathbf{d}(x)C \text{ is expanding}\}$$

is a set of allowable triples.

We shall now show that the groupoid $\mathcal{G}(C)$ is isomorphic to the groupoid constructed from the Boolean inverse monoid $\mathsf{B}(C)$ as described in [16]. We therefore need to relate ultrafilters in $\mathsf{B}(C)$ with allowable triples.

The following definitions are all expanded upon in [12, 13, 14, 16]. Let S be a Boolean inverse semigroup. Recall that an *ultrafilter* is a maximal proper filter. Let A be an ultrafilter in S. Define $\mathbf{d}(A) = (A^{-1}A)^{\uparrow}$ and $\mathbf{r}(A) = (AA^{-1})^{\uparrow}$. An *identity ultrafilter* is an ultrafilter containing an idempotent and this is equivalent to its being an inverse subsemigroup. Both $\mathbf{d}(A)$ and $\mathbf{r}(A)$ are identity ultrafilters. Observe that $A = (a\mathbf{d}(A))^{\uparrow}$ where $a \in A$. More generally, if F is any identity ultrafilter and $\mathbf{d}(a) \in F$ then $(aF)^{\uparrow}$ is an ultrafilter and if $\mathbf{r}(a) \in F$ then $(Fa)^{\uparrow}$ is an ultrafilter.

Lemma 6.1, Lemma 4.22 Proposition 4.20 and the fact that in C all tight filters are ultrafilters, we have proved the following.

Proposition 6.8. *Let C be a higher rank graph with a finite number of identities which is row finite and has no sources. For each k-tiling w of C, the set $\mathsf{P}_w := \{w(0, \mathbf{m})w(0, \mathbf{m})^{-1} : m \in \mathbb{N}^k\}^{\uparrow}\}$ is an ultrafilter in $\mathsf{E}(\mathsf{R}(C))$, and the map $w \mapsto \mathsf{P}_w$ is a bijection between k-tilings in C and ultrafilters in $\mathsf{E}(\mathsf{R}(C))$.*

We work with allowable triples.

Lemma 6.9. *Let C be a higher rank graph with a finite number of identities which is row finite and has no sources. Then there is a bijection between the set of allowable triples in C and the set of ultrafilters in $\mathsf{B}(C)$.*

Proof. Let $\tau = (A, \mathbf{n}, B)$ be an allowable triple. Then A and B are good sets, there is an $x \in A$, such that $d(x) = \mathbf{l}$, and there is $y \in B$, such that $d(y) = \mathbf{m}$, where $\mathbf{d}(x) = \mathbf{d}(y)$, $\mathbf{n} = \mathbf{l} - \mathbf{m}$ and $x^{-1}A = y^{-1}B$. Observe that xy^{-1} is a well-defined morphism in $\mathsf{R}(C)$. Define $\mathcal{F} = \mathsf{F}(B)$. Then $\mathcal{F}$ is an identity filter in $\mathsf{B}(C)$. Put

$$\mathcal{A} = \mathcal{A}_\tau = ([xy^{-1}]\mathcal{F})^{\uparrow};$$

this is the ultrafilter in $\mathsf{B}(C)$ associated with the allowable triple τ.

We now show that this ultrafilter is independent of the choices we made above. Let $x_1 \in A$ be such that $d(x_1) = \mathbf{l}_1$ and let $y_1 \in B$ be such that $d(y_1) = \mathbf{m}_1$ where

$\mathbf{d}(x_1) = \mathbf{d}(y_1)$, $\mathbf{n} = \mathbf{l}_1 - \mathbf{m}_1$ and $x_1^{-1}A = y_1^{-1}B$. We show that $\mathcal{B} = ([x_1y_1^{-1}]\mathcal{F})^{\uparrow}$ and $\mathcal{A} = ([xy^{-1}]\mathcal{F})^{\uparrow}$ are equal. Consider the product $(yx^{-1})(x_1y_1^{-1})$. Let $xC \cap x_1C = \{u_1, \ldots, u_m\}C$. Since $x, x_1 \in A$ there exists an i such that $u_i = xp_i = x_1q_i \in A$ by Lemma 4.18. Observe that $(yp_i)(y_1q_i)^{-1} \leq (yx^{-1})(x_1y_1^{-1})$. We have that $yp_i, y_1q_i \in B$ — ths follows because $p_i \in x^{-1}A = y^{-1}B$ and so $yp_i \in B$, and $q_i \in x_1^{-1}A = y_1^{-1}B$ and so $y_1q_i \in B$. We shall prove that $d(yp_i) = d(y_1q_i)$ from which it will follow that $yp_i = y_1q_i$, since B is a filter and so a good subset. But this follows from the fact that $\mathbf{l}_1 - \mathbf{m}_1 = \mathbf{l} - \mathbf{m}$. We have therefore found an idempotent in $\mathcal{F}$ below $[yx^{-1}][x_1y_1^{-1}]$. This proves that $\mathcal{A} = \mathcal{B}$.

We now go in the opposite direction. Let $\mathcal{A}$ be an ultrafilter in $\mathsf{B}(C)$. Then, using the fact that ultrafilters are prime, we may write this in the form $([xy^{-1}]\mathcal{F})^{\uparrow}$ where $[yy^{-1}] \in \mathcal{F}$ and $\mathcal{F}$ is an idempotent ultrafilter in $\mathsf{B}(C)$. The ultrafilter $\mathcal{F}$ is completely determined by the ultrafilter $\mathcal{F} \cap \mathsf{E}(\mathsf{B}(C))$ which is in $\mathsf{E}(\mathsf{B}(C))$. The ultrafilter $\mathcal{F} \cap \mathsf{E}(\mathsf{B}(C))$ arises from the maximal filter B in C via Proposition 4.20. Observe that $y \in B$. Now $([xy^{-}]\mathcal{F}[yx^{-1}])^{\uparrow}$ is an idempotent ultrafilter in $\mathsf{B}(C)$ that contains $[xx^{-1}]$. This corresponds to a maximal filter A in C that contains x. We have therefore constructed a triple $(A, d(x) - d(y), B)$. It remains to show that it is allowable. Thus we need to prove that $x^{-1}A = y^{-1}B$. let $u \in y^{-1}A$. Then $yu \in A$. It follows that $[yu(yu)^{-1}] \in \mathcal{F} = \mathbf{d}(\mathcal{A})$. But $[xy^{-1}] \in \mathcal{A}$. It follows that $[xy^{-1}][yu(yu)^{-1}] \in \mathcal{A}$. Thus $[xu(yu)^{-1}] \in \mathcal{A}$. It folows that $xu \in B$, as required. The converse is proved by symmetry. □

We now prove that the groupoid of allowable triples and the groupoid of ulrafilters are isomorphic. By Lemma 6.6, a composable pair of allowable triples has the following form:

$$(C, d(u) - d(v), B)(B, d(v) - d(w), A)$$

where $u \in C$, $v \in B$ and $w \in A$. Their product is

$$(C, d(u) - d(w)A).$$

We now turn to products of ultrafilters. Let A and B be ultrafilters such that $\mathbf{d}(A) = \mathbf{r}(B)$. Then $A \cdot B = (AB)^{\uparrow}$. Let $A = (aF)^{\uparrow}$ and $B = (bG)^{\uparrow}$ where $\mathbf{d}(A) = F$ and $\mathbf{d}(B) = G$. Observe that $A \cdot B = (abG)^{\uparrow}$. Now, $a\mathbf{r}(b) \in A\mathbf{d}(A) \subseteq A$. Similarly, $\mathbf{d}(a)b \in B$. It follows that we can assume $\mathbf{d}(a) = \mathbf{r}(b)$. The ultrafilter associated with $(C, d(u) - d(v), B)$ is $([uv^{-1}]\mathcal{F})^{\uparrow}$ where $x \in B$ iff $[xx^{-1}] \in \mathcal{F}$. The ultrafilter associated with $(B, d(v) - d(w), A)$ is $([vw^{-1}]\mathcal{G})^{\uparrow}$. The product of $([uv^{-1}]\mathcal{F})^{\uparrow}$ and $([vw^{-1}]\mathcal{G})^{\uparrow}$ is defined and equals $([uw^{-1}]\mathcal{G})^{\uparrow}$ which corresponds to the allowable triple $(C, d(u) - d(w)A)$.

We now describe the topology. Let $\mathbf{d}(x) = \mathbf{d}(y)$. Then

$$Z'(x, y) = \{(\mathsf{Pref}(xA), d(x) - d(y), \mathsf{Pref}(yA)) \colon A \subseteq \mathbf{d}(x)C\}$$

is a set of allowable triples. This corresponds to the set of ultrafilters in $\mathsf{B}(C)$ that contain the element $[xy^{-1}]$. It follows that the groupoids are isomorphic as topological groupoids.

7. Invariants of the group $\mathcal{G}(C)$

As mentioned earlier, since our group $\mathcal{G}(C)$ coincides with the topological full group of the groupoid $\mathcal{G}(C)$, which is a Hausdorff, étale, effective, minimal groupoid with unit space homeomorphic to the Cantor space, [22, Theorem 3.10] implies that if $\mathcal{G}(C) \cong \mathcal{G}(C')$, then $\mathcal{G}(C) \cong \mathcal{G}(C')$. Consequently both the K-theory of the groupoid C^*-algebra $C^*(\mathcal{G}(C))$ and the homology, in the sense of Matui, of the groupoid $\mathcal{G}(C)$ are isomorphism invariants of $\mathcal{G}(C)$. When C is a k-graph, this is particularly interesting because, as we saw in the preceding section, $\mathcal{G}(C)$ coincides with Kumjian and Pask's groupoid $\mathcal{G}_C$ [9]. So known invariants of $\mathcal{G}_C$ are also invariants of our group $\mathcal{G}(C)$. Kumjian and Pask prove in [9, Corollary 3.5(i)] that the k-graph C^*-algebra $C^*(C)$ coincides with the groupoid C^*-algebra $C^*(\mathcal{G}_C)$. Hence the K-theory of $C^*(C)$ provides an isomorphism invariant of $\mathcal{G}(C)$. We therefore have the following.

Corollary 7.1. *Let C and C' be row-finite aperiodic, cofinal higher rank graphs with finitely many vertices and no sources. If $\mathcal{G}(C) \cong \mathcal{G}(C')$ as discrete groups, then $K_*(C^*(C)) \cong K_*(C^*(C'))$.*

The difficulty here is that the K-theory of k-graph C^*-algebras has proven notoriously difficult to compute. The most general results are those of [3], but these apply in general only when $k \leq 2$. However another, closely related, invariant of the groupoid $\mathcal{G}_C$ and hence of our group $\mathcal{G}(C)$ is the homology of $\mathcal{G}_C$.

Corollary 7.2. *Let C and C' be row-finite aperiodic, cofinal higher rank graphs with finitely many vertices and no sources. If $\mathcal{G}(C) \cong \mathcal{G}(C')$ as discrete groups, then $H_*(\mathcal{G}_C) \cong H_*(\mathcal{G}_{C'})$.*

In this case, we have an explicit calculation of the invariant obtained from [5, Proposition 7.6] building on earlier work of Matui [21]. Specifically, the homology of the groupoid $\mathcal{G}_C$ is precisely the homology of a chain complex developed by Evans [3]. To describe it, we proceed as follows.

Let $\varepsilon_1, \dots, \varepsilon_k$ denote the generators of $\mathbb{Z}^k$ (so $\varepsilon_i = (0, \dots, 0, 1, 0, \dots, 0)$, with the 1 appearing in the ith coordinate). Also for each i, let M_i be the ith coordinate matrix of the k-graph C. That is, recalling that C_o is the (finite) space of identity morphisms (or vertices) of C, the matrix M_i is the $C_o \times C_o$ matrix with entries

$$M_i(e, f) = |\{a \in C : d(a) = \varepsilon_i, \mathbf{r}(\mathbf{a}) = e \text{ and } \mathbf{d}(a) = f\}|.$$

We regard each M_i as an endomorphism of the free abelian group $\mathbb{Z}C_o$. Recall that for $p \geq 1$, we write $\bigwedge^p \mathbb{Z}^k$ for the pth exterior power of $\mathbb{Z}^k$, which is generated by the elements $\varepsilon_{i_1} \wedge \cdots \wedge \varepsilon_{i_p}$ where $1 \leq i_j \leq k$ for all j. Define $D_0^C = \mathbb{Z}C_o$, and for $p \geq 1$, define $D_p^C = \left(\bigwedge^p \mathbb{Z}^k\right) \otimes \mathbb{Z}C_{\mathbf{0}}$; observe that this forces $D_p^C = \{0\}$ for $p \geq k$. For $p \geq 2$, define $\partial_p \colon D_p^C \to D_{p-1}^C$ (using the hat symbol to indicate deletion of a term) by

$$\partial_p((\varepsilon_{i_1} \wedge \cdots \wedge \varepsilon_{i_p}) \otimes \varepsilon_e) = \sum_{j=0}^{p} (-1)^{j+1} (\varepsilon_{i_1} \wedge \cdots \wedge \widehat{\varepsilon}_{i_j} \wedge \cdots \wedge \varepsilon_{i_p}) \otimes (I - M_{i_j}^t)\varepsilon_e;$$

and, finally, define $\partial_1 \colon D_1^C \to D_0^C$ by

$$\partial_1(\varepsilon_i \otimes \varepsilon_e) = (1 - M_i^t)\varepsilon_e.$$

Then (D_*^C, ∂_*) is a chain complex, and [5, Proposition 7.6] shows that $H_*(D_*^C, \partial_*)$ is isomorphic to the groupoid homology $H_*(\mathcal{G}_C)$. So we obtain the following corollary.

Corollary 7.3. *Let C and C' be row-finite aperiodic, cofinal higher rank graphs with finitely many vertices and no sources. If $\mathcal{G}(C) \cong \mathcal{G}(C')$ as discrete groups, then $H_*(D_*^C, \partial_*) \cong H_*(D_*^{C'}, \partial_*)$. In particular,*

$$H_k(D_*^C, \partial_*) = \bigcap_{i=1}^{k} \ker(I - M(C)_i^t) \cong \bigcap_{i=1}^{k} \ker(I - M(C')_i^t),$$

and

$$H_0(D_*^C, \partial_*) = \mathbb{Z}C_o / \Big(\sum_{i=1}^{k} \operatorname{im}(I - M(C)_i^t) \Big) \cong \mathbb{Z}C'_o / \Big(\sum_{i=1}^{k} \operatorname{im}(I - M(C')_i^t) \Big).$$

Proof. The first statement follows immediately from [22, Theorem 3.10] and [5, Proposition 7.6]. The second is by direct computation: the intersection of the kernels of the $I - M(C)^t_i$ is isomorphic to $H_k(D^C_*, \partial_*)$ and the quotient of $\mathbb{Z}C_o$ by the sum of their images is isomorphic to $H_0(D^C_*, \partial_*)$ (and similarly for C'). □

8. Two series of concrete examples

In this section we provide two constructions of infinite families of k-graphs C with mutually non-isomorphic groups $\mathcal{G}(C)$. In the first family, the higher-rank graphs C can all be chosen to be of the same rank $k \geq 2$, and the associated groups $\mathcal{G}(C)$ are distinguished by the finite 0th homology groups of the k-graphs. In the second family of examples, we show that for each $k \geq 1$ and each $R \geq 1$, there is an aperiodic cofinal k-graph $C_{k,R}$ that is row-finite and has finitely many vertices and whose kth homology group is of rank R.

8.1. **Examples distinguished by their 0th homology.** We shall use the results of the previous section to construct, for each $k \geq 2$ and each k-tuple of integers $(m_1, m_2, \ldots, m_k)$, two series of pairwise non-isomorphic k-graphs with two vertices and different groups

$$\mathbb{Z}C_o / \Big(\sum_{i=1}^{k} \operatorname{im}(I - M(C)^t_i) \Big).$$

Our construction consists of two steps: first, we construct a family of cube complexes with two vertices, covered by products of k trees, and second, we explain how to get a k-graph from each complex.

For background on cube complexes covered by products of k trees see [26] and references in the paper.

Step 1. Let $X_1, ..., X_k$ be distinct alphabets, such that $|X_i| = m_i$ and

$$X_i = \{x^i_1, x^i_2, ..., x^i_{m_i}\}.$$

Let F_i be the free group generated by X_i. The direct product

$$G = F_1 \times F_2 \times \ldots \times F_k$$

has presentation

$$G = \langle X_1, X_2, ..., X_k | [x^i_s, x^j_l] = 1, i \neq j = 1, ..., k; s = 1, ..., m_i; l = 1, ..., m_j \rangle,$$

where $[x, y]$ denotes the commutator $xyx^{-1}y^{-1}$.

The group G acts simply and transitively on a Cartesian product Δ of k trees $T_1, T_2, .., T_k$ of valencies $2m_1, 2m_2, ..., 2m_k$ respectively: each T_i is identified with the Cayley tree of F_i, and the action of G is the coordinatewise action of the component groups F_i.

The quotient of this action is a cube complex P with one vertex v such that the universal cover of P is Δ. The edges of the cube complex P are naturally labelled by elements of $X = X_1 \cup X_2 ... \cup X_k$, and are naturally oriented by the usual algebraic ordering on F_i. The 1-skeleton of P is a wedge of $\sum_{i=1}^k m_i$ circles.

We construct a family of double covers of P in the following way. Consider a labelling $\ell : X \to \mathbb{Z}_2$ of the elements of X (equivalently the edges of the 1-skeleton of P). We obtain a cover P^2_ℓ of P whose vertex set is $\{v\} \times \mathbb{Z}_2$ and whose set of 1-cubes is $X \times \mathbb{Z}_2$, with range and domain maps given by $r(x, i) = (r(x), i)$ and $s(x, i) = (s(x), i + \ell(x))$. Specifically, P^2_ℓ is the quotient of Δ by the action of the kernel of the homomorphism $G \to \mathbb{Z}_2$ induced by ℓ. Observe that in P^2_ℓ, for a given $x \in X$ either $(x, 0)$ and $(x, 1)$ are loops based at $(v, 0)$ and $(v, 1)$ (if $\ell(x) = 0$), or $(x, 0)$ is an edge from $(v, 1)$ to $(v, 0)$ and $(x, 1)$ is an edge from $(v, 0)$ to $(v, 1)$.

So Figure 1 illustrates the 2-cubes and part of the 1-skeleton of P corresponding to symbols $a \in X_i$ and $b \in X_j$ with $i \neq j$ in which $\ell(a) = 1$ and $\ell(b) = 0$; Figure 2 illustrates the corresponding 2-cubes and part of the 1-skeleton if $\ell(a) = \ell(b) = 1$.

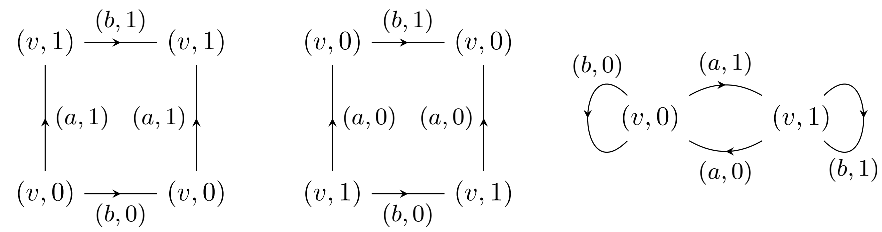

FIGURE 1. $\ell(a) = 1$ and $\ell(b) = 0$

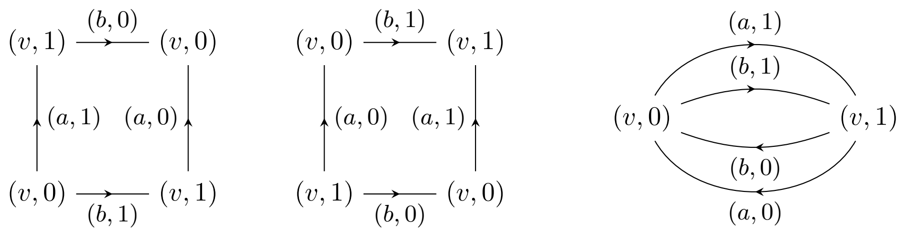

FIGURE 2. $\ell(a) = \ell(b) = 1$

We will consider two specific labellings ℓ_u and ℓ_m (the u and m stand for "uniform" and "mixed"). The uniform labelling ℓ_u is given by $\ell_u(x) = 1$ for all x, whereas the mixed labelling ℓ_m satisfies $\ell_m|_{X_1} \equiv 0$ and $\ell_m|_{X \setminus X_1} \equiv 1$. So under ℓ_u all 2-squares are as in Figure 2; but under ℓ_m squares in which b belongs to X_1 are as in Figure 1, and the remaining squares are as in Figure 2. Figure 3 illustrates a 3-cube in the cube complex for ℓ_u (left) and a 3-cube in the cube complex for ℓ_m in which a belongs to X_1 and the edges b, c belong to $X \setminus X_1$ (right).

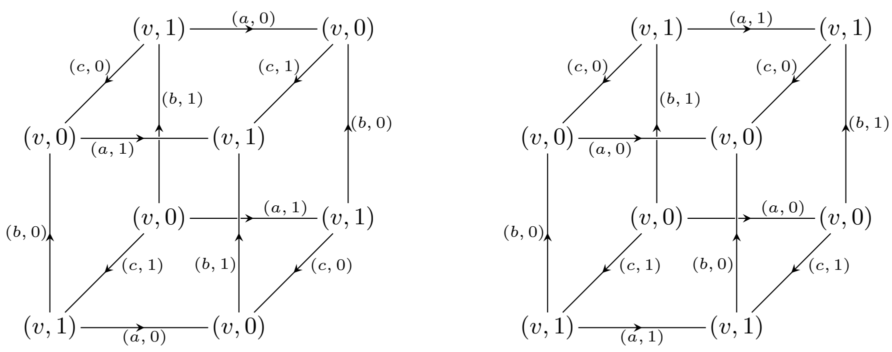

FIGURE 3. A 3-cube for ℓ_u (left) and for ℓ_m (right)

Step 2. For each of $\star = u$ and $\star = m$, we explain how to construct a k-graph $C_\star$ from $P^2_{\ell_\star}$, by specifying their skeletons and factorisation rules as in [7]. For either value of $\star$, we define $(C_\star)_o = \{(v,0),(v,1)\}$, and for each $i \le k$, we define $(C_\star)_{e_i}$ to be the set

$$X_i \times \mathbb{Z}_2 \sqcup \{(\overline{x}, j) : x \in X_i, j \in \mathbb{Z}_2\},$$

a disjoint union of two copies $X \times \mathbb{Z}_2$. The range and source maps on $(C_\star)_{e_i}$ restrict to those in $P^2_{\ell_\star}$ on $X_i \times \mathbb{Z}_2$, and we define $r(\overline{x}, j) = s(x,j)$ and $s(\overline{x}, j) = r(x,j)$. The factorisation rules are as follows: for each 2-cube

$$\begin{array}{ccc} (v,1) & \xrightarrow{(b,j_b)} & (v,1) \\ {\scriptstyle (a,i_a)}\uparrow & & \uparrow{\scriptstyle (a,j_a)} \\ (v,0) & \xrightarrow[(b,i_b)]{} & (v,0) \end{array}$$

in $P^2_{\ell_\star}$, we have four factorisation rules:

$$(b,i_b)(a,i_a) = (a,j_a)(b,j_b), \qquad (a,i_a)(\overline{b},j_b) = (\overline{b},i_b)(a,j_a),$$
$$(\overline{a},j_a)(b,i_b) = (b,j_b)(\overline{a},i_a), \qquad (\overline{b},j_b)(\overline{a},j_a) = (\overline{a},i_a)(\overline{b},i_b)$$

That these factorisation rules satisfy the associativity condition of [7] follows from a routine calculation using that the $P^2_{\ell_\star}$ is a quotient of of a direct product of trees.

To proceed, we define matrices

$$D_i := \begin{pmatrix} 2m_i & 0 \\ 0 & 2m_i \end{pmatrix} \quad \text{and} \quad T_i := \begin{pmatrix} 0 & 2m_i \\ 2m_i & 0 \end{pmatrix}.$$

Observe that in C_u the adjacency matrices $M(C)_i$ are equal to T_i, while in C_m, we have $M(C)_1 = D_1$ and $M(C)_i = T_i$ for $2 \le i \le k$. Routine calculations using this show that

$$\mathbb{Z}(C_u)_o / \Big(\sum_{i=1}^k \operatorname{im}(I - M(C_u)_i^t) \Big) \cong \mathbb{Z}/\gcd(4m_1^2 - 1, \dots, 4m_k^2 - 1)\mathbb{Z}, \quad \text{and}$$
$$\mathbb{Z}(C_m)_o / \Big(\sum_{i=1}^k \operatorname{im}(I - M(C_m)_i^t) \Big) \cong \mathbb{Z}/\gcd(2m_1 - 1, 4m_2^2 - 1, \dots, 4m_k^2 - 1)\mathbb{Z}.$$

Corollary 8.1. *There are at least two non-isomorphic groups $\mathcal{G}(C)$ for each $k \ge 2$ and infinitely many choices of alphabets $Y_1, \dots, Y_k$ of sizes $2m, \dots, 2m$.*

Proof. Take $m_1 = \cdots = m_k$ in the examples above. We obtain $H_0(D_*^{C_u}, \partial_*) \cong \mathbb{Z}/(4m^2-1)\mathbb{Z}$, and since $2m-1$ divides $4m^2-1$ we have $H_0(D_*^{C_m}, \partial_*) \cong \mathbb{Z}/(2m-1)\mathbb{Z}$. The result then follows from Corollary 7.3. □

Note that the known groups nV can be presented in our language using an n-graph C with 1×1 adjacency matrices with single entry equal 2. It is relatively easy to check that each $\ker(\partial_j)$ in Evans' complex is a subgroup of a direct sum of kernels of the maps $I - M(C)_i^t$, which in this instance are all equal to the 1×1 matrix (-1). So $H_j(C) = 0$ for $1 \le j \le n$. Also, $H_0(C) = \mathbb{Z}/\big(\sum_{i=1}^n I - M(C)_i^t \mathbb{Z}\big) = \mathbb{Z}/(-1)\mathbb{Z} = 0$. Hence the homology groups of all of these n-graphs are trivial. In particular none of the groups $\mathcal{G}(C)$ discussed above are isomorphic to the groups nV.

8.2. **Examples with nontrivial kth homology.** Fix integers $k, R \geq 1$. We will construct a k-graph $C_{k,R}$ by specifying its skeleton and factorisation rules as in [7].

The vertex set V of the skeleton E has $R+1$ elements (for example, we could take $V = \{1, \dots, R+1\}$, but to lighten notation, we will avoid choosing a particular enumeration).

For each $i \leq k$, the set of edges of E of colour i is

$$\{e^i_{v,m,w} : v \neq w \in V \text{ and } 1 \leq m \leq 2\} \cup \{e^i_{v,m,v} : v \in V \text{ and } 1 \leq m \leq 3\}$$

and the range and domain maps are given by $\mathbf{r}(e^i_{v,m,w}) = v$ and $\mathbf{d}(e^i_{v,m,w}) = w$.

So for any two distinct vertices v, w, there are 2 edges of each colour pointing from v to w (and also from w to v), and there are 3 loops of each colour at each vertex.

For example, in the skeleton of $C_{k,2}$, each of the singly-coloured subgraphs is as in Figure 4.

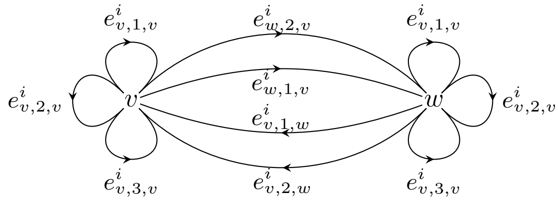

FIGURE 4. One of the singly-coloured subgraphs in the skeleton of $C_{k,2}$

We must now specify factorisation rules. For $j \neq l \leq k$, the jl-coloured paths are those of the form $e^j_{u,m,v} e^l_{v,n,w}$, and we must specify factorisation rules that determine range and source preserving bijections between jl-coloured paths and lj-coloured paths and that satisfy the associativity condition [7, Equation (3.2)].

We define them in two cases:

(F1) if $u, v \in V$ are distinct, then we define $e^i_{u,m,u} e^j_{u,n,v} = e^j_{u,n,v} e^i_{v,m,v}$.

(F2) if either $u = v = w$ or $u \neq v$ and $v \neq w$, we define $e^i_{u,m,v} e^j_{v,n,w} = e^j_{u,n,v} e^i_{v,m,w}$.

It is routine to check that these factorisation rules determine a complete collection of squares as in [7, p.578]. Routine but tedious calculations also verify the associativity condition [7, Equation (3.2)].

By [7, Theorem 4.4] there is a k-graph $C_{k,R}$ whose skeleton is the coloured graph E, and whose factorisation rules are given by (F1) and (F2). Since for all $v, w \in V$ we have $v(C_{k,R})_{e_1} w \neq \emptyset$, it is immediate that $C_{k,R}$ is cofinal. To see that it is aperiodic, let B_{R+1} be the 1-graph whose skeleton is a bouquet of $R+1$ loops at a single vertex. Fix a vertex $v \in V$ and observe that the sub-k-graph generated by the edges $\{e^i_{(v,m,v)} : i \leq k \text{ and } m \leq R+1\}$ is isomorphic to the cartesian product $\prod_{i-1}^k B_{R+1}$ of k copies of B_{R+1}. Since the 1-graph B_{R+1} is aperiodic, so is the k-graph $\prod_{i-1}^k B_{R+1}$, and so it contains an aperiodic infinite path. This is then an aperiodic infinite path in $C_{k,R}$.

We now observe that for each $i \leq k$, the matrix $(1 - M(C_{k,R})^t_i)$ is the $(R+1) \times (R+1)$ integer matrix

$$A_R := \begin{pmatrix} -2 & -2 & \cdots & -2 \\ -2 & -2 & \cdots & -2 \\ \vdots & \vdots & & \vdots \\ -2 & -2 & \cdots & -2 \end{pmatrix}.$$

Consequently $H_k(C_{k,R}) = \ker(A_R) \cong \mathbb{Z}^R$.

We have now proved the following.

Corollary 8.2. *For each positive integer k, there exists a family $\{C_{k,R} : R \geq 2\}$ of k-graphs, such that $\mathcal{G}(C_{k,R}) \cong \mathcal{G}(C_{k',R'})$ only if $k = k'$ and $R = R'$. Moreover, for any k, R, the group $\mathcal{G}(C_{k,R})$ is not isomorphic to $\mathcal{G}(\mathcal{C}')$ for any l-graph C' with $l < k$.*

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