PUSHOUTS OF EXTENSIONS OF GROUPOIDS BY BUNDLES OF ABELIAN GROUPS

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We respectfully dedicate this paper to the memory of Vaughan Jones: Extraordinary mathematician, proud New Zealander, and gracious colleague.

Abstract. We analyse extensions Σ of groupoids $\mathcal{G}$ by bundles $\mathcal{A}$ of abelian groups. We describe a pushout construction for such extensions, and use it to describe the extension group of a given groupoid $\mathcal{G}$ by a given bundle $\mathcal{A}$. There is a natural action of Σ on the dual of $\mathcal{A}$, yielding a corresponding transformation groupoid. The pushout of this transformation groupoid by the natural map from the fibre product of $\mathcal{A}$ with its dual to the Cartesian product of the dual with the circle is a twist over the transformation groupoid resulting from the action of $\mathcal{G}$ on the dual of $\mathcal{A}$. We prove that the full C^*-algebra of this twist is isomorphic to the full C^*-algebra of Σ, and that this isomorphism descends to an isomorphism of reduced algebras. We give a number of examples and applications.

Introduction

There is a significant body of literature regarding the C^*-algebras of extensions of groupoids by group bundles. The main goal of this paper is to introduce a pushout construction for extensions of groupoids by abelian group bundles and explore its applications.

Specifically, we consider a locally compact Hausdorff groupoid $\mathcal{G}$ together with an abelian group bundle $p_{\mathcal{A}} : \mathcal{A} \to \mathcal{G}^{(0)}$ where $p_{\mathcal{A}}$ a continuous, open map. Then we

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consider unit space fixing extensions

$$(\dagger)\qquad \begin{array}{ccccc} \mathcal{A} & \xrightarrow{\iota} & \Sigma & \xrightarrow{p} & \mathcal{G} \\ & \searrow\!\!\searrow & \downarrow\downarrow & \swarrow\!\!\swarrow & \\ & & \mathcal{G}^{(0)} & & \end{array}$$

where Σ is a locally compact Hausdorff groupoid, both ι and p are groupoid homomorphisms, p is a continuous, open surjection inducing a homeomorphism of $\Sigma^{(0)}$ and $\mathcal{G}^{(0)}$, ι is a homeomorphism of $\mathcal{A}$ onto $\ker p$.

A fundamental class of such examples are $\mathbf{T}$-groupoids (also called twists) introduced by the second author in [Kum83]. Then $\mathcal{A}$ is the trivial bundle $\mathcal{G}^{(0)} \times \mathbf{T}$ such that $\iota(r(\sigma), z)\sigma = \sigma\iota(s(\sigma), z)$ for all $\sigma \in \Sigma$ and $z \in \mathbf{T}$. These groupoids and their restricted groupoid C^*-algebras, $C^*(\mathcal{G}; \Sigma)$, have enjoyed considerable scrutiny [MW92, MW95, Kum83, Kum86]. As usual, in this context we often write $\dot\sigma$ in place of $p(\sigma)$.

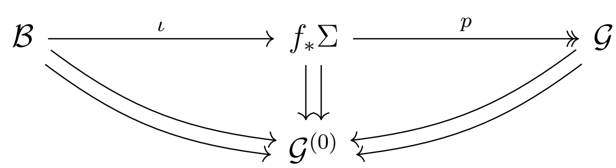
More recently, we considered more general extensions in [IKSW19] and [IKR+21] as in ($\dagger$) where it is assumed that $\mathcal{A}$ is endowed with an action of $\mathcal{G}$ and that the extension is compatible in the sense that $\sigma\iota(a)\sigma^{-1} = \iota(\dot\sigma \cdot a)$ for all $a \in \mathcal{A}$ and $\sigma \in \Sigma$ such that $p_{\mathcal{A}}(a) = s(\sigma)$.

As a consequence of the main result in [IKR+21], we showed that if Σ has a Haar system, then $C^*(\Sigma)$ can be realized as the C^*-algebra of a twist. Specifically, the action of $\mathcal{G}$ on $\mathcal{A}$ induces a natural action of $\mathcal{G}$ on $\hat{\mathcal{A}}$ (regarded as a space). We constructed a $\mathbf{T}$-groupoid $\widetilde{\Sigma}$ of the form

$$(\ddagger)\qquad \begin{array}{ccccc} \hat{\mathcal{A}} \times \mathbf{T} & \xrightarrow{i} & \widetilde{\Sigma} & \xrightarrow{j} & \hat{\mathcal{A}} \rtimes \mathcal{G}. \\ & \searrow\!\!\searrow & \downarrow\downarrow & \swarrow\!\!\swarrow & \\ & & \hat{\mathcal{A}} & & \end{array}$$

We proved ([IKR+21, Theorem 3.4]) that $C^*(\Sigma)$ is isomorphic to the restricted C^*-algebra $C^*(\hat{\mathcal{A}} \rtimes \mathcal{G}; \widetilde{\Sigma})$ of this $\mathbf{T}$-groupoid. (In [IKR+21] the $\mathbf{T}$-groupoid is denoted $\widehat{\Sigma}$, but here we use $\widetilde{\Sigma}$ to avoid possible confusion in our examples.) The $\mathbf{T}$-groupoid $\widetilde{\Sigma}$ is at the heart of the Mackey obstruction which appears in the classical "Mackey machine" of [Mac58].

The chief motivation for this article is the observation that the $\mathbf{T}$-groupoid $\widetilde{\Sigma}$ above—which was based on the construction of [MRW96, Proposition 4.3]—is derived from a natural and functorial "pushout" construction based on the second author's work in [Kum88] for étale groupoids (there called "sheaf groupoids"). Specifically, suppose we are given and extension as in ($\dagger$), and abelian group bundle $\mathcal{B}$ admitting a $\mathcal{G}$-action, and a equivariant groupoid homomorphism $f : \mathcal{A} \to \mathcal{B}$. Then there is a similar sort of extension



inducing the given $\mathcal{G}$-action on $\mathcal{B}$. In Theorem 1.5, we show that the construction producing $f_*\Sigma$ has good functorial properties that characterize the extension up to a suitable notion of isomorphism. Using these properties, we show in Theorem 2.5 that the collection $T_{\mathcal{G}}(\mathcal{A})$ of isomorphism classes of extensions of $\mathcal{A}$ by $\mathcal{G}$ forms an abelian group (see also [Tu06, §5.3]).

We close by illustrating how the pushout construction clarifies and interacts with our work in [IKSW19] and [IKR$^+$21]. In Theorem 3.2 we prove that the extension (‡) employed in [IKR$^+$21] arises from our pushout construction. Specifically, the natural pairing $(\chi, a) \mapsto \chi(a)$ from $\hat{\mathcal{A}} * \mathcal{A}$ to $\mathbf{T}$ yields a groupoid homomorphism $f : \hat{\mathcal{A}} * \mathcal{A} \to \hat{\mathcal{A}} \times \mathbf{T}$ given by $f(\chi, a) = (\chi, \chi(a))$ (see Section 3.1). There is a natural action of Σ on $\hat{\mathcal{A}}$ (compatible with that of $\mathcal{G}$ as above) and we prove that $\widetilde{\Sigma} \cong f_*(\hat{\mathcal{A}} \rtimes \Sigma)$. This allows us to realise the C^*-algebra of an extension of a groupoid $\mathcal{G}$ by an abelian group bundle $\mathcal{A}$ as the C^*-algebra of a $\mathbf{T}$-groupoid over the resulting transformation groupoid $\hat{\mathcal{A}} \rtimes \mathcal{G}$.

Several consequences flow from this observation. First suppose that A is an abelian group and that $\mathcal{A} = \mathcal{G}^{(0)} \times A$, carrying the action of $\mathcal{G}$ that is trivial in the second coordinate, so that Σ is a generalised twist. Each $\chi \in \hat{A}$ defines a homomorphism $f^\chi : \mathcal{A} \to \mathbf{T} \times \mathcal{G}^{(0)}$, so we can form the resulting pushout $f^\chi_*(\Sigma)$. We prove in Proposition 3.6 that $C^*(\Sigma)$ is the section algebra of an upper-semicontinuous C^*-bundle over $\hat{A}$ with fibres $C^*(\mathcal{G}, f^\chi_*(\Sigma))$. When A is compact, this yields a direct sum decomposition which remains valid for the corresponding reduced C^*-algebras (see Proposition 3.7). In Corollary 3.10 we extend [IKR$^+$21, Theorem 3.4] to the case that Ω is a $\mathbf{T}$-groupoid extension of Σ such that its restriction to $\iota(\mathcal{A})$ is abelian. When $\mathcal{G}$ is étale, this enables us to generalize [IKR$^+$21, Theorem 4.6] to this case (see Corollary 3.11) thereby providing criteria that guarantee that the natural abelian subalgebra of $C^*_r(\Sigma; \Omega)$ is Cartan (see also [DGN$^+$20, Theorem 5.8] and [DGN20, Theorem 4.6]).

In Subsection 3.2, we consider the case where the extension Σ is determined by an $\mathcal{A}$-valued 2-cocycle defined on $\mathcal{G}$ and show that the pushout construction is compatible with the natural change of coefficients map on cocycles. We describe the explicit construction of $\widetilde{\Sigma}$ in terms of 2-cocycles at the beginning of Subsection 3.3, and then consider various examples of this construction.

1. Pushouts of groupoid extensions

We fix a locally compact Hausdorff groupoid $\mathcal{G}$. In our applications, $\mathcal{G}$ will have a Haar system, but this is not required for the pushout construction itself. However, we do assume that $\mathcal{G}$ has open range and source maps. We call a locally compact abelian group bundle $p_{\mathcal{A}} : \mathcal{A} \to \mathcal{G}^{(0)}$ a *$\mathcal{G}$-bundle* if $p_{\mathcal{A}}$ is open and $\mathcal{G}$ acts on the left of $\mathcal{A}$ by automorphisms. For compatibility with [IKSW19]—and other examples we have in mind—we will write the group operations in the fibres of such $\mathcal{A}$ additively. An extension Σ of $\mathcal{A}$ by $\mathcal{G}$ is determined by a diagram (†) as in the introduction. Recall that Σ is a locally compact Hausdorff groupoid, p is continuous and open surjection inducing a homeomorphism from $\Sigma^{(0)}$ onto $\mathcal{G}^{(0)}$, and ι is a continuous open injective homomorphism onto $\ker p = \{\, \sigma \in \Sigma : p(\sigma) \in \mathcal{G}^{(0)} \,\}$. We call such an extension *compatible* if the action of $\mathcal{G}$ on $\mathcal{A}$ induced by conjugation is the given $\mathcal{G}$-action on $\mathcal{A}$; that is, $\sigma\iota(a)\sigma^{-1} = \iota(\dot{\sigma} \cdot a)$.

Definition 1.1. If Σ_1 and Σ_2 are compatible extensions by a locally compact abelian group $\mathcal{G}$-bundle $\mathcal{A}$, then we say that they are *properly isomorphic* if there is a groupoid isomorphism $f : \Sigma_1 \to \Sigma_2$ such that the diagram

$$\begin{array}{ccccc} & & \Sigma_1 & & \\ & \overset{\iota_1}{\nearrow} & & \overset{p_1}{\searrow} & \\ \mathcal{A} & & \downarrow f & & \mathcal{G} \\ & \underset{\iota_2}{\searrow} & & \underset{p_2}{\nearrow} & \\ & & \Sigma_2 & & \end{array} \tag{1.1}$$

commutes. We let $T_{\mathcal{G}}(\mathcal{A})$ be the collection of proper isomorphism classes of compatible extensions. We denote the equivalence class of a compatible extension Σ by $[\Sigma]$.

Remark 1.2. The second author considered extensions of this sort for étale groupoids in [Kum88, §2]. In [Tu06, §5.3], Tu denotes this set by $\operatorname{ext}(\mathcal{G}, \mathcal{A})$ and states that it forms an abelian group (see Theorem 2.5 below). Since the openness of $p_{\mathcal{A}} : \mathcal{A} \to \mathcal{G}^{(0)}$ implies that $\mathcal{A}$ has a Haar system (see [IKR+21, Lemma 2.1]), it follows that if $\mathcal{G}$ has a Haar system, then we can then equip Σ with a Haar system whenever $[\Sigma] \in T_{\mathcal{G}}(\mathcal{A})$ (see [IKR+21, Lemma 2.6]).

Of course, given $\mathcal{G}$ and a $\mathcal{G}$-bundle $\mathcal{A}$, we would like to know that $T_{\mathcal{G}}(\mathcal{A})$ is not empty. To provide a basic example, we follow [Kum88, Definition 2.1].

Example 1.3 (The Semidirect Product). We can build a fundamental compatible extension $\mathcal{A} \rtimes \mathcal{G}$ from the fibred product $\{\, (a, \gamma) \in \mathcal{A} \times \mathcal{G} : p_{\mathcal{A}}(a) = r(\gamma) \,\}$. We let $(\mathcal{A} \rtimes \mathcal{G})^{(2)} = \{\, \big((a_1, \gamma_1), (a_2, \gamma_2)\big) : s(\gamma_1) = r(\gamma_2) \,\}$, and then define

$$(a_1, \gamma_1)(a_2, \gamma_2) = (a_1 + \gamma_1 \cdot a_2, \gamma_1\gamma_2) \quad \text{and} \quad (a, \gamma)^{-1} = (-(\gamma^{-1} \cdot a), \gamma^{-1}).$$

Then we can identify the unit space of $\mathcal{A} \rtimes \mathcal{G}$ with $\mathcal{G}^{(0)}$ so that $r(a, \gamma) = r(\gamma)$ and $s(a, \gamma) = s(\gamma)$. We can then exhibit $\mathcal{A} \rtimes \mathcal{G}$ as an extension by letting $\iota(a) = (a, p_{\mathcal{A}}(a))$, and letting $p(a,\gamma)=\gamma$. Since

$$(a',\gamma)(a,p_{\mathcal{A}}(a))(-\gamma^{-1}\cdot a',\gamma^{-1})=(\gamma\cdot a,p_{\mathcal{A}}(\gamma\cdot a)),$$

$\mathcal{A}\triangleleft\mathcal{G}$ is a compatible extension as required.

Example 1.4. For $i=1,2$ let $\mathcal{A}_i$ be a locally compact abelian group $\mathcal{G}$-bundle. Note that $\mathcal{A}_1*\mathcal{A}_2=\{\,(a,a'):p_{\mathcal{A}_1}(a)=p_{\mathcal{A}_2}(a')\,\}$ is also a locally compact abelian group $\mathcal{G}$-bundle. Let Σ_i be a compatible groupoid extension of $\mathcal{G}$ by $\mathcal{A}_i$. Then as in [Kum88, §2], we may form the fibered product

$$\Sigma_1 *_{\mathcal{G}} \Sigma_2 := \{(\sigma_1,\sigma_2)\in\Sigma_1\times\Sigma_2 \mid p_1(\sigma_1)=p_2(\sigma_2)\}.$$

It is straightforward to check that $\Sigma_1 *_{\mathcal{G}} \Sigma_2$ is a compatible groupoid extension of $\mathcal{G}$ by $\mathcal{A}_1*\mathcal{A}_2$.

Assume now that $\mathcal{B}$ is another abelian group $\mathcal{G}$-bundle, and that $f:\mathcal{A}\to\mathcal{B}$ is a $\mathcal{G}$-equivariant map. Following [Kum88, Proposition 2.6], we prove that we can "pushout" Σ in a unique way to an extension of $\mathcal{G}$ by $\mathcal{B}$.

Theorem 1.5 (Pushout Construction)**.** *Let $\mathcal{A}$ and $\mathcal{B}$ be locally compact abelian group $\mathcal{G}$-bundles. Let $f:\mathcal{A}\to\mathcal{B}$ be a continuous $\mathcal{G}$-equivariant map. Assume that Σ is a compatible extension of $\mathcal{G}$ by $\mathcal{A}$. Then there is a compatible extension $f_*\Sigma$ of $\mathcal{G}$ by $\mathcal{B}$ and a homomorphism $f_*:\Sigma\to f_*\Sigma$ such that the following diagram commutes*

$$\begin{array}{ccccc} \mathcal{A} & \xrightarrow{\ \iota\ } & \Sigma & \overset{p}{\searrow} & \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f_*} & & \mathcal{G}. \\ \mathcal{B} & \xrightarrow[\ \iota_*\ ]{} & f_*\Sigma & \underset{p_*}{\nearrow} & \end{array} \tag{1.2}$$

Moreover, f_* and $f_*\Sigma$ are unique up to proper isomorphism in the sense that if Σ' is a compatible extension such that the diagram

$$\begin{array}{ccccc} \mathcal{A} & \xrightarrow{\ \iota\ } & \Sigma & \overset{p}{\searrow} & \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} & & \mathcal{G} \\ \mathcal{B} & \xrightarrow{\ \iota'\ } & \Sigma' & \underset{p'}{\nearrow} & \end{array} \tag{1.3}$$

commutes, then there is a proper isomorphism $g:f_*\Sigma\to\Sigma'$ such that $g\circ f_*=f'$.

Proof. Consider the fibred-product groupoid

$$\mathcal{D}:=(\mathcal{B}\triangleleft\mathcal{G})*_{\mathcal{G}}\Sigma=\{\,((b,\gamma),\sigma)\in(\mathcal{B}\triangleleft\mathcal{G})\times\Sigma\;:\dot{\sigma}=\gamma\,\}$$

of Example 1.4. Define $\theta:\mathcal{A}\to\mathcal{D}$ via $\theta(a)=\big((-f(a),p_{\mathcal{A}}(a)),\iota(a)\big)$. Since ι is a homeomorphism onto its closed range, $\theta(\mathcal{A})$ is a closed wide subgroupoid of $\mathcal{D}$.

Let $d = ((b,\gamma),\sigma) \in \mathcal{D}$. We claim that $d\theta(\mathcal{A}) = \theta(\mathcal{A})d$. To see this, note that

$$\begin{aligned} d\theta(a) &= \big((b,\gamma),\sigma)((-f(a),p_{\mathcal{A}}(a)),\iota(a)\big) \\ &= \big((b-\gamma\cdot f(a),\gamma),\sigma\iota(a)\big) \\ &= \big((-f(\gamma\cdot a)+p_{\mathcal{A}}(\gamma\cdot a)\cdot b,\gamma),\iota(\dot\sigma\cdot a)\sigma\big). \end{aligned}$$

Since $\dot\sigma = \gamma$, we deduce that

$$\begin{aligned} d\theta(a) &= \big((-f(\gamma\cdot a),p_{\mathcal{A}}(\gamma\cdot a)),\iota(\gamma\cdot a)\big)(b,\gamma,\sigma) \\ &= \theta(\gamma\cdot a)d. \end{aligned}$$

Let $f_*\Sigma := \mathcal{D}/\theta(\mathcal{A})$. As usual, we denote the class of $((b,\sigma),\gamma)$ in $f_*\Sigma$ by $[(b,\sigma),\gamma]$. Then $[(b,\gamma),\iota(a)\sigma] = [(b+f(a),\gamma),\sigma]$. Since $j(\mathcal{A})$ has a Haar system by Remark 1.2, $f_*\Sigma$ is a locally compact Hausdorff groupoid by [IKR+21, Lemma 2.2]. The operations are given by

$$\begin{aligned} [(b_1,\gamma_1),\sigma_1][(b_2,\gamma_2),\sigma_2] &= [(b_1+\gamma_1 b_2,\gamma_1\gamma_2),\sigma_1\sigma_2] \quad \text{and} \\ [(b,\gamma),\sigma]^{-1} &= [(-\gamma^{-1}\cdot b,\gamma^{-1}),\sigma^{-1}]. \end{aligned}$$

We can identify the unit space with $\mathcal{G}^{(0)}$ and then

$$r([(b,\gamma),\sigma]) = r(\gamma) \quad \text{and} \quad s([(b,\gamma),\sigma]) = s(\gamma).$$

To see that $f_*\Sigma$ is a compatible extension by $\mathcal{B}$, let

$$\iota_*(b) = [(b,p_{\mathcal{B}}(b)),p_{\mathcal{B}}(b)] \quad \text{and} \quad p_*([(b,\gamma),\sigma]) = \gamma.$$

It is not hard to verify that this satisfies the algebraic requirements for an extension. The most difficult one might be the inclusion $p_*^{-1}(\mathcal{G}^{(0)}) \subseteq \iota_*(\mathcal{B})$ for which we provide an outline of the proof: take $[(b,\gamma),\sigma] \in f_*\Sigma$ such that $p_*([(b,\gamma),\sigma]) = u \in \mathcal{G}^{(0)}$. Then $\gamma = u$, giving $\dot\sigma = u$. Since Σ is an extension, there exists $a \in \mathcal{A}(u)$ such that $\iota(a) = \sigma$. It follows that $[((b,u),\iota(a))] = [((b+f(a),u),u)] = \iota_*(b+f(a))$. It is easy to check that $b+f(a)$ is independent of the choice of the representative of $[(b,\gamma),\sigma]$.

Since ι_* and p_* are clearly continuous and since ι_* is easily seen to be a homeomorphism onto its range, we just need to see that p_* is open. For this, we apply Fell's Criterion (see [IKR+21, Lemma 3.1]). Suppose that $\gamma_n \to \gamma = p_*\big([(b,\sigma),\gamma]\big)$. Since $p : \Sigma \to \mathcal{G}$ is open, we can pass to a subnet, relabel, and assume that there are $\sigma_n \to \sigma$ in Σ such that $\dot\sigma_n = \gamma_n$. Since $p_{\mathcal{B}}$ is open, we can pass to subnet, relabel, and assume there are $b_n \to b$ in $\mathcal{B}$ such that $p_{\mathcal{B}}(b_n) = r(\gamma_n)$. Then $[(b_n,\gamma_n),\sigma_n] \to [(b,\gamma),\sigma]$ as required.

The map f_* is the composition of the embedding of Σ into $\mathcal{D}$ and the quotient map $\mathcal{D} \mapsto \mathcal{D}/i(\mathcal{A})$: $f_*(\sigma) = [\big((0_{r(\sigma)},p(\sigma)),\sigma\big)]$. Since f is $\mathcal{G}$-equivariant, $p_{\mathcal{B}}(f(a)) = p_{\mathcal{A}}(a)$ and

$$f_*(\iota(a)) = [(0,p_{\mathcal{A}}(a)),p_{\mathcal{A}}(a)] = [(f(a),p_{\mathcal{B}}(f(a))),p_{\mathcal{B}}(f(a))] = \iota_*(\iota(a)),$$

and (1.2) commutes as required.

Now let Σ' be an extension as in (1.3). Define $\tilde{g} : \mathcal{D} \to \Sigma'$ by $\tilde{g}((b,\gamma),\sigma) = \iota'(b)f'(\sigma)$. Since

$$\iota'(b_1)f'(\sigma_1)\iota'(b_2)f'(\sigma_2) = \iota'(b_1)\iota'(f'(\sigma_1)\cdot b_2)f'(\sigma_1)f'(\sigma_2)$$

and since $p'(f'(\sigma_1)) = \dot{\sigma}_1$, it follows that $\tilde{g}$ is a groupoid homomorphism. On the other hand,

$$\begin{aligned}\tilde{g}(\theta(a)) &= \tilde{g}((-f(a),p_{\mathcal{A}}(a)),\iota(a)) = \iota'(-f(a))f'(\iota(a)) = \iota'(-f(a))\iota'(f(a)) \\ &= \iota'(p_{\mathcal{A}}(a)).\end{aligned}$$

Hence $\tilde{g}$ factors through a homomorphism $g : f_*\Sigma \to \Sigma'$. Clearly, $g(\iota_*(b)) = \iota'(b)$ and $p' \circ g = p_*$, so g makes the diagram analogous to (1.1) commute. We have $g \circ f_* = f'$ by construction.

To see that g is a proper isomorphism, we still need to see that g is an isomorphism with a continuous inverse.

For this, fix $\alpha \in \Sigma'$. There exists $\sigma \in \Sigma$ such that $p(\sigma) = p'(\alpha)$. Using (1.3), there exists $b \in \mathcal{B}$ such that $\alpha = \iota'(b)f'(\sigma)$. So $\tilde{g}$, and hence also g, is onto.

Now suppose that $\iota'(b)f'(\sigma)$ is a unit. Then $f'(\sigma) = \iota'(-b)$. Hence $p'(f'(\sigma))$ is a unit, and $\sigma = \iota(a)$ for some $a \in \mathcal{A}$. But then $\iota'(-b) = f'(\sigma) = f'(\iota(a)) = \iota'(f(a))$. Hence, $b = -f(a)$. That is,

$$((b,p(\sigma)),\sigma) = ((-f(a),p_{\mathcal{A}}(a)),\iota(a)) \in \theta(\mathcal{A}).$$

Thus g is injective.

To see that g is an isomorphism of topological groupoids, it suffices to see that g is open. We use Fell's criterion. So suppose that $g(\alpha_i) \to g(\alpha)$ where $\alpha_i = [(b_i,p(\sigma_i)),\sigma_i]$ and $\alpha = [(b,p(\sigma)),\sigma] \in f_*\Sigma$. Since $p' \circ g = p_*$, we have $p(\sigma_i) \to p(\sigma)$. Since p is open, we can pass to a subnet, relabel, and assume there exist $a_i \in \mathcal{A}$ such that $\iota(a_i)\sigma_i \to \sigma$. But

$$\alpha_i = [(-f(a_i)+b_i),p(\sigma_i),\iota(a_i)\sigma_i],$$

and then

$$\iota'(-f(a_i)+b_i)f'(\iota(a_i)\sigma_i) \to \iota'(b)f'(\sigma).$$

It follows that

$$\iota'(-f(a_i)+b_i) \to \iota'(b).$$

Since ι' is a homeomorphism onto its range, $\alpha_i \to \alpha$ as required. □

Corollary 1.6. *Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be locally compact abelian group $\mathcal{G}$-bundles. Let $f : \mathcal{A} \to \mathcal{B}$ and $g : \mathcal{B} \to \mathcal{C}$ be continuous $\mathcal{G}$-equivariant maps. Assume that Σ is a compatible extension of $\mathcal{G}$ by $\mathcal{A}$. Then $(g \circ f)_*\Sigma$ is properly isomorphic to $g_*(f_*\Sigma)$.*

Proof. This follows from the uniqueness of $(g \circ f)_*\Sigma$ up to proper isomorphism guaranteed by Theorem 1.5. □

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2. The Extension Group $T_{\mathcal{G}}(\mathcal{A})$

As in [Kum88, §2], we can use our pushout construction to introduce a binary operation on $T_{\mathcal{G}}(\mathcal{A})$. Suppose that $[\Sigma], [\Sigma'] \in T_{\mathcal{G}}(\mathcal{A})$. Define $\nabla^{\mathcal{A}} : \mathcal{A} * \mathcal{A} \to \mathcal{A}$ by $\nabla^{\mathcal{A}}(a, a') = a + a'$. Proper isomorphisms $f : \Sigma \to \Gamma$ and $f' : \Sigma' \to \Gamma'$ of compatible extensions of $\mathcal{A}$ by $\mathcal{G}$ determine a proper isomorphism $f * f' : \Sigma * \Sigma' \to \Gamma * \Gamma'$ of extensions by $\mathcal{A} * \mathcal{A}$. The uniqueness assertion of Theorem 1.5 then yields a proper isomorphism $\nabla^{\mathcal{A}}_*(\Sigma *_{\mathcal{G}} \Sigma') \to \nabla^{\mathcal{A}}_*(\Gamma *_{\mathcal{G}} \Gamma')$. Hence the formula

$$[\Sigma] + [\Sigma'] := [\nabla^{\mathcal{A}}_*(\Sigma *_{\mathcal{G}} \Sigma')] \tag{2.1}$$

is well defined.

Example 2.1. Let $[\Sigma] \in T_{\mathcal{G}}(\mathcal{A})$. Let $\mathcal{A} \triangleleft \mathcal{G}$ be the semidirect product defined in Example 1.3. Define $g : (\mathcal{A} \triangleleft \mathcal{G}) *_{\mathcal{G}} \Sigma \to \Sigma$ by $g((a, \dot{\sigma}), \sigma) = \iota(a)\sigma$. We obtain a commutative diagram

$$\begin{array}{ccccc} \mathcal{A} * \mathcal{A} & \xrightarrow{\iota * \iota} & (\mathcal{A} \triangleleft \mathcal{G}) *_{\mathcal{G}} \Sigma & \xrightarrow{\tilde{p}} & \\ \downarrow{\scriptstyle \nabla^{\mathcal{A}}} & & \downarrow{\scriptstyle g} & & \mathcal{G}. \\ \mathcal{A} & \xrightarrow[\iota]{} & \Sigma & \xrightarrow{p} & \end{array}$$

The uniqueness assertion in Theorem 1.5 implies that $\nabla^{\mathcal{A}}_*((\mathcal{A} \triangleleft \mathcal{G}) *_{\mathcal{G}} \Sigma)$ is properly isomorphic to Σ. In other words, $[\mathcal{A} \triangleleft \mathcal{G}] + [\Sigma] = [\Sigma]$.

Example 2.2. Let $\mathcal{A} \xrightarrow{\iota} \Sigma \xrightarrow{p} \mathcal{G}$ be a compatible extension. Then we obtain another compatible extension $\mathcal{A} \xrightarrow{\iota'} \Sigma \xrightarrow{p} \mathcal{G}$ by letting $\iota'(a) = \iota(-a) = \iota(a)^{-1}$. We will write Σ^{-1} for Σ viewed as this alternate extension. Define $\theta : \mathcal{A} \to \mathcal{A}$ by $\theta(a) = -a$. Then θ is $\mathcal{G}$-invariant. Since the diagram

$$\begin{array}{ccccc} \mathcal{A} & \xrightarrow{\iota} & \Sigma & \xrightarrow{p} & \\ \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle \mathrm{id}} & & \mathcal{G} \\ \mathcal{A} & \xrightarrow{\iota'} & \Sigma^{-1} & \xrightarrow{p} & \end{array}$$

commutes, we can identify $[\theta_* \Sigma]$ with $[\Sigma^{-1}]$ by Theorem 1.5.

Example 2.3. Take $[\Sigma] \in T_{\mathcal{G}}(\mathcal{A})$ and let $\mathcal{A} \triangleleft \mathcal{G}$ be the semidirect product. The map $g : \Sigma * \Sigma^{-1} \to \mathcal{A} \triangleleft \mathcal{G}$ given by $g(\sigma, \tau) = (\iota^{-1}(\sigma\tau^{-1}), \dot{\sigma})$ is a homomorphism. Since the diagram

$$\begin{array}{ccccc} \mathcal{A} * \mathcal{A} & \xrightarrow{\iota * \iota'} & \Sigma *_{\mathcal{G}} \Sigma^{-1} & \xrightarrow{\tilde{p}} & \\ \downarrow{\scriptstyle \nabla^{\mathcal{A}}} & & \downarrow{\scriptstyle g} & & \mathcal{G} \\ \mathcal{A} & \xrightarrow[\iota]{} & \mathcal{A} \triangleleft \mathcal{G} & \xrightarrow{p} & \end{array}$$

commutes, we see that $[\Sigma] + [\Sigma^{-1}] = [\mathcal{A} \triangleleft \mathcal{G}]$ for all $\Sigma \in T_{\mathcal{G}}(\mathcal{A})$.

Example 2.4. Take $[\Sigma], [\Sigma'] \in T_{\mathcal{G}}(\mathcal{A})$. Let $\tilde{f} : \Sigma *_{\mathcal{G}} \Sigma' \to \Sigma' *_{\mathcal{G}} \Sigma$ be the flip. Similarly, let $f : \mathcal{A} * \mathcal{A} \to \mathcal{A} * \mathcal{A}$ be given by $f(a, a') = (a', a)$. The diagram

$$\begin{array}{ccccc} \mathcal{A} * \mathcal{A} & \xrightarrow{\iota * \iota'} & \Sigma *_{\mathcal{G}} \Sigma' & & \\ {\scriptstyle f}\downarrow & & {\scriptstyle \tilde{f}}\downarrow & \searrow^{\tilde{p}} & \\ \mathcal{A} * \mathcal{A} & \xrightarrow{\iota' * \iota} & \Sigma' *_{\mathcal{G}} \Sigma & \xrightarrow{\tilde{p}} & \mathcal{G} \\ {\scriptstyle \nabla^{\mathcal{A}}}\downarrow & & {\scriptstyle \nabla_*^{\mathcal{A}}}\downarrow & \nearrow_{p} & \\ \mathcal{A} & \xrightarrow{i} & \nabla_*(\Sigma' *_{\mathcal{G}} \Sigma) & & \end{array}$$

commutes. Since $\nabla^{\mathcal{A}} \circ f = \nabla^{\mathcal{A}}$, it follows from Theorem 1.5 that $[\Sigma]+[\Sigma'] = [\Sigma']+[\Sigma]$.

In Examples 2.1–2.4, we have proved much of the following theorem, which is patterned on [Kum88, Theorem 2.7].

Theorem 2.5. *Let $\mathcal{G}$ be a locally compact groupoid with open range and source maps, and let $\mathcal{A}$ be a locally compact abelian group $\mathcal{G}$-bundle. Then the binary operation $([\Sigma_1], [\Sigma_2]) \mapsto [\nabla_*^{\mathcal{A}}(\Sigma_1 *_{\mathcal{G}} \Sigma_2)]$ of (2.1) makes $T_{\mathcal{G}}(\mathcal{A})$ into an abelian group with neutral element given by the class $[\mathcal{A} \triangleleft \mathcal{G}]$ of the semidirect product of Example 1.3, and $[\Sigma]^{-1} = [\Sigma^{-1}]$ as in Example 2.2. For each continuous $\mathcal{G}$-equivariant map $f : \mathcal{A} \to \mathcal{B}$ of $\mathcal{G}$-bundles, define $T_{\mathcal{G}}(f) : T_{\mathcal{G}}(\mathcal{A}) \to T_{\mathcal{G}}(\mathcal{B})$ to be the induced map: $T_{\mathcal{G}}(f)[\Sigma] = [f_*\Sigma]$. Then $T_{\mathcal{G}}$ is a functor from the category of $\mathcal{G}$-bundles to the category of abelian groups.*

Proof. By considering diagrams similar to that in Example 2.4, we see that the operation in (2.1) is well-defined and associative. We saw that $[\mathcal{A} \triangleleft \mathcal{G}]$ acts as an identity in Example 2.1 and the statement about inverses follows from Example 2.3. The computation in Example 2.4 shows that $T_{\mathcal{G}}(\mathcal{A})$ is an abelian group.

By Corollary 1.6 we have $T_{\mathcal{G}}(f \circ g) = T_{\mathcal{G}}(f) \circ T_{\mathcal{G}}(g)$ if f and g are a composable pair of continuous $\mathcal{G}$-equivariant maps of $\mathcal{G}$-bundles. The proof that $T_{\mathcal{G}}(f)$ is a group homomorphism follows as in the proof of [Kum88, Theorem 2.7]. □

3. Applications and Examples

In this section we consider a unit space fixing extension Σ of $\mathcal{G}$ by the group bundle $\mathcal{A}$ as illustrated in the diagram (†) from the introduction. We review the basic details. We assume that all groupoids considered in this section are second-countable locally compact Hausdorff groupoids with Haar systems. The Haar system on Σ is denoted $\lambda = \{\lambda^u\}_{u \in \Sigma^{(0)}}$ and we further assume that $p_{\mathcal{A}} : \mathcal{A} \to \Sigma^{(0)}$ is a bundle of abelian groups that is a closed subgroupoid of Σ. It is equipped with a Haar system denoted $\beta = \{\beta^u\}_{u \in \Sigma^{(0)}}$ and the fibers are denoted $\mathcal{A}(u)$ for $u \in \Sigma^{(0)}$. The existence of a Haar system on $\mathcal{A}$ implies that $p_{\mathcal{A}}$ is open. It follows by [IKR+21, Lemma 2.6(c)] that there is a Haar system $\alpha = \{\alpha_u\}_{u\in\Sigma^{(0)}}$ on $\mathcal{G}$ such that for all $f \in C_c(\Sigma)$ and $u \in \Sigma^{(0)}$ we have

$$\int_\Sigma f(\sigma)\, d\lambda^u(\sigma) = \int_{\mathcal{G}} \int_{\mathcal{A}} f(\sigma a)\, d\beta^{s(\sigma)}(a)\, d\alpha^u(\dot\sigma). \tag{3.1}$$

Moreover, there is a natural action of Σ, and therefore $\mathcal{G}$, on $\mathcal{A}$.

Note that $p : \Sigma \to \mathcal{G}$ is a continuous, open surjection inducing a homeomorphism from $\Sigma^{(0)}$ onto $\mathcal{G}^{(0)}$, and $\iota : \mathcal{A} \to \Sigma$ is a homeomorphism onto $\ker p$. (Both p and ι are assumed to be groupoid morphisms).

Recall that if Σ is a $\mathbf{T}$-groupoid over $\mathcal{G}$ then

$$C_c(\mathcal{G};\Sigma) := \{f \in C_c(\Sigma) : f(t\sigma) = tf(\sigma) \text{ for all } t \in \mathbf{T}, \sigma \in \Sigma\}$$

is a *-algebra under the operations described in [MW92, §2], and $C^*(\mathcal{G};\Sigma)$ is its closure in the norm obtained by taking the supremum of the operator norm under all $*$-representations.

We may also view $C_c(\mathcal{G};\Sigma)$ as compactly supported continuous sections of the one-dimensional Fell line bundle over $\mathcal{G}$ associated to Σ. One can then construct the associated (right) Hilbert $C_0(\mathcal{G}^{(0)})$-module (see [IKR$^+$21, §1.3]) as the completion of $C_c(\mathcal{G};\Sigma)$ in the norm arising from the $C_0(\mathcal{G}^{(0)})$-valued pre-inner product given by $\langle f, g\rangle := (f^* * g)|_{\mathcal{G}^{(0)}}$ for all $f, g \in C_c(\mathcal{G};\Sigma)$. We denote the Hilbert module by $\mathcal{H}(\mathcal{G};\Sigma)$ and observe that left multiplication induces a natural $*$-homomorphism $\lambda : C_c(\mathcal{G};\Sigma) \to \mathcal{L}(\mathcal{H}(\mathcal{G};\Sigma))$. We may define the reduced norm of an element $f \in C_c(\mathcal{G};\Sigma)$ to be the operator norm of its image: $\|f\|_r := \|\lambda(f)\|$. Then $C^*_r(\mathcal{G};\Sigma)$ is the closure of $C_c(\mathcal{G};\Sigma)$ in the reduced norm.

Lemma 3.1. *With notation as above, let $F \subset \mathcal{G}^{(0)}$ be a $\mathcal{G}$-invariant clopen subset. Then F is also Σ-invariant and the reduction $\Sigma|_F$ is a twist over the reduction $\mathcal{G}|_F$. Moreover, the characteristic function of F determines a central multiplier projection p_F such that*

$$p_F C^*_r(\mathcal{G};\Sigma) \cong C^*_r(\mathcal{G}|_F;\Sigma|_F).$$

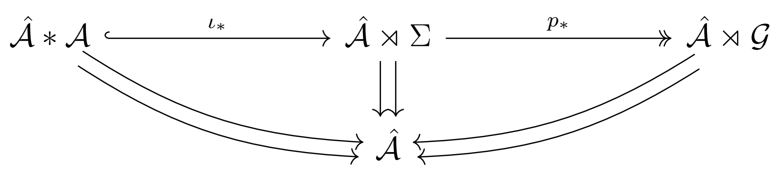
Proof. Observe that $\mathcal{H}(\mathcal{G};\Sigma)$ decomposes as the direct sum of a Hilbert $C_0(F)$-module and a Hilbert $C_0(F^c)$-module in the following way

$$\mathcal{H}(\mathcal{G};\Sigma) \cong \mathcal{H}(\mathcal{G}|_F;\Sigma|_F) \oplus \mathcal{H}(\mathcal{G}|_{F^c};\Sigma|_{F^c}).$$

Note that multiplication by the characteristic function of F, which we denote by p_F is the projection onto the first component, that p_F is in the center of the multiplier algebra of $C^*_r(\mathcal{G};\Sigma)$, and $C_c(\mathcal{G}|_F;\Sigma|_F)$ acts trivially on the second component. Hence the operator norm of $C_c(\mathcal{G}|_F;\Sigma|_F)$ acting on $\mathcal{H}(\mathcal{G}|_F;\Sigma|_F)$ coincides with that of its action on $\mathcal{H}(\mathcal{G};\Sigma)$. □

3.1. **The T-groupoid of an extension.** As noted in the introduction, we want to see that the $\mathbf{T}$-groupoid constructed in [IKR+21, §3.1] is an example of the pushout construction of Theorem 1.5. The C^*-algebra $C^*(\mathcal{A})$ is abelian and the Gelfand dual of $C^*(\mathcal{A})$ is an abelian group bundle $\hat{p} : \hat{\mathcal{A}} \to \mathcal{G}^{(0)} = \Sigma^{(0)}$ with fibres $\hat{p}^{-1}(\{u\}) \cong \mathcal{A}(u)^\wedge$ (see [MRW96, Corollary 3.4]). Furthermore, since abelian groups are amenable, it follows from [Wil19, Corollary 5.39] and [Wil07, Proposition C.10] that $\hat{p}$ is open. Therefore we can view $\hat{\mathcal{A}}$ as a right $\mathcal{G}$-bundle for the natural right action of $\mathcal{G}$ on $\hat{\mathcal{A}}$.

Since $\mathcal{G}$ and Σ both act on $\hat{\mathcal{A}}$, regarded as a topological space fibered over $\Sigma^{(0)}$, we can form the transformation groupoids $\hat{\mathcal{A}} \rtimes \mathcal{G}$ and $\hat{\mathcal{A}} \rtimes \Sigma$. Moreover, $\hat{\mathcal{A}} * \mathcal{A} = \{ (\chi, a) : \hat{p}(\chi) = p_{\mathcal{A}}(a) \}$ is a $\hat{\mathcal{A}} \rtimes \mathcal{G}$-bundle (as well as an $\hat{\mathcal{A}} \rtimes \Sigma$-bundle). Defining $\iota_* : \hat{\mathcal{A}} * \mathcal{A} \to \hat{\mathcal{A}} \rtimes \Sigma$ by $\iota_*(\chi, a) = (\chi, a)$ and $p_* : \hat{\mathcal{A}} \rtimes \Sigma \to \hat{\mathcal{A}} \rtimes \mathcal{G}$ by $p_*(\chi, \sigma) = (\chi, \dot{\sigma})$, we obtain an extension



We defined a $\mathbf{T}$-groupoid $\widetilde{\Sigma}$ associated to this extension in [IKR+21, Proposition 3.2] as follows. Define

$$\mathcal{D} = \{ (\chi, z, \sigma) \in \hat{\mathcal{A}} \times \mathbf{T} \times \Sigma : \hat{p}(\chi) = r(\sigma) \}$$

and let H be the subgroupoid of $\mathcal{D}$ consisting of triples of the form $(\chi, \overline{\chi(a)}, a)$ for $a \in \mathcal{A}(\hat{p}(\chi))$. Then H is a normal subgroupoid of $\mathcal{D}$ and we can form the locally compact Hausdorff groupoid $\widetilde{\Sigma} := \mathcal{D}/H$ (we use the notation $\widetilde{\Sigma}$, rather than the notation $\widehat{\Sigma}$ of [IKR+21], to avoid clashing with classical notational conventions when Σ is a group, for example in Remark 3.3).

Theorem 3.2. *Let Σ be the extension of $\mathcal{G}$ by the group bundle $\mathcal{A}$ as in the diagram (†) and adopt the notation established above. Let $f : \hat{\mathcal{A}} * \mathcal{A} \to \hat{\mathcal{A}} \times \mathbf{T}$ be the canonical map given by*

$$f(\chi, a) = (\chi, \chi(a)). \tag{3.2}$$

Then $\widetilde{\Sigma}$ is properly isomorphic to the pushout $f_*(\hat{\mathcal{A}} \rtimes \Sigma)$. Moreover,

$$C^*(\Sigma) \cong C^*(\hat{\mathcal{A}} \rtimes \mathcal{G}; f_*(\hat{\mathcal{A}} \rtimes \Sigma)) \quad \textit{and} \quad C^*_r(\Sigma) \cong C^*_r(\hat{\mathcal{A}} \rtimes \mathcal{G}; f_*(\hat{\mathcal{A}} \rtimes \Sigma)).$$

Proof. Theorem 1.5 implies that there is a unique (up to proper isomorphism) extension $f_*(\hat{\mathcal{A}} \rtimes \Sigma)$ of $\hat{\mathcal{A}} \rtimes \mathcal{G}$ by $\hat{\mathcal{A}} \times \mathbf{T}$ and a twist morphism that is compatible with f. In particular, $f_*(\hat{\mathcal{A}} \rtimes \Sigma)$ is a $\mathbf{T}$-groupoid. We get a natural map $g : \hat{\mathcal{A}} \rtimes \Sigma$ to $\widetilde{\Sigma}$ given by $g(\chi,\sigma)=[\chi,1,\sigma]$, and the diagram

$$\begin{array}{ccccc}
\hat{\mathcal{A}} * \mathcal{A} & \xrightarrow{\ \iota_*\ } & \hat{\mathcal{A}} \rtimes \Sigma & \xrightarrow{\ p_*\ } & \\
{\scriptstyle f}\Big\downarrow & & \Big\downarrow{\scriptstyle g} & & \hat{\mathcal{A}} \rtimes \mathcal{G} \\
\hat{\mathcal{A}} \times \mathbf{T} & \xrightarrow[\ i\ ]{} & \widetilde{\Sigma} & \xrightarrow{\ j\ } &
\end{array}$$

commutes. The proper isomorphism of $\widetilde{\Sigma}$ with $f_*(\hat{\mathcal{A}} \rtimes \Sigma)$ follows from the uniqueness guaranteed by Theorem 1.5 and the final assertion follows from [IKR+21, Theorem 3.3]. □

It follows immediately that if Σ is properly isomorphic to the semidirect product $\mathcal{A} \lhd \mathcal{G}$, then $[\hat{\mathcal{A}} \rtimes \Sigma] = [\hat{\mathcal{A}} \rtimes (\mathcal{A} \lhd \mathcal{G})] = [\mathcal{A} \lhd (\hat{\mathcal{A}} \rtimes \mathcal{G})]$ and hence $[\widetilde{\Sigma}]$ is trivial. Thus $C^*(\Sigma) \cong C^*(\hat{\mathcal{A}} \rtimes \mathcal{G})$.

Remark 3.3. As mentioned in the introduction, the twist $\widetilde{\Sigma}$ appearing in Theorem 3.2 is responsible for the Mackey obstruction of the classical normal subgroup analysis of [Mac58]. Indeed, let us apply the theorem when Σ is a locally compact group and $\mathcal{A}$ is a closed normal abelian subgroup. Then Σ and $\mathcal{G} = \Sigma/\mathcal{A}$ act on $\mathcal{A}$ by conjugation and give right actions on the space of characters $\hat{\mathcal{A}}$. The corresponding twist $\widetilde{\Sigma}$ is the quotient of the groupoid $(\hat{\mathcal{A}} \rtimes \Sigma) \times \mathbf{T}$ where $(\chi, a\sigma, \theta)$ is identified with $(\chi, \sigma, \theta\chi(a))$ for all $a \in \mathcal{A}$. We let $[\chi,\sigma,\theta]$ be the class of (χ,σ,θ) in $\widetilde{\Sigma}$. If $\chi \in \hat{\mathcal{A}}$, then let $\Sigma(\chi)$ and $\mathcal{G}(\chi)$ be the stabilizers at χ for the actions on $\hat{\mathcal{A}}$, and let $\widetilde{\Sigma}(\chi)$ be the isotropy group of $\widetilde{\Sigma}$ at χ. We observe that $\widetilde{\Sigma}(\chi)$, up to an obvious identification, is the pushout of the group extension

$$\mathcal{A} \longrightarrow \Sigma(\chi) \longrightarrow \mathcal{G}(\chi)$$

by the homomorphism $\chi : \mathcal{A} \to \mathbf{T}$. Indeed, this pushout $\chi_*(\Sigma(\chi))$ is the quotient of $\Sigma(\chi) \times \mathbf{T}$ by the equivalence relation identifying $(a\sigma, \theta)$ with $(\sigma, \theta\chi(a))$ for all $a \in \mathcal{A}$. Thus we just identify $[\chi,\sigma,\theta] \in \widetilde{\Sigma}(\chi)$ with $[\sigma,\theta] \in \chi_*(\Sigma(\chi))$. The class of $\widetilde{\Sigma}(\chi)$ in $H^2(\mathcal{G}(\chi), \mathbf{T})$ is the classical Mackey obstruction. More precisely, let L be an irreducible unitary representation of Σ. According to Theorem 3.2, we may view it as a representation of the twisted groupoid $(\hat{\mathcal{A}} \rtimes \mathcal{G}, \widetilde{\Sigma})$. Its restriction to $\hat{\mathcal{A}}$ defines a measure class which is invariant and ergodic under the action of $\mathcal{G}$. If this measure class is transitive, which will be always the case if $\mathcal{A}$ is regularly embedded, then we have a representation of a twisted transitive measured groupoid $(O \rtimes \mathcal{G}, \widetilde{\Sigma}|_O)$, where $O \subset \hat{\mathcal{A}}$ is an orbit of the action and $\widetilde{\Sigma}|_O$ is the reduction of $\widetilde{\Sigma}$ to O. We pick $\chi \in O$. Since the $(\widetilde{\Sigma}(\chi), \widetilde{\Sigma}|_O)$-groupoid equivalence $\widetilde{\Sigma}_O^\chi$ is compatible with the twists in the sense of [Ren87, Définition 5.3], it implements a bijective correspondence between the unitary representations of $(O \rtimes \mathcal{G}, \widetilde{\Sigma}|_O)$ and those of $(\mathcal{G}(\chi), \widetilde{\Sigma}(\chi))$. Therefore L is given by an irreducible unitary representation of the twisted group $(\mathcal{G}(\chi), \widetilde{\Sigma}(\chi))$.

Example 3.4. Let H be a locally compact abelian group and let $A \subset H$ be a closed subgroup. Then applying the above theorem with $\Sigma = H$ and $\mathcal{A} = A$, we conclude that $\widetilde{\Sigma}$ is a bundle of abelian groups over $\widetilde{\Sigma}^{(0)} \cong \hat{A}$ where each fiber is an extension of H/A by $\mathbf{T}$. Each of these extensions is abelian because H is abelian (and the action of H on $\hat{A}$ is trivial). Hence, each extension is determined by a symmetric $\mathbf{T}$-valued Borel 2-cocycle and any such 2-cocycle is necessarily trivial by [Kle65, Lemma 7.2]. But the twist is not trivial in general: for example, if $H = \mathbf{R}$ and $A = \mathbf{Z} \leq \mathbf{R}$, then triviality of the twist would imply $C^*(\mathbf{R}) \cong C_0(\mathbf{T} \times \mathbf{Z})$, which is nonsense.

Example 3.5 (Generalized Twists). We now consider the case where A is a locally compact abelian group, $\mathcal{A} = \mathcal{G}^{(0)} \times A$, and $\mathcal{G}$ acts on $\mathcal{A}$ by translation on the first factor. Since this simply gives us a twist when $A = \mathbf{T}$, we will say that Σ is a *generalized twist* in this case. Note that even for twists, Σ need not be a trivial extension. Generalized twists were studied in [IKSW19].

View $\hat{\mathcal{A}} := \hat{A} \times \mathcal{G}^{(0)}$ as a locally compact space. (We put the factor of $\mathcal{G}^{(0)}$ on the right, as a reminder that we are thinking of $\hat{A}$ as a space rather than as a group, and to line up with the natural identification of $\hat{\mathcal{A}} * \mathcal{A}$ with $\hat{A} \times \mathcal{G}^{(0)} \times A$, which we make without further comment). Then $\mathcal{G}$ acts on the second factor of $\hat{\mathcal{A}}$. This means we can replace $\hat{\mathcal{A}} \rtimes \mathcal{G}$ and $\hat{\mathcal{A}} \rtimes \Sigma$ with the products $\hat{A} \times \mathcal{G}$ and $\hat{A} \times \Sigma$, respectively. Under these identifications, Equation (3.2) becomes $f(\chi, u, a) = (\chi, u, \chi(a))$. Moreover we may assume that the Haar system β on $\mathcal{A} = \mathcal{G}^{(0)} \times A$ is constant in the sense that there is a fixed Haar measure μ on A such $\beta^u = \mu$ for all $u \in \mathcal{G}^{(0)}$.

If $\chi \in \hat{A}$, then we get a $\mathcal{G}$-equivariant map $f^\chi : \mathcal{G}^{(0)} \times A \to \mathcal{G}^{(0)} \times \mathbf{T}$ given by $f^\chi(u, a) = (u, \chi(a))$. Thus we can form the pushout $f^\chi_*(\Sigma)$ so that

$$\begin{array}{ccccc} \mathcal{G}^{(0)} \times A & \xrightarrow{\quad \iota \quad} & \Sigma & & \\ \downarrow{\scriptstyle f^\chi} & & \downarrow{\scriptstyle f^\chi_*} & \searrow^{p} & \\ & & & & \mathcal{G} \\ \mathcal{G}^{(0)} \times \mathbf{T} & \xrightarrow{\quad \iota' \quad} & f^\chi_*(\Sigma) & \nearrow_{p'} & \end{array}$$

commutes. Then $C^*(\mathcal{G}; f^\chi_*(\Sigma))$ is the completion of $C^\chi_c(\Sigma)$ consisting of functions $g \in C_c(\Sigma)$ such that $g(\iota(r(\sigma), a)\sigma) = \chi(a) g(\sigma)$ with the $*$-algebra structure discussed at the beginning of this section.

Proposition 3.6. *Let Σ be a generalized twist as in Example 3.5. For $\chi \in \hat{A}$, let $f^\chi : \mathcal{G}^{(0)} \times A \to \mathcal{G}^{(0)} \times \mathbf{T}$ and $f^\chi_*(\Sigma)$ be the $\mathcal{G}$-equivariant map and $\mathbf{T}$-groupoid defined above. Then with notation as above,*

$$C^*(\Sigma) \cong C^*(\hat{A} \times \mathcal{G}; f_*(\hat{A} \times \Sigma)) \tag{3.3}$$

and $C^*(\Sigma)$ is the section algebra of an upper-semicontinuous C^*-bundle over $\hat{A}$ with fiber at $\chi \in \hat{A}$ isomorphic to $C^*(\mathcal{G}; f^\chi_*(\Sigma))$.

Proof. The isomorphism in (3.3) comes from Theorem 3.2.

The map $p : \hat{A}\times\mathcal{G}^{(0)} \to \mathcal{G}^{(0)}$ is continuous and satisfies $p\circ s = p\circ r$ so that $f_*(\hat{A}\times\Sigma)$ is a groupoid bundle over $\hat{A}$ as in Appendix A. Hence we can invoke Proposition A.1 to see that $C^*(\hat{A}\times\mathcal{G}; f_*(\hat{A}\times\Sigma))$ is isomorphic to the section algebra of an upper-semicontinuous C^*-bundle over $\hat{A}$. Since we can identify $f_*(\hat{A}\times\Sigma)(\chi)$ with $f_*^{\chi}(\Sigma)$ and $(\hat{A}\times\mathcal{G})(\chi)$ with $\mathcal{G}$, the result follows. □

Proposition 3.7. *With notation as in Example 3.5, suppose that A compact. Then the dual $\hat{A}$ is discrete and*

$$C^*(\Sigma) \cong \bigoplus_{\chi\in\hat{A}} C^*(\mathcal{G}; f_*^{\chi}(\Sigma)) \quad \textit{and} \quad C_r^*(\Sigma) \cong \bigoplus_{\chi\in\hat{A}} C_r^*(\mathcal{G}; f_*^{\chi}(\Sigma)).$$

Proof. To prove the first isomorphism, note that by Proposition A.1

$$C^*(\Sigma) \cong C^*(\hat{A}\times\mathcal{G}; f_*(\hat{A}\times\Sigma))$$

is a $C_0(\hat{A})$-algebra. That is, letting $ZM(C^*(\Sigma))$ denote the center of $M(C^*(\Sigma))$, there is a σ-unital *-homomorphism $\rho : C_0(\hat{A}) \to ZM(C^*(\Sigma))$. Since $\hat{A}$ is discrete, the images of the characteristic functions of singleton sets under ρ give rise to a family $\{q_\chi\}_{\chi\in\hat{\mathcal{A}}}$ of mutually orthogonal central projections in $M(C^*(\Sigma))$ which sum to unity in the strict topology. Moreover, the summands coincide with the fibers of the upper-semicontinuous C^*-bundle over $\hat{A}$ given in Proposition 3.6 and hence

$$q_\chi C^*(\Sigma) q_\chi = q_\chi C^*(\Sigma) \cong C^*(\mathcal{G}; f_*^{\chi}(\Sigma)).$$

for all $\chi \in \hat{A}$.

For the second isomorphism, let $\pi : C^*(\Sigma) \to C_r^*(\Sigma)$ be the canonical quotient map. An argument like that of the preceding paragraph using the family $\{\pi(q_\chi)\}_{\chi\in\hat{\mathcal{A}}}$ of mutually orthogonal central projections in $M(C_r^*(\Sigma))$ gives $C_r^*(\Sigma) \cong \bigoplus_{\chi\in\hat{\mathcal{A}}} \pi(q_\chi) C_r^*(\Sigma)$. Lemma 3.1 gives $\pi(q_\chi)C_r^*(\Sigma) \cong C_r^*(\mathcal{G}; f_*^{\chi}(\Sigma))$, and the result follows. □

Remark 3.8. If $A = \mathbf{T}$ and Σ is a twist, then $\hat{A} = \mathbf{Z}$, and we have $[f_*^n(\Sigma)] = n[\Sigma]$ for $n \in \mathbf{Z}$. It follows that the central summand corresponding to $n = 1$ is isomorphic to $C^*(\mathcal{G};\Sigma)$ and thus there is central projection $q = q_1 \in M(C^*(\Sigma))$ such that

$$C^*(\mathcal{G};\Sigma) \cong qC^*(\Sigma) \quad \text{and} \quad C_r^*(G;\Sigma) \cong qC_r^*(\Sigma)$$

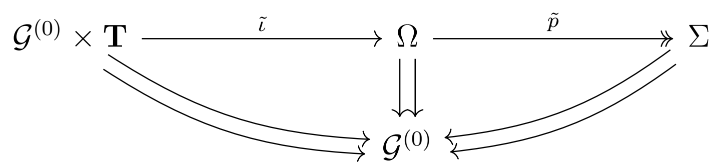
Now suppose that $\mathcal{G} = \mathcal{G}^{(0)}$ so that $\Sigma = \mathcal{A}$ is itself an abelian group bundle regarded as a groupoid with unit space $\mathcal{G}^{(0)}$ and let Λ be a $\mathbf{T}$-twist over $\mathcal{A}$. Then since $\mathcal{A}$ is amenable $C^*(\mathcal{A};\Lambda) = C_r^*(\mathcal{A};\Lambda)$ (see, for example [SW13, Thm 1]). We shall say that such a twist is *abelian* if Λ is also an abelian group bundle—that is if $\Lambda(u)$ is abelian for each $u \in \mathcal{G}^{(0)}$. Then Λ is abelian if and only if $C^*(\Lambda)$ is abelian and in that case $C^*(\Lambda) \cong C_0(\hat{\Lambda})$. Arguing as in Example 3.4, we see that such extensions must be pointwise trivial but need not be globally trivial. If Λ is determined by a continuous $\mathbf{T}$-valued 2-cocycle c, then Λ is abelian if and only if c is symmetric (cf., [DGN+20, Lemma 3.5]). Suppose now that Λ is abelian. For $n \in \mathbf{Z}$, let $V_n := \{\chi \in \hat{\Lambda} : \chi(z,u) = z^n \text{ for all } z \in \mathbf{T} \text{ and } u \in \mathcal{G}^{(0)}\}$. Then $C^*(\Lambda) \cong C_0(\hat{\Lambda})$ decomposes as a direct sum with summands of the form $C_0(V_n)$. Note that each V_n is clopen. The projection q in Remark 3.8 may then be identified with the characteristic function of $U_\Lambda := V_1$ and hence

$$C^*(\mathcal{A};\Lambda) \cong qC^*(\Lambda) \cong C_0(U_\Lambda).$$

See [DGN20, Section 3] for a related construction.

In the case that $\Lambda \cong \mathbf{T} \times \mathcal{A}$ and thus $\hat{\Lambda} \cong \mathbf{Z} \times \hat{\mathcal{A}}$, we have $U_\Lambda \cong \{1\} \times \hat{\mathcal{A}} \cong \hat{\mathcal{A}}$.

We return now to the more general situation where Σ is a unit space fixing extension of $\mathcal{G}$ by the group bundle $\mathcal{A}$ as in the diagram (†) from the introduction. Suppose that, in addition, Ω is a $\mathbf{T}$-groupoid extension of Σ



such that $\Lambda_\Omega := \tilde{p}^{-1}(\mathcal{A})$, its restriction to $\mathcal{A}$, is an abelian group bundle over $\mathcal{G}^{(0)}$. We may thus regard Ω as an extension of $\mathcal{G}$ by Λ_Ω. We assume that $\mathcal{A}$, Σ and $\mathcal{G}$ are endowed with Haar systems that satisfy (3.1), the Haar system in $\mathcal{G}^{(0)} \times \mathbf{T}$ is given by the Haar measure on $\mathbf{T}$, and the Haar system on Ω is the one naturally defined by the Haar systems on $\mathcal{G}^{(0)} \times \mathbf{T}$ and Σ. To declutter notation a little, we write $\widehat{\Lambda}_\Omega$ for the dual bundle $(\Lambda_\Omega)^\wedge$.

Corollary 3.9. *With notation as above let* $f : \widehat{\Lambda}_\Omega * \Lambda_\Omega \to \widehat{\Lambda}_\Omega \times \mathbf{T}$ *be given by* $f(\chi, a) = (\chi, \chi(a))$. *Then*

$$C^*(\Omega) \cong C^*(\widehat{\Lambda}_\Omega \rtimes \mathcal{G}; f_*(\widehat{\Lambda}_\Omega \rtimes \Omega)) \quad \textit{and}$$
$$C^*_r(\Omega) \cong C^*_r(\widehat{\Lambda}_\Omega \rtimes \mathcal{G}; f_*(\widehat{\Lambda}_\Omega \rtimes \Omega)).$$

Proof. This follows immediately from Remark 3.8, the above discussion, and Theorem 3.2 with Λ_Ω in place of $\mathcal{A}$. □

By arguing as in Remark 3.8 and Corollary 3.9 we may conclude that $C^*(\Sigma;\Omega)$ is isomorphic to the corner associated to the central projection q_Ω in

$$M(C^*(\widehat{\Lambda}_\Omega \rtimes \mathcal{G}; f_*(\widehat{\Lambda}_\Omega \rtimes \Omega)))$$

corresponding to the characteristic function of

$$U_\Omega := U_{\Lambda_\Omega} \subset \widehat{\Lambda}_\Omega = (\widehat{\Lambda}_\Omega \rtimes \mathcal{G})^{(0)}.$$

Observe that U_Ω is an invariant clopen set under the action of both $\mathcal{G}$ and Ω and thus both groupoids act on U_Ω.

Corollary 3.10. *With notation as above define* $g : U_\Omega * \Lambda_\Omega \to U_\Omega \times \mathbf{T}$ *by* $g(\chi, a) = (\chi, \chi(a))$. *Then*

$$C^*(\Sigma;\Omega) \cong C^*(U_\Omega \rtimes \mathcal{G}; g_*(U_\Omega \rtimes \Omega)) \quad \textit{and} \quad C^*_r(\Sigma;\Omega) \cong C^*_r(U_\Omega \rtimes \mathcal{G}; g_*(U_\Omega \rtimes \Omega)).$$

Proof. Observe that

$$\big(\widehat{\Lambda}_\Omega \rtimes \mathcal{G}\big)_{U_\Omega} \cong U_\Omega \rtimes \mathcal{G} \quad \text{and} \quad \big(\widehat{\Lambda}_\Omega \rtimes \Omega\big)_{U_\Omega} \cong U_\Omega \rtimes \Omega.$$

For $(\chi, a) \in U_\Omega * \Lambda_\Omega \subset \widehat{\Lambda}_\Omega * \Lambda_\Omega$,

$$f(\chi, a) = (\chi, \chi(a)) = g(\chi, a) \in U_\Omega \times \mathbf{T}$$

Therefore,

$$\big(f_*(\widehat{\Lambda}_\Omega \rtimes \Omega)\big)_{U_\Omega} \cong g_*(U_\Omega \rtimes \Omega).$$

Hence, by Remark 3.8 and Corollary 3.9

$$\begin{aligned} C^*(\Sigma;\Omega) &\cong q_\Omega C^*\big(\widehat{\Lambda}_\Omega \rtimes \mathcal{G}; f_*(\widehat{\Lambda}_\Omega \rtimes \Omega)\big) q_\Omega \\ &\cong C^*\big((\widehat{\Lambda}_\Omega \rtimes \mathcal{G})_{U_\Omega}; (f_*(\widehat{\Lambda}_\Omega \rtimes \Omega))_{U_\Omega}\big) \\ &\cong C^*\big(U_\Omega \rtimes \mathcal{G}; g_*(U_\Omega \rtimes \Omega)\big). \end{aligned}$$

The case for the reduced C^*-algebras follows by a similar argument. □

Recall that an étale groupoid $\mathcal{G}$ is said to be *effective* if the interior of the isotropy groupoid is $\mathcal{G}^{(0)}$ and *topologically principal* if the set of points with trivial isotropy is dense in $\mathcal{G}^{(0)}$. These notions are equivalent if the étale groupoid $\mathcal{G}$ is second countable (see [BCFS14, Lemma 3.1]). The above corollary allows us to generalize [IKR+21, Theorem 4.6] (see also [DGN+20, Theorem 5.8] and [DGN20, Theorem 4.6]).

Corollary 3.11. *With notation as above, suppose that* $\mathcal{G}$ *is étale and that the action groupoid* $U_\Omega \rtimes \mathcal{G}$ *is second countable and effective. Then the image of* $C^*_r(\mathcal{A}, \Lambda_\Omega)$ *under the natural embedding into* $C^*_r(\Sigma;\Omega)$ *is a Cartan subalgebra with Weyl twist* $g_*(U_\Omega \rtimes \Omega)$.

Proof. This follows from Corollary 3.10 and [Ren08, Theorem 5.2]. □

Example 3.12. Let H be a discrete abelian group and let E be a $\mathbf{T}$-twist over H—that is, a central extension by $\mathbf{T}$. Since H is discrete, there is a $\mathbf{T}$-valued skew-symmetric bicharacter ϖ on H and a set of generating unitaries $\{u_h \mid h \in H\}$ in $C^*(H;E)$ such that for all $g, h \in H$

$$u_g u_h = \varpi(g,h) u_h u_g.$$

By [Kle65, Lemma 7.2] the extension E is trivial if and only if $\varpi(g,h) = 1$ for all $g, h \in H$. Let A be a subgroup of H which is maximal amongst subgroups on which $\varpi(\cdot,\cdot)$ is identically 1. It is shown in [Kum86, Example 1.12] that the C^*-subalgebra B generated by $\{u_a \mid a \in A\}$ is a diagonal subalgebra of $C^*(H;E)$. We now show that this also follows from Corollary 3.11 with $\Sigma := H$, $\mathcal{A} := A$, $\mathcal{G} = H/A$ and $\Omega := E$.

Since the restriction of ϖ to A is trivial the extension E is trivial on A and thus Λ is trivial as a $\mathbf{T}$-twist. Hence, $B \cong C^*(A)$ and $U_\Lambda \cong \hat{A}$. There is a continuous homomorphism $\varpi_A : H \to \hat{A}$ such that for all $h \in H$, $a \in A$

$$(\varpi_A(h))(a) = \varpi(h, a).$$

Moreover, $A = \ker \varpi$ and thus ϖ induces an injection $H/A \to \hat{A}$. The action of H/A on $\hat{A}$ is then given by translation and, hence, is free. Since H/A is étale and its action on $U_\Omega \cong \hat{A}$ is principal, the image of $C^*_r(\mathcal{A}, \Lambda_\Omega) \cong C^*(A)$ under the natural embedding into $C^*_r(\Sigma; \Omega) = C^*(H; E)$ is a diagonal subalgebra.

3.2. **Extensions by 2-cocycles.** Extensions associated to groupoid 2-cocycles yield some nice applications of the pushout construction. For convenience, we review the basics here. (For more details, see [IKSW19, Appendix A].) Assume that $p_{\mathcal{A}} : \mathcal{A} \to \mathcal{G}^{(0)}$ is a $\mathcal{G}$-bundle. As before we write $\mathcal{A}(u)$ for $p_{\mathcal{A}}^{-1}(u)$ for $u \in \mathcal{G}^{(0)}$. Assume that $\varphi : \mathcal{G}^{(2)} \to \mathcal{A}$ is a continuous normalized 2-cocycle. That is, $\varphi(\gamma_1, \gamma_2) \in \mathcal{A}(r(\gamma_1))$ for all $(\gamma_1, \gamma_2) \in \mathcal{G}^{(2)}$, $\varphi(\gamma_0, \gamma_1) + \varphi(\gamma_0\gamma_1, \gamma_2) = \gamma_0 \cdot \varphi(\gamma_1, \gamma_2) + \varphi(\gamma_0, \gamma_1\gamma_2)$ for all $(\gamma_0, \gamma_1), (\gamma_1, \gamma_2) \in \mathcal{G}^{(2)}$, and $\varphi(\gamma, u) = \varphi(u, \gamma) = 0_u$ for all $\gamma \in \mathcal{A}(u)$ and $u \in \mathcal{G}^{(0)}$. Then the extension Σ_φ of $\mathcal{G}$ by $\mathcal{A}$ determined by φ is obtained by giving the fibered product $\mathcal{A} * \mathcal{G}$ the groupoid structure where $(a_1, \gamma_1)(a_2, \gamma_2) = (a_1 + \gamma_1 \cdot a_2 + \varphi(\gamma_1, \gamma_2), \gamma_1\gamma_2)$ if $(\gamma_1, \gamma_2) \in \mathcal{G}^{(2)}$ and $(a, \gamma)^{-1} = (-\gamma^{-1} \cdot a - \varphi(\gamma^{-1}, \gamma), \gamma^{-1})$. We exhibit Σ_φ as an extension of $\mathcal{G}$ by $\mathcal{A}$ via $i(a) = (a, p_{\mathcal{A}}(a))$ and $p(a, \gamma) = \gamma$.

Example 3.13. If $\mathcal{A} = \mathcal{G}^{(0)} \times A$ is the trivial bundle (with trivial action), then an $\mathcal{A}$-valued cocycle is given by a continuous A-valued 2-cocycle σ on $\mathcal{G}$ via the formula $\varphi(\gamma_1, \gamma_2) = (\sigma(\gamma_1, \gamma_2), r(\gamma_1))$.

Example 3.14. Let φ be a continuous normalized $\mathbf{T}$-valued 2-cocycle and let Σ_φ be the $\mathbf{T}$-twist associated to φ. Then by Proposition 3.7 and Remark 3.8, and the fact that $\Sigma_{\varphi^n} \cong n_*(\Sigma_\varphi)$ for all $n \in \mathbf{Z}$, we have

$$C^*(\Sigma_\varphi) \cong \bigoplus_{n \in \mathbf{Z}} C^*(\mathcal{G}; \Sigma_{\varphi^n}).$$

This recovers [BaH14, Theorem 3.2].

Example 3.15 (Transformation groupoids). Let $\mathcal{G}$ be a groupoid acting on the right of a locally compact Hausdorff space X. Recall that the transformation groupoid $X \rtimes \mathcal{G}$ is obtained by endowing the fibered product $X * \mathcal{G}$ with the groupoid operations $(x, \gamma_1)(x \cdot \gamma_1, \gamma_2) = (x, \gamma_1\gamma_2)$ if $(\gamma_1, \gamma_2) \in \mathcal{G}^{(2)}$ and $(x, \gamma)^{-1} = (x \cdot \gamma, \gamma^{-1})$.

Assume that $\varphi : \mathcal{G}^{(2)} \to \mathcal{A}$ is a 2-cocycle as above. Then one can define a natural 2-cocycle $\tilde{\varphi} : (X \rtimes \mathcal{G})^{(2)} \to X * \mathcal{A}$ via $\tilde{\varphi}((x, \gamma_1), (x \cdot \gamma_1, \gamma_2)) = (x, \varphi(\gamma_1, \gamma_2))$. The extension $\Sigma_{\tilde{\varphi}}$ of $X \rtimes \mathcal{G}$ defined by $\tilde{\varphi}$ is isomorphic to the extension $X \rtimes \Sigma_\varphi$, where Σ_φ is the extension of $\mathcal{G}$ defined by φ. To see this, note that $\Sigma_{\tilde{\varphi}} = \{ \big((x, a), (x, \gamma)\big) : x \in X, a \in \mathcal{A}^x, \gamma \in \mathcal{G}^x \}$ with the operations

$$\big((x, a_1), (x, \gamma_1)\big)\big((x \cdot \gamma_1, a_2), (x \cdot \gamma_1, \gamma_2)\big) = \big((x, a_1 + \gamma_1 a_2 + \varphi(\gamma_1, \gamma_2)), (x, \gamma_1\gamma_2)\big)$$

and

$$\big((x,a),(x,\gamma)\big)^{-1} = \big((x\cdot\gamma, -\gamma^{-1}a - \varphi(\gamma^{-1},\gamma)),(x\cdot\gamma,\gamma^{-1})\big).$$

On the other hand, $X \rtimes \Sigma_\varphi = \{\, (x,(a,\gamma)) : x \in X, a \in \mathcal{A}^x, \gamma \in \mathcal{G}^x \,\}$ with the operations

$$(x,(a_1,\gamma_1))(x\cdot\gamma_1,(a_2,\gamma_2)) = (x,(a_1+\gamma_1 a_2 + \varphi(\gamma_1,\gamma_2),\gamma_1\gamma_2))$$

and

$$(x,(a,\gamma))^{-1} = (x\cdot\gamma,(-\gamma^{-1}\cdot a - \varphi(\gamma^{-1},\gamma),\gamma^{-1})).$$

Therefore the map $V : \Sigma_{\tilde\varphi} \to X \rtimes \Sigma_\varphi$ defined by $V\big((x,a),(x,\gamma)\big) = (x,(a,\gamma))$ is a groupoid isomorphism.

Suppose that $p_{\mathcal{B}} : \mathcal{B} \to \mathcal{G}^{(0)}$ is another abelian $\mathcal{G}$-bundle and that $f : \mathcal{A} \to \mathcal{B}$ is an equivariant map such that $f|_{\mathcal{A}(u)} : \mathcal{A}(u) \to \mathcal{B}(u)$ is a continuous group homomorphism for all $u \in \mathcal{G}^{(0)}$. There is a $\mathcal{B}$-valued 2-cocycle $f_*(\varphi) : \mathcal{G}^{(2)} \to \mathcal{B}$ given by $f_*(\varphi)(\gamma_1,\gamma_2) = f(\varphi(\gamma_1,\gamma_2))$.

Lemma 3.16. *Let $\Sigma_{f_*(\varphi)}$ be the extension of $\mathcal{G}$ by $\mathcal{B}$ determined by $f_*(\varphi)$. Then $f_*\Sigma_\varphi$ is properly isomorphic to $\Sigma_{f_*(\varphi)}$.*

Proof. Define $g : \Sigma_\varphi \to \Sigma_{f_*\varphi}$ by $g(a,\gamma) = (f(a),\gamma)$. The diagram

$$\begin{array}{ccccc} \mathcal{A} & \xrightarrow{\ i\ } & \Sigma_\varphi & \xrightarrow{\ p\ } & \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} & & \mathcal{G} \\ \mathcal{B} & \xrightarrow[\ i\ ]{} & \Sigma_{f_*(\varphi)} & \xrightarrow{\ p\ } & \end{array}$$

commutes. Therefore the lemma follows from Theorem 1.5. □

3.3. **The T-groupoid defined by a 2-cocycle.** We continue to assume the setting from Section 3.2: $\mathcal{A}$ is an abelian $\mathcal{G}$-bundle, $\varphi : \mathcal{G}^{(2)} \to \mathcal{A}$ is a 2-cocycle, and Σ_φ is the extension defined by φ. Then, as in Example 3.15 there is a 2-cocycle

$$\tilde\varphi : (\hat{\mathcal{A}} \rtimes \mathcal{G})^{(2)} \to \hat{\mathcal{A}} * \mathcal{A}$$

defined by

$$\tilde\varphi\big((\chi,\gamma_1),(\chi\cdot\gamma_1,\gamma_2)\big) = \big(\chi,\varphi(\gamma_1,\gamma_2)\big) \tag{3.4}$$

if $(\gamma_1,\gamma_2) \in \mathcal{G}^{(2)}$. Therefore we can identify $\hat{\mathcal{A}} \rtimes \Sigma_\varphi$ with $\Sigma_{\tilde\varphi}$, the extension of $\hat{\mathcal{A}} \rtimes \mathcal{G}$ determined by $\tilde\varphi$. Consider the 2-cocycle $\hat\varphi := f_*\tilde\varphi : (\hat{\mathcal{A}} \rtimes \mathcal{G})^{(2)} \to \hat{\mathcal{A}} \times \mathbf{T}$ defined via

$$\hat\varphi\big((\chi,\gamma_1),(\chi,\gamma_2)\big) = \big(\chi,\chi(\varphi(\gamma_1,\gamma_2))\big).$$

Lemma 3.16 and Theorem 3.2 imply that $\widetilde{\Sigma}_\varphi$ is isomorphic to the $\mathbf{T}$-groupoid defined by $\hat\varphi$ and $C^*(\Sigma_\varphi)$ is isomorphic to $C^*(\hat{\mathcal{A}} \rtimes \mathcal{G}; \Sigma_{\hat\varphi})$.

Example 3.17. The following example was studied in [IKSW19]. Let X be a second-countable locally compact Hausdorff space, and G a second-countable locally compact abelian group. Let $\mathscr{G}$ denote the sheaf of germs of continuous G-valued functions on X, and let $c \in Z^2(\mathscr{U}, \mathscr{G})$ be a normalized Čech two cocycle for some locally finite cover $\mathscr{U} = \{U_i\}_{i\in I}$ of X by precompact open sets. The blow-up groupoid $\mathcal{G}_{\mathscr{U}}$ with respect to the natural map from $\bigsqcup_i U_i$ into X is

$$\mathcal{G}_{\mathscr{U}} = \{(i,x,j) \,:\, x \in U_{ij} := U_i \cap U_j\}$$

with $(i,x,j)(j,x,k) = (i,x,k)$ and $(i,x,j)^{-1} = (j,x,i)$. As noted in [IKSW19, Remark 3.3], the Čech 2-cocycle c defines a groupoid 2-cocycle $\varphi_c : \mathcal{G}_{\mathscr{U}}^{(2)} \to G$ via

$$\varphi_c\big((i,x,j),(j,x,k)\big) = c_{ijk}(x).$$

Let Σ_c be the extension of $\mathcal{G}_{\mathscr{U}}$ by the 2-cocycle φ_c. Define

$$\hat{\varphi} : \big((\hat{G} \times \bigsqcup_i U_i) \rtimes \mathcal{G}_{\mathscr{U}}\big)^{(2)} \to \mathbf{T} \times \hat{G} \times \bigsqcup_i U_i$$

by

$$\hat{\varphi}\big((\tau,(i,x,j)),(\tau,(j,x,k))\big) = \big(\overline{\tau(c_{ijk}(x))}, \tau\big)$$

for $\tau \in \hat{G}$ and $\big((i,x,j),(j,x,k)\big) \in (\mathcal{G}_{\mathscr{U}})^{(2)}$. Then $\hat{\varphi}$ is a groupoid 2-cocycle, and the pushout groupoid $\widetilde{\Sigma}$ is isomorphic to the $\mathbf{T}$-groupoid that is the extension of $(\hat{G} \times \bigsqcup_i U_i) \rtimes \mathcal{G}_{\mathscr{U}}$ defined by $\hat{\varphi}$.

Let $\mathscr{V} = \{\hat{G} \times U_i\}_{i\in I}$ be the locally finite cover of $\hat{G} \times X$, let $\mathscr{S}$ be the sheaf of germs of continuous $\mathbf{T}$-valued functions, and define $\nu^c = \{\, \nu^c_{ijk} \,\} \in Z^2(\mathscr{V}, \mathscr{S})$ by

$$\nu^c\big((\tau,(i,x,j)),(\tau,(j,x,k))\big) = \overline{\tau(c_{ijk}(x))}.$$

Then the 2-cocycle $\hat{\varphi}$ is defined by the Čech 2-cocycle $\nu^c \in Z^2(\mathscr{V}, \mathscr{S})$.

That is, ν^c is the normalized 2-cocycle considered in [IKSW19, Equation (3.4)]. Hence the generalized Raeburn–Taylor C^*-algebra $A(\nu)$ studied in [IKSW19] is isomorphic to the restricted C^*-algebra of the $\mathbf{T}$-groupoid defined by the 2-cocycle ν^c.

By [IKSW19, Lemma 5.2], $A(\nu)$ is a continuous-trace C^*-algebra with spectrum $\hat{G} \times X$ with Dixmier–Douady invariant $\delta(A(\nu)) = [\nu^c]$. For a concrete example, let $G = \mathbf{Z}$ and choose a Čech 2-cocycle c associated to any line bundle.

Example 3.18. This example is an expansion of [IKR+21, Example 4.10]. Let $\Gamma = \mathbf{Z}$ act on $\mathbf{T}$ via rotation by $\alpha \in \mathbf{Q}$: $z \cdot k := ze^{2\pi i k\alpha}$. If $\alpha = m/n$ with m and n relatively prime, then $n\mathbf{Z}$ fixes the action. We have a short exact sequence of groups

$$n\mathbf{Z} \hookrightarrow \mathbf{Z} \xrightarrow{p} \mathbf{Z}_n. \tag{3.5}$$

The action on $\mathbf{T}$ leads to an extension of groupoids

$$n\mathbf{Z} \times \mathbf{T} \xhookrightarrow{i} \mathbf{T} \rtimes \mathbf{Z} \xrightarrow{\pi} \mathbf{T} \rtimes \mathbf{Z}_n. \tag{3.6}$$

Thus, using the notation from the previous section, $\mathcal{A} = \mathbf{T} \times n\mathbf{Z}$, $\Sigma = \mathbf{T} \rtimes \mathbf{Z}$, and $\mathcal{G} = \mathbf{T} \rtimes \mathbf{Z}_n$. The C^*-algebra $C^*(\mathbf{T} \rtimes \mathbf{Z})$ is the rational rotation C^*-algebra $\mathcal{A}_\alpha$ (see, for example, [DB84]). The groupoid $\mathcal{D}$ is the cartesian product $\mathbf{T} \times \mathbf{T}_n \times \mathbf{T} \times \mathbf{Z}$, where $\mathbf{T}_n = \mathbf{T}/\mathbf{Z}_n$ is the dual of $n\mathbf{Z}$. The extension $\widetilde{\Sigma}$ is the quotient of $\mathcal{D}$ where we identify $(\omega, \chi, z, nl+k)$ with $(\omega, \chi^{nl}, z, k)$. Therefore the rational rotation algebra $\mathcal{A}_\alpha$ is the completion of continuous functions F on $\mathbf{T}\times\mathbf{T}_n\times\mathbf{Z}$ such that $F(\omega, \chi, nl+k) = \chi^{nl}F(\omega, \chi, k)$ for all $l \in \mathbf{Z}$.

The extension $\widetilde{\Sigma}$ is properly isomorphic to the one defined by a 2-cocycle. Indeed, let $\sigma = e^{2\pi i\alpha} \in \mathbf{T}$ and view σ as a character on $\mathbf{Z}$. Thus we can identify $\mathbf{Z}_n$ with $\sigma(\mathbf{Z})$ and then the map p in the short exact sequence (3.5) equals σ. Choose $s \in \mathbf{Z}$ such that $sm = 1 \pmod{n}$. Then the map $\tau : \mathbf{Z}_n \to \mathbf{Z}$ defined by $\tau(k) = sk$ defines a cross-section of σ. In particular, $\mathbf{Z}$ is properly isomorphic to the extension $n\mathbf{Z}\times_\omega\mathbf{Z}_n$ by a two cocycle $\omega : \mathbf{Z}_n\times\mathbf{Z}_n \to n\mathbf{Z}$ defined by τ. Using the proof of [IKSW19, Proposition A.6], $\omega(\dot{k}_1, \dot{k}_2) = \tau(\dot{k}_1) + \tau(\dot{k}_2) - \tau(\dot{k}_1 + \dot{k}_2)$. A quick computation shows that

$$\omega(\dot{k}_1, \dot{k}_2) = \begin{cases} 0 & \text{if } \dot{k}_1 + \dot{k}_2 < n \\ ns & \text{if } \dot{k}_1 + \dot{k}_2 \geq n, \end{cases}$$

which recovers the 2-cocycle used in Step 2 of the proof of [DB84, Proposition 1].

The map $\underline{\tau} : \mathbf{T} \rtimes \mathbf{Z}_n \to \mathbf{T} \rtimes \mathbf{Z}$ defined by $\underline{\tau}(z, k) = (z, \tau(k))$ is a cross-section of the extension of the groupoids (3.6). Hence $\mathbf{T} \rtimes \mathbf{Z}$ is properly isomorphic to the extension given by the 2-cocycle $\varphi \in Z^2(\mathbf{T} \rtimes \mathbf{Z}_n, \mathbf{T} \times n\mathbf{Z})$ defined by $\varphi\big((w, \dot{k}_1), (w \cdot \dot{k}_1, \dot{k}_2)\big) = (w, \omega(\dot{k}_1, \dot{k}_2))$. The extension of the 2-cocycle φ is $\Sigma_\varphi = \mathbf{T} \times n\mathbf{Z} \times \mathbf{Z}_n$ with operations $(w, nl_1, \dot{k}_1)(w \cdot \dot{k}_1, nl_2, \dot{k}_2) = (w, nl_1 + nl_2 + \omega(\dot{k}_1, \dot{k}_2), \dot{k}_1 + \dot{k}_2)$ and $(w, nl, \dot{k})^{-1} = (w, -nl - \omega(-\dot{k}, \dot{k}), -\dot{k})$. Following the proof of [IKSW19, Proposition A.6] the isomorphism between Σ_φ and $\mathbf{T} \rtimes \mathbf{Z}$ is given by $(w, nl, \dot{k}) \mapsto (w, nl + \tau(\dot{k}))$.

We have that $\hat{\mathcal{A}} \simeq \mathbf{T}_n\times\mathbf{T}$ and $\hat{\mathcal{A}}*\mathcal{A} \simeq \mathbf{T}_n\times\mathbf{T}\times n\mathbf{Z}$. The action of $\mathcal{G} = \mathbf{T} \rtimes \mathbf{Z}_n$ on $\hat{\mathcal{A}}$ is given via $(\chi, w) \cdot (w, \dot{k}) = (\chi, w \cdot k) = (\chi, w\sigma^k)$. Therefore we can identify $\hat{\mathcal{A}} \rtimes \mathcal{G}$ with $\mathbf{T}_n \times \mathbf{T} \rtimes \mathbf{Z}_n := \{(\chi, w, \dot{k}) \in \mathbf{T}_n \times \mathbf{T} \times \mathbf{Z}_n\}$, where $(\chi, w, \dot{k}_1) \cdot (\chi, w \cdot k_1, \dot{k}_2) = (\chi, w, \dot{k}_1 + \dot{k}_2)$ and $(\chi, w, \dot{k})^{-1} = (\chi, w \cdot k, -\dot{k})$. Thus the 2-cocycle $\tilde{\varphi} : \big(\mathbf{T}_n \times \mathbf{T} \rtimes \mathbf{Z}_n\big)^{(2)} \to \mathbf{T}_n \times \mathbf{T} \times n\mathbf{Z}$ of (3.4) is defined by

$$\tilde{\varphi}\big((\chi, w, \dot{k}_1), (\chi, w \cdot \dot{k}_1, \dot{k}_2)\big) = \big(\chi, w, \omega(\dot{k}_1, \dot{k}_2)\big).$$

By Lemma 3.16, $\widetilde{\Sigma}$ is properly isomorphic to the extension by the 2-cocycle $\hat{\varphi}$ which is the pushout of $\tilde{\varphi}$. Therefore $\hat{\varphi} : \big(\mathbf{T}_n \times \mathbf{T} \rtimes \mathbf{Z}_n\big)^{(2)} \to \mathbf{T}_n \times \mathbf{T} \times \mathbf{T}$ is defined by

$$\hat{\varphi}((\chi, w, \dot{k}_1), (\chi, w \cdot \dot{k}_1, \dot{k}_2)) = (\chi, w, \chi^{\omega(\dot{k}_1, \dot{k}_2)}).$$

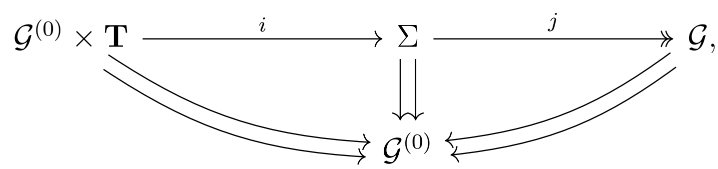
Hence the rotation algebra $\mathcal{A}_\alpha$ is isomorphic to $C^*(\mathbf{T}_n \times \mathbf{T} \rtimes \mathbf{Z}_n; \Sigma_{\hat{\varphi}})$. For $\chi \in \mathbf{T}_n$, define $\chi_*(\varphi) : (\mathbf{T} \rtimes Z_n)^{(2)} \to \mathbf{T}$ by

$$\chi_*(\varphi)((w, \dot{k}_1), (w \cdot \dot{k}_1, \dot{k}_2) = (w, \chi^{\omega(\dot{k}_1, \dot{k}_2)}).$$

Then Proposition 3.6 implies that $\mathcal{A}_\alpha$ is the section algebra of an upper-semicontinuous C^*-bundle over $\mathbf{T}_n$ with fiber at $\chi \in \mathbf{T}_n$ isomorphic to $C^*(\mathbf{T} \rtimes Z_n; \Sigma_{\chi_*(\varphi)})$.

Appendix A. Bundles of Twists

Let Σ be a twist over $\mathcal{G}$. Alternatively, Σ is a $\mathbf{T}$-groupoid so that we have the following diagram



where as usual we have identified $\Sigma^{(0)}$ and $\mathcal{G}^{(0)}$. In particular, if $F \subset \mathcal{G}^{(0)}$ is $\mathcal{G}$-invariant, then it is Σ-invariant and the reduction $\Sigma|_F$ is also a twist over the reduction $\mathcal{G}|_F$.

Suppose that $p : \mathcal{G}^{(0)} \to T$ is a continuous map such that $p \circ r = r \circ s$. Then we say that Σ is a groupoid bundle over T.[1] Then $p^{-1}(t)$ is invariant for all $t \in T$. We write $\Sigma(t)$ and $\mathcal{G}(t)$ for the restrictions to $p^{-1}(t)$, respectively. Then $\Sigma(t)$ is a twist over $\mathcal{G}(t)$.

Proposition A.1. *Suppose that $\mathcal{G}$ is a second countable locally compact Hausdorff groupoid with a Haar system and that Σ is a twist over $\mathcal{G}$. If $p : \mathcal{G}^{(0)} \to T$ is a continuous map such that $p \circ r = p \circ s$, then $C^*(\mathcal{G}; \Sigma)$ is a $C_0(T)$-algebra. Let $\Sigma(t)$ be the twist over $\mathcal{G}(t)$ defined above. Then $C^*(\mathcal{G}; \Sigma)$ is (isomorphic to) the section algebra of an upper-semicontinuous C^*-bundle over T. The fibre $C^*(\mathcal{G}; \Sigma)(t)$ is isomorphic to $C^*(\mathcal{G}(t); \Sigma(t))$.*

Proof. Recall that $C^*(\mathcal{G}; \Sigma)$ is the C^*-algebra $C^*(\mathcal{G}, \mathcal{B})$ of a Fell bundle $q : \mathcal{B} \to \mathcal{G}$ as described in [MW08, Example 2.9]. Similarly, $C^*(\mathcal{G}(t); \Sigma(t))$ is the C^*-algebra $C^*(\mathcal{G}(t), \mathcal{B})$ of $q|_{q^{-1}(\mathcal{G}(t))}$. Let $U(t) = \mathcal{G}^{(0)} \setminus p^{-1}(t)$. Using [IW12, Theorem 3.7] (as in [SW13, Lemma 9]), we obtain a short exact sequence

$$0 \longrightarrow C^*(\mathcal{G}|_{U(t)}, \mathcal{B}) \xrightarrow{\;i\;} C^*(\mathcal{G}, \mathcal{B}) \xrightarrow{\;j\;} C^*(\mathcal{G}(t), \mathcal{B}) \longrightarrow 0$$

where i identifies $C^*(\mathcal{G}|_{U(t)}, \mathcal{B})$ with the completion in $C^*(\mathcal{G}, \mathcal{B})$ of the ideal of sections in $\Gamma_c(\mathcal{G}, \mathcal{B})$ that vanish off $\mathcal{G}|_{U(t)}$, and j is given on $\Gamma_c(\mathcal{G}, \mathcal{B})$ by restriction to $p^{-1}(t)$. Now exactly as in [Wil19, Proposition 5.37], we see that $C^*(\mathcal{G}, \mathcal{B})$ is a $C_0(T)$-algebra with fibres $C^*(\mathcal{G}, \mathcal{B})(t)$ identified with $C^*(\mathcal{G}(t), \mathcal{B})$. □

[1] The third author defined groupoid bundles in [Ren15, Definition 3.3] where it is also required that p be open.

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