AN ALGEBRAIC ANALOGUE OF EXEL–PARDO C^*-ALGEBRAS

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Abstract. We introduce an algebraic version of the Katsura C^*-algebra of a pair A, B of integer matrices and an algebraic version of the Exel–Pardo C^*-algebra of a self-similar action on a graph. We prove a Graded Uniqueness Theorem for such algebras and construct a homomorphism of the latter into a Steinberg algebra that, under mild conditions, is an isomorphism. Working with Steinberg algebras over non-Hausdorff groupoids we prove that in the unital case, our algebraic version of Katsura C^*-algebras are all isomorphic to Steinberg algebras.

Introduction

Recently in [15] Exel and Pardo introduced C^*-algebras $\mathcal{O}_{G,E}$ giving a unified treatment of two classes of C^*-algebras which have attracted significant recent attention, namely Katsura C^*-algebras [17] and Nekrashevych's self-similar group C^*-algebras [21, 22]. Katsura C^*-algebras are important as they provide concrete models for all UCT Kirchberg algebras [18]; self-similar C^*-algebras have provided the first known example of a groupoid whose C^*-algebra is simple and whose Steinberg algebra over some fields is non-simple [9]. This suggests that algebraic analogues of Exel–Pardo C^*-algebras may be a source of interesting new examples. Partial results have already been established by Clark, Exel and Pardo in [8] who introduced $\mathcal{O}^{\mathrm{alg}}_{(G,E)}(R)$ for finite graphs E. However, as far as we know this article is the first study of algebraic analogues of Exel–Pardo C^*-algebras for infinite graphs.

In this paper we introduce an algebraic version of Exel–Pardo C^*-algebras $\mathcal{O}_{G,E}$ providing novel results both when E is infinite and when it is finite. We focus on graphs E that are row-finite with no sources. Up to Morita equivalence, they include all Exel–Pardo C^*-algebras $\mathcal{O}_{G,E}$ [16, Theorem 3.2].

The paper is structured as follows. In Section 1 we introduce the *-algebras $L_R(G,E)$ as an algebraic version of Exel–Pardo C^*-algebras. The main ingredient is an action of a group G on a graph E which

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incorporates a notion of a remainder. More specifically, we take a countable discrete group G, an action

$$(g, v) \mapsto g \cdot v, \quad (g, e) \mapsto g \cdot e$$

of G on a row-finite graph $E = (E^0, E^1, r, s)$ with no sources, and a one-cocycle $\varphi : G \times E^1 \to G$ for the action of G on the edges of E. Following [15], with this data and a few natural axioms we get an action of G on the space of finite paths E^* which satisfies the following "self-similarity" equation

$$g \cdot (\alpha\beta) = (g \cdot \alpha)(\varphi(g, \alpha) \cdot \beta), \quad \text{for all } \ g \in G, \alpha\beta \in E^*. \tag{0.1}$$

We refer to Notation 1.2 for careful exposition of this setup.

Our algebraic analogue of the Exel–Pardo algebras is described by generators and relations. Specifically, given a self-similar action of G on E as above, we consider *-algebras generated by elements $p_{v,h}$ and $s_{e,g}$ indexed by vertices v and edges e of E, and by elements g, h of G under relations that ensure that: the p_{v,e_G} and s_{e,e_G} form an E-family in the sense of Leavitt-path algebras; and for each v, the map $g \mapsto p_{v,g}$ is a unitary representation of G; and multiplication amongst the $p_{v,h}$ and $s_{e,g}$ reflect the structure of the self-similar action (see Definition 1.5). We prove in Theorem 1.6 that up to *-isomorphism there exists a unique *-algebra $L_R(G, E)$ over a commutative unital ring R universal for these generators and relations.

The analogous C^*-algebras $\mathcal{O}_{G,E}$ were first studied for finite graphs in [15] and then for countably infinite graphs in [16]; our generators and relations are modelled on the latter, and determine the same *-algebra over R, though we omit the proof of this assertion. For finite graphs, an algebraic analogue $\mathcal{O}^{\text{alg}}_{(G,E)}(R)$ has also been studied in [8]. Our generators and relations appear different to those in [8] because we include a representation $g \mapsto p_{v,g}$ of G for each $v \in E^0$, rather than a single representation $g \mapsto u_g$ of G. This is to avoid the use of multiplier rings, which would otherwise be necessary in the setting of graphs with infinitely many vertices; but we show in Proposition 1.9 that our construction coincides with that of [8] when E^0 is finite.

After Proposition 1.9, we consider the algebraic analogue of Katsura C^*-algebras which we denote $\mathcal{O}^{\text{alg}}_{A,B}(R)$ and also revisit the algebraic analogue of graph algebras, the Leavitt path algebras $L_R(E)$ [2]. We prove in Proposition 1.13 and Proposition 1.15 that both of these are special cases of Exel–Pardo *-algebras. Beyond $\mathcal{O}^{\text{alg}}_{(G,E)}(R)$, $\mathcal{O}^{\text{alg}}_{A,B}(R)$, $L_R(E)$, there are other examples of Exel–Pardo *-algebras, but these examples give a good indication of the scope of the class of algebras we consider. We refer to Figure 1 for a schematic comparison of the examples in Section 1.

In Section 2 we study the graded structure of $L_R(G,E)$. In Lemma 2.5 we prove that $L_R(G,E)$ admits a $\mathbb{Z}$-grading and we use it to prove the Graded Uniqueness Theorem A:

Theorem A (Graded Uniqueness). *Let (G,E,φ) be as in Notation 1.2. Let R be a unital commutative *-ring. Let $\pi\colon L_R(G,E)\to B$ be a $\mathbb{Z}$-graded *-algebra homomorphism into a $\mathbb{Z}$-graded *-algebra B. Suppose that*

$$\pi(a)\neq 0 \quad \textit{for all} \quad a\in \operatorname{span}_R\{p_{v,f} : v\in E^0, f\in G\}\setminus\{0\},$$

then π is injective.

We then study the subalgebra $\mathcal{D} := \operatorname{span}_R\{p_{v,f} : v\in E^0, f\in G\}$ inside of $L_R(G,E)$. We provide a structural characterisation of $\mathcal{D}$ as a direct sum of matrix algebras of certain *-algebras over R (see Theorem 2.10).

In Section 3 we revisit and generalise some of the work of Exel, Pardo and Clark in [8]. There they considered (among other things) an algebraic analogue of the well known isomorphism of [15] between $\mathcal{O}_{G,E}$ and the groupoid C^*-algebra $C^*(\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E}))$ associated to a groupoid of germs constructed from G and E (see Definition 3.5). In particular, they proved that $\mathcal{O}^{\text{alg}}_{(G,E)}(R)$ is isomorphic to the Steinberg algebra $A_R(\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E}))$ whenever E is finite and $R=\mathbb{C}$ (see Remark 3.8). In Proposition 3.7 we prove that there always exists a canonical *-homomorphism

$$\pi_{G,E}\colon L_R(G,E)\to A_R(\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})).$$

When the groupoid is Hausdorff we prove that this $\pi_{G,E}$ is an isomorphism:

Theorem B (The Hausdorff case). *Let (G,E,φ) be as in Notation 1.2. Let R be a unital commutative *-ring. If $\mathcal{G}_{tight}(\mathcal{S}_{G,E})$ is Hausdorff then*

$$\pi_{G,E}\colon L_R(G,E)\to A_R(\mathcal{G}_{tight}(\mathcal{S}_{G,E}))$$

from Proposition 3.7 is a *-isomorphism.

Using Theorem B we can apply existing machinery [3, 14, 15] to describe precisely when $L_R(G,E)$ is simple and provide sufficient conditions for $L_R(G,E)$ to be simple and purely infinite. We do this in Proposition 3.14 and Proposition 3.15.

We have not proved a general non-Hausdorff version of Theorem B, but we obtain partial results in Section 4. In particular we show that the *-algebra $\mathcal{O}^{\text{alg}}_{A,B}(R)$ associated to finite integer matrices A,B is always isomorphic to the associated Steinberg algebra:

Theorem C (Steinberg–Katsura *-algebras). *Fix $N\in\mathbb{N}$ and matrices $A,B\in M_N(\mathbb{Z})$ such that $A_{ij}\geq 0$ and $\sum_j A_{ij}>0$ for all i. Let $\mathcal{G}_{A,B}$ be* *the groupoid of germs for the Katsura triple $(\mathbb{Z}, E, \varphi)$ associated to A and B as in Definition 1.12 and Definition 3.5. Then*

$$L_R(\mathbb{Z}, E) \cong \mathcal{O}^{alg}_{A,B}(R) \cong A_R(\mathcal{G}_{A,B}).$$

Theorem C is obtained by proving $L_R(G, E) \cong A_R(\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E}))$ for a broad class of self-similar actions (G, E, φ) (see Theorem 4.2) and then applying this result to Katrura triples.

1. The algebraic version of Exel–Pardo C^*-algebras

The C^*-algebras $\mathcal{O}_{G,E}$ unify many previously known classes of C^*-algebras, including graph C^*-algebras, Katsura C^*-algebras and C^*-algebras associated to self-similar groups [15]. In the algebraic setting much less is known. In this section we define the Exel–Pardo *-algebra $L_R(G, E)$ as an algebraic analogue of the Exel–Pardo C^*-algebra and compare it to other known algebras. Exel, Clark and Pardo have already made a definition of $\mathcal{O}^{\text{alg}}_{(G,E)}(R)$ when E is finite and we show that our definition is a genuine generalisation of theirs (see Proposition 1.9). We do not attempt to give an exhaustive list of examples, but we will consider how $L_R(G, E)$ relates to an algebraic analogue of Katsura C^*-algebras associated to infinite matrices, and an algebraic analogue of graph C^*-algebras of infinite graphs, the Levitt path algebras $L_R(E)$.

We start with a few definitions. Following [19], a *directed graph* E consists of countable sets E^0, E^1 of vertices and edges, and maps $r, s\colon E^1 \to E^0$ describing the range and source of edges. The graph is *row-finite* if $vE^1 := r^{-1}(v)$ is finite for each $v \in E^0$, and has *no sources* if vE^1 is non-empty for each $v \in E^0$.

A *-algebra over a *-ring R is an algebra equipped with a map $a \mapsto a^*$ called an *involution* satisfying that $(a^*)^* = a$, $(ab)^* = b^*a^*$ and $(ra + b)^* = r^*a^* + b^*$. Let A be such an algebra. We call $p \in A$ a *projection* if $p = p^* = p^2$, we call $s \in A$ a *partial isometry* if $s = ss^*s$ and we call $u \in A$ a *partial unitary* if $u^*u = uu^* = (u^*u)^2$. Two projections are mutually *orthogonal* if their product is zero.

In this paper we use the convention from [23] where paths read from right to left when defining graph algebras, hence the adjusted Definition 1.1.

Definition 1.1 ([19])**.** If E is a row-finite directed graph, an *E-family* in a *-algebra A consists of a set $\{P_v : v \in E^0\}$ of mutually orthogonal projections and a set $\{S_e : e \in E^1\}$ of partial isometries in A such that

$$S_e^* S_f = \delta_{e,f} P_{s(e)} \text{ for all } e, f \in E^1, \text{ and} \tag{1.1}$$

$$P_v = \sum_{e \in vE^1} S_e S_e^* \text{ for all } v \in r(E^1). \tag{1.2}$$

Let E be a directed graph. Following [15], by an *automorphism* of E we mean a bijective map $\sigma\colon E^0 \sqcup E^1 \to E^0 \sqcup E^1$ such that $\sigma(E^i) = E^i$, for $i = 0, 1$ and such that $r \circ \sigma = \sigma \circ r$, and $s \circ \sigma = \sigma \circ s$. By an *action* of a group G on E we shall mean a group homomorphism $g \mapsto \sigma_g$ from G to the group of all automorphisms of E. We often write $g \cdot e$ instead of $\sigma_g(e)$. The unit in a group G is denoted e_G.

Let X be a set, and let σ be an action of a group G on X (i.e., a homomorphism from G to the group of bijections from X to X). A map $\varphi\colon G \times X \to G$ is a *one-cocycle* for σ if

$$\varphi(gh, x) = \varphi(g, \sigma_h(x))\varphi(h, x)$$

for all $g, h \in G$, and all $x \in X$, see [15].

Notation 1.2. The quadruple (G, E, σ, φ), sometimes written as a triple (G, E, φ) or a pair (G, E), will denote a countable discrete group G, a row-finite graph E with no sources, an (occasionally unnamed) action σ of G on E and a one-cocycle $\varphi\colon G \times E^1 \to G$ for the restriction of σ to E^1 such that for all $g \in G, e \in E^1, v \in E^0$

$$\varphi(g, e) \cdot v = g \cdot v. \tag{1.3}$$

Remark 1.3. The axiom (1.3) implies the apparently more general self-similarity condition (0.1) (see Lemma 3.1(8)). However, (0.1) can also be obtained if we only assume the condition that $\varphi(g, e) \cdot s(e) = g \cdot s(e)$ whenever $g \in G, e \in E^1$. Thus, the constraint (1.3) might seem unnatural. However, as shown in [15], the most prominent classes of examples satisfy this constraint, see [15, p. 1049]. To remove this constraint it is arguably more natural to work in the setting of self-similar action of groupoids as in [20].

Remark 1.4. In this section we consider 3 types of triples (G, E, φ):

(1) the triples (G, E, φ) where E is finite.
(2) the Katsura triples as defined in Definition 1.12.
(3) the triples (G, E, φ) where G is trivial.

In subsection 1.1–1.3 we will see how they generate 3 important classes of algebras that we schematically illustrate on Figure 1. Each of these serves as an example of our more general construction.

We now present a bit more terminology and then state Theorem 1.6, which asserts the existence and uniqueness of the *-algebra $L_R(G, E)$ of the triple (G, E, φ).

Definition 1.5. Let (G, E, φ) be as in Notation 1.2. Let R be a unital commutative *-ring. A (G, E)-*family* in a *-algebra A over R is a set

$$\{P_{v,f} : v \in E^0, f \in G\} \cup \{S_{e,g} : e \in E^1, g \in G\} \subseteq A$$

such that

(a) $\{P_{v,e_G} : v \in E^0\} \cup \{S_{e,e_G} : e \in E^1\}$ is an E-family in A,
(b) $(P_{v,f})^* = P_{f^{-1} \cdot v, f^{-1}}$,
(c) $P_{v,f}P_{w,h} = \delta_{v,f \cdot w}P_{v,fh}$

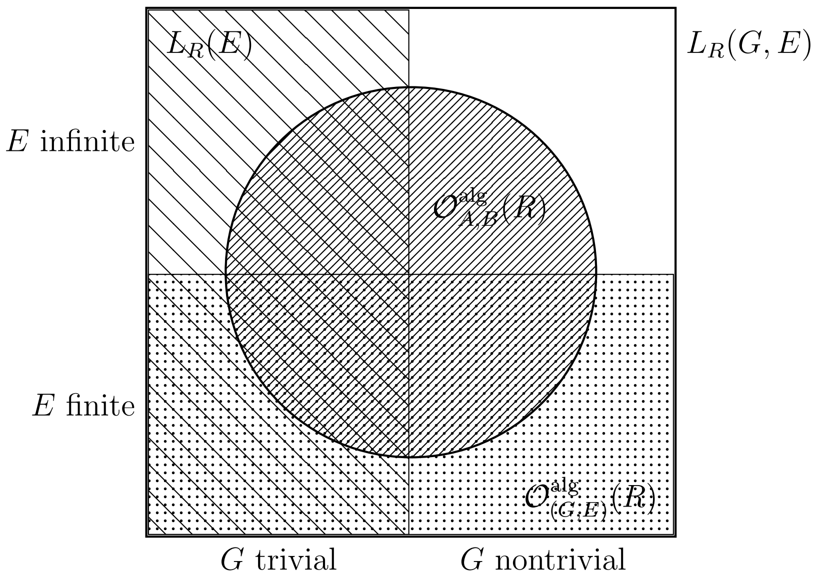


FIGURE 1. The dotted area indicates that each unital Exel–Pardo *-algebra is isomorphim to some $\mathcal{O}^{\mathrm{alg}}_{(G,E)}(R)$.

(d) $P_{v,f}S_{e,g} = \delta_{v,r(f\cdot e)}S_{f\cdot e,\varphi(f,e)g}$, and
(e) $S_{e,g}P_{v,f} = \delta_{g\cdot v,s(e)}S_{e,gf}$.

We shall often abbreviate a (G,E)-family as $\{P_{v,f}, S_{e,g}\}$.

Theorem 1.6 (Universal Property)**.** *Let (G,E,φ) be as in Notation 1.2. Let R be a unital commutative *-ring. Then there is a *-algebra $L_R(G,E)$ over R generated by a (G,E)-family $\{p_{v,f}, s_{e,g}\}$ with the following universal property: whenever $\{P_{v,f}, S_{e,g}\}$ is a (G,E)-family in a *-algebra A over R, there is a unique *-homomorphism $\pi_{G,E}\colon L_R(G,E) \to A$ such that*

$$\pi_{G,E}(p_{v,f}) = P_{v,f}, \quad \textit{and} \quad \pi_{G,E}(s_{e,g}) = S_{e,g}$$

for all $v \in E^0$, $e \in E^1$ and $f,g \in G$.

We call the *-algebra $L_R(G,E)$ of Theorem 1.6 the *Exel–Pardo *-algebra of* (G,E), and we call $\{p_{v,f}, s_{e,g}\}$ the *universal* (G,E)*-family.*

Proof of Theorem 1.6. Let

$$X := \{P_{v,f}, S_{e,g}, (P_{v,f})^*, (S_{e,g})^* : v \in E^0, e \in E^1, f,g \in G\}$$

be a set of formal symbols and $Y := w(X)$ the set of all finite words in the alphabet X. Let $\mathbb{F}_R(Y)$ be the free R-module generated by Y, that is $\mathbb{F}_R(Y)$ is the set of formal sums $\sum_{y\in Y} r_y y$ in which all but finitely many coefficients $r_y \in R$ are zero. We equip $\mathbb{F}_R(Y)$ with the multiplication

$$\Big(\sum_{x\in Y} r_x x\Big)\Big(\sum_{y\in Y} s_y y) := \sum_{z\in Y}\sum_{\{x,y\in Y:\ xy=z\}} r_x s_y z.$$

We define $((P_{v,f})^*)^* = P_{v,f}$ and $((S_{e,g})^*)^* = S_{e,g}$. For $x = x_1x_2\dots x_n \in Y$, we define $x^* := x_n^* x_{n-1}^* \dots x_1^*$. We then define ${}^*\colon \mathbb{F}_R(Y) \to \mathbb{F}_R(Y)$ by

$$\Big(\sum_{x\in Y} r_x x\Big)^* := \sum_{x\in Y} r_x^* x^*.$$

This makes $\mathbb{F}_R(Y)$ it into a *-algebra over R.

Let I be the two-sided ideal of $\mathbb{F}_R(Y)$ generated be the union of the following nine sets and their set adjoints:

$$
\begin{aligned}
&\{P_{v,e_G} - P_{v,e_G}^*,\ P_{v,e_G}^2 - P_{v,e_G} : v \in E^0\},\\
&\{S_{e,e_G}S_{e,e_G}^*S_{e,e_G} - S_{e,e_G} : e \in E^1\},\\
&\{S_{e,e_G}^*S_{e,e_G} - P_{s(e),e_G} : e \in E^1\},\\
&\{P_{v,e_G} - \sum_{e\in vE^1} S_{e,e_G}S_{e,e_G}^* :\ v \in r(E^1)\},\\
&\{(S_{e,e_G}S_{e,e_G}^*)(S_{f,e_G}S_{f,e_G}^*) : v \in r(E^1),\, e,f \in vE^1, e \neq f\},\\
&\{(P_{v,f})^* - P_{f^{-1}\cdot v,f^{-1}} : v \in E^0, f \in G\},\\
&\{P_{v,f}P_{w,h} - \delta_{v,f\cdot w}P_{v,fh} : v,w \in E^0, f,h \in G\},\\
&\{P_{v,f}S_{e,g} - \delta_{v,r(f\cdot e)}S_{f\cdot e,\varphi(f,e)g} : v \in E^0, e \in E^1, f,g \in G\},\\
&\{S_{e,g}P_{v,f} - \delta_{g\cdot v,s(e)}S_{e,gf} : v \in E^0, e \in E^1, f,g \in G\}.
\end{aligned}
\tag{1.4}
$$

We now define $L_R(G,E) := \mathbb{F}_R(Y)/I$ and let $\{p_{v,f}, s_{e,g}\}$ be the image of $\{P_{v,f}, S_{e,g}\}$ via the quotient map $q\colon \mathbb{F}_R(Y) \to L_R(G,E)$. By construction the collection $\{p_{v,f}, s_{e,g}\}$ is a (G,E)-family in $L_R(G,E)$.

Now let $\{\tilde{P}_{v,f}, \tilde{S}_{e,g}\}$ be any (G,E)-family in a *-algebra A over R. Define $\phi\colon X \to A$ by $\phi(P_{v,f}) = \tilde{P}_{v,f}$, $\phi(S_{e,g}) = \tilde{S}_{e,g}$, $\phi((P_{v,f})^*) = (\tilde{P}_{v,f})^*$, and $\phi((S_{e,g})^*) = (\tilde{S}_{e,g})^*$. Extend ϕ to a map also denoted $\phi\colon Y \to A$ via $\phi(x_1x_2\dots x_n) := \phi(x_1)\phi(x_2)\dots\phi(x_n)$ for $x_i \in X$. Recall, we may identify Y with a subset of $\mathbb{F}_R(Y)$ via $y \mapsto 1y$. Now, by the universal property of the free R-module $\mathbb{F}_R(Y)$, there exists a unique R-module homomorphism $\Phi\colon \mathbb{F}_R(Y) \to A$ extending ϕ. Since ϕ is compatible with *, Φ is a *-homomorphism. Since $\Phi(I) = 0$, there is a *-homomorphism $pi_{G,E}\colon L_R(G,E) \to A$ such that $\pi_{G,E}(x + I) = \Phi(x)$ for all $x \in \mathbb{F}_R(Y)$. For each $v \in E^0, f \in G$ we have

$$\pi_{G,E}(p_{v,f}) = \pi_{G,E}(P_{v,f} + I) = \Phi(P_{v,f}) = \phi(P_{v,f}) = \tilde{P}_{v,f}.$$

Similarly, for each $e \in E^1, g \in G$, we have $\pi_{G,E}(s_{e,g}) = \tilde{S}_{e,g}$. □

Remark 1.7. We note $L_R(G,E)$ satisfies the following:

(1) When E^0 is finite, $L_R(G,E)$ is unital with unit $\sum_{v\in E^0} p_{v,e_G}$.

(2) We will show in Proposition 3.7 that the generators $p_{v,f}$, $s_{e,g}$ of $L_R(G,E)$ are all nonzero.

In the following subsection we consider the triples (G,E,φ) of Remark 1.4 and their associated algebras as illustrated on Figure 1.

1.1. **The unital case.** For finite graphs E, algebraic versions of the Exel–Pardo C^*-algebras $\mathcal{O}_{G,E}$ were introduced in [8]. Here we show that our definition yields the same algebras as those defined in [8].

First we recall some definitions. Let A be a unital *-algebra. A *unitary representation* of a discrete group G on A corresponds to a collection $\{u^g : g \in G\}$ of unitaries in A satisfying $u^g u^h = u^{gh}$ (for all $g,h \in G$). It follows that $u^{e_G} = 1$ and $(u^g)^* = u^{(g^{-1})}$.

Let (G,E,φ) be as in Notation 1.2 and suppose that E is finite. Let R be a unital commutative ring. In [8, Definition 6.2], $\mathcal{O}^{\mathrm{alg}}_{(G,E)}(R)$ is defined to be the universal *-algebra over R with the following generators and relations:

(1) Generators: $\{p_v : v \in E^0\} \cup \{s_e : e \in E^1\} \cup \{u^g : g \in G\}$.
(2) Relations:
   (a) $\{p_v : v \in E^0\} \cup \{s_e : e \in E^1\}$ is an E-family.
   (b) The map $u : G \to \mathcal{O}^{\mathrm{alg}}_{(G,E)}(R)$ defined by the rule $g \mapsto u^g$ is a unitary representation of G.
   (c) $u^g s_e = s_{g\cdot e} u^{\varphi(g,e)}$ for every $g \in G, e \in E^1$.
   (d) $u^g p_v = p_{g\cdot v} u^g$ for every $g \in G, v \in E^0$.
   (e) $\sum_{v\in E^0} p_v = u^{e_G}$.

Remark 1.8. The last relation in the definition of $\mathcal{O}^{\mathrm{alg}}_{(G,E)}(R)$ does not explicitly appear in [8], but was certainly intended. We need this property at the end of the proof of Proposition 1.9.

Proposition 1.9. *Let (G,E,φ) be as in Notation 1.2 and suppose that E is finite. Equip R with the trivial involution. Then*

$$\mathcal{O}^{alg}_{(G,E)}(R) \cong L_R(G,E).$$

Proof. We first build a homomorphism $\pi_1 \colon L_R(G,E) \to \mathcal{O}^{\mathrm{alg}}_{(G,E)}(R)$ by invoking the universal property of $L_R(G,E)$. Let $\{p_v, s_e, u^g\}$ denote the generators for $\mathcal{O}^{\mathrm{alg}}_{(G,E)}(R)$. For each $v \in E^0$, $e \in E^1$ and $f,g \in G$ define

$$P_{v,f} := p_v u^f \quad \text{and} \quad S_{e,g} := s_e u^g.$$

Routine calculations using that $s_e = p_{r(e)} s_e p_{s(e)}$ show that $\{P_{v,f}, S_{e,g}\}$ is a (G,E)-family. The universal property of $L_R(G,E)$ now yields a *-algebra homomorphism

$$\pi_1 \colon L_R(G,E) \to \mathcal{O}^{\mathrm{alg}}_{(G,E)}(R)$$

such that $\pi_1(p_{v,f}) = P_{v,f}$ and $\pi_1(s_{e,g}) = S_{e,g}$.

To construct an inverse for π_1 we will use the universal property of $\mathcal{O}^{\mathrm{alg}}_{(G,E)}(R)$. Using the generators $\{p_{v,f}, s_{e,g}\}$ for $L_R(G,E)$, for each $v \in E^0$, $e \in E^1$ and $g \in G$ define

$$P_v := p_{v,e_G}, \quad S_e := s_{e,e_G}, \quad U^g := \sum_{v\in E^0} p_{v,g}.$$

Again, routine calculations using that $e_G \cdot v = v$ show that $\{P_v, S_e, U^g\}$ satisfy the relations (2a)–(2e) in the definition of $\mathcal{O}^{\mathrm{alg}}_{(G,E)}(R)$. The universal property of $\mathcal{O}^{\mathrm{alg}}_{(G,E)}(R)$ provides a *-algebra homomorphism

$$\pi_2 \colon \mathcal{O}^{\mathrm{alg}}_{(G,E)}(R) \to L_R(G,E)$$

such that $\pi_2(p_v) = P_v$, $\pi_2(s_e) = S_e$ and $\pi_2(u^g) = U^g$. By computing $\pi_i \circ \pi_j$ $(i \neq j)$ on generators we see that π_1 is an inverse for π_2; for the generator u^g in $\mathcal{O}^{\mathrm{alg}}_{(G,E)}(R)$ we use (2e) from the definition of $\mathcal{O}^{\mathrm{alg}}_{(G,E)}(R)$ to see that

$$\pi_1 \circ \pi_2(u^g) = \sum_{v\in E^0} p_v u^g = u^g. \quad \square$$

1.2. **The vertex-trivial case.** In [17] Katsura introduced C^*-algebras $\mathcal{O}_{A,B}$ which we call Katsura C^*-algebras. Here we consider an algebraic analogue, denoted $\mathcal{O}^{\mathrm{alg}}_{A,B}(R)$, and prove that all such *-algebras are Exel–Pardo *-algebras using the translation of the matrices A, B into an action of $\mathbb{Z}$ on a graph discovered by Exel and Pardo in [14] (see Definition 1.12).

We recall the relevant notation needed to introduce Katsura C^*-algebras $\mathcal{O}_{A,B}$. Fix $N \in \mathbb{N} \cup \{\infty\}$. Let $I := \{1, 2, \ldots, N\}$ for N finite and $I := \mathbb{N}$ otherwise. With A_{ij} denoting the ij-entry of an $I \times I$ nonnegative integer matrix A define $\Omega_A := \{(i,j) \in I \times I : A_{ij} > 0\}$, and $\Omega_A(i) := \{j : (i,j) \in \Omega_A\}$ for $i \in I$.

Definition 1.10 ([14, Definition 2.3],[17, Definition 2.2])**.** Fix $N \in \mathbb{N} \cup \{\infty\}$ and row-finite matrixes $A, B \in M_N(\mathbb{Z})$ such that A has nonnegative entries and no zero rows. Let $I := \{1, 2, \ldots, N\}$ for N finite and $I := \mathbb{N}$ otherwise. The C^*-algebra $\mathcal{O}_{A,B}$ is the universal C^*-algebra generated by mutually orthogonal projections $(q_k)_{k\in I}$, partial unitaries $(u_k)_{k\in I}$ with $q_k = u_k u_k^*$, and partial isometries $(s_{ijn})_{(i,j)\in\Omega_A, n\in\mathbb{Z}}$ such that

(†)
(i) $s_{ijn} u_j = s_{ij(n+A_{ij})}, u_i s_{ijn} = s_{ij(n+B_{ij})}$ for $(i,j) \in \Omega_A, n \in \mathbb{Z}$,
(ii) $s^*_{ijn} s_{ijn} = q_j$ for $(i,j) \in \Omega_A$, $n \in \mathbb{Z}$, and
(iii) $q_i = \sum_{j\in\Omega_A(i), 1\le n\le A_{ij}} s_{ijn} s^*_{ijn}$ $(i \in I)$.

Definition 1.11. Similarly to the construction of $L_R(G,E)$, it makes perfect sense to consider the universal *-algebra over a unital commutative *-ring R with the same generators and relations as those for $\mathcal{O}_{A,B}$ but with the additional relations that if $j \neq j'$ then $s^*_{ijn} s_{ij'n} = 0$ for all n and that if $1 \le n < n' \le A_{ij}$ then $s^*_{ijn} s_{ijn'} = 0$. These relations follow automatically from the others in a C^*-algebra, but must be imposed separately in an abstract *-algebra. We denote this universal *-algebra by $\mathcal{O}^{\mathrm{alg}}_{A,B}(R)$.

Definition 1.12 ([15, Remark 18.3])**.** Fix N, A, B and I as in Definition 1.10. Let E be the directed graph with vertices $\{v_i : i \in I\}$ and edges $\{e_{ijn} : i, j \in I, 0 \leq n \leq A_{ij} - 1\}$ with $r(e_{ijn}) = v_i$ and $s(e_{ijn}) = v_j$. By construction E is row-finite and has no sources.

It is straightforward to check that we can define an action σ of $\mathbb{Z}$ on E^1 and a one-cocycle $\varphi\colon \mathbb{Z} \times E^1 \to \mathbb{Z}$ as follows: For any $i, j \in \Omega_A$, $n \in \{0, \dots, A_{ij} - 1\}$ and $m \in \mathbb{Z}$ there are a unique $\hat{n} \in \{0, \dots, A_{ij} - 1\}$ and a unique $\hat{k} \in \mathbb{Z}$ such that $mB_{ij} + n = \hat{k}A_{ij} + \hat{n}$. We define $\sigma_m(e_{ijn}) := e_{ij\hat{n}}$ and $\varphi(m, e_{ijn}) := \hat{k}$. Since σ_m permutes parallel edges, σ extends to an action on E such that $\sigma_m(v) = v$ for all $m \in \mathbb{Z}$ and $v \in E^0$. We call $(\mathbb{Z}, E, \varphi)$ the *Katsura triple* associated to A, B.

Proposition 1.13. *Take $N \in \mathbb{N} \cup \{\infty\}$, and let $A, B \in M_N(\mathbb{Z})$ be as in Definition 1.10. Let $(\mathbb{Z}, E, \varphi)$ be the Katsura triple associated to A, B. Then*

$$\mathcal{O}^{alg}_{A,B}(R) \cong L_R(\mathbb{Z}, E),$$

as *-algebras over R.

Proof. We first use the universal property of $L_R(\mathbb{Z}, E)$ to obtain a homomorphism $\pi_1\colon L_R(\mathbb{Z}, E) \to \mathcal{O}^{\mathrm{alg}}_{A,B}(R)$. For this let $\{q_k, u_k, s_{ijn}\}$ denote the generators for $\mathcal{O}^{\mathrm{alg}}_{A,B}(R)$. Let $I := \{1, 2, \dots, N\}$ for N finite and $I := \mathbb{N}$ otherwise. For each $k \in I$, and each $m \in \mathbb{Z}$, set

$$u_k^m := \begin{cases} (u_k^*)^{-m} & \text{if } m < 0, \\ q_k & \text{if } m = 0, \\ (u_k)^m & \text{if } m > 0. \end{cases}$$

For $v = v_k \in E^0$, $e = e_{ijn} \in E^1$ and $m, l \in \mathbb{Z}$ define

$$P_{v,m} := u_k^m, \quad S_{e,l} := s_{ijn} u_j^l.$$

Routine calculations show that $\{P_{v,m}, S_{e,l}\}$ is a $(\mathbb{Z}, E)$-family. So the universal property of $L_R(\mathbb{Z}, E)$ provides a *-algebra homomorphism

$$\pi_1\colon L_R(\mathbb{Z}, E) \to \mathcal{O}^{\mathrm{alg}}_{A,B}(R)$$

such that $\pi_1(p_{v,m}) = P_{v,m}$ and $\pi_1(s_{e,l}) = S_{e,l}$.

To construct an inverse for π_1, let $\{p_{v,m}, s_{e,l}\}$ denote the generators for $L_R(\mathbb{Z}, E)$. For each $(i, j) \in \Omega_A$ and $m \in \mathbb{Z}$, there exists unique elements $n \in \{0, \dots, A_{ij} - 1\}$ and $k \in \mathbb{Z}$ such that $m = n + kA_{ij}$. Define

$$Q_k := p_{v_k,0}, \quad S_{ijm} := s_{e_{ijn},k}, \quad U_k := p_{v_k,1}.$$

It is routine to see that the Q_k are mutually orthogonal projections, the S_{ijm} are partial isometries and $U_k^* U_k = Q_k = U_k U_k^*$ for each $k \in I$. It remains to show that $\{Q_k, S_{ijm}, U_k\}$ satisfy (†).

First we show (†)(i). Fix $(i,j) \in \Omega_A, m \in \mathbb{Z}$. Let $n \in \{0,\dots,A_{ij}-1\}$ and $k \in \mathbb{Z}$ be the elements such that $m = n + kA_{ij}$. We get the first equality of (†)(i) by

$$S_{ij(m+A_{ij})} = S_{ij(n+(k+1)A_{ij})} = s_{e_{ijn},k+1} = s_{e_{ijn},k}p_{v_j,1} = S_{ijm}U_j.$$

For the second equality in (†)(i) consider the same (i,j) and m. Let $\hat{n} \in \{0,\dots,A_{ij}-1\}$ and $\hat{k} \in \mathbb{Z}$ be the elements such that $B_{ij} + n = \hat{k}A_{ij} + \hat{n}$. By Definition 1.12, $\sigma_1(e_{ijn}) = e_{ij\hat{n}}$ and $\varphi(1, e_{ijn}) = \hat{k}$. So

$$S_{ij(m+B_{ij})} = S_{ij(n+kA_{ij}+B_{ij})} = S_{ij((\hat{k}+k)A_{ij}+\hat{n})} = s_{e_{ij\hat{n}},\hat{k}+k}.$$

Hence

$$U_iS_{ijm} = p_{v_i,1}s_{e_{ijn},k} = \delta_{v_i,r(1\cdot e_{ijn})}s_{1\cdot e_{ijn},\varphi(1,e_{ijn})+k} = S_{ij(m+B_{ij})}.$$

To verify (†)(ii) let $m = n + kA_{ij}$ as above. Then

$$S_{ijm}S_{ijm}^* = s_{e_{ijn},0}p_{v_j,k}p_{v_j,k}^*s_{e_{ijn},0}^* = s_{e_{ijn},0}s_{e_{ijn},0}^* = p_{s(e_{ijn}),0} = Q_j,$$

as required. Finally (†)(iii) follows from the fact that $\{p_{v,0}, s_{e,0}\}$ is an E-family in $L_R(\mathbb{Z}, E)$.

The universal property of $\mathcal{O}_{A,B}^{\mathrm{alg}}(R)$ provides a *-algebra homomorphism

$$\pi_2 \colon \mathcal{O}_{A,B}^{\mathrm{alg}}(R) \to L_R(\mathbb{Z}, E)$$

such that $\pi_2(q_k) = Q_k$, $\pi_2(u_k) = U_k$ and $\pi_2(s_{ijm}) = S_{ijm}$. Direct computation on generators shows that π_1 and π_2 are mutually inverse. □

1.3. **The trivial group case.** As our final example for this section we consider the case where the group $G = \{0\}$. When $G = \{0\}$ the C^*-algebra $\mathcal{O}_{G,E}$ is isomorphic to the graph C^*-algebra $C^*(E)$, see [15, 16]. In the algebraic setting we show that for any row-finite graph E with no sources, if $G = \{0\}$ then the Exel–Pardo *-algebra $L_R(G, E)$ is isomorphic to the Leavitt path algebra of E.

We start by introducing Leavitt path R-algebras, although we reverse the usual edge-direction convention to match the rest of the paper. Let R be a unital commutative ring. Let E be a row-finite graph. As in [4, p. 161], the *Leavitt path R-algebra* $L_R(E)$ of E with coefficients in R is the R-algebra generated by elements $\{p_v, x_e, y_e : v \in E^0, e \in E^1\}$ such that

$$\begin{aligned}
&p_vp_{v'} = \delta_{v,v'}p_v \text{ for all } v, v' \in E^0,\\
&p_{r(e)}x_e = x_ep_{s(e)} = x_e \text{ for all } e \in E^1,\\
&p_{s(e)}y_e = y_ep_{r(e)} = y_e \text{ for all } e \in E^1,\\
&y_ex_{e'} = \delta_{e,e'}p_{s(e)} \text{ for all } e, e' \in E^1, \text{ and}\\
&p_v = \sum_{\{e: r(e)=v\}} x_ey_e \text{ for all } v \in r(E^1).
\end{aligned} \tag{1.5}$$

As pointed out in [1, p. 70] this definition coincides with the one by Abrams and Aranda Pino in [2, Definition 1.3]. Below we show that every Leavitt path algebra, regarded as a *-algebra under the involution such that $(rx_e)^* = r^*y_e$, is an Exel–Pardo *-algebra. For this we firstly confirm that the mentioned property defines an involution on $L_R(E)$.

Lemma 1.14. *Let E be a row-finite graph with no sources and R unital commutative *-ring. Then there is a unique involution on $L_R(E)$ such that $(rx_e)^* = r^*y_e$ for all $r \in R$ and $e \in E^0$.*

Proof. Note that $(rp_v)^* = r^*p_v$ using the last equality of (1.5) and the fact that E is row-finite with no sources. □

For the following proposition we note that for the self-similar actions (G, E, σ, φ) considered in this paper, if $G = \{0\}$ then necessarily $\varphi = 0$ and $\sigma = \mathrm{id}_E$.

Proposition 1.15. *Let E be a row-finite graph with no sources, and consider the quadruple $(\{0\}, E, id_E, 0)$ as in Notation 1.2. Let R be a unital commutative ring. Then there is an R-algebra isomorphism $\pi_1 \colon L_R(E) \to L_R(\{0\}, E)$ such that*

$$\pi_1(p_v) = p_{v,0}, \quad \pi_1(x_e) = s_{e,0}, \textit{and} \quad \pi_1(y_e) = s_{e,0}^*.$$

Proof. The defining relations for $L_R(\{0\}, E)$ are

(a) $\{p_{v,0} : v \in E^0\} \cup \{s_{e,0} : e \in E^1\}$ is an E-family in $L_R(\{0\}, E)$,
(b) $(p_{v,0})^* = p_{v,0}$,
(c) $p_{v,0}p_{w,0} = \delta_{v,w}p_{v,0}$,
(d) $p_{v,0}s_{e,0} = \delta_{v,r(e)}s_{e,0}$, and
(e) $s_{e,0}p_{v,0} = \delta_{v,s(e)}s_{e,0}$.

Note that $\{p_{v,0}, s_{e,0}, s_{e,0}^*\}$ satisfy all the relations satisfied by $\{p_v, x_e, y_e\}$ in $L_R(E)$, see (1.5). Therefore, the universal property of $L_R(E)$ provides an R-algebra homomorphism

$$\pi_1 \colon L_R(E) \to L_R(\{0\}, E)$$

such that $\pi_1(p_v) = p_{v,0}$, $\pi_1(x_e) = s_{e,0}$ and $\pi_1(y_e) = s_{e,0}^*$.

We now construct a map in the opposite direction. For this we need all the elements in $L_R(E)$ to have an adjoint. With the trivial adjoint on R we turn $L_R(E)$ into a *-algebra using the adjoint of Lemma 1.14. Then $\{p_v, x_e, y_e\}$ satisfy the relations (a)–(e) with $p_{v,0}, s_{e,0}, s_{e,0}^*$ replaced by p_v, x_e, y_e. The universal property of $L_R(\{0\}, E)$ provides a *-algebra homomorphism

$$\pi_2 \colon L_R(\{0\}, E) \to L_R(E)$$

such that $\pi_2(p_{v,0}) = p_v$, $\pi_2(s_{e,0}) = x_e$ and $\pi_2(s_{e,0}^*) = y_e$. We deduce that π_1 is an R-isomorphism with inverse π_2. □

2. Proof of Theorem A

In this section we prove the Graded Uniqueness Theorem A and the structure result Theorem 2.10. Much of the work here is inspired by Tomforde who proved the Graded Uniqueness Theorem [25, Theorem 5.3] for Leavitt path algebras. Tomforde proved that a graded homomorphism out of a Leavitt path algebra is injective if it is injective on $\operatorname{span}_R\{p_v : v \in E^0\}$. In our Theorem A we need the graded homomorphism π to be injective on $\mathcal{D} := \operatorname{span}_R\{p_{v,g} : v \in E^0, g \in G\}$; that is, for each v we need to insist that π is injective on the image of the group ring RG under the representation $g \mapsto p_{v,g}$.

Theorem 2.10 characterises $\mathcal{D}$ as a direct sum of matrix algebras over certain R-algebras $\mathcal{W}_v$ defined for each $v \in E^0$. We show that each such R-algebra is generated by unitaries $\{W_v^g : g \cdot v = v\}$ inside of a corner of $\mathcal{D}$. When looking into how these unitaries behave it turns out that the possibilities are virtually endless. For example, even when $G = \mathbb{Z}$ there are cases where all the generators $\{W_v^g : g \cdot v = v\}$ are pairwise distinct and other cases where they all coincide. This has important implications in terms of applying Theorem A to decide if π is injective on $\mathcal{D}$, we must first determine the amount of "collapsing" that takes place in the canonical homomorphism $g \mapsto p_{v,g}$ and this will vary from example to example.

We now introduce the notation needed to prove Theorem A. Let G be a discrete group. Following [5], a ring A (possibly without unit) is *G-graded* if as an additive group it can be written as $A = \bigoplus_{g \in G} A_g$, such that each $A_f A_g \subseteq A_{fg}$. The group A_g is called the *g-homogeneous component* of A. If A is an algebra over a ring R, then A is *G-graded* if A is a G-graded ring and each A_g is a R-submodule of A (i.e., A_g satisfies $RA_g \subseteq A_g$). The elements of $\bigcup_{g \in G} A_g$ in a G-graded ring A are called *homogeneous elements* of A. The nonzero elements of A_g are called *homogeneous of degree g* and we write $\deg(a) = g$ for $a \in A_g \setminus \{0\}$. If $\pi : A \to B$ is a homomorphism between two G-graded algebras over a ring R, then π is a *G-graded homomorphism* if $\pi(A_g) \subseteq B_g$ for all $g \in G$.

Let E be a directed graph. We declare vertices to be paths of length 0 with $r(v) = v = s(v)$. By a path α in E of length $|\alpha| = n \geq 1$, as in [15, Part 2.3], we shall mean any finite sequence of the form $\alpha = \alpha_1\alpha_2 \dots \alpha_n$ such that $\alpha_i \in E^1$ and $s(\alpha_i) = r(\alpha_{i+1})$ for all i (this convention agrees with [15] rather than, for example, [1]). Here $s(\alpha) := s(\alpha_n)$ and $r(\alpha) := r(\alpha_1)$. For $n \geq 0$ we let E^n denote the set of all paths of length n. We let E^* denote the set of all finite paths, so $E^* := \bigcup_{m \geq 0} E^m$. If $\alpha, \beta \in E^*$ satisfy $s(\alpha) = r(\beta)$ we let $\alpha\beta$ be their concatenation.

Definition 2.1. Let (G, E, φ) be as in Notation 1.2. Let R be a unital commutative *-ring. For each $g \in G$ and $\alpha = \alpha_1 \dots \alpha_n \in E^*$ we define

$$s_{\alpha,g} := \begin{cases} s_{\alpha_1,e_G} s_{\alpha_2,e_G} \dots s_{\alpha_{n-1},e_G} s_{\alpha_n,g} & \text{if } n > 0, \\ p_{\alpha,g} & \text{if } n = 0. \end{cases}$$

Remark 2.2. Let (G, E, φ) be as in Notation 1.2. Let R be a unital commutative *-ring. Using the relations of a (G, E)-family we have

$$s_{\alpha,g} = s_{\alpha,g} p_{g^{-1}.s(\alpha),e_G}, \quad s_{\alpha,g} = p_{r(\alpha),e_G} s_{\alpha,g}, \tag{2.1}$$

$$s_{\alpha,g} = s_{\alpha,e_G} p_{s(\alpha),g}, \tag{2.2}$$

$$s_{\alpha,g} s_{\beta,h}^* = s_{\alpha,gh^{-1}} s_{\beta,e_G}^*. \tag{2.3}$$

for each $\alpha, \beta \in E^*$ and $g, h \in G$.

Lemma 2.3. *Let (G, E, φ) be as in Notation 1.2. Let R be a unital commutative *-ring. Then*

$$L_R(G, E) = \operatorname{span}_R\{s_{\alpha,g} s_{\beta,e_G}^* : \alpha, \beta \in E^*, g \in G, s(\alpha) = g \cdot s(\beta)\}.$$

Proof. Fix $\alpha, \beta \in E^*$ and $g \in G$. By (2.1) we have

$$s_{\alpha,g} s_{\beta,e_G}^* = \delta_{s(\alpha),g\cdot s(\beta)} s_{\alpha,g} s_{\beta,e_G}^*,$$

so the requirement $s(\alpha) = g \cdot s(\beta)$ is clear. Let M denote the set $\operatorname{span}_R\{s_{\mu,e_G} s_{\nu,e_G}^* : \mu, \nu \in E^*\}$. Fix any $\alpha, \beta \in E^*$ and $g, h \in G$. By (2.1)–(2.2) we have

$$s_{\alpha,g} = s_{\alpha,e_G} p_{s(\alpha),g} = s_{\alpha,e_G} p_{s(\alpha),e_G} p_{s(\alpha),g} = s_{\alpha,e_G} s_{s(\alpha),e_G}^* p_{s(\alpha),g} \in M p_{s(\alpha),g}$$

and similarly $s_{\beta,h} \in M p_{s(\beta),h}$. Using properties of an E-family we have $M^* M \subseteq M$. Hence

$$s_{\alpha,g}^* s_{\beta,h} \in (p_{s(\alpha),g})^* M p_{s(\beta),h} \subseteq \operatorname{span}_R\{s_{\mu,g} s_{\nu,h}^* : \mu, \nu \in E^*, g, h \in G\}.$$

The desired equality now follows from (2.3). □

Remark 2.4. Using (2.2) and borrowing notation from Section 3 (Definition 3.2), $L_R(G, E) = \operatorname{span}_R\{s_{\alpha,e_G} p_{s(\alpha),g} s_{\beta,e_G}^* : (\alpha, g, \beta) \in \mathcal{S}_{G,E}\}$.

Lemma 2.5. *(cf.* [25, Proposition 4.7]*) Let (G, E, φ) be as in Notation 1.2. Let R be a unital commutative *-ring. For each $n \in \mathbb{Z}$ define*

$$A_n := \operatorname{span}_R\{s_{\alpha,g} s_{\beta,e_G}^* : \alpha, \beta \in E^*, g \in G, s(\alpha) = g \cdot s(\beta), |\alpha| - |\beta| = n\}.$$

Then $(A_n)_{n\in\mathbb{Z}}$ is a $\mathbb{Z}$-grading on $L_R(G, E)$.

Proof. Define the symbols $X := \{P_{v,f}, S_{e,g}, (P_{v,f})^*, (S_{e,g})^*\}$ and the words $Y := \{x_1 \dots x_n : n \geq 1, x_i \in X\}$. Recall that from the proof of Thorem 1.6, $L_R(G, E)$ is the quotient of the free *-algebra $\mathbb{F}_R(Y)$ by the ideal I generated by the elements of the sets (1.4) and their adjoints.

The *-algebra $\mathbb{F}_R(Y)$ has a unique $\mathbb{Z}$-grading for which the elements $P_{v,f}$, $S_{e,g}$, $(P_{v,f})^*$, $(S_{e,g})^*$ of X have degrees $0, 1, 0$ and -1, respectively.

Moreover, each generator of I is homogeneous of degree 0. It follows that I is a graded ideal, in the sense of [25, Definition 4.6], i.e.,

$$I = \bigoplus_{n\in\mathbb{Z}} (I \cap \mathbb{F}_R(Y)_n).$$

Hence $L_R(G,E)$ admits a natural $\mathbb{Z}$-grading such that the quotient map $q\colon \mathbb{F}_R(Y) \to L_R(G,E)$ a $\mathbb{Z}$-graded homomorphism. We see that $A_n = q(\mathbb{F}_R(Y)_n)$ for all n, so $(A_n)_{n\in\mathbb{Z}}$ is a $\mathbb{Z}$-grading on $L_R(G,E)$. □

With Lemma 2.3 and Lemma 2.5 at our disposal we are in position to prove Theorem A.

Proof of Theorem A. To ease notation we define $A := L_R(G,E)$. Suppose that $\pi(a) = 0$. We must show that $a = 0$. Write $a = \sum_{n\in\mathbb{Z}} a_n$ such that $a_n \in A_n$ for each $n \in \mathbb{Z}$. Since π and B are graded, each $\pi(a_n) = 0$. Since $(A_n)^* = A_{-n}$, it suffices to show $a_n = 0$ for each $n \geq 0$. Fix such n and for convenience set $d := a_n$. We may write d as a finite sum as follows

$$d = \sum_{i\in F} r_i s_{\alpha_i,g_i} s^*_{\beta_i,e_G} = \sum_{i\in F} r_i s_{\alpha_i,e_G} p_{s(\alpha_i),g_i} s^*_{\beta_i,e_G}$$

with $\alpha_i, \beta_i \in E^*, g_i \in G$ such that $|\alpha_i| - |\beta_i| = n$. For each $i \in F$ set $v_i := g_i^{-1} \cdot s(\alpha_i)$. We have

$$p_{s(\alpha_i),g_i} = p_{s(\alpha_i),g_i} p_{v_i,e_G} = \sum_{e\in v_iE^1} p_{s(\alpha_i),g_i} s_{e,e_G} s^*_{e,e_G} = \sum_{e\in v_iE^1} s_{g_i\cdot e,\varphi(g_i,e)} s^*_{e,e_G},$$

so we may assume there are $m_1, m_2 \in \mathbb{N}$ such that $\alpha_i \in E^{m_1}$ and $\beta_i \in E^{m_2}$ for all $i \in F$. For each $j \in F$ set $F(j) := \{i \in F : (\alpha_i, \beta_i) = (\alpha_j, \beta_j)\}$. Since $s^*_{\alpha,e_G} s_{\beta,e_G} = \delta_{\alpha,\beta} p_{s(\alpha),e_G}$ for $\alpha, \beta \in E^*$ such that $|\alpha| = |\beta|$ and since $s_{\alpha_j,g_i} = s_{\alpha_j,e_G} p_{s(\alpha_j),g_i}$ we get

$$s^*_{\alpha_j,e_G} d s_{\beta_j,e_G} = \sum_{i\in F(j)} r_i (s^*_{\alpha_j,e_G} s_{\alpha_j,e_G}) p_{s(\alpha_j),g_i} (s^*_{\beta_j,e_G} s_{\beta_j,e_G}) \tag{2.4}$$

$$= \sum_{i\in F(j)} r_i (p_{s(\alpha_j),e_G}) p_{s(\alpha_j),g_i} (p_{s(\beta_j),e_G}) \in \mathcal{D}. \tag{2.5}$$

Since $\pi(d) = \pi(a_n) = 0$, we have $\pi(s^*_{\alpha_j,e_G} d s_{\beta_j,e_G}) = 0$. By injectivity of π on $\mathcal{D}$ we have $s^*_{\alpha_j,e_G} d s_{\beta_j,e_G} = 0$. Since each s_{μ,e_G} is a partial isometry and $s_{\alpha_j,g_i} = s_{\alpha_j,e_G} p_{s(\alpha_j),g_i}$ we conclude that

$$s_{\alpha_j,e_G} s^*_{\alpha_j,e_G} d s_{\beta_j,e_G} s^*_{\beta_j,e_G} = \sum_{i\in F(j)} r_i s_{\alpha_i,g_i} s^*_{\beta_i,e_G} = 0.$$

As F is a disjoint union of subsets of the form $F(j)$, we deduce $a_n = d = 0$ as requested. □

We now turn to the proof of Theorem 2.10 describing the R-algebra $\mathcal{D}$ used in the statement of the Graded Uniqueness Theorem A. The proof essentially boils down to identifying the appropriate matrix units and algebraic tensor products inside of $\mathcal{D}$. We start by recalling the notion of "matrix units".

Definition 2.6. Let X be a non-empty set. Write $M_X(R)$, or just M_X, for the universal *-algebra over R generated by elements

$$\{\eta_{x,y} : x, y \in X\}$$

satisfying $\eta_{x,y}^* = \eta_{y,x}$ and $\eta_{x,y}\eta_{w,z} = \delta_{y,w}\eta_{x,z}$. We call the $\eta_{x,y}$ the *matrix units* for M_X.

Lemma 2.7. *Let (G, E, φ) be as in Notation 1.2. Let R be a unital commutative *-ring and let $v \in E^0$ be any vertex. Let $g_v := e_G$. For each $w \in G \cdot v \setminus \{v\}$, fix $g_w \in G$ such that $w = g_w \cdot v$.*

(1) For each $w, w' \in G \cdot v$ define

$$e_{w,w'} := p_{w,g_w} p^*_{w',g_{w'}} = p_{w,g_w(g_{w'})^{-1}}.$$

Then $\mathcal{M}_v := \operatorname{span}_R\{e_{w,w'} : w, w' \in G \cdot v\}$ is isomorphic to $M_{G\cdot v}$ via the map sending $e_{w,w'}$ to $\eta_{w,w'}$.

(2) For each $w, w' \in G \cdot v$ and $g \in \operatorname{Stab}_G(v) := \{g \in G : g \cdot v = v\}$ define

$$e^g_{w,w'} := p_{w,g_w} p_{v,g} p^*_{w',g_{w'}} = p_{w,g_w g (g_{w'})^{-1}}.$$

Then $e^g_{w,w'} e^h_{u,u'} = \delta_{w',u} e^{gh}_{w,u'}$ and $(e^g_{w,w'})^* = e^{g^{-1}}_{w',w}$ for all $g, h \in G$ and $w, w', u, u' \in G \cdot v$.

(3) Suppose $G \cdot v$ is finite. For each $g \in \operatorname{Stab}_G(v)$ define

$$W^g_v := \sum_{w \in G\cdot v} e^g_{w,w}, \quad (W^g_v)^* := \sum_{w\in G\cdot v} e^{g^{-1}}_{w,w}.$$

Then $\mathcal{W}_v := \operatorname{span}_R\{W^g_v : g \in \operatorname{Stab}_G(v)\}$ is a *-algebra over R, and $W^g_v W^h_v = W^{gh}_v$ for all $g, h \in \operatorname{Stab}_G(v)$.

Proof. First we prove (1): Since $g_w^{-1} \cdot w = v = (g_{w'})^{-1} \cdot w'$, properties (b)–(c) of a (G, E)-family (Definition 1.5) give

$$e_{w,w'} = p_{w,g_w} p_{(g_{w'})^{-1}\cdot w',(g_{w'})^{-1}} = \delta_{w,g_w\cdot((g_{w'})^{-1}\cdot w')} p_{w,g_w(g_{w'})^{-1}} = p_{w,g_w(g_{w'})^{-1}}.$$

Clearly $e^*_{w,w'} = e_{w',w}$ for $w, w' \in G \cdot v$. For $w, w', u, u' \in G \cdot v$,

$$e_{w,w'} e_{u,u'} = p_{w,g_w(g_{w'})^{-1}} p_{u,g_u(g_{u'})^{-1}} = \delta_{w,g_w(g_{w'})^{-1}\cdot u} p_{w,g_w(g_{w'})^{-1} g_u (g_{u'})^{-1}}.$$

Now, the equality $w = g_w(g_{w'})^{-1} \cdot u$ simplifies to $v = (g_{w'})^{-1} \cdot u$ or equivalently $w' = g_{w'} \cdot v = u$. So $e_{w,w'} e_{u,u'} = \delta_{w',u} p_{w,g_w(g_{u'})^{-1}} = \delta_{w',u} e_{w,u'}$. Hence $(e_{w,w'})_{w,w'\in G\cdot v}$ form matrix units with $e_{w,w} = p_{w,e_G}$ and $e_{w,v} = p_{w,g_w}$. By the universal property of $M_{G\cdot v}$ there exists a *-algebra homomorphism $\pi\colon M_{G\cdot v} \to \mathcal{M}_v$ such that $\pi(\eta_{w,w'}) = e_{w,w'}$.

The map π is surjective by linearity. We prove π is injective. Suppose that $\pi(\sum_{w,w'\in G\cdot v} r_{w,w'}\eta_{w,w'}) = 0$. For any $w', w'' \in G \cdot v$,

$$r_{w',w''} e_{w',w''} = e_{w',w'}\Big(\sum_{w,u\in G\cdot v} r_{w,u} e_{w,u}\Big) e_{w'',w''} = 0.$$

By Remark 3.7 we know that $r_{w',w''} = 0$, so π is injective. Hence

$$M_{G\cdot v} \cong \mathcal{M}_v.$$

To prove (2) we use that the collection $(e_{w,w'})_{w,w' \in G\cdot v}$ forms matrix units with $e_{w,w} = p_{w,e_G}$ and $e_{w,v} = p_{w,g_w}$. This gives the equalities $e^g_{w,w'} = e_{w,v}p_{v,g}e_{v,w'}$, $e_{v,w'}e_{u,v} = \delta_{w',u}e_{v,v} = \delta_{w',u}p_{v,e_G}$, and $p_{v,g}p_{v,e_G}p_{v,h} = p_{v,gh}$. It follows that

$$e^g_{w,w'}e^h_{u,u'} = e_{w,v}(p_{v,g}(e_{v,w'}e_{u,v})p_{v,h})e_{v,u'} = e_{w,v}\delta_{w',u}p_{v,gh}e_{v,u'} = \delta_{w',u}e^{gh}_{w,u'}.$$

Finally $e^g_{w,w'} = e_{w,v}e^g_{v,v}e_{v,w'}$, $e^g_{v,v} = p_{v,g}$ and $(p_{v,g})^* = p_{v,g^{-1}}$, so $(e^g_{w,w'})^* = e^{g^{-1}}_{w',w}$.

The final property (3) follows from the computation

$$\begin{aligned} W^g_v W^h_v &= \Big(\sum_{w\in G\cdot v} e^g_{w,w}\Big)\Big(\sum_{w'\in G\cdot v} e^h_{w',w'}\Big) = \sum_{w\in G\cdot v}\Big(\sum_{w'\in G\cdot v} e^g_{w,w}e^h_{w',w'}\Big), \\ &= \sum_{w\in G\cdot v}(e^g_{w,w}e^h_{w,w}) = \sum_{w\in G\cdot v}(e^{gh}_{w,w}) = W^{gh}_v. \end{aligned}$$

□

Definition 2.8. Let R be a ring. The *algebraic tensor product* $A\otimes B$ of *-algebras A and B over R is the universal *-algebra over R generated by elements $\{a\otimes b : a \in A, b \in B\}$ subject to the relations

$$\begin{aligned} &(a_1 \otimes b_1)(a_2 \otimes b_2) = a_1a_2 \otimes b_1b_2, \\ &r(a_1 \otimes b_1) = (ra_1) \otimes b_1 = a_1 \otimes (rb_1), \\ &(a_1 \otimes b_1)^* = a_1^* \otimes b_1^*, \quad a_1 \otimes (b_1 + b_2) = (a_1 \otimes b_1) + (a_1 \otimes b_2), \\ &(a_1 + a_2) \otimes b_1 = (a_1 \otimes b_1) + (a_2 \otimes b_1). \end{aligned} \tag{2.6}$$

Note that $A\otimes B$ is spanned by the elements $\{a\otimes b : a \in A, b \in B\}$.

Proposition 2.9. *Let (G, E, φ) be as in Notation 1.2. Let R be a unital commutative *-ring. Suppose that $V \subseteq E^0$ has the property that $E^0 = \bigsqcup_{v\in V} G\cdot v$ is a partition of E^0. For $v \in V$, set $B_v := \operatorname{span}_R\{p_{w,g} : w \in G\cdot v, g \in G\}$.*

(1) Each B_v is a *-algebra over R and with $\mathcal{D}$ as in Theorem A,

$$\mathcal{D} = \bigoplus_{v\in V} B_v.$$

(2) Suppose that $v \in V$ satisfies $|G\cdot v| < \infty$. With $\mathcal{W}_v, (e^g_{w,w'}), W^g_v$ as in Lemma 2.7, there exists a surjective *-algebra homomorphism

$$\psi_v : M_{G\cdot v} \otimes \mathcal{W}_v \to B_v$$

sending $\eta_{w,w'} \otimes W^g_v$ to $e^g_{w,w'}$.

(3) Suppose that $G\cdot v$ is finite for each $v \in V$. Then each ψ_v is an isomorphism. In particular

$$\mathcal{D} \cong \bigoplus_{v\in V} (M_{G\cdot v} \otimes \mathcal{W}_v).$$

Proof. First we prove (1): Fix $v \in V$. For any $g, g' \in G$ and any $w, w' \in G \cdot v$

$$(p_{w,g})^* = p_{g^{-1}\cdot w, g^{-1}} \in B_v, \quad p_{w,g}p_{w',g'} = \delta_{w,g\cdot w'}p_{w,gg'} \in B_v, \tag{2.7}$$

so B_v is a *-algebra over R. Clearly $\mathcal{D} := \operatorname{span}_R\{p_{v,f} : v \in E^0, f \in G\}$ is the R-span of elements in $\bigcup_{v\in V} B_v$. Moreover, elements from two distinct algebras $B_v, B_{v'}$ have product zero, so $\mathcal{D} \cong \bigoplus_{v\in V} B_v$.

To prove (2) fix $v \in V$. For any $a_i := e_{w_i,w_i'} \in \mathcal{M}_v$, and any $b_j := W_v^{g_j} \in \mathcal{W}_v$ with $i, j \in \{1, 2\}$ define

$$a_i \Box b_j := a_i b_j.$$

Since $a_i b_j = b_j a_i$ we get

$$\begin{aligned} &(a_1\Box b_1)(a_2\Box b_2) = a_1a_2\Box b_1b_2, \quad r(a_1\Box b_1) = (ra_1)\Box b_1 = a_1\Box(rb_1),\\ &(a_1\Box b_1)^* = a_1^*\Box b_1^*, \quad a_1\Box(b_1 + b_2) = (a_1\Box b_1) + (a_1\Box b_2),\\ &(a_1 + a_2)\Box b_1 = (a_1\Box b_1) + (a_2\Box b_1). \end{aligned}$$

Notice that $a_i\Box b_j = e_{w,w'}W_v^{g_j} = e^{g_j}_{w_i,w_i'} \in B_v$. By R-linearity the above properties extend to any elements $a_i \in \mathcal{M}_v$ and $b_j \in \mathcal{W}_v$. Let $\varphi\colon \mathcal{M}_v \times \mathcal{W}_v \to \mathcal{M}_v \otimes \mathcal{W}_v$ denote the R-bilinear map $(a, b) \mapsto a \otimes b$. Define $h\colon \mathcal{M}_v \times \mathcal{W}_v \to B_v$ by $h(a, b) = a\Box b$. Since h is R-bilinear the universal property of tensor products gives a unique R-linear homomorphism

$$\tilde{h}\colon \mathcal{M}_v \otimes \mathcal{W}_v \to B_v$$

satisfying $h = \tilde{h} \circ \varphi$. With $\operatorname{Stab}_G(v) := \{g \in G : g \cdot v = v\}$ as in Lemma 2.7 we have

$$B_v = \operatorname{span}_R\{e_{w,w'}W_v^g : w, w' \in G \cdot v, g \in \operatorname{Stab}_G(v)\}. \tag{2.8}$$

Thus $\tilde{h}$ is surjective. Since $\tilde{h}$ is multiplicative, preserves adjoints on elementary tensors and is R-linear, it is multiplicative and *-preserving on all of $\mathcal{M}_v \otimes \mathcal{W}_v$. Hence $\tilde{h}$ is a surjective homomorphism of *-algebras over R, sending $e_{w,w'} \otimes W_v^g$ to $e_{w,w'}\Box W_v^g$. The isomorphism $\mathcal{M}_v \cong M_{G\cdot v}$ of Lemma 2.7 now provides the desired map $\psi_v\colon M_{G\cdot v} \otimes \mathcal{W}_v \to B_v$.

For (3), fix $v \in V$. Each element $d \in B_v$ may be written as

$$d = \sum_{w,w'\in G\cdot v}\ \sum_{g\in F_{w,w'}} r_{(g,w,w')}e^g_{w,w'},$$

for some finite subsets $F_{w,w'} \subseteq \operatorname{Stab}_G(v)$ and coefficients $r_{(g,w,w')} \in R$. Let $J := \ker(\psi_v)$. Then $J = M_{G\cdot v} \otimes I \cong M_{G\cdot v}(I)$ where $I \subseteq \mathcal{W}_v$ is given by

$$\begin{aligned}
I &:= \{a_{v,v} : a \in J\} \\
&= \Big\{a_{v,v} : \psi_v(a) = 0, a = \sum_{w,w' \in G\cdot v} \sum_{g \in F_{w,w'}} r_{(g,w,w')}\, \eta_{w,w'} \otimes W_v^g \Big\} \\
&= \Big\{a_{v,v} : \psi_v(a) = 0, a = \sum_{w,w' \in G\cdot v} \eta_{w,w'} \otimes \sum_{g \in F_{w,w'}} r_{(g,w,w')} W_v^g \Big\} \\
&= \Big\{ \sum_{g \in F_{v,v}} r_{(g,v,v)} W_v^g : \sum_{w,w' \in G\cdot v} \sum_{g \in F_{w,w'}} r_{(g,w,w')} e^g_{w,w'} = 0 \Big\} \\
&= \Big\{ \sum_{g \in F_{v,v}} r_{(g,v,v)} W_v^g : \sum_{g \in F_{v,v}} r_{(g,v,v)} e^g_{v,v} = 0 \Big\} \\
&= \Big\{ \sum_{g \in F_{v,v}} r_{(g,v,v)} W_v^g : \sum_{g \in F_{v,v}} r_{(g,v,v)} W^g_v = 0 \Big\} \\
&= \{0\}.
\end{aligned}$$

Hence $J = M_{G\cdot v} \otimes I = \{0\}$, so $\psi_v \colon M_{G\cdot v} \otimes \mathcal{W}_v \to B_v$ is injective, and hence an isomorphism. □

We are now in position to describe the R-algebra $\mathcal{D}$ appearing in the statement of the Graded Uniqueness Theorem A.

Theorem 2.10. *Let (G, E, φ) be as in Notation 1.2. Let R be a unital commutative *-ring. Fix $V \subseteq E^0$ such that $E^0 = \bigsqcup_{v\in V} G \cdot v$ is a partition of E^0. For each $v \in V$ let $\mathcal{W}_v$ be the *-algebra of Lemma 2.7. If E is finite or for each $v \in V$, $G \cdot v$ is finite then*

$$\mathcal{D} \cong \bigoplus_{v \in V} \left(M_{G\cdot v} \otimes \mathcal{W}_v\right).$$

Proof. Since each $G\cdot v, v \in V$ is finite (by assumption) the result follows from Proposition 2.9. □

3. Proof of Theorem B

It was proved in [15] that the C^*-algebra $\mathcal{O}_{G,E}$ is isomorphic to the groupoid C^*-algebra of the groupoid $\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$ as defined in Definition 3.5, cf. [15, Corollary 6.4]. In this section we prove Theorem B, which establishes an algebraic analogue of this C^*-algebraic result.

We need a number of preliminary results before proving the theorem. The proof of Theorem B starts on page 26.

To make sense of the following Lemma 3.1 we recall the notion of actions on the paths and on sets. Let G be countable discrete group G, and E a row-finite graph with no sources. An action σ of G on E is a group homomorphism $g \mapsto \sigma_g$ from G to the group of all automorphisms of E (i.e., bijections σ_g of $E^0 \sqcup E^1$ such that $\sigma_g(E^i) = E^i$, for $i = 0, 1$ and such that $r \circ \sigma_g = \sigma_g \circ r$, and $s \circ \sigma_g = \sigma_g \circ s$). An action σ of

G on E^* is a homomorphism $g \mapsto \sigma_g$ from G to the group of bijections from E^* to E^*. We often write $g \cdot \alpha$ instead of $\sigma_g(\alpha)$.

Lemma 3.1 ([15, Proposition 2.4]). *Let (G, E, φ) be as in Notation 1.2. Then σ, φ extend to an action $\sigma\colon G \times E^* \to E^*$, $(g, \mu) \mapsto g \cdot \mu$ of G on E^* (viewed as a set) and a one-cocycle $\varphi\colon G \times E^* \to G$ for σ such that:*

(1) $g \cdot (E^n) \subseteq E^n$,
(2) $(gh) \cdot \alpha = g \cdot (h \cdot \alpha)$,
(3) $\varphi(gh, \alpha) = \varphi(g, h \cdot \alpha)\varphi(h, \alpha)$, hence $\varphi(e_G, \alpha) = e_G$,
(4) $\varphi(g, x) = g$,
(5) $r(g \cdot \alpha) = g \cdot r(\alpha)$,
(6) $s(g \cdot \alpha) = g \cdot s(\alpha)$,
(7) $\varphi(g, \alpha) \cdot x = g \cdot x$,
(8) $g \cdot (\alpha\beta) = (g \cdot \alpha)\varphi(g, \alpha) \cdot \beta$,
(9) $\varphi(g, \alpha\beta) = \varphi(\varphi(g, \alpha), \beta)$.

for all $g, h \in G$, $n \geq 0$, $x \in E^0$ and all $\alpha, \beta \in E^*$ with $s(\alpha) = r(\beta)$.

Recall that a semigroup $\mathcal{S}$ is an inverse semigroup if for each $s \in \mathcal{S}$ there is a unique s^* such that $s^* = s^*ss^*$ and $s = ss^*s$. A zero in $\mathcal{S}$ is an element $0 \in \mathcal{S}$ such that $0s = s0 = 0$ for all $s \in \mathcal{S}$.

Definition 3.2 ([15, Definition 2.4]). Let (G, E, φ) be as in Notation 1.2. Define

$$\mathcal{S}_{G,E} := \{(\alpha, g, \beta) \in E^* \times G \times E^* : s(\alpha) = g \cdot s(\beta)\} \cup \{0\}.$$

The proof of [15, Proposition 4.3] shows that under the multiplication

$$(\alpha, g, \beta)(\gamma, h, \delta) = \begin{cases} (\alpha g \cdot \varepsilon, \quad \varphi(g, \varepsilon)h, \quad \delta), & \text{if } \gamma = \beta\varepsilon, \\ (\alpha, \; g\varphi(h^{-1}, \varepsilon)^{-1}, \; \delta(h^{-1}) \cdot \varepsilon), & \text{if } \beta = \gamma\varepsilon, \\ 0, & \text{otherwise.} \end{cases}$$

$\mathcal{S}_{G,E}$ is an inverse semigroup, in which $(\alpha, g, \beta)^* = (\beta, g^{-1}, \alpha)$ and where 0 acts as a zero in $\mathcal{S}_{G,E}$.

Recall that an idempotent in a semigroup is an element s such that $s^2 = s$. A semilattice is a partially ordered set X such that each pair $s, t \in X$ has a greatest lower bound $s \wedge t$. Using the order on $\mathcal{S}_{G,E}$ given by $s \leq t \Leftrightarrow s = ts^*s$, the set

$$\mathcal{E} := \{(\alpha, e_G, \alpha) : \alpha \in E^*\} \cup \{0\} \tag{3.1}$$

of all idempotents in $\mathcal{S}_{G,E}$ is a semilattice of mutually commuting elements with $s \wedge t = st$.

Let X be any partially ordered set with minimum element 0. A filter in X is a nonempty subset $\xi \subseteq X$, such that $0 \notin \xi$, if $x \in \xi$ and $x \leq y$, then $y \in \xi$, and if $x, y \in \xi$, there exists $z \in \xi$, such that $z \leq x, y$.

Lemma 3.3 ([13, Definition 10.1, Theorem 12.9, Proposition 19.11]). *Let (G, E, φ) be as in Notation 1.2. Then*

(1) *The semicharacter space (or the Stone spectrum of $\mathcal{E}$)*

$$\widehat{\mathcal{E}} := \{\varphi\colon \mathcal{E} \to \{0,1\} : \varphi(st) = \varphi(s)\varphi(t), \varphi \neq 0\}$$

with respect to the topology of pointwise convergence[1] *is a locally compact Hausdorff topological space.*

(2) *The character space $\widehat{\mathcal{E}}_0 := \{\varphi \in \widehat{\mathcal{E}} : \varphi(0) = 0\}$ is a closed (hence locally compact) subset of $\widehat{\mathcal{E}}$. There is a bijection from $\widehat{\mathcal{E}}_0$ to the space of filters on $\mathcal{E}$ given by*

$$\varphi \mapsto \xi_\varphi := \{x \in \mathcal{E} : \varphi(x) = 1\}.$$

The inverse of this map bijection is given by $\xi \mapsto \varphi_\xi := 1_{\{x\in\xi\}}$.

(3) *Given an infinite path $x = e_1e_2e_3\cdots \in E^\infty$, we define $\mathcal{F}_x \subseteq \mathcal{E}$ by $\mathcal{F}_x := \{(e_1\ldots e_n, e_G, e_1\ldots e_n) : n \geq 1\} \cup \{(r(e_1), e_G, r(e_1))\}$. Then the space $\widehat{\mathcal{E}}_\infty := \{\varphi \in \widehat{\mathcal{E}}_0 : \xi_\varphi$ is an ultra-filter$\}$ is homeomorphic to the space E^∞ (equiped with the product topology) of one-sided infinite paths on E via the map $x \mapsto 1_{\mathcal{F}_x}$.*

(4) *The tight spectrum $\widehat{\mathcal{E}}_{\text{tight}}$* [13, Definition 12.8] *in $\widehat{\mathcal{E}}_0$ satisfies*

$$\overline{\widehat{\mathcal{E}}_\infty} = \widehat{\mathcal{E}}_{\text{tight}}.$$

As the following shows, for our setting the tight spectrum $\widehat{\mathcal{E}}_{\text{tight}}$ corresponds to the infinite path space. We refer to [15, p. 1074] and [14, Proposition 5.12] for special cases of this result.

Lemma 3.4 ([16, Proposition 4.1])**.** *Let (G, E, φ) be as in Notation 1.2. Then $\widehat{\mathcal{E}}_{tight} = \widehat{\mathcal{E}}_\infty$.*

Let (G, E, φ) be as in Notation 1.2. Recall that the canonical action of $g \in G$ on $x = e_1e_2\cdots \in E^\infty$ is given by $x \mapsto g\cdot x$ where $g\cdot x$ is the unique infinite path such that $(g\cdot x)_1 \ldots (g\cdot x)_n = g\cdot(e_1\ldots e_n)$ for all n. Identifying $\widehat{\mathcal{E}}_\infty \cong E^\infty$ the action of $\mathcal{S}_{G,E}$ on E^∞ is given as follows: each $s = (\alpha, g, \beta) \in \mathcal{S}_{G,E}$ acts on elements of $Z(\beta) := \{\beta x : x \in s(\beta)E^\infty\}$ by $s\cdot(\beta x) := \alpha(g\cdot x)$.

Definition 3.5 ([13, Definition 4.6])**.** Let (G, E, φ) be as in Notation 1.2. We let $\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$ be the groupoid of germs of the action of $\mathcal{S}_{G,E}$ on $\widehat{\mathcal{E}}_{\text{tight}} \cong \widehat{\mathcal{E}}_\infty \cong E^\infty$.

As a set $\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$ is given by

$$\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E}) := \{[(\alpha, g, \beta), x] : (\alpha, g, \beta) \in \mathcal{S}_{G,E}, x \in Z(\beta)\}, \tag{3.2}$$

where for $s, t \in \mathcal{S}_{G,E}$, $[s, x] = [t, y]$ if and only if $x = y$ and there exists nonzero $e = (\gamma, e_G, \gamma) \in \mathcal{E}$ such that $e\cdot x = x$ (or equivalently $x \in Z(\gamma)$) and $se = te$. The unit space

$$\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})^{(0)} = \{[(\alpha, e_G, \alpha), x] : x \in Z(\alpha)\},$$

[1] Equivalently the relative topology from the product space $\{0,1\}^{\mathcal{E}}$.

is identified with E^∞ via$[(\alpha, e_G, \alpha), x] \mapsto x$, and the source and range maps are given by $s([(\alpha, g, \beta), x]) = x$ and $r([(\alpha, g, \beta), \beta y]) = \alpha(g \cdot y)$. The groupoid $\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$ is equiped with a topology making it locally compact and ample with a Hausdorff unit space, cf. [13, Proposition 4.14].

Definition 3.6. Let $\mathcal{G}$ be a locally compact and ample groupoid with Hausdorff unit space and R a unital commutative *-ring. The *Steinberg algebra* $A_R(\mathcal{G})$ is defined as

$$A_R(\mathcal{G}) := \operatorname{span}_R\{1_B : B \text{ is a compact open bisection}\} \subseteq R^{\mathcal{G}}.$$

We endow $A_R(\mathcal{G})$ with pointwise addition. Multiplication and adjoint are given by $fg(\gamma) := \sum_{\alpha\beta=\gamma} f(\alpha)g(\beta)$ and $f^*(\gamma) := f(\gamma^{-1})^*$.

When $\mathcal{G}$ is Haurdorff $A_R(\mathcal{G})$ is just the *-algebra of locally constant functions with compact support from $\mathcal{G}$ to R.

Let (G, E, φ) be as in Notation 1.2. By [13, Proposition 4.18], for each $s \in \mathcal{S}_{G,E}$ the set $\Theta_s := \{[s, x] : x \in Z(\beta)\}$ is a compact open bisection and such sets form a basis for the topology on $\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$. It follows that

$$A_R(\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})) = \operatorname{span}_R\{1_{\Theta_s} : s \in \mathcal{S}_{G,E}\}.$$

We now prove that every Exel–Pardo *-algebra admits an homomorphism into a Steinberg algebra.

Proposition 3.7. *Let (G, E, φ) be as in Notation 1.2. Let R be a unital commutative *-ring. Then there exists a unique *-homomorphism $\pi_{G,E} \colon L_R(G, E) \to A_R(\mathcal{G}_{tight}(\mathcal{S}_{G,E}))$ such that*

$$\pi_{G,E}(p_{v,f}) = 1_{\Theta_{(v,f,f^{-1}\cdot v)}} \quad \text{and} \quad \pi_{G,E}(s_{e,g}) = 1_{\Theta_{(e,g,g^{-1}\cdot s(e))}},$$

for any $v \in E^0$, $e \in E^1$ and $f, g \in G$. In particular, the generators $\{p_{v,f}, s_{e,g}\}$ of $L_R(G, E)$ are nonzero.

Proof. Using [13, Proposition 7.4] and [24, Proposition 4.5(3)] we see that $1_{\Theta_s} 1_{\Theta_t} = 1_{\Theta_{st}}$ and $(1_{\Theta_s})^* = 1_{\Theta_s^{-1}} = 1_{\Theta_{s^*}}$ for $s, t \in \mathcal{S}_{G,E}$. A tedious but straightforward computation using Lemma 3.1 shows that the elements

$$P_{v,f} := 1_{\Theta_{(v,f,f^{-1}\cdot v)}} \quad \text{and} \quad S_{e,g} := 1_{\Theta_{(e,g,g^{-1}\cdot s(e))}}$$

form a (G, E)-family in the Steinberg algebra $A_R(\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E}))$. The result now follows from the universal property of $L_R(G, E)$ (Theorem 1.6). □

Remark 3.8. We do not assert that the map of Proposition 3.7 is injective. In the unital case, an interesting approach was presented in [8, Theorem 6.4] intending to show that the map $\pi_{G,E}$ in Proposition 3.7 is an isomorphism, but we believe this argument is valid only for $R = \mathbb{C}$. The idea was to build an inverse

$$\phi\colon A_R(\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})) \to \mathcal{O}^{\text{alg}}_{(G,E)}(R)$$

using the construction of a map $\psi\colon C^*(\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})) \to C^*_{\text{tight}}(\mathcal{S}_{G,E})$ from the proof of [12, Theorem 2.4]. We outline relevant details:

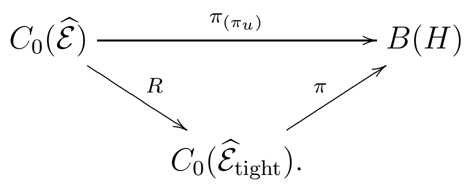
Let $\mathcal{S}$ be an inverse semigroup with zero. A representation σ of $\mathcal{S}$ in a unital C^*-algebra A is a zero- and *-preserving multiplicative map

$$\sigma : \mathcal{S} \to A.$$

If $A = B(H)$, then σ is called a representation of $\mathcal{S}$ on H ([2]). For $e \in \mathcal{E} := \{s \in \mathcal{S} : s^2 = s\}$, we define $D_e := \{\varphi \in \widehat{\mathcal{E}} : \varphi(e) = 1\}$. This is a clopen subset of $\widehat{\mathcal{E}}$. For each representation $\sigma\colon \mathcal{S} \to B(H)$ there exists a unique *-homomorphism $\pi_\sigma\colon C_0(\widehat{\mathcal{E}}) \to B(H)$ such that for each $e \in \mathcal{E}$, $\pi_\sigma(1_{D_e}) = \sigma_e$ ([13, Proposition 10.6]). It is a tight representation precisely if π_σ vanishes on $C_0(\widehat{\mathcal{E}} \setminus \widehat{\mathcal{E}}_{\text{tight}})$ ([13, Theorem 13.2]). The tight C^*-algebra of $\mathcal{S}$, denoted $C^*_{\text{tight}}(\mathcal{S})$, is defined as the universal C^*-algebra generated by a universal tight representation π_u of $\mathcal{S}$, so any tight representation π of $\mathcal{S}$ in a unital C^*-algebra A induces a unital *-homomorphism

$$\psi\colon C^*_{\text{tight}}(\mathcal{S}) \to A$$

such that $\psi \circ \pi_u = \pi$. Viewing $C^*_{\text{tight}}(\mathcal{S})$ as an algebra of operators on a Hilbert space H via some faithful representation, $\pi_u\colon \mathcal{S} \to C^*_{\text{tight}}(\mathcal{S})$ may be regarded as a tight representation of $\mathcal{S}$ on H. Since π_u is tight $\pi_{(\pi_u)}$ factors through $C_0(\widehat{\mathcal{E}}_{\text{tight}})$ giving a representation π of $C_0(\widehat{\mathcal{E}}_{\text{tight}})$ on H. Denoting $R : C_0(\widehat{\mathcal{E}}) \to C_0(\widehat{\mathcal{E}}_{\text{tight}})$ for the restriction map, we get the following commuting diagram:



Define $\mathcal{G} := \mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$ and $D_e^{\widehat{\mathcal{E}}_{\text{tight}}} := \{\varphi \in \widehat{\mathcal{E}}_{\text{tight}} : \varphi(e) = 1\}$. Since the collection $\{\Theta_s : s \in \mathcal{S}_{G,E}\}$ is a basis for the topology on $\mathcal{G}$ we may define $\pi_u \times \pi\colon C_c(\mathcal{G}) \to B(H)$ via

$$\begin{aligned}(\pi_u \times \pi)(1_{\Theta_s}) &:= \pi_u(s)\pi(1_{D_{s^*s}^{\widehat{\mathcal{E}}_{\text{tight}}}}) \\ &= \pi_u(s)\pi_{\pi_u}(1_{D_{s^*s}}) = \pi_u(s)\pi_u(s^*s) = \pi_u(s).\end{aligned}$$

and extend by linearity. The map $\pi_u \times \pi$ is well-defined (see [13, Lemma 8.4] but we will give a sketch below) and norm decreasing (see the proof of [13, Theorem 8.5]). Its range is contained in $C^*_{\text{tight}}(\mathcal{S})$,

[2]The zero-preserving property does not appear in [12, Definition 10.4], but presumably $\sigma_0 = 0$ was intended for inverse semigroups with 0.

so it extends to a homomorphism $\psi\colon C^*(\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})) \to C^*_{\text{tight}}(\mathcal{S}_{G,E})$. Restricting this map to the Steinberg algebra and identifying its image we obtain

$$\phi\colon A_{\mathbb{C}}(\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})) \to \mathcal{O}^{\text{alg}}_{(G,E)}(\mathbb{C}).$$

To see why $\pi_u \times \pi$ is well-defined, let $\tilde{\pi}$ denote the weakly continuous extension of π to the algebra $\mathcal{B}(\widehat{\mathcal{E}})$ of all bounded Borel measurable functions on $\widehat{\mathcal{E}}$. Let $s\colon \mathcal{G} \to \mathcal{G}^{(0)}$ be the source map and fix $\xi, \eta \in H$. For each $s \in \mathcal{S}_{G,E}$ a finite Borel measure $\mu_s := \mu_{s,\xi,\eta}$ on Θ_s is defined (in the proof of [13, Lemma 8.4]) by

$$\mu_s(A) := \langle \pi_u(s)\tilde{\pi}(1_{s(A)})\xi, \eta\rangle$$

for every Borel measurable $A \subseteq \Theta_s$. Fix $f := \sum_{s\in J} c_s 1_{\Theta_s} \in C_c(\mathcal{G})$ with $c_s \in \mathbb{C}$ and $|J| < \infty$. Let $M := \bigcup_{s\in J} \Theta_s$ and let μ be a measure on M such that $\mu(A) = \mu_s(A)$ for every $s \in \mathcal{S}_{G,E}$ and every measurable $A \subseteq \Theta_s$. Since

$$\Big\langle \sum_{s\in J} c_s \pi_u(s)\xi, \eta \Big\rangle = \int_M f d\mu,$$

it follows that if $f = 0$ in $C_c(\mathcal{G})$ then $\sum_{s\in J} c_s\pi_u(s) = 0$ in $B(H)$. Therefore $\pi_u \times \pi$ is well-defined. The point is that constructing the inverse ϕ uses Hilbert space arguments. As such it is less clear if the map $\pi_{G,E}$ in Proposition 3.7 has an inverse for $R \neq \mathbb{C}$.

When $\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$ is Hausdorff the result of Proposition 3.7 can be substantially improved, resulting in Theorem B. Before giving the proof of Theorem B we need two preliminary lemmas.

Lemma 3.9. *Let (G, E, φ) be as in Notation 1.2. Let R be a unital commutative *-ring. Let $\mathcal{B}$ be the set of compact open bisections of $\mathcal{G}_{tight}(\mathcal{S}_{G,E})$. Let*

$$\mathcal{J} := \left\{ J \subseteq \mathcal{S}_{G,E} : |J| < \infty, \bigsqcup_{s\in J} \Theta_s = \bigcup_{s\in J} \Theta_s \in \mathcal{B} \right\}.$$

Then

(1) For each $U \in \mathcal{B}$ there exists $J \in \mathcal{J}$ such that $U = \bigsqcup_{s\in J} \Theta_s$.
(2) For each $J \in \mathcal{J}$ define $t^J := \sum_{(\alpha,g,\beta)\in J} s_{\alpha,g} s^*_{\beta,e_G}$. Then

$$\forall\ I, J \in \mathcal{J} : \bigsqcup_{s\in J} \Theta_s = \bigsqcup_{s\in I} \Theta_s \implies t^J = t^I.$$

(3) For each $U \in \mathcal{B}$ there exists a unique $t_U \in L_R(G, E)$ such that $t_U = \sum_{(\alpha,g,\beta)\in J} s_{\alpha,g} s^*_{\beta,e_G}$ whenever $U = \bigsqcup_{s\in J} \Theta_s$ and $|J| < \infty$.

Proof. (1) For $s = (\alpha, g, \beta) \in \mathcal{S}_{G,E}$, $\Theta_s := \{[(\alpha, g, \beta), x] : x \in Z(\beta)\}$ is a compact open bisection and such sets form a basis for the topology on $\mathcal{G} := \mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$. We prove that for each $U \in \mathcal{B}$ there exists a finite $J \subseteq \mathcal{S}_{G,E}$ such that $U = \bigsqcup_{s\in J} \Theta_s$ is a disjoint union: Compactness of U

gives $U = \bigcup_{t\in J} \Theta_t$ for some finite $J \subseteq \mathcal{S}_{G,E}$. Since $s(\Theta_{(\alpha,g,\beta)}) = Z(\beta)$, any two sets from $\{s(\Theta_t) : t \in J\}$ are either disjoint or contained in one another. Since U is a bisection $s|_U$ is a homeomorphism, so any two sets from $\{\Theta_t : t \in J\}$ are either disjoint or contained in one another. Now remove the superfluous $t \in J$.

(2) To ease terminology, we view $J \in \mathcal{J}$ as a partition of U via the equality $U = \bigsqcup_{s\in J} \Theta_s$. We claim that for two partitions I, J of the same $U \in \mathcal{B}$ we have $t^J = t^I$. The prove strategy is first to show that if $\{\Theta_{(\alpha,g,\beta)} : (\alpha, g, \beta) \in J\}$ is a partition of $U \in \mathcal{B}$ and if $n \geq \max_{(\alpha,g,\beta)\in J} |\beta|$ then the refinement

$$J' := \{(\alpha g \cdot \lambda, \varphi(g, \lambda), \beta\lambda) : (\alpha, g, \beta) \in J, \lambda \in s(\beta)E^{(n-|\beta|)}\}$$

of U satisfies $t^J = t^{J'}$. One then shows that if J, I are two partitions of the same $U \in \mathcal{B}$, then for large enough n the refinements J', I' just described above are equal.

Following this strategy, take any nonempty $U \in \mathcal{B}$ and a partition $U = \bigsqcup_{s\in J} \Theta_s$ of U. Fix any

$$n \geq \max_{(\alpha,g,\beta)\in J} |\beta|.$$

For each $r = (\alpha, g, \beta) \in J$ and each $m \in \mathbb{N}$ calculations show that

$$\begin{aligned}\Theta_r &= \Theta_r\Theta_{r^*r} = \Theta_r\Theta_{(\beta,e_G,\beta)}\\ &= \Theta_r \bigsqcup_{\lambda\in s(\beta)E^m} \Theta_{(\beta\lambda,e_G,\beta\lambda)} = \bigsqcup_{\lambda\in s(\beta)E^m} \Theta_{(\alpha g\cdot\lambda,\varphi(g,\lambda),\beta\lambda)}\end{aligned}$$

and similarly

$$s_{\alpha,g}s^*_{\beta,e_G} = \sum_{\lambda\in s(\beta)E^m} s_{\alpha g\cdot\lambda,\varphi(g,\lambda)}s^*_{\beta\lambda,e_G}.$$

Hence $t^{\{r\}} = t^{\{(\alpha g\cdot\lambda,\varphi(g,\lambda),\beta\lambda):\lambda\in s(\beta)E^m\}}$. In particular, we also have that $t^{\{r\}} = t^{\{(\alpha g\cdot\lambda,\varphi(g,\lambda),\beta\lambda):\lambda\in s(\beta)E^{(n-|\beta|)}\}}$. With

$$J' := \{(\alpha g \cdot \lambda, \varphi(g, \lambda), \beta\lambda) : (\alpha, g, \beta) \in J, \lambda \in s(\beta)E^{(n-|\beta|)}\}$$

it follows that

$$t^J = \sum_{s\in J} t^{\{s\}} = \sum_{s\in J} t^{\{(\alpha g\cdot\lambda,\varphi(g,\lambda),\beta\lambda):\, s=(\alpha,g,\beta),\lambda\in s(\beta)E^{(n-|\beta|)}\}} = t^{J'}.$$

Now suppose $U = \bigsqcup_{s\in I} \Theta_s$ is another partition of U. Let $n := \max_{(\alpha,g,\beta)\in J\cup I} |\beta|$. By the above, $J' := \{(\alpha g \cdot \lambda, \varphi(g, \lambda), \beta\lambda) : (\alpha, g, \beta) \in J, \lambda \in s(\beta)E^{(n-|\beta|)}\}$ and $I' := \{(\gamma h \cdot \lambda, \varphi(h, \lambda), \delta\lambda) : (\gamma, h, \delta) \in I, \lambda \in s(\delta)E^{(n-|\delta|)}\}$ satisfy $t^J = t^{J'}$ and $t^I = t^{I'}$. We have

$$s(U) = \bigsqcup_{\{\nu\in E^n:(\mu,f,\nu)\in J'\}} Z(\nu) = \bigsqcup_{\{\nu\in E^n:(\mu,f,\nu)\in I'\}} Z(\nu).$$

Hence $\{\nu : (\mu, f, \nu) \in J'\} = \{\nu : (\mu, f, \nu) \in I'\}$. Since U is a bisection, we deduce that $I' = J'$, hence $t^J = t^{J'} = t^{I'} = t^I$.

(3) For each $U \in \mathcal{B}$ use (1) to find $I \in \mathcal{J}$ such that $U = \bigsqcup_{s\in I} \Theta_s$. The desired result now follows from property (2). □

Let $\mathcal{G}$ be a locally compact, ample groupoid with a Hausdorff unit space. Let $\mathcal{B}$ be the family of all compact open bisections of $\mathcal{G}$. Let R be a unital commutative *-ring and let B be a *-algebra over R. For Hausdorff groupoids (regarded as carrying the trivial grading), [10, Definition 3.10]([3]) defines a *representation of* $\mathcal{B}$ in B as a family $\{t_U : U \in \mathcal{B}\} \subseteq B$ satisfying

(1) $t_\emptyset = 0$;
(2) $t_U t_V = t_{UV}$ for all $U, V \in \mathcal{B}$; and
(3) $t_U + t_V = t_{U\cup V}$ whenever U and V are disjoint elements of $\mathcal{B}$ such that $U \cup V$ is a bisection.

For non-Hausdorff $\mathcal{G}$, it is not clear that this is an appropriate definition of a representation, but we will nevertheless want to refer to (R1)–(R3) in this context.

Lemma 3.10. *Let* (G, E, φ) *be as in Notation 1.2. Let* R *be a unital commutative* **-ring. Then the family* $\{t_U : U \in \mathcal{B}\}$ *of Lemma 3.9 satisfies* (R1)–(R3).

Proof. Property (R1) follows from $t_\emptyset = t^\emptyset = 0$.

For (R3), take any pair of disjoint sets $U, V \in \mathcal{B}$ such that $U \cup V$ is a bisection. Choose $J, I \in \mathcal{J}$ such that $U = \bigsqcup_{s\in J} \Theta_s$ and $V = \bigsqcup_{s\in I} \Theta_s$. Then $t_U = t^J$ and $t_V = t^I$. Since $U \sqcup V = \bigsqcup_{s\in I\sqcup J} \Theta_s$ we get $t^{I\sqcup J} = t_{U\sqcup V}$, so

$$t_U + t_V = t^I + t^J = t^{I\sqcup J} = t_{U\sqcup V}.$$

Finally we consider (R2). Fix any $U = \bigsqcup_{s\in J} \Theta_s$, $V = \bigsqcup_{t\in I} \Theta_t$ in $\mathcal{B}$. If (s, t), (s', t') are distinct elements of $J \times I$, then $\Theta_s\Theta_t \cap \Theta_{s'}\Theta_{t'} = \emptyset$ because both U and V are bisections. So $UV = \bigsqcup_{s\in J,t\in I} \Theta_s\Theta_t$. Since $\Theta_s\Theta_t = \Theta_{st}$ for any $s, t \in \mathcal{S}_{G,E}$ (see [13, Proposition 7.4]), we get $UV = \bigsqcup_{s\in J,t\in I} \Theta_{st}$. Using the relations in $L_R(G, E)$ we have $t_{\Theta_s} t_{\Theta_t} = t_{\Theta_{st}}$ for any $s \in J, t \in I$. Hence

$$t_{UV} = \sum_{s\in J,t\in I} t_{\Theta_{st}} = \sum_{s\in J,t\in I} t_{\Theta_s} t_{\Theta_t} = \Big(\sum_{s\in J} t_{\Theta_s}\Big)\Big(\sum_{t\in I} t_{\Theta_t}\Big) = t_U t_V. \quad \square$$

Having constructed a homomorphism from the Exel–Pardo *-algebra to the Steinberg algebra of $\mathcal{G}_{\mathrm{tight}}(\mathcal{S}_{G,E})$ and a representation of $\mathcal{B}$ in $L_R(G, E)$ we can now prove Theorem B.

Proof of Theorem B. Let $\mathcal{B}$ denote the set of compact open bisections of $\mathcal{G}_{\mathrm{tight}}(\mathcal{S}_{G,E})$. By Lemma 3.10 the family $\{t_U : U \in \mathcal{B}\}$ of Lemma 3.9 is a representation of $\mathcal{B}$ in $L_R(G, E)$. So [7, Proposition 2.3] shows that $\pi_{G,E}$ is an R-algebra isomorphism. One then verify the map is *-preserving (cf. [10, Theorem 3.11]). □

[3]Condition (R3) in [10] is missing "such that $B \cup D$ is a bisection".

Remark 3.11. In Theorem B the inverse map $\pi_{G,E}^{-1}$ from $A_R(\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E}))$ into $L_R(G,E)$ satisfies

$$\pi_{G,E}^{-1}(1_{\Theta_{(\alpha,g,\beta)}}) = t_{\Theta_{(\alpha,g,\beta)}}, \quad \text{where} \quad t_{\Theta_{(\alpha,g,\beta)}} := s_{\alpha,g}s_{\beta,e_G}^*. \tag{3.3}$$

We remark that for *any* triple (G,E,φ) as in Notation 1.2, $\pi_{G,E}$ is a *-isomorphism if and only if (3.3) extends by linearity to a well-defined R-linear map on $A_R(\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E}))$.

Due to work in [16] it is known when the groupoid $\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$ is Hausdorff. We recall the relevant terminology. A path $\alpha \in E^*$ is *strongly fixed* by $g \in G$ if $g \cdot \alpha = \alpha$ and $\varphi(g,\alpha) = e_G$. In addition if no prefix (i.e., initial segment) of α is strongly fixed by g we say α is a *minimal strongly fixed* path for g ([15, Definition 5.2]).

Proposition 3.12 ([16, Theorem 4.2])**.** *Let (G,E,φ) be as in Notation 1.2. Then the following properties are equivalent:*

(1) For every $g \in G$, and every $v \in E^0$ there are at most finitely many minimal strongly fixed paths for g with range v.
(2) $\mathcal{G}_{tight}(\mathcal{S}_{G,E})$ is Hausdorff.

Combining Proposition 3.12 and Theorem B we get:

Corollary 3.13. *Let (G,E,φ) be as in Notation 1.2. Let R be a unital commutative *-ring. Suppose that for every $g \in G$, and every $v \in E^0$ there are at most finitely many minimal strongly fixed paths for g with range v. Then*

$$L_R(G,E) \cong A_R(\mathcal{G}_{tight}(\mathcal{S}_{G,E})).$$

3.1. **Simplicity and pure infiniteness for $L_R(G,E)$.** Having a Steinberg algebra realisation of Exel–Pardo *-algebras we can use known results on Steinberg algebras to say something about $L_R(G,E)$. In particular the results in [15, 16] apply to our setting giving the two propositions below.

The terminology used in Proposition 3.14 and Proposition 3.15 was introduced in [15]. More specifically, for the definition of a weakly-G-transitive directed graph E, the notion of a G-circuit in E having an entry and the definiton of a group element $g \in G$ being slack at a vertex v, see [15, Definition 13.4], [15, Definition 14.4] and [15, Definition 14.9] respectively.

Proposition 3.14 (cf. [16, Theorem 4.5])**.** *Let (G,E,φ) be as in Notation 1.2. Let R be a unital commutative *-ring. Suppose that for every $g \in G$, and every $v \in E^0$ there are at most finitely many minimal strongly fixed paths for g with range v. Then $L_R(G,E)$ is simple if and only if R is simple and*

(1) the graph E is weakly-G-transitive;
(2) every G-circuit has an entry; and

(3) for every vertex v, and every $g \in G$ that fixes $Z(v)$ pointwise, g is slack at v.

Proposition 3.15 (cf. [16, Theorem 4.7]). *Let (G, E, φ) be as in Notation 1.2. Let R be a unital commutative *-ring. Suppose that for every $g \in G$, and every $v \in E^0$ there are at most finitely many minimal strongly fixed paths for g with range v and that $L_R(G, E)$ is simple. If E contains at least one G-circuit, then $L_R(G, E)$ is purely infinite (simple).*

Proof. Use translates of a G-circuit to construct a infinite path and proceed as in [16]. □

4. Proof of Theorem C

In this section we prove Theorem C studying Steinberg algebras of non-Hausdorff groupoids. In this setting it is not clear when the *-homomorphism $\pi_{G,E} \colon L_R(G, E) \to A_R(\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E}))$ of Proposition 3.7 is a *-isomorphism. We still know that the family $\{t_U : U \in \mathcal{B}\}$ of Lemma 3.9 satisfies (R1)–(R3), but we cannot conclude immediately that $\pi_{G,E}$ admits an inverse. By considering actions with an appropriate amount of "strongly fixed" paths one can nevertheless get an inverse. We now introduce such paths and state Theorem 4.2, giving Theorem C as a corollary.

Definition 4.1. Let E be as in Notation 1.2. Let $\beta \in E^* \setminus E^0$ be a finite path in E. We say β is *strongly fixed* if β is strongly fixed by some $g \in G \setminus \{e_G\}$. We say β is *minimal* strongly fixed if no prefix (i.e., initial segment) of β is strongly fixed. Let $x \in E^\infty$ be an infinite path in E. We say x is *strongly fixed* if some initial segment $\beta \in r(x)E^* \setminus \{r(x)\}$ of x is strongly fixed.

Theorem 4.2. *Let (G, E, φ) be as in Notation 1.2. Let R be a unital commutative *-ring. For $u \in E^0$, let $\mathcal{F}_u$ be the set of all minimal strongly fixed paths with range u. Suppose that $Z(\gamma) \cap Z(\gamma') = \emptyset$ whenever $\gamma \neq \gamma' \in \mathcal{F}_u$. If for each $u \in E^0$,*

- *there exist $x \in Z(u)$ that is not strongly fixed, or*
- *$\mathcal{F}_u$ is finite,*

then $L_R(G, E) \cong A_R(\mathcal{G}_{tight}(\mathcal{S}_{G,E}))$, via the map $\pi_{G,E}$ of Proposition 3.7.

The proof of Theorem 4.2 is essentially contained in the five lemmas Lemma 4.5–Lemma 4.9. Lemma 4.5 establishes a graded structure of $A_R(\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E}))$ allowing us to use the Graded Uniqueness Theorem A. Lemma 4.6 further reduces the problem, so we only need to prove injectivity of $\pi_{G,E}$ on each $\operatorname{span}_R\{p_{u,g}, g \in G\}$. We then consider two complementary cases:

(1) There is an infinite path with range u that is not strongly fixed.
(2) All infinite paths with range u are strongly fixed.

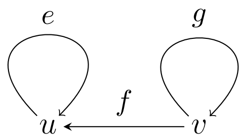


FIGURE 2. Smallest graph of a non-Hausdorff groupoid $\mathcal{G}_{A,B}$.

In case (1) we prove that the elements $\{p_{u,g}, g \in G\}$ are linearly independent (Lemma 4.7). In case (2) we introduce a certain disjointification of $p_{u,g}$ and $\pi_{G,E}(p_{u,g})$ relative to suitable strongly fixed paths (Lemma 4.8). We use this disjointification to show that the elements $\{p_{u,g}, g \in G\}$ are "sufficiently" linearly independent (Lemma 4.9). We finally combine these results in the proof of Theorem 4.2.

Before getting more technical we present two examples of non-Hausdorff groupoids of germs $\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$ illustrating how these two complementary cases may arise.

Example 4.3. Let $\mathcal{G}_{A,B}$ be the groupoid of germs for the Katsura triple $(\mathbb{Z}, E, \varphi)$ associated to the matrices

$$A = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), \quad B = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$$

(see Definition 1.12 and Definition 3.5). The graph E associated to A is illustrated on Figure 2.

Notice that $\mathcal{G}_{A,B}$ is non-Hausdorff: With N, A, B, I and E as in Definition 1.12 and with $K_j^{\alpha,l} := \frac{lB_{r(\alpha_1)s(\alpha_1)} \cdots B_{r(\alpha_j)s(\alpha_j)}}{A_{r(\alpha_1)s(\alpha_1)} \cdots A_{r(\alpha_j)s(\alpha_j)}}$ defined for each $\alpha = \alpha_1\alpha_2 \dots \alpha_{|\alpha|} \in E^*$, $l \in \mathbb{Z}$, and $j \in \{1, \dots, |\alpha|\}$ we see that

$$M_l^i = \{\alpha \in iE^* \setminus \{i\} \ : \ K_1^{\alpha,l}, \dots, K_{|\alpha|-1}^{\alpha,l} \in \mathbb{Z} \ \setminus \ \{0\}, \, K_{|\alpha|}^{\alpha} = 0\}$$

is the set of all minimal strongly fixed paths for l with range i (see [15, Lemma 18.4]). Here $M_1^u = \{f, ef, eef, eeef, \dots\}$ is infinite, so $\mathcal{G}_{A,B}$ is non-Hausdorff by Proposition 3.12.

Using Theorem 4.2 (or Theorem C) proved later in this section we know that $\mathcal{O}_{A,B}^{\text{alg}}(R) \cong A_R(\mathcal{G}_{A,B})$. However, for this example the arguments simplify as follows:

Here the proof comes down to showing that for each vertex $w \in E^0$ the indicator functions on the sets $\{\Theta_{(w,m,w)} : m \in \mathbb{Z}\}$ are linearly independent in $A_R(\mathcal{G}_{A,B})$, That is, for each finite subset F of $\mathbb{Z}$,

$$\sum_{m \in F} r_m 1_{\Theta_{(w,m,w)}} = 0 \implies \text{each } r_m = 0.$$

It turns out that a direct inspection of the sets $U_m := \Theta_{(u,m,u)}$ and $V_m := \Theta_{(v,m,v)}$ suffices. For each $m \in \mathbb{Z}$, we have $V_m \cap W = \emptyset$ for all $W \in \{U_n : n \in \mathbb{Z}\} \cup \{V_n : n \neq m\}$, and $[(u, m, u), eee\dots] \in U_m \setminus \big(\bigcup_{n \neq m} U_n \cup \bigcup_{n \in \mathbb{Z}} V_n\big)$. So it is easy to see that the functions 1_{U_m}

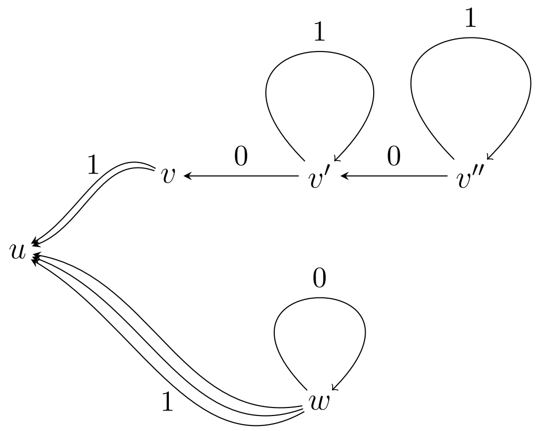


FIGURE 3. Graph of a non-Hausdorff groupoid $\mathcal{G}_{A,B}$ without linear independence amongst $\{1_{\Theta_{(u,m,u)}} : m \in \mathbb{Z}\}$.

and 1_{V_m} are linearly independent. The infinite path $eee\dots$ with range u is not strongly fixed, so this is an instance of case (1).

Example 4.4. Let $\mathcal{G}_{A,B}$ be the groupoid of germs for the Katsura triple $(\mathbb{Z}, E, \varphi)$ associated to the matrices

$$A = \begin{pmatrix} 0&2&0&0&3 \\ 0&0&1&0&0 \\ 0&0&1&1&0 \\ 0&0&0&1&0 \\ 0&0&0&0&1 \end{pmatrix}, \quad B = \begin{pmatrix} 0&1&0&0&1 \\ 0&0&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&0 \end{pmatrix}.$$

The graph E is illustrated on Figure 3 and the matrixes are given with respect to the ordering u, v, v', v'', w of the vertices. Once again the groupoid $\mathcal{G}_{A,B}$ is non-Hausdorff (see Proposition 3.12) but nevertheless we know that $\mathcal{O}^{\mathrm{alg}}_{A,B}(R) \cong A_R(\mathcal{G}_{A,B})$. Here it is the vertex u that makes the arguments more challenging. All infinite paths with range u are strongly fixed, cf. case (2). With $U_m := \Theta_{(u,m,u)}$ for $m \in \mathbb{Z}$ we have

$$1_{U_0} + 2 \cdot 1_{U_1} + 1_{U_2} = 1_{U_3} + 2 \cdot 1_{U_4} + 1_{U_5}. \tag{4.1}$$

In particular the indicator functions $1_{U_0}, \dots, 1_{U_5}$ in $A_R(\mathcal{G}_{A,B})$ are not linearly independent and the analysis used in Example 4.3 is insufficient. Regardless, the lack of linear independence is compensated by a nice structure amongs the sets U_m. For $o \in \{v, w\}$, set $U^o_m := \{[(u, m, u), x] : x \in Z(u), x(1) = o\}$. Each U_m admits a partition $U_m = U^v_m \sqcup U^w_m$ corresponding to minimal strongly fixed paths passing though v and w respectively. Any two of these sets $\{U^v_m, U^w_m : m = 0, \dots, 5\}$

are either equal or disjoint([4]). This mutual disjointness powers the proof of injectivity of $\pi_{\mathbb{Z},E}$ as we shall see in the proof of Lemma 4.9.

We now return back to the proofs of Lemma 4.5 to Lemma 4.9.

Lemma 4.5. *Let (G, E, φ) be as in Notation 1.2. Let R be a unital commutative *-ring. Then the *-algebra $A_R(\mathcal{G}_{tight}(\mathcal{S}_{G,E}))$ and the *-homomorphism $\pi_{G,E} \colon L_R(G, E) \to A_R(\mathcal{G}_{tight}(\mathcal{S}_{G,E}))$ of Proposition 3.7 are $\mathbb{Z}$-graded.*

Proof. Lemma 3.1 of [11] generalises to not necessarily Hausdorff groupoids $\mathcal{G} := \mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$. Indeed, as in [8] the map

$$c\colon \mathcal{G} \to \mathbb{Z}$$

given by $c([(\alpha, g, \beta), \mu]) := |\alpha| - |\beta|$ is well-defined and satisfies the cocycle identity $c(\gamma_1)c(\gamma_2) = c(\gamma_1\gamma_2)$ for any valid product $\gamma_1\gamma_2 \in \mathcal{G}$. Now using that $A_R(\mathcal{G}) = \operatorname{span}_R\{1_{\Theta_s} : s \in \mathcal{S}_{G,E}\}$ and the fact that c is constant on each bisection Θ_s one can adopt the proof of [11, Lemma 3.1(1)] as follows.

Fix any function $f = \sum_{s\in I} r_s 1_{\Theta_s} \in A_R(\mathcal{G})$ with $I \subseteq \mathcal{S}_{G,E}$ finite and each $r_s \neq 0$. For each $n \in \mathbb{Z}$ define

$$f_n := \sum_{s\in I,\, c(s)=n} r_s 1_{\Theta_s}.$$

If follows that $f = \sum_n f_n$ is a sum of functions each in a graded component. Since the open supports $\operatorname{supp}(f_n) := \{x \in \mathcal{G} : f_n(x) \neq 0\}$ of the functions f_n are disjoint we deduce that $f = 0$ if and only if each $f_n = 0$. It follows that the *-algebra $A_R(\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E}))$ and the *-homomorphism $\pi_{G,E} \colon L_R(G, E) \to A_R(\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E}))$ of Proposition 3.7 are both $\mathbb{Z}$-graded. □

Lemma 4.6. *Let (G, E, φ) be as in Notation 1.2. Let R be a unital commutative *-ring. Fix $u \in E^0$ and $n, m \in G$. Then*

(1) For each $\gamma \in E^*$ and $x \in E^\infty$ we have $x \in Z(\gamma)$ if and only if $m \cdot x \in Z(m \cdot \gamma)$.
(2) If $y = [(u, m, m^{-1} \cdot u), z] = [(u, n, n^{-1} \cdot u), z]$ for some $z \in E^\infty$, then $m \cdot z = n \cdot z$ and there exists $\alpha \in uE^* \setminus \{u\}$ that is strongly fixed by nm^{-1} and satisfies $m \cdot z \in Z(\alpha)$.
(3) Suppose that $\gamma \in uE^* \setminus \{u\}$ is strongly fixed by nm^{-1}. Then

$$[(u, m, m^{-1} \cdot u), z] = [(u, n, n^{-1} \cdot u), z] \text{ for all } z \in Z(m^{-1} \cdot \gamma).$$

(4) For any $v \in E^0 \setminus \{u\}$ we have $\Theta_{(u,m,m^{-1}\cdot u)} \cap \Theta_{(v,n,n^{-1}\cdot v)} = \emptyset$.

[4] This is how we see (4.1): we have $U^v_0 = U^v_2 = U^v_4$, $U^v_1 = U^v_3 = U^v_5$, $U^w_0 = U^w_3$, $U^w_1 = U^w_4$, $U^w_2 = U^w_5$, so $1_{U_0} + 2\cdot 1_{U_1} + 1_{U_2} = 2\cdot 1_{U^v_0} + 2\cdot 1_{U^v_1} + 1_{U^w_0} + 2\cdot 1_{U^w_1} + 1_{U^w_2} = 1_{U_3} + 2\cdot 1_{U_4} + 1_{U_5}$.

Proof. (1) Take any $\gamma \in E^*$ and $x \in Z(\gamma)$. By definition of $m \cdot x$ we have $(m \cdot x)(0, |\gamma|) = m \cdot (x(0, |\gamma|)) = m \cdot \gamma$, so $m \cdot x \in Z(m \cdot \gamma)$. The converse is similar.

(2) Firstly we consider the case $m = n$. Then clearly $m \cdot z = n \cdot z$, and since $z \in Z(m^{-1} \cdot u)$ part (1) gives $m \cdot z \in Z(u)$. The element $\alpha := (m \cdot z)_1 \in uE^* \setminus \{u\}$ is strongly fixed by nm^{-1} and satisfies $m \cdot z \in Z(\alpha)$.

Suppose that $y = [(u, m, m^{-1} \cdot u), z] = [(u, n, n^{-1} \cdot u), z]$ for some $z \in E^\infty$ and $m \neq n$. With $s := (u, m, m^{-1} \cdot u)$ and $t := (u, n, n^{-1} \cdot u)$ we know that $[s, z] = [t, z]$ so there exists $e := (\beta, 0, \beta)$ with $\beta \in E^*$ such that $se = te$ and $z \in Z(\beta)$. With $\beta_1 := m \cdot \beta$ and $\beta_2 := n \cdot \beta$ we get

$$se = (\beta_1, \varphi(m^{-1}, \beta_1)^{-1}, m^{-1} \cdot \beta_1), \text{ and } te = (\beta_2, \varphi(n^{-1}, \beta_2)^{-1}, n^{-1} \cdot \beta_2),$$

so $\beta_1 = \beta_2$ and $\varphi(m^{-1}, \beta_1) = \varphi(n^{-1}, \beta_2)$. Let $\alpha := \beta_1$. Then $n^{-1} \cdot \alpha = m^{-1} \cdot \alpha$ and $\varphi(n^{-1}, \alpha) = \varphi(m^{-1}, \alpha)$, so α is strongly fixed by nm^{-1} [15, Proposition 5.6]. Since $z \in Z(\beta)$ it follows that $m \cdot z \in Z(m \cdot \beta) = Z(\alpha)$. Now since $z \in Z(m^{-1} \cdot u)$ we get $m \cdot z \in Z(u)$ and $r(\alpha) = u$. The element $\alpha \in uE^*$ can not be a vertex, because that would imply $m = \varphi(m^{-1}, \alpha)^{-1} = \varphi(n^{-1}, \alpha)^{-1} = n$. So $\alpha \in uE^* \setminus \{u\}$. Now we prove that $m \cdot z = n \cdot z$. Using that $\varphi(g^{-1}, \gamma) = \varphi(g, g^{-1} \cdot \gamma)^{-1}$, see [15, Proposition 2.6], we get $\varphi(m, \beta) = \varphi(m^{-1}, (m^{-1})^{-1} \cdot \beta)^{-1} = \cdots = \varphi(n, \beta)$. Also $m \cdot \beta = n \cdot \beta$. Since $z \in Z(\beta)$, $m \cdot z = m \cdot (\beta z') = (m \cdot \beta)(\varphi(m, \beta) \cdot z') = \cdots = n \cdot z$ for suitable $z' \in E^\infty$.

(3) Suppose that $\gamma \in uE^* \setminus \{u\}$ is strongly fixed by nm^{-1}. Take any $z \in Z(m^{-1} \cdot \gamma)$. Set $s := (u, m, m^{-1} \cdot u)$ and $t := (u, n, n^{-1} \cdot u)$. Since $r(\gamma) = u$ and since γ is strongly fixed by nm^{-1}, we have $z \in Z(m^{-1} \cdot u) = Z(n^{-1} \cdot u)$. Hence both $[s, z]$ and $[t, z]$ make sense. We claim that $[s, z] = [t, z]$. To see this, set $e := (m^{-1} \cdot \gamma, 0, m^{-1} \cdot \gamma)$. As γ is strongly fixed by nm^{-1}, we have $e = (n^{-1} \cdot \gamma, 0, n^{-1} \cdot \gamma)$, and hence

$$se = (\gamma, \varphi(m^{-1}, \gamma)^{-1}, m^{-1} \cdot \gamma) = (\gamma, \varphi(n^{-1}, \gamma)^{-1}, n^{-1} \cdot \gamma) = te.$$

Since $z \in Z(m^{-1} \cdot \gamma) = Z(n^{-1} \cdot \gamma)$ it follows that $[s, z] = [t, z]$.

(4) Elements of $\Theta_{(u,m,m^{-1} \cdot u)}$ have range in $Z(u)$ ([13]), so $\Theta_{(u,m,m^{-1} \cdot u)} \cap \Theta_{(v,n,n^{-1} \cdot v)} \neq \emptyset$ implies $Z(u) \cap Z(v) \neq \emptyset$ and $u = v$. □

Lemma 4.7. *Let (G, E, φ) be as in Notation 1.2. Let R be a unital commutative *-ring. Fix $u \in E^0$. Suppose $x \in Z(u)$ is not strongly fixed. Then the indicator functions on the sets $\{\Theta_{(u,m,m^{-1} \cdot u)}, m \in G\}$ are linearly independent, i.e., for each finite subset F of G,*

$$\sum_{m \in F} r_m 1_{\Theta_{(u,m,m^{-1} \cdot u)}} = 0 \implies r_m = 0 \quad \textit{for each } m \in F.$$

Proof. Set $h := \sum_{m \in F} r_m 1_{\Theta_{(u,m,m^{-1} \cdot u)}}$ and for each $n \in G$, set $x^{(n)} := [(u, n, n^{-1} \cdot u), n^{-1} \cdot x]$. Suppose that $x^{(n)} \in \Theta_{(u,m,m^{-1} \cdot u)}$ for some $n, m \in G$. It follows that $[(u, m, m^{-1} \cdot u), z] = [(u, n, n^{-1} \cdot u), z]$ for $z = n^{-1} \cdot x \in$

E^∞ . By Lemma 4.6 we have $x = m \cdot z = n \cdot z$ and there exists $\alpha \in uE^*$ that is strongly fixed by nm^{-1} and satisfies $x \in Z(\alpha)$. But x is not strongly fixed, so $n = m$. Hence $x^{(n)} \notin \Theta_{(u,m,m^{-1}\cdot u)}$ for $m \neq n$, so $h(x^{(m)}) = r_m$. □

Lemma 4.8. *Let (G, E, φ) be as in Notation 1.2. Let R be a unital commutative *-ring. Fix $u \in E^0$. For $(m, \gamma) \in G \times uE^*$ define*

$$U_{(m,\gamma)} := \{[(u, m, m^{-1} \cdot u), x] : x \in Z(m^{-1} \cdot \gamma)\}.$$

Let $\mathcal{F}$ be the set of all minimal strongly fixed paths with range u (see Definition 4.1). Suppose that $Z(\gamma) \cap Z(\gamma') = \emptyset$ whenever $\gamma \neq \gamma' \in \mathcal{F}$. Set $\mathcal{P} := \{U_{(m,\gamma)} : (m, \gamma) \in G \times \mathcal{F}\}$. Then

(1) The sets in $\mathcal{P}$ are compact open bisections.
(2) For $U := U_{(m,\gamma)}$ and $V := U_{(n,\eta)} \in \mathcal{P}$, the following are equivalent:
  (a) $U \cap V \neq \emptyset$;
  (b) $\gamma = \eta$ and γ is strongly fixed by nm^{-1}; and
  (c) $U = V$.
(3) For any $V_1, V_2 \in \mathcal{P}$ either $V_1 = V_2$ or $V_1 \cap V_2 = \emptyset$.
(4) For any $V_i = U_{(m,\gamma_i)} \in \mathcal{P}$ we have $V_1 \cap V_2 = \emptyset \Leftrightarrow \gamma_1 \neq \gamma_2$.
(5) If $\mathcal{F}$ is finite and every $x \in Z(u)$ is strongly fixed, then

$$1_{\Theta_{(u,m,m^{-1}\cdot u)}} = \sum_{\gamma\in\mathcal{F}} 1_{U_{(m,\gamma)}}, \quad \textit{for each } m \in G.$$

(6) For each $U := U_{(m,\gamma)} \in \mathcal{P}$ set $p_U := s_{\gamma,\varphi(m^{-1},\gamma)^{-1}} s^*_{m^{-1}\cdot\gamma,e_G}$. With $\pi_{G,E}$ as in Proposition 3.7,

$$\pi_{G,E}(p_U) = 1_{\Theta_{(\gamma,\varphi(m^{-1},\gamma)^{-1},m^{-1}\cdot\gamma)}} = 1_U.$$

(7) For $U := U_{(m,\gamma)}$ and $V := U_{(n,\eta)} \in \mathcal{P}$, the following are equivalent:
  (c) $U = V$; and
  (d) $p_U = p_V$.
(8) If $\mathcal{F}$ is finite and every $x \in Z(u)$ is strongly fixed, then

$$p_{u,m} = \sum_{\gamma\in\mathcal{F}} p_{U_{(m,\gamma)}} \quad \textit{for each } m \in G.$$

Proof. (1) Notice that $Z(m^{-1} \cdot \gamma)$ is an open subset of $Z(m^{-1} \cdot u)$. The result now follows from [13, Proposition 4.18].

(2a)⇒(2b) Suppose that $y \in U \cap V$. Then $y = [(u, m, m^{-1} \cdot u), z] = [(u, n, u; n^{-1} \cdot u), z]$ for some $z \in E^\infty$. By Lemma 4.6 we have $x := m \cdot z = n \cdot z$ and there exists $\alpha \in uE^*$ that is strongly fixed by nm^{-1} with $x \in Z(\alpha)$. Since $z \in Z(m^{-1} \cdot \gamma)$, Lemma 4.6 gives $x = m \cdot z \in Z(m \cdot (m^{-1} \cdot \gamma)) = Z(\gamma)$. Similarly $x \in Z(\eta)$, so $\gamma = \eta$. We may assume $n \neq m$. Since $x \in Z(\alpha) \cap Z(\gamma)$ we deduce that $\gamma = \alpha$ is strongly fixed by nm^{-1}.

(2b)⇒(2c) Now suppose that that $\gamma = \eta$ is strongly fixed by nm^{-1}. Take any $y \in U$, say $y = [(u, m, m^{-1} \cdot u), z']$ for some $z' \in Z(m^{-1} \cdot \gamma)$. Using γ is strongly fixed by nm^{-1} it follows from Lemma 4.6 that

$$[(u, m, m^{-1} \cdot u), z] = [(u, n, n^{-1} \cdot u), z] \text{ for all } z \in Z(m^{-1} \cdot \gamma).$$

Since $Z(m^{-1} \cdot \gamma) = Z(n^{-1} \cdot \gamma)$ we get $y \in V$. By symmetry $U = V$.

(2c)⇒(2a) is trivial, completing the proof of (2). Both (3) and (4) follow from (2).

(5) Recall that $\Theta_{(u,m,m^{-1}\cdot u)} = \{[(u, m, m^{-1} \cdot u), x] : x \in Z(m^{-1} \cdot u)\}$. Take any element $y := [(u, m, m^{-1} \cdot u), z] \in \Theta_{(u,m,m^{-1}\cdot u)}$, so $z \in Z(m^{-1} \cdot u)$. Then $x := m \cdot z \in Z(u)$. Since $x \in Z(u)$ is strongly fixed, there exists some initial segment $\gamma \in uE^* \setminus \{u\}$ of x such that γ is minimal strongly fixed. Clearly $\gamma \in \mathcal{F}$. Moreover, $z = m^{-1} \cdot x \in Z(m^{-1} \cdot \gamma)$, so $y \in U_{(m,\gamma)}$. Conversely, for each $\gamma \in \mathcal{F}$ we have $Z(m^{-1} \cdot \gamma) \subseteq Z(m^{-1} \cdot u)$, so

$$\Theta_{(u,m,m^{-1}\cdot u)} = \bigcup_{\gamma \in \mathcal{F}} U_{(m,\gamma)}.$$

So (4) implies that the $U_{(m,\gamma)}$ are mutually disjoint giving the desired result.

(6) By direct computation (cf. Remark 3.11) one can verify that $\pi_{G,E}(p_U) = 1_{\Theta_{(\gamma,\varphi(m^{-1},\gamma)^{-1},m^{-1}\cdot\gamma)}}$. With $s := (u, m, m^{-1} \cdot u)$ and $e := (m^{-1} \cdot \gamma, 0, m^{-1} \cdot \gamma)$ we have $[s, x] = [se, x]$ for each $x \in Z(m^{-1} \cdot \gamma)$. It follows that

$$\begin{aligned} U_{(m,\gamma)} &= \{[s, x] : x \in Z(m^{-1} \cdot \gamma)\} = \{[se, x] : x \in Z(m^{-1} \cdot \gamma)\} \\ &= \{[(\gamma, \varphi(m^{-1}, \gamma)^{-1}, m^{-1} \cdot \gamma), x] : x \in Z(m^{-1} \cdot \gamma)\} \\ &= \Theta_{(\gamma,\varphi(m^{-1},\gamma)^{-1},m^{-1}\cdot\gamma)}. \end{aligned}$$

(7) If $p_U = p_V$ then $U = V$ by (6). Conversely if $U = V$ then part (2) gives that $\gamma = \eta$ and γ is strongly fixed by nm^{-1}. Hence $m^{-1} \cdot \gamma = n^{-1} \cdot \gamma$ and $\varphi(m^{-1}, \gamma) = \varphi(n^{-1}, \gamma)$, see [15, Proposition 5.6]. So $s_{\gamma,\varphi(m^{-1},\gamma)^{-1}} s^*_{m^{-1}\cdot\gamma,e_G} = s_{\eta,\varphi(n^{-1},\eta)^{-1}} s^*_{n^{-1}\cdot\eta,e_G}$ and $p_U = p_V$.

(8) Fix $m \in G$. Define $J := \{(\gamma, \varphi(m^{-1}, \gamma)^{-1}, m^{-1} \cdot \gamma) : \gamma \in \mathcal{F}\}$ and $I := \{(u, m, m^{-1} \cdot u)\}$. Then (5) and (6) give

$$\bigsqcup_{s \in J} \Theta_s = \bigsqcup_{\gamma \in \mathcal{F}} \Theta_{(\gamma,\varphi(m^{-1},\gamma)^{-1},m^{-1}\cdot\gamma)} = \bigsqcup_{\gamma \in \mathcal{F}} U_{(m,\gamma)} = \Theta_{(u,m,m^{-1}\cdot u)} = \bigsqcup_{s \in I} \Theta_s.$$

With $\mathcal{J}$ as in Lemma 3.9 we have $I, J \in \mathcal{J}$. Using Lemma 3.9 we get

$$\sum_{(\alpha,g,\beta) \in I} s_{\alpha,g} s^*_{\beta,e_G} = \sum_{(\alpha,g,\beta) \in J} s_{\alpha,g} s^*_{\beta,e_G}.$$

It follows that

$$p_{u,m} = \sum_{(\alpha,g,\beta) \in J} s_{\alpha,g} s^*_{\beta,e_G} = \sum_{\gamma \in \mathcal{F}} p_{U_{(m,\gamma)}}.$$

as claimed. □

Lemma 4.9. *Let (G, E, φ) be as in Notation 1.2. Let R be a unital commutative *-ring. Fix $u \in E^0$. Suppose that every $x \in Z(u)$ is strongly fixed. Let $\mathcal{F}$ be the set of all minimal strongly fixed paths with range u. Suppose that $\mathcal{F}$ is finite and $Z(\gamma) \cap Z(\gamma') = \emptyset$ whenever $\gamma \neq \gamma'$ in $\mathcal{F}$. Then for each finite subset F of G,*

$$\sum_{m\in F} r_m 1_{\Theta_{(u,m,m^{-1}\cdot u)}} = 0 \implies \sum_{m\in F} r_m p_{m,u} = 0.$$

Proof. Suppose that $h := \sum_{m\in F} r_m 1_{\Theta_{(u,m,m^{-1}\cdot u)}}$ is the zero function. For each $(m,\gamma) \in F \times \mathcal{F}$ define $r_{(m,\gamma)} := r_m$. Using Lemma 4.8

$$h = \sum_{m\in F} r_m \sum_{\gamma\in\mathcal{F}} 1_{U_{(m,\gamma)}} = \sum_{(m,\gamma)\in F\times\mathcal{F}} r_m 1_{U_{(m,\gamma)}} = \sum_{p\in F\times\mathcal{F}} r_p 1_{U_p}.$$

For each $p \in F \times \mathcal{F}$ set $I(p) := \{p' \in F \times \mathcal{F} : U_p = U_{p'}\}$. Since $F \times \mathcal{F}$ is finite there exist a smallest set $P_{\min} \subseteq F \times \mathcal{F}$ such that $U_p \neq U_{p'}$ for $p \neq p' \in P_{\min}$ and $F \times \mathcal{F} = \bigcup_{p\in P_{\min}} I(p)$. For each $p \in P_{\min}$ set $s_p = \sum_{p'\in I(p)} r_{p'}$. Then

$$h = \sum_{p\in P_{\min}} \sum_{p'\in I(p)} r_{p'} 1_{U_p} = \sum_{p\in P_{\min}} s_p 1_{U_p}.$$

If $p_1 \neq p_2 \in P_{\min}$ then $U_{p_1} \neq U_{p_2} \in \mathcal{P}$, so $U_{p_1} \cap U_{p_2} = \emptyset$ by Lemma 4.8. So for $p \in P_{\min}$, we have $h(\gamma) = s_p$ for all $\gamma \in U_p$. Thus $s_p = 0$ for each $p \in P_{\min}$. Hence

$$0 = \sum_{p\in P_{\min}} s_p p_{U_p} = \sum_{p\in P_{\min}} \sum_{p'\in I(p)} r_{p'} p_{U_p}.$$

For each $p' \in I(p)$ we have $U_p = U_{p'} \in \mathcal{P}$, so Lemma 4.8 gives $p_{U_p} = p_{U_{p'}}$. Thus

$$0 = \sum_{p\in P_{\min}} \sum_{p'\in I(p)} r_{p'} p_{U_p} = \sum_{p\in P_{\min}} \sum_{p'\in I(p)} r_{p'} p_{U_{p'}} = \sum_{p'\in F\times\mathcal{F}} r_{p'} p_{U_{p'}}.$$

Lemma 4.8 gives

$$\sum_{m\in F} r_m p_{m,u} = \sum_{m\in F} r_m \sum_{\gamma\in\mathcal{F}} p_{U_{(m,\gamma)}} = \sum_{(m,\gamma)\in F\times\mathcal{F}} r_m p_{U_{(m,\gamma)}} = 0$$

as claimed. □

We are now able to prove Theorem 4.2.

Proof of Theorem 4.2. By the Graded Uniqueness Theorem A, $\pi_{G,E}$ is injective if and only if its restriction to $\mathcal{D}$ is injective, that is, if and only if for each finite subset P of $E^0 \times G$,

$$\sum_{(u,m)\in P} r_{(u,m)} 1_{\Theta_{(u,m,m^{-1}\cdot u)}} = 0 \implies \sum_{(u,m)\in P} r_{(u,m)} p_{u,m} = 0.$$

By the last item of Lemma 4.6 it suffices to consider one vertex at a time and show that: for each $u \in E^0$ and each finite subset F of G,

$$\sum_{m \in F} r_m 1_{\Theta_{(u,m,m^{-1}\cdot u)}} = 0 \implies \sum_{m \in F} r_m p_{m,u} = 0.$$

So suppose that $h := \sum_{m \in F} r_m 1_{\Theta_{(u,m,m^{-1}\cdot u)}} = 0$.

Firstly suppose that there exists $x \in Z(u)$ that is not strongly fixed. By Lemma 4.7 the indicator functions of the sets $\{\Theta_{(u,m,m^{-1}\cdot u)}, m \in F\}$ are linearly independent. Thus each $r_m = 0$, and so $\sum_{m \in F} r_m p_{u,m} = 0$.

Secondly suppose that $\mathcal{F}_u$, the set of all minimal strongly fixed paths with range u, is finite. Then Lemma 4.9 gives $\sum_{m \in F} r_m p_{u,m} = 0$. □

Having Theorem 4.2 at our disposal we can now prove Theorem C by simply verifying that each Katsura triple $(\mathbb{Z}, E, \varphi)$ with E finite satisfies the conditions set out in Theorem 4.2. To do this we recall some terminology. Let E be any directed graph and let A, B be integer valued $E^0 \times E^0$ matrices. Recall that E^* denotes the set of finite paths in E. For a path $\alpha \in E^*$ and $i \in \{1, \dots, |\alpha|\}$ we let α_i be the ith edge of α so $\alpha = \alpha_1\alpha_2 \dots \alpha_{|\alpha|}$. For $l \in \mathbb{N}$ and $i \in \{1, \dots, |\alpha|\}$ define $K_i^{\alpha,l} := l\,\frac{B_{r(\alpha_1)s(\alpha_1)} \cdots B_{r(\alpha_i)s(\alpha_i)}}{A_{r(\alpha_1)s(\alpha_1)} \cdots A_{r(\alpha_i)s(\alpha_i)}}$, cf. [14, 15]. We finally proceed with the proof of Theorem C.

Proof of Theorem C. Fix any vertex $u \in E^0$. We must show that $Z(\gamma) \cap Z(\gamma') = \emptyset$ whenever $\gamma \neq \gamma' \in \mathcal{F}_u$ and that $\mathcal{F}_u$ is finite whenever every $x \in Z(u)$ is strongly fixed and $N = |E^0| < \infty$.

Fix $\gamma \neq \gamma'$ of $\mathcal{F}_u$. Since γ is minimal strongly fixed, it is minimal strongly fixed by some $l \geq 1$. Hence

$$\varphi(l, \gamma_1 \dots \gamma_i) = K_i^{\gamma,l} \in \mathbb{Z}\backslash\{0\} \quad \text{for } i < |\gamma|, \text{ and}$$
$$\varphi(l, \gamma) = K_{|\gamma|}^{\gamma,l} = 0.$$

It follows that $B_{r(\gamma_i)s(\gamma_i)} \neq 0$ for all $i < |\gamma|$ and $B_{r(\gamma_i)s(\gamma_i)} = 0$ for $i = |\gamma|$. By symmetry, if one of γ, γ' is an initial segment of the other then they must have the same length. Hence $Z(\gamma) \cap Z(\gamma') \neq \emptyset$ implies $\gamma = \gamma'$, so $Z(\gamma) \cap Z(\gamma') = \emptyset$ whenever $\gamma \neq \gamma'$.

Now suppose that every $x \in Z(u)$ is strongly fixed and $N = |E^0| < \infty$. Fix $\beta \in uE^N$. We claim that β is strongly fixed. To see this let $x \in E^\infty$ be an infinite path having β as an initial segment. Find the shortest initial segment $\beta_x \in E^* \setminus \{u\}$ of x that is strongly fixed. Say β_x is fixed by $m \neq 0$. We suppose that $|\beta_x| > N$ and derive a contradiction. Since $|\beta_x| > N$, any initial segment β_n of x of length $n \in \{1, \dots, N\}$ must satisfy $\varphi(m, \beta_n) \neq 0$ because β_x is the shortest segment that is strongly fixed by m. Since $N = |E^0|$, one of the paths β_n has the form $\alpha\gamma$ where γ is a loop. By construction $\varphi(m, \alpha\gamma) \neq 0$ so $B_{r(\alpha_i)s(\alpha_i)} \neq 0$ for $i \leq |\alpha|$ and $B_{r(\gamma_i)s(\gamma_i)} \neq 0$ for all $i \leq |\gamma|$. Hence $x := \alpha\gamma\gamma\gamma \cdots \in Z(u)$ is not strongly fixed. This contradicts that every

$x \in Z(u)$ is strongly fixed. We conclude that $|\beta_x| \leq N$. Since $|\beta| = N$, β_x as an initial segment of β. So β is strongly fixed. Since uE^N is finite and $\mathcal{F}_u$ is a subset of uE^N we deduce that $\mathcal{F}_u$ is finite. The result now follows from Theorem 4.2. □

Remark 4.10. It may happen that all the sets $\{\Theta_{(w,m,w)} : m \in \mathbb{Z}\}$ are identical. This is the case, for example, for $w \in E^0$ in Example 4.4. In this situation, the corresponding row of B is identically 0.

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