PREFERRED TRACES ON C^*-ALGEBRAS OF SELF-SIMILAR GROUPOIDS ARISING AS FIXED POINTS

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Abstract. Recent results of Laca, Raeburn, Ramagge and Whittaker show that any self-similar action of a groupoid on a graph determines a 1-parameter family of self-mappings of the trace space of the groupoid C^*-algebra. We investigate the fixed points for these self-mappings, under the same hypotheses that Laca et al. used to prove that the C^*-algebra of the self-similar action admits a unique KMS state. We prove that for any value of the parameter, the associated self-mapping admits a unique fixed point, which is in fact a universal attractor. This fixed point is precisely the trace that extends to a KMS state on the C^*-algebra of the self-similar action.

There has been a lot of recent interest in the structure of KMS states for the natural gauge actions on C^*-algebras associated to algebraic and combinatorial objects (see, for example, [1, 2, 3, 6, 8, 9, 10, 11, 17]). The theme is that there is a critical inverse temperature below which the system admits no KMS states, and above this critical inverse temperature the structure of the KMS simplex reflects some of the underlying combinatorial data. For example, for C^*-algebras of strongly-connected finite directed graphs, the critical inverse temperature is the logarithm of the spectral radius of the graph, there is a unique KMS state at this inverse temperature, and at supercritical inverse temperatures the extreme KMS states are parameterised by the vertices of the graph [5, 8].

A particularly striking instance of this phenomenon appeared recently in the context of C^*-algebras associated to self-similar groups [14, 12] and, more generally, self-similar actions of groupoids on graphs [13]. Roughly speaking a self-similar action of a groupoid on a finite directed graph E consists of a discrete groupoid $\mathcal{G}$ with unit space identified with E^0 and an action of $\mathcal{G}$ on the left of the path-space of E with the property that for each groupoid element g and each path μ for which $g \cdot \mu$ is defined, there is a unique groupoid element $g|_\mu$ such that $g \cdot (\mu\nu) = (g \cdot \mu)(g|_\mu \cdot \nu)$ for any other path ν.

In [13], the authors first show that at supercritical inverse temperatures, the KMS states on the Toeplitz algebra $\mathcal{T}(\mathcal{G}, E)$ of the self-similar action are determined by their restrictions to the embedded copy of $C^*(\mathcal{G})$. They then show that the self-similar action can be used to transform an arbitrary trace on $C^*(\mathcal{G})$ into new trace that extends to a KMS state, and that this transformation is an isomorphism of the trace simplex of $C^*(\mathcal{G})$ onto the KMS-simplex of $\mathcal{T}(\mathcal{G}, E)$. The transformation is quite natural: given a trace τ on $C^*(\mathcal{G})$ and given $g \in \mathcal{G}$, the value of the transformed trace at the generator u_g is a weighted infinite sum of the values of the original trace on restrictions $g|_\mu$ of g such

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that $g \cdot \mu = \mu$; so the transformed trace at u_g reflects the proportion—as measured by the initial trace—of the path-space of E that is fixed by g. Building on this analysis, Laca, Raeburn, Ramagge and Whittaker proved that if E is strongly connected and the self-similar action satisfies an appropriate finite-state condition, then $\mathcal{T}(\mathcal{G}, E)$ admits a unique KMS state at the critical inverse temperature and this is the only state that factors through the quotient $\mathcal{O}(\mathcal{G}, E)$ determined by the Cuntz–Krieger relations for E. So the KMS structure picks out a "preferred trace" on the groupoid C^*-algebra $C^*(\mathcal{G})$. Some enlightening examples of this are discussed in [13, Section 9].

This paper is motivated by the observation that the transformation described in the preceding paragraph for a given inverse temperature β is a self-mapping χ_β of the simplex of normalised traces of $C^*(\mathcal{G})$, and so can be iterated. This raises a natural question: for which initial traces τ and at which supercritical inverse temperatures does the sequence $(\chi_\beta^n(\tau))_{n=1}^\infty$ converge, and what information about the self-similar action do the limit traces—that is, the fixed points for χ_β—encode? Our main result, Theorem 2.1, gives a very satisfactory answer to this question: the hypotheses of [13] (namely that E is strongly connected and the action satisfies the finite-state condition) seem to be exactly the hypotheses needed to guarantee that χ_β admits a unique fixed point for every supercritical β, that this fixed point is a universal attractor, and that it is precisely the preferred trace that extends to a KMS state at the critical inverse temperature.

1. Preliminaries

1.1. **KMS states.** Consider a C^*-algebra A together with a strongly continuous homomorphism $\alpha : \mathbb{R} \to \operatorname{Aut}(A)$. An element $x \in A$ is called *analytic* if the function $t \mapsto \alpha_t(x)$ extends to an analytic function from $\mathbb{C}$ to A. The set A^a of analytic elements is a dense *-subalgebra of A (see for example [15, Chapter 8]).

We say that a state ϕ of A satisfies the Kubo–Martin–Schwinger (KMS) condition at inverse temperature $\beta \in (0, \infty)$ if it satisfies

$$\phi(xy) = \phi(y\alpha_{i\beta}(x)) \qquad \text{for all analytic } x, y \in A.$$

We call such a ϕ a *KMS$_\beta$ state* for (A, α). It is well-known that a state ϕ is KMS$_\beta$ if and only if there exists a set S of analytic elements such that $\operatorname{span} S$ is an α-invariant dense subspace of A, and ϕ satisfies the KMS condition at all $x, y \in S$.

1.2. **Self-similar groupoids.** A *groupoid* is a countable small category $\mathcal{G}$ with inverses. In this paper, we will use d and t for the domain and terminus maps $\mathcal{G} \to \mathcal{G}^{(0)}$ to distinguish them from the range and source maps on directed graphs. For $u \in \mathcal{G}^{(0)}$, we write $\mathcal{G}_u = \{g \in \mathcal{G} : d(g) = u\}$ and $\mathcal{G}^u = \{g \in \mathcal{G} : t(g) = u\}$.

Consider a finite directed graph $E = (E^0, E^1, r, s)$. For $n \geq 2$, write E^n for the paths of length n in E; that is $E^n = \{e_1 e_2 \dots e_n : e_i \in E^1, r(e_{i+1}) = s(e_i)\}$. We write $E^* := \bigcup_{n \geq 0} E^n$. We can visualise the set E^* as indexing the vertices of a forest $T = T_E$ given by $T^0 = E^*$ and $T^1 = \{(\mu, \mu e) \in E^* : \mu \in E^*, e \in E^1 \text{ and } s(\mu) = r(e)\}$. Throughout this paper, we write A_E for the integer matrix with entries $A_E(v, w) = |vE^1 w|$.

We are interested in self-similar actions of groupoids on directed graphs E as introduced and studied in [13]. To describe these, first recall that a *partial isomorphism* of the forest T_E corresponding to a directed graph E as above consists of a pair $(v, w) \in E^0 \times E^0$ and a bijection $g : vE^* \to wE^*$ such that

(1) $g|_{vE^k} : vE^k \to wE^k$ is bijective for $k \geq 1$.

(2) $g(\mu e) \in g(\mu)E^1$ for $\mu \in vE^*$ and $e \in E^1$ with $r(e) = s(\mu)$.

The set of partial isomorphisms of T_E forms a groupoid $\mathrm{PIso}(T_E)$ with unit space E^0 [13, Proposition 3.2]: the identity morphisms are the partial isomorphisms $\mathrm{id}_v : vE^* \to vE^*$ given by the identity map on vE^*, the inverse of $g : vE^* \to wE^*$ is the standard inverse map $g^{-1} : wE^* \to vE^*$, and the groupoid multiplication is composition.

Definition 1.1. Let E be a directed graph, and let $\mathcal{G}$ be a groupoid with unit space E^0. A *faithful action* of $\mathcal{G}$ on T_E is an injective groupoid homomorphism $\theta : \mathcal{G} \to \mathrm{PIso}(T_E)$ that is the identity map on unit spaces. We write $g \cdot \mu$ rather than $\theta_g(\mu)$ for $g \in \mathcal{G}$ and $\mu \in E^*$ with $d(g) = r(\mu)$. The action θ is *self-similar* if for each $g \in \mathcal{G}$ and $\mu \in d(g)E^*$ there exists $g|_\mu \in \mathcal{G}$ such that $d(g|_\mu) = s(\mu)$ and

$$g \cdot (\mu\nu) = (g \cdot \mu)(g|_\mu \cdot \nu) \quad \text{for all } \nu \in s(\mu)E^*. \tag{1.1}$$

The faithfulness condition ensures that for each $g \in \mathcal{G}$ and $\mu \in E^*$ with $d(g) = r(\mu)$, there is a *unique* element $g|_\mu \in \mathcal{G}$ satisfying (1.1). Throughout the paper, we will write $\mathcal{G} \curvearrowright E$ to indicate that the groupoid $\mathcal{G}$ acts faithfully on the directed graph E.

By Proposition 3.6 of [13], self-similar groupoid actions have the following properties, which we will use without comment henceforth: for $g, h \in \mathcal{G}$, $\mu \in d(g)E^*$, and $\nu \in s(\mu)E^*$,

(1) $g|_{\mu\nu} = (g|_\mu)|_\nu$,
(2) $\mathrm{id}_{r(\mu)}|_\mu = \mathrm{id}_{s(\mu)}$,
(3) if $(h, g) \in \mathcal{G}^{(2)}$, then $\big(h|_{g\cdot\mu}, g|_\mu\big) \in \mathcal{G}^{(2)}$, and $(hg)|_\mu = h|_{g\cdot\mu} g|_\mu$, and
(4) $(g^{-1})|_\mu = (g|_{g^{-1}\cdot\mu})^{-1}$.

We say that a self-similar action $\mathcal{G} \curvearrowright E$ is *finite-state* if for every element $g \in \mathcal{G}$, the set $\{g|_\mu : \mu \in d(g)E^*\}$ is a finite subset of $\mathcal{G}$.

1.3. **The C^*-algebras of a self-similar groupoid.** The Toeplitz algebra of a self-similar action $\mathcal{G} \curvearrowright E$ is defined in [13] as follows. A *Toeplitz representation* (v, q, t) of $(\mathcal{G}, E)$ in a unital C^*-algebra B is a triple of maps $v : \mathcal{G} \to B$, $q : E^0 \to B$, $t : E^1 \to B$ such that

(1) (q, t) is a Toeplitz–Cuntz–Krieger family in B such that $\sum_{w\in E^0} q_w = 1_B$;
(2) $\{v_g : g \in \mathcal{G}\}$ is a family of partial isometries on B satisfying $v_g v_h = \delta_{d(g),t(h)} v_{gh}$ and $v_{g^{-1}} = v_g^*$ for all $g, h \in \mathcal{G}$, and $v_w = q_w$ for $w \in \mathcal{G}^{(0)} = E^0$;
(3) $v_g t_e = \delta_{d(g),r(e)} t_{g\cdot e} v_{g|_e}$ for $g \in \mathcal{G}$ and $e \in E^1$; and
(4) $v_g q_w = \delta_{d(g),w} q_{g\cdot w} v_g$ for all $g \in \mathcal{G}$ and $w \in E^0$.

Standard arguments show that there exists a universal C^*-algebra $\mathcal{T}(\mathcal{G}, E)$ generated by a Toeplitz representation $\{u, p, s\}$. We have $\mathcal{T}(\mathcal{G}, E) = \overline{\mathrm{span}}\{s_\mu u_g s_\nu^* : \mu, \nu \in E^*, g \in \mathcal{G}_{s(\nu)}^{s(\mu)}\}$. We call $\mathcal{T}(\mathcal{G}, E)$ the *Toeplitz algebra* of the self-similar action $\mathcal{G} \curvearrowright E$. The argument of the paragraph following the statement of [13, Theorem 6.1] applied with π_τ replaced by a faithful representation of $C^*(\mathcal{G})$ shows that $C^*(\mathcal{G})$ embeds in $\mathcal{T}(\mathcal{G}, E)$ via $\delta_g \mapsto u_g$.

Following [13, Proposition 4.7], the *Cuntz–Pimsner* algebra of $(\mathcal{G}, E)$, denoted $\mathcal{O}(\mathcal{G}, E)$, is defined to be the quotient of $\mathcal{T}(\mathcal{G}, E)$ by the ideal I generated by $\big\{p_v - \sum_{e\in vE^1} s_e s_e^* : v \in E^0\big\}$. We have $1_{\mathcal{O}(\mathcal{G},E)} = \sum_{\mu\in E^n} s_\mu s_\mu^*$ for any n.

1.4. **Dynamics on $\mathcal{T}(\mathcal{G}, E)$ and $\mathcal{O}(\mathcal{G}, E)$.** The universal property of $\mathcal{T}(\mathcal{G}, E)$ yields a dynamics $\sigma : \mathbb{R} \to \mathrm{Aut}(\mathcal{T}(\mathcal{G}, E))$ such that

$$\sigma_t(u_g) = u_g, \quad \sigma_t(q_w) = q_w, \quad \text{and} \quad \sigma_t(t_e) = e^{it} t_e$$

for all $t \in \mathbb{R}, g \in \mathcal{G}, w \in E^0$, and $e \in E^1$. Since each $p_v - \sum_{e \in vE^1} s_e s_e^*$ is fixed by σ, the dynamics σ descends to a dynamics, also denoted σ, on $\mathcal{O}(\mathcal{G}, E)$.

Let $\rho(A_E)$ denote the spectral radius of the adjacency matrix A_E. Proposition 5.1 of [13] shows that there are no KMS$_\beta$ states for $(\mathcal{T}(\mathcal{G}, E), \sigma)$ for $\beta < \log \rho(A_E)$. In [13, Theorem 6.1], given a trace τ on the groupoid algebra $C^*(\mathcal{G})$, the authors show that for $\beta > \log \rho(A_E)$, the series

$$Z(\beta, \tau) := \sum_{k=0}^{\infty} e^{-\beta k} \sum_{\mu \in E^k} \tau(u_{s(\mu)})$$

converges to a positive real number, and that there is a KMS$_\beta$ state $\Psi_{\beta,\tau}$ on the Toeplitz algebra $\mathcal{T}(\mathcal{G}, E)$ given by

$$\Psi_{\beta,\tau}(s_\mu u_g s_\nu^*) = \delta_{\mu,\nu} e^{-\beta|\mu|} Z(\beta, \tau)^{-1} \sum_{k=0}^{\infty} e^{-\beta k} \Big(\sum_{\lambda \in s(\mu)E^k,\, g\cdot\lambda = \lambda} \tau(u_{g|_\lambda}) \Big). \tag{1.2}$$

They show that the map $\tau \mapsto \Psi_{\beta,\tau}$ is an isomorphism from the simplex of tracial states of $C^*(\mathcal{G})$ to the KMS$_\beta$-simplex of $\mathcal{T}(\mathcal{G}, E)$.

2. A fixed-point theorem, and the preferred trace on $C^*(\mathcal{G})$

Consider a self-similar action $\mathcal{G} \curvearrowright E$ and a number $\beta > \log \rho(A_E)$. The starting point for our analysis is that if τ is a trace on $C^*(\mathcal{G})$ and $\Psi_{\beta,\tau}$ is the associated KMS$_\beta$-state of $\mathcal{T}(\mathcal{G}, E)$ given by (1.2), then $\Psi_{\beta,\tau}|_{C^*(\mathcal{G})}$ is again a trace on $C^*(\mathcal{G})$. So there is an operator $\chi_\beta : \operatorname{Tr}(C^*(\mathcal{G})) \to \operatorname{Tr}(C^*(\mathcal{G}))$ given by

$$\chi_\beta(\tau) = \Psi_{\beta,\tau}|_{C^*(\mathcal{G})}. \tag{2.1}$$

Our main theorem is the following; its proof occupies the remainder of the paper.

Theorem 2.1. *Let E be a finite strongly connected graph, suppose that $\mathcal{G} \curvearrowright E$ is a faithful self-similar action of a groupoid $\mathcal{G}$ on E, and suppose that $\beta > \log \rho(A_E)$. If $\mathcal{G} \curvearrowright E$ is finite state, then*

1. *the map $\chi_\beta : \operatorname{Tr}(C^*(\mathcal{G})) \to \operatorname{Tr}(C^*(\mathcal{G}))$ of (2.1) has a unique fixed point θ;*
2. *for any $\tau \in \operatorname{Tr}(C^*(\mathcal{G}))$ we have $\chi_\beta^n(\tau) \overset{w^*}{\to} \theta$;*
3. *θ is the unique trace on $C^*(\mathcal{G})$ that extends to a KMS$_{\log \rho(A_E)}$-state of $\mathcal{T}(\mathcal{G}, E)$.*

We start with a straightforward observation about the map χ_β of (2.1).

Lemma 2.2. *Let $\mathcal{G} \curvearrowright E$ be a faithful self-similar action of a groupoid on a finite strongly connected graph, and suppose that $\beta > \log \rho(A_E)$. Then the map χ_β is weak*-continuous. If $\tau \in \operatorname{Tr}(C^*(\mathcal{G}))$ satisfies $\chi_\beta^n(\tau) \overset{w^*}{\to} \theta$, then $\theta \in \operatorname{Tr}(C^*(\mathcal{G}))$ and $\chi_\beta(\theta) = \theta$.*

Proof. The map $\tau \mapsto \Psi_{\beta,\tau}$ is a homeomorphism and hence continuous, and restriction of states to a subalgebra is clearly continuous, so χ_β is continuous. Hence if $\chi_\beta^n(\tau) \overset{w^*}{\to} \theta$, then $\theta \in \operatorname{Tr}(C^*(\mathcal{G}))$ because the trace simplex of a unital C^*-algebra is weak*-compact, and then $\chi_\beta(\theta) = \chi_\beta(\lim_n \chi_\beta^n(\tau)) = \lim_n \chi_\beta^{n+1}(\tau) = \theta$. $\square$

Proposition 2.3. *Let $\mathcal{G} \curvearrowright E$ be a faithful self-similar action of a groupoid on a finite graph, and fix $\beta > \log \rho(A_E)$. Let $\chi_\beta : \operatorname{Tr}(C^*(\mathcal{G})) \to \operatorname{Tr}(C^*(\mathcal{G}))$ be the map (2.1). For $\tau \in \operatorname{Tr}(C^*(\mathcal{G}))$, define*

$$N(\beta, \tau) := e^\beta (1 - Z(\beta, \tau)^{-1}).$$

(1) *If $\tau \in \operatorname{Tr}(C^*(\mathcal{G}))$ is a fixed point for χ_β, then for each $g \in \mathcal{G}$, we have*

$$N(\beta,\tau)^n \tau(u_g) = \sum_{\mu \in E^n,\, g\cdot\mu=\mu} \tau(u_{g|_\mu}) \qquad \textit{for all } n \geq 1. \tag{2.2}$$

(2) *If E is strongly connected with adjacency matrix A_E, and $\tau \in \operatorname{Tr}(C^*(\mathcal{G}))$ satisfies (2.2), then $m := (\tau(u_v))_{v\in E^0}$ is the Perron–Frobenius eigenvector of A_E, and $N(\beta,\tau) = \rho(A_E)$.*

Proof. (1) For each $g \in \mathcal{G}$ we have

$$\begin{aligned}\tau(u_g) = \chi_\beta(\tau)(u_g) &= Z(\beta,\tau)^{-1} \sum_{k=0}^\infty e^{-\beta k}\Big(\sum_{\mu\in E^k,\, g\cdot\mu=\mu} \tau(u_{g|_\mu})\Big)\\ &= Z(\beta,\tau)^{-1}\Big[\tau(u_g) + e^{-\beta}\sum_{k=0}^\infty e^{-\beta k}\Big(\sum_{\mu\in E^{k+1},\, g\cdot\mu=\mu} \tau(u_{g|_\mu})\Big)\Big].\end{aligned}$$

The map $(e,\nu) \mapsto e\nu$ is a bijection

$$\{(e,\nu) \in E^1 \times E^k : s(e) = r(\nu), g\cdot e = e \text{ and } g|_e \cdot \nu = \nu\} \longrightarrow \{\mu \in E^{k+1} : g\cdot\mu = \mu\}.$$

So the definition of $\Psi_{\beta,\tau}$ yields

$$\begin{aligned}\tau(u_g) &= Z(\beta,\tau)^{-1}\tau(u_g) + \sum_{e\in E^1,\, g\cdot e=e}\Big(Z(\beta,\tau)^{-1}e^{-\beta}\sum_{k=0}^\infty e^{-\beta k}\Big(\sum_{\nu\in s(e)E^k,\, g|_e\cdot\nu=\nu}\tau(u_{(g|_e)|_\nu})\Big)\Big)\\ &= Z(\beta,\tau)^{-1}\tau(u_g) + \sum_{e\in E^1,\, g\cdot e=e}\Psi_{\beta,\tau}(s_e u_{g|_e} s_e^*).\end{aligned} \tag{2.3}$$

We have $\Psi_{\beta,\tau}(s_e u_{g|_e} s_e^*) = \delta_{s(e),t(g)}\delta_{s(e),d(g)} e^{-\beta}\Psi_{\beta,\tau}(u_{g|_e}) = e^{-\beta}\chi_\beta(\tau)(u_{g|_e})$; applying this and rearranging (2.3) gives

$$e^\beta\big(1 - Z(\beta,\tau)^{-1}\big)\tau(u_g) = \sum_{e\in E^1,\, g\cdot e=e}\chi_\beta(\tau)(u_{g|_e}) = \sum_{e\in E^1,\, g\cdot e=e}\tau(u_{g|_e}).$$

Statement (1) now follows from an induction on n.

(2) Using (2.2) for τ with $n = 1$ at the first step, we see that for $v \in E^0$,

$$m_v = N(\beta,\tau)^{-1}\sum_{e\in vE^1}\tau(u_{s(e)}) = N(\beta,\tau)^{-1}\sum_{w\in E^0}A_E(v,w)\tau(u_w) = N(\beta,\tau)^{-1}(A_E m)_v.$$

Hence, since $1 = \tau(1) = \sum_v \tau(u_v)$, the vector m is a unimodular nonnegative eigenvector for the irreducible matrix A_E and has eigenvalue $N(\beta,\tau)$. So the Perron–Frobenius theorem [16, Theorem 1.6] shows that m is the Perron–Frobenius eigenvector and $N(\beta,\tau) = \rho(A_E)$. □

We now turn our attention to the situation where E is strongly connected, and $\mathcal{G} \curvearrowright E$ is finite-state, and aim to show that χ_β admits a unique fixed point. The strategy is to show that if $C^*(\mathcal{G})$ admits a trace θ satisfying (2.2), then for any other trace τ we have $\chi_\beta^n(\tau) \to \theta$. From this it will follow first that χ_β^n admits at most one fixed point, and second that a trace θ is fixed point if and only if it satisfies (2.2). We start with an easy result from Perron–Frobenius theory.

Lemma 2.4. *Let $A \in M_n(\mathbb{R})$ be an irreducible matrix, and take $\beta > \log(\rho(A))$.*

(1) *The matrix* $I - e^{-\beta}A$ *is invertible, and* $A_{vN} := (I - e^{-\beta}A)^{-1}$ *is primitive; indeed, every entry of* A_{vN} *is strictly positive.*
(2) *Let* m^A *be the Perron–Frobenius eigenvector of* A*. Then* m^A *is also the Perron–Frobenius eigenvector of* A_{vN}*, and* $\rho(A_{vN}) = (1 - e^{-\beta}\rho(A))^{-1}$*.*

Proof. (1) The matrix $I - e^{-\beta}A$ is invertible because $e^{\beta} > \rho(A)$ and so does not belong to the spectrum of A. As in, for example, [4, Section VII.3.1], we have

$$A_{vN} := (I - e^{-\beta}A)^{-1} = \sum_{k=0}^{\infty} e^{-k\beta}A^k.$$

Fix $i, j \leq n$. Since A is irreducible, we have $A^k_{i,j} > 0$ for some $k \geq 0$, and since $A^l_{i,j} \geq 0$ for all l, we deduce that $(A_{vN})_{i,j} \geq e^{-\beta k}A^k_{i,j} > 0$.

(2) We compute $A_{vN}^{-1}m^A = (I - e^{-\beta}A)m^A = (1 - e^{-\beta}\rho(A))m^A$. Multiplying through by $(1 - e^{-\beta}\rho(A))^{-1}A_{vN}$ shows that m^A is a positive eigenvector of m^A with eigenvalue $(1 - e^{-\beta}\rho(A))^{-1}$, so the result follows from uniqueness of the Perron–Frobenius eigenvector of A_{vN}. □

Notation 2.5. Henceforth, given a self-similar action $\mathcal{G} \curvearrowright E$ of a groupoid on a finite graph and a trace $\tau \in \mathrm{Tr}(C^*(\mathcal{G}))$, we denote by $x^\tau \in [0,1]^{E^0}$ the vector

$$x^\tau = \big(\tau(u_v)\big)_{v \in E^0}.$$

Proposition 2.6. *Let* $\mathcal{G} \curvearrowright E$ *be a faithful self-similar action of a groupoid on a finite strongly connected graph. Fix* $\beta > \log \rho(A_E)$*, and let* $A_{vN} := (I - e^{-\beta}A_E)^{-1}$*. Let* $\chi_\beta : \mathrm{Tr}(C^*(\mathcal{G})) \to \mathrm{Tr}(C^*(\mathcal{G}))$ *be the map* (2.1)*. Fix* $\tau \in \mathrm{Tr}(C^*(\mathcal{G}))$*. Then*

$$x^{\chi^n_\beta(\tau)} = \|A^n_{vN}x^\tau\|_1^{-1} A^n_{vN}x^\tau. \tag{2.4}$$

Proof. For $v \in E^0$, the definition of χ_β gives

$$\begin{aligned}\chi_\beta(\tau)(u_v) &= Z(\beta,\tau)^{-1}\sum_{k=0}^{\infty} e^{-\beta k}\Big(\sum_{\mu \in vE^k}\tau(u_{s(\mu)})\Big)\\ &= Z(\beta,\tau)^{-1}\sum_{k=0}^{\infty} e^{-\beta k}(A^k_E x^\tau)_v = Z(\beta,\tau)^{-1}(A_{vN}x^\tau)_v\end{aligned}$$

So an induction gives $x^{\chi^n_\beta(\tau)} = Z(\beta, \chi^{n-1}_\beta(\tau))^{-1} \cdots Z(\beta,\tau)^{-1}A^n_{vN}x^\tau$.

Since both $x^{\chi^n_\beta(\tau)}$ and x^τ have unit 1-norm, we have $Z(\beta, \chi^{n-1}_\beta(\tau))^{-1} \cdots Z(\beta,\tau)^{-1} = \|A^n_{vN}x^\tau\|_1^{-1}$, and the result follows. □

Our next result shows that for any $\tau \in \mathrm{Tr}(C^*(\mathcal{G}))$, the sequence $x^{\chi^n_\beta(\tau)}$ converges exponentially fast to the Perron–Frobenius eigenvector of A_E.

Theorem 2.7. *Let* $\mathcal{G} \curvearrowright E$ *be a faithful self-similar action of a groupoid on a finite strongly connected graph. Fix* $\beta > \log \rho(A_E)$*. Let* $\chi_\beta : \mathrm{Tr}(C^*(\mathcal{G})) \to \mathrm{Tr}(C^*(\mathcal{G}))$ *be the map* (2.1)*. Fix* $\tau \in \mathrm{Tr}(C^*(\mathcal{G}))$*. Let* $m = m^E$ *be the Perron–Frobenius eigenvector of* A_E*. Then* $x^{\chi^n_\beta(\tau)} \to m^E$ *exponentially quickly, and* $Z(\beta, \chi^n_\beta(\tau)) \to \rho(A_{vN})$ *exponentially quickly.*

Proof. Since E is strongly connected, Lemma 2.4 shows that m is the (right) Perron–Frobenius eigenvector of $A_{vN} := (I - e^{-\beta} A_E)^{-1}$. Write $\tilde{m}$ for the left Perron–Frobenius eigenvector of A_{vN} such that $\widetilde{m} \cdot m = 1$.

Let $r := \widetilde{m} \cdot x^\tau$. Then $r > 0$ because every entry of $\widetilde{m}$ is strictly positive, and x^τ is a nonnegative nonzero vector.

Proposition 2.6 implies that

$$x_v^{\chi_\beta^n(\tau)} - m_v = \frac{\rho(A_{vN})^n}{\big\|A_{vN}^n x^\tau\big\|_1}\Big[\big(\rho(A_{vN})^{-n} A_{vN}^n x^\tau - rm\big)_v + \big(r - \big\|\big(\rho(A_{vN})^{-n} A_{vN}^n x^\tau\big)\big\|_1\big) m_v\Big]. \tag{2.5}$$

By [16, Theorem 1.2], there exist a real number $0 < \lambda < 1$, a positive constant C, and an integer $s \geq 0$ such that for large n we have $\rho(A_{vN})^{-n} A_{vN}^n - m \cdot \widetilde{m}^t \leq C n^s \lambda^n$. Indeed, since $C n^s (\lambda'/\lambda)^n \to 0$ for any $0 < \lambda' < \lambda < 1$, we can take $C = 1$ and $s = 0$. So for large n, we have

$$\big|\rho(A_{vN})^{-n} (A_{vN}^n x^\tau)_v - r m_v\big| \leq \lambda^n$$

Since $v \in E^0$ was arbitrary, we deduce that

$$\big|r - \rho(A_{vN})^{-n} \|A_{vN}^n x^\tau\|_1\big| \leq |E^0| \lambda^n.$$

Hence $\rho(A_{vN})^{-n} \big\|A_{vN}^n x^\tau\big\|_1 \overset{n}{\to} r$ exponentially quickly. Making this approximation twice in (2.5), we obtain

$$|x_v^{\chi_\beta^n(\tau)} - m_v| \leq \frac{(1 + |E^0|)}{\rho(A_{vN})^{-n} \big\|A_{vN}^n x^\tau\big\|_1} \lambda^n,$$

which converges exponentially quickly to 0. Hence $x^{\chi_\beta^n(\tau)} \to m$ exponentially quickly.

For the second statement, using Proposition 2.6 at the third equality, we calculate

$$\begin{aligned} Z(\beta, \chi_\beta^n(\tau)) &= \sum_{k=0}^\infty e^{-\beta k} \sum_{\mu \in E^k} \chi_\beta^n(\tau)(u_{s(\mu)}) \\ &= \big\|A_{vN} x^{\chi_\beta^n(\tau)}\big\|_1 = \frac{\|A_{vN}^{n+1} x^\tau\|_1}{\|A_{vN}^n x^\tau\|_1} = \frac{\rho(A_{vN})^{-(n+1)} \big\|A_{vN}^{n+1} x^\tau\big\|_1}{\rho(A_{vN})^{-n} \big\|A_{vN}^n x^\tau\big\|_1} \rho(A_{vN}). \end{aligned}$$

We saw that $\rho(A_{vN})^{-(n+1)} \big\|A_{vN}^{n+1} x^\tau\big\|_1$ converges to $r > 0$ exponentially quickly, so the ratio $\frac{\rho(A_{vN})^{-(n+1)} \|A_{vN}^{n+1} x^\tau\|_1}{\rho(A_{vN})^{-n} \|A_{vN}^n x^\tau\|_1}$ converges exponentially quickly to 1. □

The following estimate is needed for our key technical result, Theorem 2.9.

Lemma 2.8. *Let $\mathcal{G} \curvearrowright E$ be a faithful finite-state self-similar action of a groupoid on a finite strongly connected graph. Let $A_{vN} := (I - e^{-\beta} A_E)^{-1}$, and let $m = m^E$ be the unimodular Perron–Frobenius eigenvector of A_E. For $g \in \mathcal{G} \setminus E^0$, $v \in E^0$, and $k \geq 0$, define*

$$\mathcal{G}_g^k(v) := \{\mu \in d(g) E^k v : g \cdot \mu = \mu\} \quad \text{and} \quad \mathcal{F}_g^k(v) := \{\mu \in \mathcal{G}_g^k(v) : g|_\mu = v\}.$$

Then for $\beta > \log \rho(A_E)$ and $g \in \mathcal{G}$, we have

$$\sum_{k=0}^\infty e^{-\beta k} \sum_{v \in E^0} |\mathcal{G}_g^k(v) \setminus \mathcal{F}_g^k(v)| m_v < \rho(A_{vN}) m_{d(g)}.$$

Proof. The argument of [13, Lemma 8.7] shows that there exists $k(g) > 0$ such that

$$\sum_{v \in E^0} |\mathcal{G}_g^{nk(g)}(v) \setminus \mathcal{F}_g^{nk(g)}(v)| m_v \leq (\rho(A_E)^{k(g)} - 1)^n m_{d(g)}$$

for all $n \geq 0$. For each $k \in \mathbb{N}$ we also have

$$\sum_{v \in E^0} |\mathcal{G}_g^k(v)| m_v \leq \sum_{v \in E^0} |d(g) E^k v| m_v = (A_E^k m)_{d(g)} = \rho(A_E)^k m_{d(g)}.$$

Combining these estimates and using Lemma 2.4(2) at the final step, we obtain

$$\begin{aligned}
&\sum_{k=0}^{\infty} e^{-\beta k} \sum_{v \in E^0} |\mathcal{G}_g^k(v) \setminus \mathcal{F}_g^k(v)| m_v \\
&\quad = \sum_{k \neq k(g)} e^{-\beta k} \sum_{v \in E^0} |\mathcal{G}_g^k(v) \setminus \mathcal{F}_g^k(v)| m_v + e^{-\beta k(g)} \sum_{v \in E^0} |\mathcal{G}_g^{k(g)}(v) \setminus \mathcal{F}_g^{k(g)}(v)| m_v \\
&\quad \leq \sum_{k \neq k(g)} e^{-\beta k} \rho(A_E)^k m_{d(g)} + e^{-\beta k(g)} (\rho(A_E)^{k(g)} - 1) m_{d(g)} \\
&\quad < \sum_{k=0}^{\infty} e^{-\beta k} \rho(A_E)^k m_{d(g)} \\
&\quad = \rho(A_{vN}) m_{d(g)}.
\end{aligned}$$

□

We are now ready to prove a converse to Proposition 2.3(1), under the hypotheses that E is strongly connected and the action of $\mathcal{G}$ on E is finite-state.

Theorem 2.9. *Let $\mathcal{G} \curvearrowright E$ be a faithful finite-state self-similar action of a groupoid on a finite strongly connected graph. Fix $\beta > \log \rho(A_E)$. Let $\chi_\beta : \mathrm{Tr}(C^*(\mathcal{G})) \to \mathrm{Tr}(C^*(\mathcal{G}))$ be the map (2.1). Suppose that $\theta \in \mathrm{Tr}(C^*(\mathcal{G}))$ satisfies (2.2). Then for any $\tau \in \mathrm{Tr}(C^*(\mathcal{G}))$, we have $\lim_n \chi_\beta^n(\tau) = \theta$. In particular, θ is a fixed point for χ_β.*

Proof. We will prove that for each $g \in \mathcal{G}$ there are constants $0 < \lambda < 1$ and $K, D > 0$ such that $|\chi_\beta^n(\tau)(u_g) - \theta(u_g)| < (nK + D)K\lambda^{n-1}$ for all $n \geq 0$. Since $(nK + D)\lambda^{n-1} \to 0$ exponentially quickly in n, the first statement will then follow from an $\varepsilon/3$-argument.

To simplify notation, define $\tau_0 := \tau$ and $\tau_n := \chi_\beta^n(\tau)$ for $n \geq 1$. For $g \in \mathcal{G}$ and $n \geq 0$, let

$$\Delta_n(g) := \tau_n(u_g) - \theta(u_g).$$

Fix $g \in \mathcal{G}$; if $t(g) \neq d(g)$, then $\tau_n(u_g) = \theta(u_g) = 0$ by [13, Proposition 7.2], so we may assume that $t(g) = d(g)$. Since the action is finite-state, the set $\{g|_\mu : \mu \in d(g)E^*\}$ is finite. By Lemma 2.8, there is a constant $\alpha < 1$ such that

$$\sum_{k=0}^{\infty} e^{-\beta k} \sum_{v \in E^0} |\mathcal{G}_{g|_\mu}^k(v) \setminus \mathcal{F}_{g|_\mu}^k(v)| m_v < \alpha \rho(A_{vN}) m_{d(g|_\mu)} \tag{2.6}$$

for all $\mu \in E^*$.

Since θ satisfies (2.2), we have

$$\theta(u_g) = N(\beta, \theta)^{-k} \sum_{\mu \in E^k,\, g \cdot \mu = \mu} \theta(u_{g|_\mu}) \quad \text{for all } k \geq 0.$$

Consequently,

$$\sum_{k=0}^{\infty} e^{-\beta k} \sum_{\mu\in E^k,\, g\cdot\mu=\mu} \theta(u_{g|_\mu}) = \sum_{k=0}^{\infty} e^{-\beta k} N(\beta,\theta)^k \theta(u_g) = \big(1 - e^{-\beta} N(\beta,\theta)\big)^{-1}\theta(u_g).$$

Since $N(\beta,\theta) = e^{\beta}(1 - Z(\beta,\theta)^{-1})$ by definition, we can rearrange to obtain

$$\theta(u_g) = Z(\beta,\theta)^{-1} \sum_{k=0}^{\infty} e^{-\beta k} \sum_{\mu\in E^k,\, g\cdot\mu=\mu} \theta(u_{g|_\mu}).$$

Using this, and applying the definition of χ_β at the third equality, we calculate

$$\begin{aligned}\Delta_{n+1}(g) &= \tau_{n+1}(u_g) - \theta(u_g)\\ &= \chi_\beta(\tau_n)(u_g) - Z(\beta,\theta)^{-1} \sum_{k=0}^{\infty} e^{-\beta k} \sum_{\mu\in E^k,\, g\cdot\mu=\mu} \theta(u_{g|_\mu})\\ &= Z(\beta,\tau_n)^{-1} \sum_{k=0}^{\infty} e^{-\beta k} \sum_{\mu\in E^k,\, g\cdot\mu=\mu} \tau_n(u_{g|_\mu}) - Z(\beta,\theta)^{-1} \sum_{k=0}^{\infty} e^{-\beta k} \sum_{\mu\in E^k,\, g\cdot\mu=\mu} \theta(u_{g|_\mu}).\end{aligned}$$

Since the sums are absolutely convergent, we can rewrite each $\theta(u_{g|_\mu})$ as $\tau_n(u_{g|_\mu}) - \Delta_n(g|_\mu)$ and rearrange to obtain

$$\begin{aligned}\Delta_{n+1}(g) &= \big(Z(\beta,\tau_n)^{-1} - Z(\beta,\theta)^{-1}\big) \sum_{k=0}^{\infty} e^{-\beta k}\Big(\sum_{\mu\in E^k,\, g\cdot\mu=\mu} \tau_n(u_{g|_\mu})\Big)\\ &\quad + Z(\beta,\theta)^{-1} \sum_{k=0}^{\infty} e^{-\beta k} \sum_{\mu\in E^k,\, g\cdot\mu=\mu} \Delta_n(g|_\mu).\end{aligned} \tag{2.7}$$

Since θ satisfies (2.2), Proposition 2.3(2) combined with the definition of $N(\beta,\theta)$ imply that $Z(\beta,\theta) = \big(1 - e^{-\beta}N(\beta,\theta)\big)^{-1} = \big(1 - e^{-\beta}\rho(A)\big)^{-1}$, and then Lemma 2.4(2) gives $Z(\beta,\theta) = \rho(A_{vN})$. Also, by definition of χ_β, we have $\sum_{k=0}^{\infty} e^{-\beta k} \sum_{\mu\in E^k,\, g\cdot\mu=\mu} \tau_n(u_{g|_\mu}) = Z(\beta,\tau_n)\tau_{n+1}(u_g)$. Making these substitutions in (2.7), we obtain

$$\begin{aligned}\Delta_{n+1}(g) &= \big(Z(\beta,\tau_n)^{-1} - \rho(A_{vN})^{-1}\big) Z(\beta,\tau_n)\tau_{n+1}(u_g)\\ &\quad + \rho(A_{vN})^{-1} \sum_{k=0}^{\infty} e^{-\beta k} \sum_{\mu\in E^k,\, g\cdot\mu=\mu} \Delta_n(g|_\mu).\end{aligned}$$

With $\mathcal{G}_g^k(v)$ and $\mathcal{F}_g^k(v)$ defined as in Lemma 2.8, the preceding expression for $\Delta_{n+1}(g)$ becomes

$$\begin{aligned}\Delta_{n+1}(g) &= \big(Z(\beta,\tau_n)^{-1} - \rho(A_{vN})^{-1}\big) Z(\beta,\tau_n)\tau_{n+1}(u_g)\\ &\quad + \rho(A_{vN})^{-1} \sum_{k=0}^{\infty} e^{-\beta k} \sum_{v\in E^0} \Big(\sum_{\mu\in\mathcal{G}_g^k(v)\setminus\mathcal{F}_g^k(v)} \Delta_n(g|_\mu) + \sum_{\mu\in\mathcal{F}_g^k(v)} \Delta_n(g|_\mu)\Big).\end{aligned} \tag{2.8}$$

The Cauchy–Schwarz inequality implies that for any $h \in \mathcal{G}$,

$$|\tau_{n+1}(u_h)|^2 = |\tau_{n+1}(u_h^* u_{t(h)})|^2 \leq \tau_{n+1}(u_h^* u_h)\tau(u_{t(h)}^* u_{t(h)}) = \tau_{n+1}(u_{d(h)})\tau_{n+1}(u_{t(h)}).$$

Since our fixed g satisfies $d(g) = t(g)$, taking square roots in the preceding estimate gives $|\tau_{n+1}(u_g)| \le \tau_{n+1}(u_{d(g)})$. Applying this combined with the triangle inequality to the right-hand side of (2.8), we obtain

$$\begin{aligned}|\Delta_{n+1}(g)| \le \big|Z(\beta,\tau_n)^{-1} - \rho(A_{vN})^{-1}\big| Z(\beta,\tau_n)\tau_{n+1}(u_{d(g)}) \\ + \rho(A_{vN})^{-1}\sum_{k=0}^{\infty} e^{-\beta k}\sum_{v\in E^0}\Big(\sum_{\mu\in\mathcal{G}_g^k(v)\setminus\mathcal{F}_g^k(v)}\big|\Delta_n(g|_\mu)\big| + \sum_{\mu\in\mathcal{F}_g^k(v)}\big|\Delta_n(g|_\mu)\big|\Big),\end{aligned}$$

which, using that $g|_\mu = v$ for $\mu \in \mathcal{F}_g^k(v)$, becomes

$$\begin{aligned}|\Delta_{n+1}(g)| &\le \big|Z(\beta,\tau_n)^{-1} - \rho(A_{vN})^{-1}\big| Z(\beta,\tau_n)\tau_{n+1}(u_{d(g)}) \\ &\quad + \rho(A_{vN})^{-1}\sum_{k=0}^{\infty} e^{-\beta k}\sum_{v\in E^0}\sum_{\mu\in\mathcal{G}_g^k(v)\setminus\mathcal{F}_g^k(v)}\big|\Delta_n(g|_\mu)\big| \\ &\quad + \rho(A_{vN})^{-1}\sum_{k=0}^{\infty} e^{-\beta k}\sum_{v\in E^0}\sum_{\mu\in\mathcal{F}_g^k(v)}|\Delta_n(v)|.\end{aligned}$$

Since $\big(Z(\beta,\tau_n)^{-1} - \rho(A_{vN})^{-1}\big)Z(\beta,\tau_n) = \rho(A_{vN})^{-1}\big(\rho(A_{vN}) - Z(\beta,\tau_n)\big)$, we obtain

$$\begin{aligned}|\Delta_{n+1}(g)| &\le \rho(A_{vN})^{-1}\big|\rho(A_{vn}) - Z(\beta,\tau_n)\big|\tau_{n+1}(u_{d(g)}) \\ &\quad + \rho(A_{vN})^{-1}\sum_{k=0}^{\infty} e^{-\beta k}\sum_{v\in E^0}\sum_{\mu\in\mathcal{G}_g^k(v)\setminus\mathcal{F}_g^k(v)}\big|\Delta_n(g|_\mu)\big| \\ &\quad + \rho(A_{vN})^{-1}\sum_{\mu\in d(g)E^*} e^{-\beta|\mu|}|\Delta_n(s(\mu))|.\end{aligned}$$

By Theorem 2.7 there are positive constants λ_0, K_1 and K_2 with $\lambda_0 < 1$ such that $|\rho(A_{vN}) - Z(\beta,\tau_n)| < K_1\lambda_0^n$ for all n and $|\Delta_n(v)| = |\tau_n(u_v) - m_v| < K_2\lambda_0^n$ for all $v \in E^0$ and $n \ge 0$. Thus we obtain

$$\begin{aligned}|\Delta_{n+1}(g)| &\le K_1\lambda_0^n\rho(A_{vN})^{-1}\tau_{n+1}(u_{d(g)}) \\ &\quad + \rho(A_{vN})^{-1}\sum_{k=0}^{\infty} e^{-\beta k}\sum_{v\in E^0}\sum_{\mu\in\mathcal{G}_g^k(v)\setminus\mathcal{F}_g^k(v)}\big|\Delta_n(g|_\mu)\big| \\ &\quad + K_2\lambda_0^n\rho(A_{vN})^{-1}\sum_{\mu\in d(g)E^*} e^{-\beta|\mu|}.\end{aligned}$$

Theorem 3.1(a) of [8] shows that $\sum_{\mu\in d(g)E^*} e^{-\beta|\mu|}$ converges, and since the entries of the Perron–Frobenius eigenvector m are strictly positive, $l := \max_v m_v^{-1}$ is finite. So $K := 2l\rho(A_{vN})^{-1}\max\{K_1, K_2\sum_{\mu\in E^*} e^{-\beta|\mu|}\}$ satisfies

$$|\Delta_{n+1}(g)| \le K\lambda_0^n m_{d(g)} + \rho(A_{vN})^{-1}\sum_{k=0}^{\infty} e^{-\beta k}\sum_{v\in E^0}\sum_{\mu\in\mathcal{G}_g^k(v)\setminus\mathcal{F}_g^k(v)}\big|\Delta_n(g|_\mu)\big|. \tag{2.9}$$

Since both λ_0 and the constant α of (2.6) are less than 1, the quantity $\lambda := \max\{\lambda_0, \alpha\}$ is less than 1.

Let $D := l\max_{\mu\in d(g)E^*}\big(|\tau(u_{g|_\mu})| + |\theta(u_{g|_\mu})|\big)$, which is finite because $\mathcal{G} \curvearrowright E$ is finite state. Let $g|_{E^*} := \{g|_\mu : \mu \in E^*\} \subseteq \mathcal{G}$. We will prove by induction that $|\Delta_n(h)| \le$

$(nK+D)\lambda^{n-1}m_{d(h)}$ for all n and for all $h \in g|_{E^*}$. The base case $n=0$ is trivial because each $|\Delta_0(h)| = |\tau(u_h) - \theta(u_h)| \le |\tau(u_h)| + |\theta(u_h)| \le Dl^{-1} \le \lambda^{-1}Dm_{d(h)}$. Now suppose as an inductive hypothesis that $|\Delta_n(h)| \le (nK+D)\lambda^{n-1}m_{d(h)}$ for all $h \in g|_{E^*}$. Fix $h \in g|_{E^*}$. Applying the inductive hypothesis on the right-hand side of (2.9), and then using that $h|_{E^*} \subseteq g|_{E^*}$ and invoking (2.6) gives

$$\begin{aligned}|\Delta_{n+1}(h)| &\le K\lambda_0^n m_{d(h)} + (nK+D)\lambda^{n-1}\rho(A_{vN})^{-1}\sum_{k=0}^{\infty} e^{-\beta k}\sum_{v\in E^0}\sum_{\nu\in\mathcal{G}_h^k(v)\setminus\mathcal{F}_h^k(v)} m_{d(h|_\nu)} \\ &= K\lambda_0^n m_{d(h)} + (nK+D)\lambda^{n-1}\rho(A_{vN})^{-1}\sum_{k=0}^{\infty} e^{-\beta k}\sum_{v\in E^0}|\mathcal{G}_h^k(v)\setminus\mathcal{F}_h^k(v)|m_v \\ &\le K\lambda_0^n m_{d(h)} + (nK+D)\lambda^{n-1}\alpha m_{d(h)},\end{aligned}$$

and since $\lambda_0, \alpha < \lambda$ we deduce that

$$|\Delta_{n+1}(h)| \le ((n+1)K+D)\lambda^n m_{d(h)}.$$

The claim follows by induction, and in particular we have $|\Delta_n(g)| \le (nK+D)\lambda^{n-1}m_{d(g)}$ for all n as claimed. This proves the first statement.

The second statement follows immediately from Lemma 2.2. □

Proof of Theorem 2.1. (1) Let $m = m^E$ be the Perron–Frobenius eigenvector of A_E. For $v \in \mathcal{G}^{(0)} = E^0$, let $c_v := m_v$. Fix $g \in \mathcal{G} \setminus E^0$. By [13, Proposition 8.2], the sequence

$$\Big(\rho(A_E)^{-n}\sum_{v\in E^0}\big|\{\mu \in E^n : g\cdot\mu = \mu, g|_\mu = v\}\big|m_v\Big)_{n=1}^{\infty}$$

converges to some $c_g \in [0, m_{d(g)}]$. By [13, Theorem 8.3], there is a $\mathrm{KMS}_{\log\rho(A_E)}$-state ψ of $\mathcal{T}(\mathcal{G}, E)$ that factors through $\mathcal{O}(\mathcal{G}, E)$. This ψ satisfies

$$\psi(s_\mu u_g s_\nu^*) = \begin{cases}\rho(A_E)^{-|\mu|}c_g & \text{if } \mu = \nu \text{ and } d(g) = t(g) = s(\mu)\\ 0 & \text{otherwise.}\end{cases}$$

In particular, $\theta := \psi|_{C^*(\mathcal{G})}$ belongs to $\mathrm{Tr}(C^*(\mathcal{G}))$.

We claim that θ is a fixed point for χ_β. By the final statement of Theorem 2.9, it suffices to show that θ satisfies (2.2). Proposition 8.1 of [13] shows that $x^\theta = \big(\theta(u_v)\big)_{v\in E^0}$ is equal to m. Using this, we see that

$$\begin{aligned}Z(\beta,\theta) &= \sum_{v\in E^0}\sum_{k=0}^{\infty} e^{-k\beta}\sum_{\mu\in vE^k}\theta(s(\mu)) = \Big\|\sum_{k=0}^{\infty}(e^{-k\beta}A_E^k x)\Big\|_1 \\ &= \Big\|\sum_{k=0}^{\infty}(e^{-k\beta}\rho(A_E)^k)x\Big\|_1 = (1 - e^{-\beta}\rho(A_E))^{-1}.\end{aligned}$$

Hence $N(\beta,\theta) = \rho(A_E)$.

Since $1_{\mathcal{O}(\mathcal{G},E)} = \sum_{v\in E^0} p_v = \sum_{e\in E^1} s_e s_e^*$, we have

$$\begin{aligned}\theta(u_g) = \psi(u_g) &= \sum_{e\in E^1}\psi(u_g s_e s_e^*) = \sum_{e\in E^1}\delta_{d(g),r(e)}\psi(s_{g\cdot e}u_{g|_e}s_e^*) \\ &= \sum_{e\in E^1}\delta_{d(g),r(e)}\delta_{g\cdot e,e}\delta_{d(g|_e),s(e)}\delta_{t(g|_e),s(e)}\rho(A_E)^{-1}\theta(u_{g|_e}) = N(\beta,\theta)^{-1}\sum_{e\in E^1,\, g\cdot e = e}\theta(u_{g|_e}).\end{aligned}$$

Now an easy induction shows that θ satisfies relation (2.2).

It remains to prove that θ is the unique fixed point for χ_β. For this, suppose that θ' is a fixed point for χ_β, so $\theta' = \lim_n \chi_\beta^n(\theta')$. Since θ satisfies (2.2), Theorem 2.9 shows that $\lim_n \chi_\beta^n(\theta') = \theta$. So $\theta' = \theta$.

(2) This follows immediately from Theorem 2.9 because θ satisfies (2.2).

(3) The trace θ of part (1) extends to a $\mathrm{KMS}_{\log\rho(A_E)}$ state of $\mathcal{T}(\mathcal{G}, E)$ by construction. If ϕ is any $\mathrm{KMS}_{\log\rho(A_E)}$-state of $\mathcal{T}(\mathcal{G}, E)$, then it restricts to a $\mathrm{KMS}_{\log\rho(A_E)}$-state of the subalgebra $\mathcal{T}C^*(E)$, so it follows from [8, Theorem 4.3(a)] that ϕ agrees with ψ on $\mathcal{T}C^*(E)$, and in particular $(\phi(u_v))_{v\in E^0}$ is equal to the Perron–Frobenius eigenvector m^E. So [13, Proposition 8.1] shows that ϕ factors through $\mathcal{O}(\mathcal{G}, E)$. By construction, ψ also factors through $\mathcal{O}(\mathcal{G}, E)$. By [13, Theorem 8.3(2)], there is a unique KMS state on $\mathcal{O}(\mathcal{G}, E)$, and we deduce that $\phi = \psi$. In particular, $\phi|_{C^*(\mathcal{G})} = \psi|_{C^*(\mathcal{G})} = \theta$. □

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