UNITIZATIONS AND CROSSED PRODUCTS

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Abstract. These are course notes for an honours course offered at the University of Wollongong in Spring of 2014. They are a modified and updated version of the notes for a similar course delivered in 2012. They are still in rough-draft form, especially the final subsection on Cuntz-Pimsner algebras. So they are likely to contain lots of typo's and possibly more serious errors. Please let me know of any you find. Thanks go to Michael Mampusti for pointing out a number of typos.

1. Unitizations

1.1. **Adjoining a unit.** In this section we investigate how to embed a non-unital C^*-algebra into a unital one. The most straightforward approach is just to adjoin a unit. We will see that this has both advantages and disadvantages.

Definition 1.1. Let A be a complex *-algebra. Let A^+ denote the cartesian product $A \times \mathbb{C}$ endowed with operations $(a, \lambda)(b, \mu) := (ab + \lambda a + \mu b, \lambda\mu)$ and $(a, \lambda)^* := (a^*, \overline{\lambda})$.

Lemma 1.2. *With operations as above, A^+ is a *-algebra with identity element $(0, 1)$. The map $i : a \mapsto (a, 0)$ is an injective *-homomorphism of A into A^+. If A is a normed *-algebra, then $\|(a, \lambda)\|_1 := \|a\| + |\lambda|$ defines a *-algebra norm on A^+ under which i is isometric, and A^+ is complete in this norm if and only if A is complete.*

Proof. Routine calculations check the axioms of a *-algebra. We have $(0, 1)(a, \lambda) = (0a + \lambda 0 + 1a, 1\lambda) = (a, \lambda)$ and similarly $(a, \lambda)(0, 1) = (a, \lambda)$, and so $(0, 1)$ is an identity for A^+. It is clear that i is injective and linear, and elementary calculations show that it preserves involution and multiplication. The direct sum of two normed spaces is always a normed space under the 1-norm, and we have

$$\|(a, \lambda)^*\|_1 = \|(a^*, \overline{\lambda})\|_1 = \|a^*\| + |\overline{\lambda}| = \|a\| + |\lambda| = \|(a, \lambda)\|_1,$$

and

$$\begin{aligned}\|(a, \lambda)(b, \mu)\|_1 &\leq \|a\|\|b\| + |\mu|\|a\| + |\lambda|\|b\| + |\lambda||\mu| \\ &= (\|a\| + |\lambda|)(\|b\| + |\mu|) = \|(a, \lambda)\|_1\|(b, \mu)\|_1.\end{aligned}$$

Hence $\|\cdot\|_1$ is a *-algebra norm. The inclusion $i : A \to A^+$ is isometric from $\|\cdot\|$ to $\|\cdot\|_1$ by definition of the latter. It follows that if A^+ is complete, then A is also. For the converse, suppose that A is complete and that $(a_n, \lambda_n)_{n=1}^\infty$ is a Cauchy sequence in A^+. Then each of $(a_n)_{n=1}^\infty$ and $(\lambda_n)_{n=1}^\infty$ is Cauchy, and so these sequences converge to limits a and λ, and we then have $(a_n, \lambda_n) \to (a, \lambda)$ by a simple $\varepsilon/2$-argument. □

So we are done, right? No. The 1-norm on A^+ does not satisfy the C^*-identity, even when A is a C^*-algebra. To see this, suppose that $a \in A$ is nonzero and self-adjoint. Then

$$\|(a, i)\|_1^2 = (\|a\| + 1)^2 = \|a\|^2 + 2\|a\| + 1,$$

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whereas
$$\|(a,i)^*(a,i)\|_1 = \|(a^*a + ia^* + \bar{i}a, i\bar{i})\|_1 = \|a\|^2 + 1.$$

To describe a C^*-norm on A^+, we regard its elements as operators on A. Recall from the Complex and Functional Analysis course (Lemma 1.5 in Chapter 2 of the 2013 course notes) that the space of all bounded linear operators $B(X)$ on a Banach space X is a Banach algebra.

Lemma 1.3. *Let A be a C^*-algebra without an identity. For each $a \in A$ and $\lambda \in \mathbb{C}$, we have*
$$\sup_{\|b\|=1} \|ab + \lambda b\| = \sup_{\|c\|=1} \|ca + \lambda c\|.$$
Moreover, $\|(a,\lambda)\| := \sup_{\|b\|=1} \|ab + \lambda b\|$ defines a C^*-norm on A^+ with respect to which $i : a \mapsto (a, 0)$ is isometric.

Proof. Since A is a C^*-algebra, for each $c \in A$, submultiplicativity of the norm and C^*-identity (applied to c^*) shows that $\|c\| = \sup_{\|c'\|=1} \|cc'\|$. Hence for fixed $b \in A$ with $\|b\| = 1$,
$$\|ab + \lambda b\| = \sup_{\|c\|=1} \|c(ab + \lambda b)\| = \sup_{\|c\|=1} \|(ca + \lambda c)b\| \leq \sup_{\|c\|=1} \|(ca + \lambda c)\|.$$
Taking the supremum over b gives $\sup_{\|b\|=1} \|ab+\lambda b\| \leq \sup_{\|c\|=1} \|ca+\lambda c\|$, and the reverse inequality follows by symmetry.

For $(a,\lambda) \in A^+$, define $L := L_{(a,\lambda)} \in B(A)$ by $L(b) := ab + \lambda b$. This L is a bounded linear operator on A. If $L = 0$ then $ab = -\lambda b$ for all $b \in A$, forcing either $a = \lambda = 0$ or $(-\lambda^{-1}a)b = b$ for all b. The latter would imply that $(-\lambda^{-1}a)$ was an identity for A, which does not exist by assumption. So the map $(a,\lambda) \mapsto L_{(a,\lambda)}$ is an injection of A^+ into $B(A)$. This shows that $\|\cdot\|$ is positive definite on A^+. It is routine to check that $(a,\lambda) \mapsto L_{(a,\lambda)}$ is a homomorphism. Since the norm on $B(A)$ is an algebra norm, it follows that $\|\cdot\|$ is an algebra norm on A^+. We have
$$\|(a,\lambda)^*\| = \sup_{\|b\|=1} \|a^*b + \overline{\lambda}b\| = \sup_{\|b\|=1} \|(b^*a + \lambda b^*)^*\| = \sup_{\|c\|=1} \|ca + \lambda c\| = \|(a,\lambda)\|.$$
In particular, $\|(a,\lambda)^*(a,\lambda)\| \leq \|(a,\lambda)^*\|\|(a,\lambda)\| = \|(a,\lambda)\|^2$, and so to check the C^*-identity, we just need the reverse inequality. We calculate:
$$\begin{aligned}
\|(a,\lambda)\|^2 &= \sup_{\|b\|=1} \|ab + \lambda b\|^2 \\
&= \sup_{\|b\|=1} \|(ab + \lambda b)^*(ab + \lambda b)\| \\
&= \sup_{\|b\|=1} \|(b^*a^* + \overline{\lambda}b^*)(ab + \lambda b)\| \\
&= \sup_{\|b\|=1} \|b^* L_{(a,\lambda)^*(a,\lambda)}(b)\| \\
&\leq \|L_{(a,\lambda)^*(a,\lambda)}\| \\
&= \|(a,\lambda)^*(a,\lambda)\|.
\end{aligned}$$
It remains only to check that i is isometric and that A^+ is complete in $\|\cdot\|$. We have $\|(a,0)\| = \|L_{(a,0)}\| = \sup_{\|b\|=1} \|ab\| = \|a\|$, and so i is isometric.

To see that A^+ is complete, observe that its completion $\overline{A}$ is a C^*-algebra with an identity. Since i is isometric, $i(A)$ is closed in A^+, and since $(a,\lambda)i(b) = (ab + \lambda b, 0) =$

$i(ab + \lambda b)$, it is a closed 2-sided ideal of $\overline{A}$. Since A does not have an identity, $(0, 1)$ does not belong to $i(A)$, and so $q_{i(A)}((0, 1)) \neq 0$. Hence $z \mapsto (0, z) + i(A)$ is an injection of $\mathbb{C}$ onto $\overline{A}/i(A)$. Fix a Cauchy sequence (a_n, λ_n) in A^+. Then $(a_n, \lambda_n) \to m$ for some $m \in \overline{A}$. Since $q_{i(A)}$ is a C^*-homomorphism between unital C^*-algebras, it is norm-decreasing, and so $(q_{i(A)}(a_n, \lambda_n))$ is Cauchy. Thus (λ_n) is Cauchy in $\mathbb{C}$ so converges to some λ. But now $q_{i(A)}(m) = \lambda q_{i(A)}(0, 1)$, and so $m - (0, \lambda) \in \ker(q_{i(A)}) = i(A)$: say $m - (0, \lambda) = (a, \mu)$. Then $m = (a, \lambda + \mu) \in A^+$, and so A^+ is complete. □

Corollary 1.4. *Let A be a C^*-algebra. There is a unique norm on A^+ under which it is a C^*-algebra. The inclusion $i : A \to A^+$ is isometric, and $i(A)$ is an ideal of A^+.*

Proof. Existence of a norm under which A^+ is a C^*-algebra will imply uniqueness because of the uniqueness of norm result for unital C^*-algebras. When A has no unit, the result follows from Lemma 1.3. If A already has a unit, then routine calculations show that the map $\phi : (a, \lambda) \mapsto (a + \lambda 1_A, \lambda)$ is a *-isomorphism of A^+ onto the direct sum C^*-algebra $A \oplus \mathbb{C}$. This is a C^*-algebra with norm $\|(a, \lambda)\| = \max\{\|a\|, |\lambda|\}$. Hence $\|(a, \lambda)\| := \max\{\|a + \lambda 1_A\|, |\lambda|\}$ is a C^*-norm on A^+.

For $a \in A$ and $(b, \lambda) \in A^+$, we have $i(a)(b, \lambda) = (a, 0)(b, \lambda) = (ab + \lambda a, 0) = i(ab + \lambda a)$ and similarly $(b, \lambda)i(a) = i(ba + \lambda a)$, and so $i(A)$ is an ideal in A. □

We want to deduce from our construction of A^+ that every C^*-algebra has a faithful representation on Hilbert space. For this, we first wish to know that each homomorphism of a C^*-algebra A extends to a homomorphism of A^+.

Corollary 1.5.
(1) *Let A be a C^*-algebra, let B be a C^*-algebra with 1 and let $\phi : A \to B$ be a C^*-homomorphism. Then there is a unique unital homomorphism $\phi^+ : A^+ \to B$ such that $\phi^+(a, 0) = \phi(a)$. If A does not have an identity then ϕ^+ is injective if and only if ϕ is.*
(2) *Let $\phi : A \to B$ be a C^*-homomorphism. Then there is a unique unital homomorphism $\phi^+ : A^+ \to B^+$ such that $\phi^+(a, 0) = (\phi(a), 0)$ for all $a \in A$.*

Proof. (1) For the first statement, define $\phi^+(a, \lambda) = \phi(a) + \lambda 1_B$. It is straightforward to check that this is a *-homomorphism, and then automatic continuity kicks in. This is clearly the only homomorphism compatible with $\phi^+(a, 0) = \phi(a)$ and $\phi^+(0, 1) = 1_B$. If ϕ is not injective, then nor is ϕ^+. For the final statement, we suppose that ϕ is injective and ϕ^+ is not, and show that A has a unit. We must have $\phi^+(a, \lambda) = 0$ for some nonzero (a, λ), and since ϕ is injective, we have $\lambda \neq 0$. Hence $\phi(-\lambda^{-1}a) - 1_B = \phi^+(-\lambda^{-1}a, -1) = -\lambda^{-1}\phi^+(a, \lambda) = 0$. Hence

$$\phi((-\lambda^{-1}a)b) = \phi(-\lambda^{-1}a)\phi(b) = 1_B\phi(b) = \phi(b)$$

for all $b \in A$. Since ϕ is injective, this forces $(-\lambda^{-1}a)b = b$ for all b, and hence $(-\lambda^{-1}a)$ is an identity for A.

(2) Apply the first assertion to the homomorphism $a \mapsto (\phi(a), 0)$ from A to B^+. □

The preceding corollary says that if A has no unit, then A^+ is minimal in the sense that any unital C^*-algebra containing a copy of A contains a copy of A^+. For this reason, A^+ is sometimes called the "minimal unitization" of A when A is nonunital. One must be a little careful: if A contains a unit, then A^+ is strictly larger than A, and in this case, A itself is a strictly smaller unital C^*-algebra containing a copy of A. To sidestep this subtlety, it is common to write $\widetilde{A}$ for the minimal unitization of A, which is defined to be equal to A^+ if A is nonunital, and to A if A is unital.

Corollary 1.6. *Let A be a C^*-algebra. Then A has a faithful nondegenerate representation on Hilbert space.*

Proof. If A is unital, this is the Gelfand-Naimark theorem for unital C^*-algebras. If not, then we first apply the Gelfand-Naimark theorem to obtain a faithful nondegenerate representation π of A^+ on a Hilbert space $\mathcal{H}$. Now $\pi \circ i$ is a faithful representation of A. The set $\mathcal{H}' := \pi(i(A))\mathcal{H} = \{\pi((a,0))h : a \in A, h \in \mathcal{H}\}$ is an invariant subspace, and $\pi(i(a)) = 0$ if and only if $\pi(i(a))|_{\mathcal{H}'} = 0$. Hence $\pi' := \pi(i(\cdot))|_{\mathcal{H}'}$ is a faithful nondegenerate representation as claimed.

□

Recall that a *state* of a C^*-algebra is a positive linear functional of norm 1. A linear functional f on a unital C^*-algebra is a state if and only if $f(1) = 1$.

Corollary 1.7. *Let A be a C^*-algebra. For $a \in A$, the following are equivalent:*

(1) *$a = b^*b$ for some $b \in A$;*
(2) *$(a, 0) = b^*b$ for some $b \in A^+$;*
(3) *a is positive in A^+ in the sense that $\sigma_{A^+}((a, 0)) \subseteq [0, \infty)$;*
(4) *$f(a) \geq 0$ for every state f of A;*

Proof. The implication (1) $\implies$ (2) is trivial. For the reverse, observe that if $(a, 0) = b^*b$ for some $b \in A^+$, we may write $b = (c, \lambda)$ and observe that $(a, 0) = b^*b = (c^*c + \lambda c^* + \overline{\lambda}c, |\lambda|^2)$ forces $\lambda = 0$ and hence $a = c^*c$. The equivalence of (2) and (3) was established in [7], and (3) $\implies$ (4) is part of the definition of a state. It remains to establish that (4) implies (3). For this, and fix a faithful nondegenerate representation π of A^+ on $\mathcal{H}$. Identify A with $A \times \{0\} \subseteq A^+$. Each $h \in \mathcal{H}$ with $\|h\| = 1$ determines a vector state $b \mapsto (\pi(b)h|h)$ of A^+, and so $(\pi(a)h|h) \geq 0$ for every $h \in \mathcal{H}$. Hence $\pi(a)$ is a positive operator on $\mathcal{H}$. By an exercise from [7], $\pi(a)$ is then a positive element of $\mathcal{B}(\mathcal{H})$. That is, $\sigma_{\mathcal{B}(\mathcal{H})}(\pi(a)) \subseteq [0, \infty)$. Since π is nondegenerate, $1_{\pi(A^+)} = 1_{\mathcal{B}(\mathcal{H})}$, and so spectral permanence implies that $\sigma_{\pi(A^+)}(\pi(a)) \subseteq [0, \infty)$, and the result follows since π is injective. □

We take the equivalent conditions of the preceding corollary as the definition of positivity of elements of nonunital C^*-algebras.

Example 1.8. Let X be a locally compact Hausdorff space. That is, X is a Hausdorff space, and for every $x \in X$ there is an open set $U \subseteq X$ such that $x \in U$ and $\overline{U}$ is compact.

We say that a continuous function $f : X \to \mathbb{C}$ *vanishes at infinity* if, for every $\varepsilon > 0$ there exists a compact set $K \subseteq X$ such that $|f(x)| < \varepsilon$ for all $x \notin K$. Define

$$C_0(X) := \{f \in C(X) : f \text{ vanishes at infinity}\}.$$

We claim that $C_0(X)$ is a commutative C^*-algebra under the supremum norm $\|f\|_\infty = \sup\{|f(x)| : x \in X\}$ and pointwise operations. It is routine to check that $C_0(X)$ is closed under all the algebraic operations. To see that $\|\cdot\|_\infty$ defines a norm, we must first check that it is finite for every f. For this fix $f \in C_0(X)$. There is a compact K such that $|f(x)| < 1$ for $x \notin K$. Since f is continuous and K is compact, f is bounded on K, say

$f(x) \leq b$ for all $x \in K$. We then have $\|f\|_\infty \leq \max\{b, 1\}$. To see that $C_0(X)$ is complete, observe that if $(f_n)_{n=1}^\infty$ is a Cauchy sequence in $C_0(X)$, then $(f_n(x))_{n=1}^\infty$ is Cauchy in $\mathbb{C}$ and hence converges to some $f(x)$ for each $x \in X$. We claim that $f_n \to f$ in $\|\cdot\|_\infty$ and that $f \in C_0(X)$. To see that $f_n \to f$, fix $\varepsilon > 0$. Choose N such that $\|f_m - f_n\| < \varepsilon/2$ whenever $m, n \geq N$. For each $x \in X$ there exists $n_x \geq N$ such that $|f_n(x) - f(x)| < \varepsilon/2$, and hence

$$|f_m(x) - f(x)| \leq |f_m(x) - f_{n_x}(x)| + |f_{n_x}(x) - f(x)| \leq \varepsilon$$

for all $m \geq N$. Hence $\|f_m - f\|_\infty < \varepsilon$ for all $m \geq N$. Now we show that $f \in C_0(X)$. For every compact set K, the restriction $f|_K$ is the uniform limit of the continuous functions $f_n|_K$ and hence continuous. Since X is locally compact, this implies that f is continuous at every point in X. We show that f vanishes at infinity. Fix $\varepsilon > 0$. Choose m such that $\|f_m - f\|_\infty < \varepsilon/2$, and choose a compact set K such that $|f_m(x)| < \varepsilon/2$ for $x \notin K$. Then for $x \notin K$, we have $|f(x)| \leq |f(x) - f_m(x)| + |f_m(x)| \leq \varepsilon$. The C^*-identity holds for $C_0(X)$ because it holds in $\mathbb{C}$. Hence $C_0(X)$ is a C^*-algebra as claimed.

We claim that if X is not compact, then $C_0(X)$ is not unital. To see this, we establish the contrapositive. Suppose that $f \in C_0(X)$ satisfies $fg = g$ for all $g \in C_0(X)$, then $g(x) = (fg)(x) = f(x)g(x)$ for all x, and so $f(x) = 1$ whenever there exists $g \in C_0(X)$ with $g(x) \neq 0$. The Tietze Extension Theorem implies that for every $x \in X$ there exists $g \in C_0(X)$ with $g(x) = 1$. So we have $f(x) = 1$ for all x. Since $f \in C_0(X)$, the set $\{x : |f(x)| \geq 1/2\}$ is compact, and this shows that X is compact.

We claim next that the unitisation $C_0(X)^+$ of $C_0(X)$ can be identified with the collection $C_{\mathrm{cnv}}(X)$ of functions f which "converge at infinity" in the sense that there exists $\lim_{x\to\infty} f(x) \in \mathbb{C}$ such that for every $\varepsilon > 0$ there is a compact set K such that $|f(x) - \lim_{x\to\infty} f(x)| < \varepsilon$ for all $x \in X \setminus K$. To see this, observe that $1 \in C_{\mathrm{cnv}}(X)$, and so the latter is unital. Moreover, $C_0(X)$ includes as the subalgebra $\{f \in C_{\mathrm{cnv}}(X) : \lim_{x\to\infty} f(x) = 0\}$ via the inclusion ι such that $\iota(f)(x) = x$ for $x \in X$ and $\iota(f)(\infty) = 0$. Corollary 1.5 implies that there is a unique unital map $i^+ : C_0(X)^+ \to C_{\mathrm{cnv}}(X)$ extending the inclusion map, and that this map is injective. To see that it is surjective, observe that for $f \in C_{\mathrm{cnv}}(X)$, we have $f_0 := f - (\lim_{x\to\infty} f(x))1 \in C_0(X)$, so that $f = i^+(f_0, \lim_{x\to\infty} f(x))$.

Now consider the space $X \cup \{\infty\}$ where the point ∞ is some element not already belonging to X. The collection

$$B := \{U : U \subset X \text{ is open}\} \cup \{(X \setminus K) \cup \{\infty\} : K \subseteq X \text{ is compact}\}$$

is a base for a topology on $X \cup \{\infty\}$. This topology is Hausdorff: each $x, y \in X$ can be separated by open sets in X which are also open in $X \cup \{\infty\}$, and for each $x \in X$ we can fix a compact neighbourhood K of x, and so there is an open U with $x \in U \subseteq K$, and then U and $(X \setminus K) \cup \{\infty\}$ are disjoint open neighbourhoods of x and ∞ respectively. The space is also compact: If $\mathcal{U}$ is a cover of X by open sets, then there exists $U_0 \in \mathcal{U}$ such that $\infty \in U$. The complement K of U_0 is a compact subset of X by definition of B. The collection $\{V \setminus \{\infty\} : V \in \mathcal{U}\}$ is an open cover of K, and so has a finite subcover $\{U_1 \setminus \{\infty\}, \ldots U_n \setminus \{\infty\}\}$. The collection $\{U_0, U_1, \ldots, U_n\}$ is then a finite subcover of $\mathcal{U}$ for X.

We claim that there is a unique isomorphism $\phi : C_0(X)^+ \cong C(X \cup \{\infty\})$ such that $\phi((f, \lambda))(x) = f(x) + \lambda$ for $x \in X$ and $\phi((0, 1)) = 1_{X\cup\{\infty\}}$. We prove that there is an isomorphism $\psi : C_{\mathrm{cnv}}(X) \to C(X \cup \{\infty\})$ satisfying $\psi(f)(x) = f(x)$ for $x \in X$ and $\psi(f)(\infty) = \lim_{x\to\infty} f(x)$. The definition of $C_{\mathrm{cnv}}(X)$ guarantees that this formula determines a map $\psi : C_{\text{cnv}}(X) \to C(X \cup \{\infty\})$. The restriction map $\text{res} : f \mapsto f|_X$ is a homomorphism from $C(X \cup \{\infty\})$ to $C_{\text{cnv}}(X)$. It clearly satisfies $\text{res}(\psi(f)) = f$ for all f. The reverse composition $\psi(\text{res}(f))$ agrees with f on X by definition; and it agrees with f at ∞ because f is continuous at ∞. So res and ψ are mutually inverse and hence both isomorphisms. Now composing ψ with the isomorphism $i^+ : C_0(X)^+ \to C_{\text{cnv}}(X)$ gives the desired isomorphism $\phi : C_0(X)^+ \cong C(X \cup \{\infty\})$.

The Gelfand-Naimark theorem [7, Theorem 6.1] implies that the spectrum Δ of $C_0(X)^+$ is, up to homeomorphism, the unique compact Hausdorff space such that $C_0(X)^+ \cong C(\Delta)$. The preceding example shows that Δ is homeomorphic to $X \cup \{\infty\}$: the maximal ideal space of $C(X \cup \{\infty\})$ consists of the evaluation functionals $\{\epsilon_y : y \in X \cup \{\infty\}\}$ and these specify maximal ideals δ_y of $C_0(X)^+$ by $\delta_y(a) = \epsilon_y(\phi^{-1}(a))$. That is, $\delta_x(f, \lambda) = f(x) + \lambda$ for $x \in X$, and $\delta_\infty(f, \lambda) = \lambda$.

1.2. **Hilbert modules.** This section of the course is based on [8, Chapter 2].

Definition 1.9. Let A be a C^*-algebra. A right inner-product A-module is a right A-module X with a map $\langle \cdot, \cdot \rangle_A : X \times X \to A$ such that for all $x, y, z \in X$, $a \in A$ and $\lambda, \mu \in \mathbb{C}$, we have

(1) $\langle x, \lambda y + \mu z \rangle_A = \lambda \langle x, y \rangle_A + \mu \langle x, z \rangle_A$;
(2) $\langle x, y \cdot a \rangle_A = \langle x, y \rangle_A a$;
(3) $\langle x, y \rangle_A = \langle y, x \rangle_A^*$;
(4) $\langle x, x \rangle_A \geq 0$ as an element of A; and
(5) $\langle x, x \rangle_A = 0$ implies $x = 0$.

Lemma 1.10. *Let A be a C^*-algebra and let X be a right inner-product A-module. Then for all $x, y, z \in X$, $a \in A$ and $\lambda, \mu \in \mathbb{C}$ we have*

$$\langle \lambda x + \mu y, z \rangle_A = \overline{\lambda} \langle x, z \rangle_A + \overline{\mu} \langle y, z \rangle_A, \qquad \text{and}$$
$$\langle x \cdot a, y \rangle_A = a^* \langle x, y \rangle_A.$$

Moreover, $I_X := \operatorname{span}\{\langle x, y \rangle_A : x, y \in X\}$ is a two-sided *-ideal in A.

Proof. We calculate

$$\langle \lambda x + \mu y, z \rangle_A = \langle z, \lambda x + \mu y \rangle_A^* = \overline{\lambda} \langle z, x \rangle_A^* + \overline{\mu} \langle z, y \rangle_A^* = \overline{\lambda} \langle x, z \rangle_A + \overline{\mu} \langle y, z \rangle_A,$$

and

$$\langle x \cdot a, y \rangle_A = \big(\langle y, x \rangle_A a\big)^* = a^* \langle x, y \rangle_A.$$

The set $\operatorname{span}\{\langle x, y \rangle_A : x, y \in X\}$ is certainly a *-closed subspace of A, and for $x, y \in X$ and $a \in A$, we have

$$a \langle x, y \rangle_A = \langle x \cdot a^*, y \rangle_A \quad \text{and} \quad \langle x, y \rangle_A a = \langle x, y \cdot a \rangle_A.$$

Hence I_X is an ideal. □

Example 1.11. Inner-product $\mathbb{C}$-modules are the usual inner-product spaces over $\mathbb{C}$ under the "physicist's convention" that inner-products are conjugate-linear in the *first* variable. In these notes, we shall use angle-brackets $\langle \cdot, \cdot \rangle$ for such inner-products, and reserve $(\cdot \mid \cdot)$ for $\mathbb{C}$-valued inner-products which are conjugate-linear in the second variable.

Example 1.12. Let A be a C^*-algebra. Define a right-action of A on $A_A := A$ by right multiplication: $x \cdot b := xb$ for $x \in A_A$ and $b \in A$. Then the map $\langle x, y\rangle_A := x^*y$ determines an A-valued inner-product on A_A: the first four axioms are straightforward, and the last one follows from the C^*-identity:

$$\langle x, x\rangle_A = 0 \implies x^*x = 0 \implies \|x\|^2 = 0 \implies x = 0.$$

In both of these examples, the formula $\|x\| := \|\langle x, x\rangle_A\|^{1/2}$ determines a norm on X. It turns out that this always happens, but to prove it, we need a version of the Cauchy-Schwarz inequality

Lemma 1.13 (The Cauchy-Schwarz inequality)**.** *Let A be a C^*-algebra and let X be a right inner-product A-module. For $x, y \in X$ we have*

$$\langle x, y\rangle_A^*\langle x, y\rangle_A \le \|\langle x, x\rangle_A\|\langle y, y\rangle_A$$

as elements of the C^*-algebra A.

To prove the lemma, we need the following standard result.

Lemma 1.14. *Let A be a C^*-algebra. Suppose that $c \in A$ is positive. Then for each $b \in A$, we have $b^*cb \le \|c\|b^*b$ as elements of A.*

Proof. Express $c = d^*d$. Then $b^*cb = (db)^*db$ is positive. Choose a faithful representation π of A on a Hilbert space $\mathcal{H}$. Then for $h \in \mathcal{H}$, we have

$$\begin{aligned}(\pi(b^*cb)h \mid h) &= (\pi(d)(\pi(b)h) \mid \pi(d)(\pi(b)h))\\ &= \|\pi(d)(\pi(b)h)\|^2 \le \|d\|^2(\pi(b)h, \pi(b)h) = \|c\|(\pi(b^*bh|h)).\end{aligned}$$

Hence $\pi(\|c\|b^*b - b^*cb)$ is a positive operator on $\mathcal{H}$. It follows that $\|c\|b^*b - b^*cb$ is positive also. □

The following elegant proof comes from [4].

Proof of Lemma 1.13. If $x = 0$ then the result is trivial. Suppose that $x \neq 0$. Let $N := \|\langle x, x\rangle_A\|^{1/2}$, and let $z := N^{-1}x$. Then $\|\langle z, z\rangle_A\| = 1$, and

$$\begin{aligned}0 &\le \langle z \cdot \langle z, y\rangle_A - y, z \cdot \langle z, y\rangle_A - y\rangle_A\\ &= \langle z \cdot \langle z, y\rangle_A, z \cdot \langle z, y\rangle_A\rangle_A - \langle z \cdot \langle z, y\rangle_A, y\rangle_A - \langle y, z \cdot \langle z, y\rangle_A\rangle_A + \langle y, y\rangle_A\\ &= \langle z, y\rangle_A^*\langle z, z\rangle_A\langle z, y\rangle_A - 2\langle z, y\rangle_A^*\langle z, y\rangle_A + \langle y, y\rangle_A.\end{aligned}$$

Applying Lemma 1.14 and using that $\|\langle z, z\rangle_A\| = 1$, we obtain $\langle y, y\rangle_A - \langle z, y\rangle_A^*\langle z, y\rangle_A \ge 0$. Hence

$$\langle x, y\rangle_A^*\langle x, y\rangle_A = N^2\langle z, y\rangle_A^*\langle z, y\rangle_A \le N^2\langle y, y\rangle_A = \|\langle x, x\rangle_A\|\langle y, y\rangle_A.$$

□

Corollary 1.15. *Let A be a C^*-algebra and let X be a right inner-product A-module. Then $\|x\| := \|\langle x, x\rangle_A\|^{1/2}$ satisfies $\|\langle x, y\rangle_A\| \le \|x\|\|y\|$ for all x, y. The map $\|\cdot\|$ is a norm on X such that $\|x \cdot a\| \le \|x\|\|a\|$ for all $a \in A$.*

Proof. The first assertion is obtained by taking norms on both sides of the Cauchy-Schwarz inequality, applying the C^*-identity on the left, and then taking square roots.

It is straightforward that $\|\lambda x\| = |\lambda|\|x\|$ for $\lambda \in \mathbb{C}$ and $x \in X$. Condition 4 of Definition 1.9 implies that $\|x\| \geq 0$ for $x \in X$, and condition 5 implies that $\|x\| = 0$ only when $x = 0$. For the triangle inequality, we compute

$$\|x+y\|^2 = \|\langle x+y, x+y\rangle_A\| \leq \|\langle x,x\rangle_A\| + \|\langle x,y\rangle_A\| + \|\langle x,y\rangle_A^*\| + \|\langle y,y\rangle_A\|.$$

Applying the first assertion to the middle two terms, we obtain

$$\|x+y\|^2 \leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 = (\|x\| + \|y\|)^2.$$

For the final inequality, we just observe that

$$\|x \cdot a\|^2 = \|\langle x\cdot a, x \cdot a\rangle_A\| = \|a^*\langle x,x\rangle_A a\| \leq \|a^*\|\|x\|^2\|a\| = \|x\|^2\|a\|^2. \qquad \square$$

Corollary 1.16. *Let A be a C^*-algebra and let X be a right inner-product A-module. For each $x \in X$ we have $\|x\| = \sup\{\|\langle x,y\rangle_A\| : y \in X, \|y\| \leq 1\}$, and in particular $x = z$ in X if and only if $\langle x,y\rangle_A = \langle z,y\rangle_A$ for every $y \in X$.*

Proof. The formula for $\|x\|$ is trivial if $x = 0$. Suppose $x \neq 0$. The Cauchy-Schwarz inequality implies that $\|x\| \leq \|\langle x,y\rangle_A\|$ whenever $\|y\| \leq 1$. Thus $\|x\| \leq \sup\{\|\langle x,y\rangle_A\| : y \in X, \|y\| \leq 1\}$. For the reverse inequality, observe that

$$\sup\{\|\langle x,y\rangle_A\| : y \in X, \|y\| \leq 1\} \geq \|\langle x, \|x\|^{-1}x\rangle_A\| = \|x\|.$$

The "only if" implication in the second statement is trivial. For the "if" direction observe that if $\langle x,y\rangle_A = \langle z,y\rangle_A$ for every $y \in X$, then $\|\langle x-z,y\rangle_A\| = 0$ for all $y \in X$, and then the first statement gives $\|x-z\| = 0$ and hence $x = z$. $\square$

Definition 1.17. Let A be a C^*-algebra. A *Hilbert A-module* is a right inner-product A-module X which is complete in the norm $\|x\| = \|\langle x,x\rangle_A\|^{1/2}$. We say that it is *full* if $1_X = \operatorname{span}\{\langle x,y\rangle_A : x,y \in X\}$ is dense in A.

Example 1.18. The Hilbert $\mathbb{C}$-modules are the Hilbert spaces with inner products conjugate linear in the first variable. Every nonzero Hilbert $\mathbb{C}$ module is trivially full.

Example 1.19. If A is a C^*-algebra, then A_A is a Hilbert A-module with $x \cdot b := xb$ and $\langle x,y\rangle_A = x^*y$. The C^*-identity ensures that the norm coming from the inner product coincides with the C^*-norm on A. To see that A is full, fix $a \in A$ and apply [7, Lemma 8.7] to the ideal $A \subseteq \widetilde{A}$ to obtain a sequence of self-adjoint elements e_n of A such that $e_n a \to a$ and $ae_n \to a$. We then have $a = \lim_{n\to\infty} e_n a = \lim_{n\to\infty}\langle e_n, a\rangle_A$.

Example 1.20. Let I be an ideal of A. Then I is a Hilbert A-module with right action $x \cdot b = xb$ and inner product $\langle x,y\rangle_A := x^*y$. This Hilbert module is not full unless $I = A$: we have $\overline{\operatorname{span}}\{\langle x,y\rangle_A : x,y \in I\} = I$.

1.3. Adjointable operators.

Our real interest in Hilbert modules is in the operators on them. For Hilbert spaces, we study bounded linear operators and prove that they all have adjoints. For Hilbert modules, this does not work: there exist bounded linear operators which do not have an adjoint with respect to the A-valued inner product. So we focus on the operators which do have an adjoint, and prove that they are necessarily bounded and linear.

Definition 1.21. Let A be a C^*-algebra, and let X be a Hilbert A-module. We say that $T : X \to X$ is *adjointable* if there exists $S : X \to X$ such that $\langle Tx, y\rangle_A = \langle x, Sy\rangle_A$ for all $x, y \in X$. We call such an S an adjoint for T.

Lemma 1.22. *Let X be a Hilbert A-module. Then each adjointable operator T on X has a unique adjoint T^*, which is itself adjointable with $T^{**} = T$. Each adjointable operator T on X is bounded, linear and A-linear, and we have $\|T\| = \|T^*\|$. If S, T are adjointable operators on X, then ST is adjointable with $(ST)^* = T^*S^*$.*

Proof. Let R and S be adjoints for T. Fix $x, y \in X$. We have

$$\langle Rx - Sx, y\rangle_A = \langle y, Rx\rangle_A^* - \langle y, Sx\rangle_A^* = \langle Ty, x\rangle_A^* - \langle Ty, x\rangle_A^* = 0.$$

Allowing y to vary, we deduce from the final statement of Corollary 1.16 that $Rx = Sx$. Allowing x to vary also, we obtain $R = S$. Thus T has a unique adjoint T^*.

We have $\langle T^*x, y\rangle_A = \langle y, T^*x\rangle_A^* = \langle Ty, x\rangle_A^* = \langle x, Ty\rangle_A$ for all $x, y \in X$, and hence T^* is adjointable with $T^{**} = T$.

To see that T is linear, we fix $x, y, z \in X$ and $\lambda \in \mathbb{C}$ and calculate

$$\langle T(\lambda x + y), z\rangle_A = \langle \lambda x + y, T^*z\rangle_A = \overline{\lambda}\langle x, T^*z\rangle_A + \langle y, T^*z\rangle_A = \langle \lambda Tx + Ty, z\rangle_A.$$

Allowing z to vary and applying Corollary 1.16 shows that $T(\lambda x + y) = \lambda Tx + Ty$. To see that T is A-linear, fix $x, y \in X$ and $a \in A$, and calculate

$$\langle T(x \cdot a), y\rangle_A = \langle x \cdot a, T^*y\rangle_A = a^*\langle x, T^*y\rangle_A = \langle (Tx) \cdot a, y\rangle_A;$$

allowing y to vary and applying Corollary 1.16 gives $T(x \cdot a) = (Tx) \cdot a$.

To see that T is bounded, we invoke the closed-graph theorem[1]. Suppose that $x_n \to x$ and $Tx_n \to z$ in X. For $y \in X$, we have

$$\langle z, y\rangle_A = \lim\langle Tx_n, y\rangle_A = \lim\langle x_n, T^*y\rangle_A = \langle x, T^*y\rangle_A = \langle Tx, y\rangle_A.$$

Hence $Tx = z$ and we deduce that the range of T is closed. Hence T has a closed graph and is therefore bounded.

We have

$$\begin{aligned}\|T\| &= \sup_{\|x\|\le 1} \|Tx\| = \sup_{\|x\|,\|y\|\le 1} \|\langle Tx, y\rangle_A\| \\ &= \sup_{\|x\|,\|y\|\le 1} \|\langle T^*y, x\rangle_A^*\| = \sup_{\|x\|,\|y\|\le 1} \|\langle T^*y, x\rangle_A\| = \sup_{\|y\|\le 1} \|T^*y\| = \|T^*\|,\end{aligned}$$

so that $T \mapsto T^*$ is isometric as claimed.

For adjointable S, T and for $x, y \in X$, we have

$$\langle STx, y\rangle_A = \langle S(Tx), y\rangle_A = \langle Tx, S^*y\rangle_A = \langle x, T^*S^*y\rangle_A,$$

so ST is adjointable with $(ST)^* = T^*S^*$ as claimed. □

Notation 1.23. If X is a Hilbert A-module, then we write $\mathcal{L}(X)$ for the collection of all adjointable operators on X.

Lemma 1.22 implies that $\mathcal{L}(X)$ is a subalgebra of the Banach algebra $\mathcal{B}(X)$ of all bounded linear operators on X, and that $T \mapsto T^*$ is an isometric involution on $\mathcal{L}(X)$.

---

[1]Any continuous map T from a topological space X to a Hausdorff space Y has the property that the graph $\{(x, T(x)) : x \in X\}$ of T is a closed subset of the product space $X \times Y$. The closed graph theorem in functional analysis says that if $T : X \to Y$ is a linear operator between Banach spaces and the graph $\{(x, Tx) : x \in X\}$ is a closed subset of the product space $X \times Y$, then T is continuous. Since a linear map between normed spaces is continuous if and only if it is bounded, it follows that a linear map T between Banach spaces is bounded if and only if it has closed graph.

Proposition 1.24. *If X is a Hilbert A-module, then $\mathcal{L}(X)$ is a unital C^*-algebra under composition of operators, pointwise addition and the involution $T \mapsto T^*$ of Lemma 1.22. The identity element is the identity operator $1_X : x \mapsto x$.*

Proof. We saw above that involution is isometric, and it follows that if T_n is a sequence of adjointable operators with $T_n \to T$, then the T_n^* form a Cauchy sequence. Since $\mathcal{B}(X)$ is a Banach algebra, then T_n^* converge to some S, and continuity implies that S is an adjoint for T. Thus $\mathcal{L}(X)$ is a closed subset of $\mathcal{B}(X)$ and hence complete in norm.

We verify the C^*-identity. Using Corollary 1.16, we calculate

$$\begin{aligned}\|T\|^2 = \|T^*\|\|T\| \geq \|T^*T\| &= \sup_{\|x\|,\|y\|=1} \|\langle T^*Tx, y\rangle_A\| \\ &\geq \sup_{\|x\|=1} \|\langle Tx, Tx\rangle_A\| = \|T\|^2.\end{aligned}$$

Hence we have equality throughout, which gives the C^*-identity.

Since $\langle 1_X x, y\rangle_A = \langle x, y\rangle_A = \langle x, 1_X y\rangle_A$, the identity operator is adjointable, and is clearly an identity element for $\mathcal{L}(X)$. □

Corollary 1.25. *Let A be a C^*-algebra and let X be a right Hilbert A-module. For $T \in \mathcal{L}(X)$ and $x \in X$, we have*

$$\langle Tx, Tx\rangle_A \leq \|T\|^2 \langle x, x\rangle_A$$

as elements of the C^*-algebra A.

Proof. The element $\|T\|^2 1_X - T^*T$ is a positive element of the C^*-algebra $\mathcal{L}(X)$, and so can be written as S^*S for some $S \in \mathcal{L}(X)$. Hence

$$\|T\|^2\langle x, x\rangle_A - \langle Tx, Tx\rangle_A = \langle (\|T\|^2 1_X - T^*T)x, x\rangle_A = \langle Sx, Sx\rangle_A \geq 0.$$ □

We now introduce the algebra of "generalised compact operators" on a Hilbert module X. This standard terminology is, unfortunately, somewhat misleading since these operators are typically not compact in the classical sense of mapping the unit ball to a compact set (see Remark 1.28 below).

Lemma 1.26. *Let A be a C^*-algebra and let X be a right Hilbert A-module. Fix $x, y \in X$. The map $\Theta_{x,y} : X \to X$ determined by $\Theta_{x,y}(z) = x \cdot \langle y, z\rangle_A$ belongs to $\mathcal{L}(X)$ with adjoint $\Theta_{x,y}^* = \Theta_{y,x}$.*

Proof. Fix $w, z \in X$ and calculate:

$$\langle w, \Theta_{y,x}(z)\rangle_A = \langle w, y \cdot \langle x, z\rangle_A\rangle_A = \langle w, y\rangle_A \langle x, z\rangle_A = \langle x \cdot \langle w, y\rangle_A^*, z\rangle_A = \langle \Theta_{x,y}(w), z\rangle_A.$$

Hence $\Theta_{y,x}$ is an adjoint for $\Theta_{x,y}$. □

Definition 1.27. Let A be a C^*-algebra and let X be a right Hilbert A-module. We define $\mathcal{K}(X) := \overline{\operatorname{span}}\{\Theta_{x,y} : x, y \in X\}$. We call $\mathcal{K}(X)$ the algebra of *compact operators on X*.

Remark 1.28. The terminology "compact operators" for elements of $\mathcal{K}(X)$ is by analogy with the Hilbert-space setting: if X is a right Hilbert $\mathbb{C}$-module (that is, a Hilbert space), then it is a theorem that T belongs to $\overline{\operatorname{span}}\{\Theta_{h,k} : h, k \in \mathcal{H}\}$ if and only if is a compact operator in the classical sense that the image $T(B[0; 1])$ of the closed unit ball under T is compact.

Unfortunately, the analogy doesn't carry over, and so the terminology is a little misleading. For example, consider the discrete topological space $\mathbb{N}$, and let A be the C^*-algebra c_c of convergent sequences under pointwise operations and the supremum norm (exercise: A can also be described as $C_0(\mathbb{N})^+$). Consider the module A_A of Example 1.19. We claim that the closed unit ball $B_A[0,1]$ in A_A is not compact. To see this, for each $n \in \mathbb{N}$, let $1_n \in X$ be the sequence such that $1_n(m) = 1$ for $m \leq n$ and $1_n(m) = 0$ for $m > n$. We saw in Example 1.19 that the module norm on A_A agrees with the C^*-norm on A, and so $\|1_n\| = 1$ for all n, and $\|1_m - 1_n\| = 1$ for $m \neq n$. So $(1_n)_{n=1}^\infty$ is a sequence in $B_A[0,1]$ with no convergent subsequence. Hence $B_A[0,1]$ is not compact. Now let 1_∞ denote the sequence $1_\infty(n) = 1$ for all n; this is the identity element of A. Then $\Theta_{1_\infty,1_\infty}$ is the identity operator on A_A, and so $\Theta_{1_\infty,1_\infty}(B[0,1]) = B[0,1]$ is not compact.

Proposition 1.29. *Let A be a C^*-algebra and let X be a right Hilbert A-module. If $T \in \mathcal{L}(X)$ and $x, y \in X$, then $T\Theta_{x,y} = \Theta_{Tx,y}$. Hence $\mathcal{K}(X)$ is a closed 2-sided ideal of $\mathcal{L}(X)$*

Proof. Fix $z \in X$. We have

$$T\Theta_{x,y}(z) = T(x \cdot \langle y, z\rangle) = (Tx) \cdot \langle y, z\rangle = \Theta_{Tx,y}(z).$$

Taking adjoints gives $\Theta_{x,y}T = (T^*\Theta_{y,x})^* = \Theta^*_{T^*y,x} = \Theta_{x,T^*y}$. Hence $\mathcal{K}(X)$ is a 2-sided ideal in $\mathcal{L}(X)$. It is closed by definition. □

Example 1.30. Let A be a C^*-algebra, and consider the module A_A of Example 1.19. Each $a \in A$ determines a map $L_a : A_A \to A_A$ by $L_a x := ax$. For $a \in A$ and $x, y \in A_A$, we have

$$\langle L_a x, y\rangle_A = (ax)^* y = x^*(a^* y) = \langle x, L_{a^*} y\rangle_A,$$

so L_a is adjointable with adjoint $L_a^* = L_{a^*}$. Moreover $L_a L_b x = a(bx) = (ab)x = L_{ab}x$, and so $a \mapsto L_a$ is a homomorphism from A to $\mathcal{L}(A_A)$. We have $\|L_a x\| = \|ax\| \leq \|a\|\|x\|$, and so $\|L_a\| \leq \|a\|$; and since $\|L_a a^*\| = \|aa^*\| = \|a\|\|a^*\|$, we deduce that $\|L_a\| = \|a\|$. So $a \mapsto L_a$ is an isomorphism of A onto a C^*-subalgebra of $\mathcal{L}(A_A)$.

We claim that each $L_a \in \mathcal{K}(A_A)$. To see this, fix $a \in A$. By [7, Lemma 8.7] there is a sequence e_n of positive elements in A such that $\|ae_n - a\| \to 0$ as $n \to \infty$. So for $b \in A_A$, we have $\|L_a b - \Theta_{a,e_n} b\| = \|(ab) - (ae_n)b\| \leq \|a - ae_n\|\|b\|$ and hence $\Theta_{a,e_n} \to L_a$. Thus $L_a \in \mathcal{K}(A_A)$. Since each $\Theta_{x,y} = L_{xy^*}$, it follows that $L : a \mapsto L_a$ is an isomorphism of C^*-algebras $A \cong \mathcal{K}(A_A)$.

1.4. **Approximate identities.** Before proceeding, we need to know that every C^*-algebra has an approximate identity. For separable C^*-algebras it would suffice to consider sequences, but to deal with non-separable C^*-algebras we need to work with nets. Our development follows that of [6, Section 3.1]

Definition 1.31. A *partial order* on a set Λ is a relation $\leq$ on Λ which is reflexive, antisymmetric and transitive. A partially-ordered set $(\Lambda, \leq)$ is *directed* if, for all $\mu, \nu \in \Lambda$ there exists $\lambda \in \Lambda$ such that $\mu \leq \lambda$ and $\nu \leq \lambda$.

Example 1.32. Every totally ordered set Λ is a directed set: if $\mu, \nu \in \Lambda$ then $\max\{\mu, \nu\}$ is an upper bound for μ and ν. In particular, $\mathbb{N}$ is a directed set under the usual ordering.

Example 1.33. Let X be a set. Let $\mathcal{F}(X) := \{S \subseteq X : S \text{ is finite}\}$. Define a relation $\leq$ on $\mathcal{F}(X)$ by $S \leq T$ if and only if $S \subseteq T$. This is a partial order on $\mathcal{F}(X)$. Moreover, $\mathcal{F}(X)$ is directed: if $S, T \in \mathcal{F}(X)$, then $S \cup T \in \mathcal{F}(X)$ is an upper bound for S and T.

Example 1.34. Let X be a set. Then the power set $\mathcal{P}(X)$ is a directed set with respect to $\subseteq$.

We are interested in directed sets because we can use them in place of the natural numbers as indexing sets. This is how we get from sequences to nets.

Definition 1.35. Let Λ be a directed set, and let X be a topological space. A function $x : \Lambda \to X$ is called a *net*, and we write it $(x_\lambda)_{\lambda\in\Lambda}$. We say that the net $(x_\lambda)_{\lambda\in\Lambda}$ *converges* to $x \in X$, and write $x_\lambda \to x$ if, for every open neighbourhood U of x, there exists $\lambda_0 \in \Lambda$ such that $\lambda \geq \lambda_0$ implies $x_\lambda \in U$. We say that the net $(x_\lambda)_{\lambda\in\Lambda}$ *converges* if there exists $x \in X$ such that $x_\lambda \to x$.

Lemma 1.36. *Let X be a Hausdorff space, and suppose that $(x_\lambda)_{\lambda\in\Lambda}$ is a net in X. If $x_\lambda \to x$ and $x_\lambda \to y$ in X, then $x = y$.*

Proof. We prove the equivalent statement that if $x_\lambda \to x$ and $x \neq y$, then $x_\lambda \not\to y$. Suppose that $x \neq y$. Since X is Hausdorff, there are disjoint open sets U, V such that $x \in U$ and $y \in V$. Since $x_\lambda \to x$, there exists $\lambda_0 \in \Lambda$ such that $x_\lambda \in U$ whenever $\lambda_0 \leq \lambda$. Fix $\mu \in \Lambda$. Since Λ is directed, there exists $\nu \in \Lambda$ such that $\mu, \lambda_0 \leq \nu$. We then have $x_\nu \in U$ and hence $x_\nu \notin V$. So the open neighbourhood V of y has the property that for every $\mu \in \Lambda$ there exists $\nu \geq \mu$ such that $x_\nu \notin V$. Hence $x_\lambda \not\to y$. □

In view of Lemma 1.36, it makes sense to discuss *the* limit of a convergent net $(x_\lambda)_{\lambda\in\Lambda}$ in a Hausdorff space X and to denote the limit by $\lim_{\lambda\in\Lambda} x_\lambda$.

Lemma 1.37. *Let $f : X \to Y$ be a continuous map between topological spaces. Suppose that $(x_\lambda)_{\lambda\in\Lambda}$ is a net in X with $x_\lambda \to x$. Then $f(x_\lambda) \to f(x)$ in Y.*

Proof. Fix an open neighbourhood U of $f(x)$. Then $f^{-1}(U)$ is an open neighbourhood of x, and so there exists $\lambda_0 \in \Lambda$ such that $x_\lambda \in f^{-1}(U)$ whenever $\lambda \geq \lambda_0$. Hence $\lambda \geq \lambda_0$ implies $f(x_\lambda) \in U$. □

Corollary 1.38 (Algebra of limits for nets)**.** *Let X be a C^*-algebra, let Λ be a directed set, and let $(x_\lambda)_{\lambda\in\Lambda}$ and $(y_\lambda)_{\lambda\in\Lambda}$ be convergent nets in X with $x_\lambda \to x$ and $y_\lambda \to y$. Then*
1. $\|x_\lambda\| \to \|x\|$.
2. $\alpha x_\lambda + y_\lambda \to \alpha x + y$ *for all* $\alpha \in \mathbb{C}$*;*
3. $x_\lambda y_\lambda \to xy$*;*
4. $x_\lambda^* \to x^*$*; and*

Proof. Item (1) is by definition of the topology on X.

Item (2) is trivial if $\alpha = 0$. If $\alpha \neq 0$, fix $\varepsilon > 0$, and choose λ_x such that $\lambda \geq \lambda_x$ implies $\|x - x_\lambda\| \leq \varepsilon/2|\alpha|$ and λ_y such that $\lambda \geq \lambda_y$ implies $\|y - y_\lambda\| \leq \varepsilon/2$. Since Λ is directed, there exists $\lambda_0 \geq \lambda_x, \lambda_y$. Now $\lambda \geq \lambda_0$ implies

$$\|\alpha x + y - \alpha x_\lambda + y_\lambda\| \leq |\alpha|\|x - x_\lambda\| + \|y - y_\lambda\| \leq \varepsilon/2.$$

For (3), fix $\varepsilon > 0$. Apply (1) to obtain λ_y such that $\lambda \geq \lambda_y$ implies $\|y_\lambda\| \leq \|y\| + 1$. Now fix λ_x such that $\lambda \geq \lambda_x$ implies $\|x - x_\lambda\| < \varepsilon/2(\|y\| + 1)$ and fix λ'_y such that $\lambda \geq \lambda'_y$ implies $\|y - y_\lambda\| \leq \varepsilon/2\|x\|$. Since Λ is directed, there exists $\lambda_0 \geq y_\lambda, x_\lambda, y'_\lambda$, and for $\lambda \geq \lambda_0$, we have

$$\|xy - x_\lambda y_\lambda\| \leq \|x(y - y_\lambda)\| + \|(x - x_\lambda)y_\lambda\| \leq \|y - y_\lambda\|\|x\| + \|x - x_\lambda\|(\|y\| + 1) \leq \varepsilon.$$

Finally for (4), observe that $\|x^* - x_\lambda^*\| = \|(x - x_\lambda)^*\| = \|x - x_\lambda\|$. □

Definition 1.39. Let A be a C^*-algebra. An *approximate identity* for A is a net $(e_\lambda)_{\lambda\in\Lambda}$ of positive elements in the closed unit ball of A such that $e_\lambda \le e_\mu$ in A whenever $\lambda \le \mu \in \Lambda$, and such that $ae_\lambda \to a$ for all $a \in A$.

Remark 1.40. Suppose that $(e_\lambda)_{\lambda\in\Lambda}$ is an approximate identity for a C^*-algebra A. Then for $a \in A$, part (4) of the algebra of limits gives $e_\lambda a = (a^* e_\lambda)^* \to (a^*)^* = a$.

Theorem 1.41. *Let A be a C^*-algebra. Let Λ denote the set of all positive $a \in A$ such that $\|a\| < 1$, and let $\le$ be the usual order relation on positive elements. Then Λ is a directed set. The net $(e_a)_{a\in\Lambda}$ defined by $e_a := a$ is an approximate identity for A, called the* canonical approximate identity *for A.*

To prove the theorem, we need to know that the usual order relation on positive elements is a partial order and behaves well with respect to inverses.

Lemma 1.42. *Let A be a C^*-algebra. The relation $a \le b$ if and only if $b - a$ is positive is a partial order on the set of positive elements of A. If a, b are self-adjoint in A and $a \le b$ then for every $c \in A$ we have $c^*ac \le c^*bc$. Moreover, if a and b are invertible and $a \le b$ then a^{-1} and b^{-1} are positive with $b^{-1} \le a^{-1}$.*

Proof. The relation $\le$ is reflexive because $a - a = 0 = 0^*0$. To see that it is transitive, suppose that $a \le b$ and $b \le c$, then for every state f of A we have $f(a) \le f(b) \le f(c)$, and so Corollary 1.7(4) implies that $a \le c$, and so $\le$ is transitive. Suppose that $a \le b$ and $b \le a$. Fix a faithful representation π of A on Hilbert space. Each unit vector $h \in \mathcal{H}$ gives a state $c \mapsto (\pi(c)h \mid h)$ of A, and so $0 \le (\pi(a) - \pi(b)h \mid h) = -(\pi(b) - \pi(a)h \mid h) \le 0$ for all $h \in \mathcal{H}$. Hence $(\pi(a) - \pi(b)h \mid h) = 0$ for every $h \in \mathcal{H}$, and since $\pi(a) - \pi(b)$ is positive, this forces $\pi(a) - \pi(b) = 0$. Since π is faithful, this implies that $a = b$. Hence $\le$ is antisymmetric.

Suppose that $a \le b$ are self-adjoint and fix $c \in A$. Since $b - a$ is positive, we can write $b - a = d^*d$ and then $c^*(b - a)c = (dc)^*(dc)$ is also positive.

For the final assertion, first note that if $c \ge 1$, then the functional calculus carries c to a strictly positive function $\Gamma(c)$ on $\sigma(c)$. The formula $g(x) := \Gamma(c)(x)^{-1}$ defines a continuous positive function on the spectrum of c which is an inverse to $\Gamma(c)$. Hence $\Gamma^{-1}(g)$ is an inverse for c and is positive. Now suppose that $0 \le a \le b$ are invertible elements of A. Then $1 = a^{-1/2}aa^{-1/2} \le a^{-1/2}ba^{-1/2}$. Applying the functional calculus to the map $z \mapsto z^{-1}$ on $C^*(a^{-1/2}ba^{-1/2})$ shows that $(a^{-1/2}ba^{-1/2})^{-1} \le 1$. That is, $a^{1/2}b^{-1}a^{1/2} \le 1$. Hence the previous assertion applied with $c = a^{-1/2}$ gives

$$b^{-1} = a^{-1/2}(a^{1/2}b^{-1}a^{1/2})a^{-1/2} \le a^{-1/2}1a^{-1/2} = a^{-1}. \qquad \square$$

Remark 1.43. For the next proof, we need to decompose elements of C^*-algebras as linear combinations of positive elements. Let A be a C^*-algebra and fix $b \in A$. It is easy to check that $b_R := (b + b^*)/2$ and $b_I := (b - b^*)/2i$ are self-adjoint and that $b = b_R + ib_I$. Define functions $x \mapsto x^+$ and $x \mapsto x^-$ from $\mathbb{R}$ to $\mathbb{R}$ by

$$x^+ = \begin{cases} x & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad x^- = \begin{cases} -x & \text{if } x < 0 \\ 0 & \text{otherwise} \end{cases}$$

Both functions are continuous on $\mathbb{R}$ and $x = x^+ - x^-$ for all x. Applying the functional calculus to each of b_R and b_I we obtain positive elements b_R^+, b_R^-, b_I^+ and b_I^- such that

$$b = b_R^+ + ib_I^+ - b_R^- - ib_I^-.$$

In particular, A is spanned by its positive elements.

Proof of Theorem 1.41. Lemma 1.42 implies that $\leq$ is a partial order. We check that Λ is directed under $\leq$. We first show that if a is positive in A, then $1+a$ is invertible in $\widetilde{A}$, the minimal unitization of A, and $a(1+a)^{-1} \in \Lambda$. Applying the functional calculus to $1+a \in C^*(a)$ yields a strictly positive function, and so $1+a$ is invertible with positive inverse. Indeed the image of $1+a$ under the functional calculus is pointwise strictly greater than the image of a, and hence $a(1+a)^{-1}$ is pointwise strictly less than 1, giving $a(1+a)^{-1} \in \Lambda$.

For $a \geq 0$ we have have $a(1+a)^{-1} = ((1+a)-1)(1+a)^{-1} = 1-(1+a)^{-1}$. We claim that

$$0 \leq a \leq b \in A \quad \text{implies that } a(1+a)^{-1} \leq b(1+b)^{-1}.$$

For if $0 \leq a \leq b$, then $1+a \leq 1+b$ and hence $(1+a)^{-1} \geq (1+b)^{-1}$ by Lemma 1.42. Hence $a(1+a)^{-1} = 1-(1+a)^{-1} \leq 1-(1+b)^{-1} = b(1+b)^{-1}$ as claimed.

Now suppose that a, b are positive elements of A with $\|a\|, \|b\| < 1$. Then $1-a$ and $1-b$ are within norm 1 of the identity and so invertible. Let $a' := a(1-a)^{-1}$ and $b' := b(1-b)^{-1}$. Let $c := (a'+b')(1+(a'+b'))^{-1}$; we saw above that $c \in \Lambda$. We show that $a, b \leq c$. Observe that

$$\begin{aligned} a'(1+a')^{-1} = 1-(1+a')^{-1} &= 1-(1+a(1-a)^{-1})^{-1} \\ &= 1-(((1-a)+a)(1-a)^{-1})^{-1} = 1-(1-a) = a. \end{aligned}$$

Since $a' \leq a'+b'$, the claim therefore gives $a = a'(1+a')^{-1} \leq (a'+b')(1+a'+b')^{-1} = c$. Symmetrically $b \leq c$. Hence Λ is directed.

We show that $e_a b \to b$ for all $b \in A$. By Remark 1.43, it suffices to consider $b \in \Lambda$. Fix $b \in \Lambda$ and let Ω be the spectrum of b. Let $\phi : C^*(b) \to C(\Omega)$ be the Gelfand transform, and let $f := \phi(b)$; that is $f(x) = x$ for all $x \in \Omega$. Define $g : \mathbb{C} \to [0,1]$ by

$$g(x) = \begin{cases} |x|/\varepsilon & \text{if } |x| < \varepsilon \\ 1 & \text{if } |x| \geq \varepsilon. \end{cases}$$

Then g restricts to a continuous function on Ω. The set $K := \{x \in \Omega : f(x) \geq \varepsilon\}$ is compact, and $g|_K \equiv 1$. Let $\delta := 1-\varepsilon/2$. Since $b \in \Lambda$, we have $\|f\| < 1$. Thus for $x \in K$ we have $|f(x) - \delta g(x) f(x)| \leq (\varepsilon/2)\|f\| < \varepsilon$; and for $x \notin K$ we have $|f(x) - \delta g(x) f(x)| \leq |f(x)| < \varepsilon$. Hence $\|f - \delta g f\| < \varepsilon$. By construction we have $\|\phi^{-1}(\delta g)\| \leq 1-\varepsilon/2 < 1$, and so $\lambda_0 := \phi^{-1}(\delta g) \in \Lambda$. Moreover, $\|b - e_{\lambda_0} b\| = \|f - \delta g f\| < \varepsilon$ by construction. Suppose that $\lambda \in \Lambda$ satisfies $\lambda_0 \leq \lambda$. Then $1 - e_\lambda \leq 1 - e_{\lambda_0}$, giving $b(1-e_\lambda)b \leq b(1-e_{\lambda_0})b$. Thus

$$\|b - e_\lambda b\|^2 = \|(1-e_\lambda)^{1/2}(1-e_\lambda)^{1/2} b\|^2 \leq \|(1-e_\lambda)^{1/2} b\|^2 = \|b(1-e_\lambda)b\| \leq \|b(1-e_{\lambda_0})b\| < \varepsilon.$$

Hence $e_\lambda b \to b$ as required. □

Remark 1.44. If A is separable, then it has an approximate unit which is a sequence. See [6, Remark 3.1.1] for details.

1.5. **Multiplier algebras.**

We are now ready to discuss the second standard unitization of a C^*-algebra: its *maximal* unitization, known as the multiplier algebra. We begin by making formal sense of what we mean by a unitization in the first place.

Definition 1.45. An ideal I in a C^*-algebra A is *essential* if the only ideal J satisfying $J \cap I = \{0\}$ is $J = \{0\}$.

Lemma 1.46. *An ideal I of a C^*-algebra A is essential if and only if $aI = \{0\}$ implies $a = 0$ for $a \in A$.*

Proof. Suppose that $aI = \{0\}$ implies $a = 0$ for all $a \in A$, and fix an ideal J of A such that $J \cap I = \{0\}$. For $a \in J$, we have $aI = \{ab : b \in I\} \subseteq I \cap J = \{0\}$, and hence $a = 0$. Thus $J = \{0\}$.

Now suppose that $J \cap I = \{0\}$ implies $J = \{0\}$ and fix $a \in A$ with $aI = \{0\}$. Then $\overline{AaA} := \overline{\operatorname{span}}\{bac : b, c \in A\}$ is an ideal of A. For each $b, c \in A$ we have $(bac)I = b(a(cI)) \subseteq b \cdot (aI) = \{0\}$. Linearity and continuity force $\overline{AaA}I = \{0\}$ and hence $\overline{AaA} = \{0\}$. By [7, Lemma 8.7] there is a sequence e_n of positive elements in A such that $\|ae_n - a\| \to 0$ as $n \to \infty$. We have $a^*ae_n \in \overline{AaA} = \{0\}$ for all n, and hence $\|a\|^2 = \lim_{n\to\infty} \|a^*ae_n\| = 0$. □

Definition 1.47. Let A be a C^*-algebra. A *unitization* of A is an injective homomorphism $i : A \to B$ of A onto an essential ideal of a unital C^*-algebra B.

Lemma 1.48. *If A has an identity, then, up to isomorphism, the only unitization of A is $\mathrm{id} : A \to A$. If A does not have an identity, then $i : A \to A^+$ is a unitization of A.*

Proof. Suppose that A has an identity 1_A, and that $i : A \to B$ is an injective homomorphism of A onto an essential ideal of a unital C^*-algebra B. We show that i is an isomorphism of A onto B. Indeed, we have $(i(1_A) - 1_B)i(a) = i(a) - i(a) = 0$ for all $a \in A$. Since $i(A)$ is an essential ideal, it follows from Lemma 1.46 that $i(1_A) = 1_B$. Now for each $b \in B$ we have $b = 1_B b = i(1_A)b \in i(A)$. So i is surjective and hence an isomorphism of C^*-algebras.

Now suppose that A does not have an identity. We saw already that $i(A)$ is an ideal in A^+. It therefore suffices to show that it is essential. Fix $(b, \lambda) \in A^+ \setminus \{0\}$; we must find $a \in A$ such that $(b, \lambda)i(a) \neq 0$. We consider two cases: $\lambda = 0$ and $\lambda \neq 0$. If $\lambda = 0$, then $\|(b, \lambda)i(b^*)\| = \|(b, 0)(b^*, 0)\| = \|b^*b\| \neq 0$. Now suppose that $\lambda \neq 0$. Since A does not have an identity, there exists $a \in A$ such that $(-\lambda^{-1}b)a \neq a$. Rearranging gives $ba + \lambda a \neq 0$. Hence $(b, \lambda)i(a) = (ba + \lambda a, 0) \neq 0$. □

Example 1.49. Let X be a locally compact Hausdorff space. We say that a compact Hausdorff space Y is a *compactification* of X if there is an injection $i : X \to Y$ which is a homeomorphism onto a dense open subset of Y. We claim that the map $i^* : C_0(X) \to C(Y)$ given by

$$i^*(f)(y) = \begin{cases} 0 & \text{if } y \notin i(X) \\ f(x) & \text{if } y = i(x) \text{ for some } x \in X \end{cases}$$

is then a unitization of $C_0(X)$. To see this, first observe that i^* is well-defined because i is injective. Fix $f \in C_0(X)$. To see that $i^*(f)$ is continuous, we prove that the preimage of a closed set under $i^*(f)$ is closed. Fix a closed $C \subset \mathbb{C}$. Then

$$i^*(f)^{-1}(C) = \begin{cases} i(f^{-1}(C)) & \text{if } 0 \notin C \\ i(f^{-1}(C)) \cup Y \setminus i(X) & \text{if } 0 \in C. \end{cases}$$

To see that $i^*(f)^{-1}(C)$ is closed, we consider the two cases separately. First suppose that $0 \notin C$ then there exists $\varepsilon > 0$ such that $z \in C \implies |z| \geq \varepsilon$. Since $f \in C_0(X)$, there is a compact set K such that $|f(x)| < \varepsilon$ for $x \notin K$, and it follows that $f^{-1}(C) \subseteq K$ is a closed subset of a compact set and hence compact. Its image $i^*(f)^{-1}(C)$ under the continuous map i is therefore also compact, and hence closed in the Hausdorff space Y. Now suppose that $0 \in C$. Then

$$Y \setminus i^*(f)^{-1}(C) = i(X) \setminus i(f^{-1}(C)) = i(X \setminus f^{-1}(C)).$$

The set $f^{-1}(C)$ is closed in X. Since i is a homeomorphism of X onto $i(X)$ with the relative topology, it follows that $i(X \setminus f^{-1}(C))$ is open in the relative topology on $i(X) \subseteq Y$. That is, $i(X \setminus f^{-1}(C)) = i(X) \cap U$ for some open $U \subseteq Y$. Since $i(X)$ is open, it follows that $i(X \setminus f^{-1}(C))$ is open also.

The image of i^* is precisely the ideal $I_X := \{g \in C(Y) : g(y) = 0 \text{ for all } y \in Y \setminus i(X)\}$. To see that this is an essential ideal, suppose $h \in C(Y)$ satisfies $hg = 0$ for all $g \in I_X$. Fix $x \in X$. There exists $f \in C_0(X)$ such that $f(x) = 1$, and then $i^*(f)(i(x)) = 1$ also. Since $hi^*(f) = 0$, it follows that $h(i(x)) = 0$. So $h|_{i(X)} = 0$. Since $i(X)$ is dense in Y, continuity of h implies that $h = 0$.

Proposition 1.50. *Let A be a C^*-algebra. Let $L : A \to \mathcal{K}(A_A)$ be the isomorphism of Example 1.30. Then $L : A \to \mathcal{L}(A_A)$ is a unitization of A.*

Proof. We have seen that L is an injective homomorphism of A onto $\mathcal{K}(A_A)$ and that $\mathcal{K}(A_A)$ is an ideal of $\mathcal{L}(A_A)$, which is certainly a unital C^*-algebra. So we just have to show that $\mathcal{K}(A_A)$ is essential in $\mathcal{L}(A_A)$. So fix $T \in \mathcal{L}(A_A)$, and suppose that $TL(A) = \{0\}$; we must show that $T = 0$. For $x, y \in A$, we have

$$0 = T\Theta_{x,y} = \Theta_{Tx,y}.$$

Fix $x \in A$. Putting $y = Tx$, we have $\Theta_{Tx,Tx} = 0$. By [7, Lemma 8.7], there is a sequence e_n of positive elements of A such that $(Tx)^* e_n \to (Tx)^*$ as $n \to \infty$, and therefore

$$\|Tx\|^2 = \lim_{n\to\infty} \|Tx(Tx)^* e_n\| = 0.$$

Since x was arbitrary, we deduce that $T = 0$. □

Our aim is to show that this unitization is maximal.

Definition 1.51. Let A be a C^*-algebra and let $i : A \to B$ be a unitization of A. Then i is *maximal* if, for every injective homomorphism $j : A \hookrightarrow C$ of A onto an essential ideal of a unital C^*-algebra C, there is a homomorphism $\phi : C \to B$ such that

$$\phi \circ j = i. \tag{1}$$

Theorem 1.52. *Let A be a C^*-algebra. The unitization $L : A \to \mathcal{L}(A_A)$ of Example 1.30 is a maximal unitization. It is the unique maximal unitization of A: if $i : A \to B$ is a maximal unitization of A, then there is a unique isomorphism $\phi : B \to \mathcal{L}(A_A)$ such that $\phi(i(a)) = L_a$ for all $a \in A$.*

Definition 1.53. We call $\mathcal{L}(A_A)$ the *multiplier algebra* of A and denote it $\mathcal{M}(A)$.

To prove our theorem, we need to be able to extend homomorphisms into $\mathcal{L}(X)$ from ideals up to their enveloping algebras. This works provided that the homomorphism in question is nondegenerate in the following sense.

Definition 1.54. Let B be a C^*-algebra and let X be a Hilbert A-module. A homomorphism $\alpha : B \to \mathcal{L}(X)$ is *nondegenerate* if $\overline{\operatorname{span}}\{\alpha(b)x : b \in B \text{ and } x \in X\} = X$.

Proposition 1.55. *Let A, B, and C be C^*-algebras and let X be a Hilbert A-module. Suppose that $i : B \to C$ is an injective homomorphism of B onto an ideal of C. Suppose that $\alpha : B \to \mathcal{L}(X)$ is nondegenerate. Then there is a unique homomorphism $\bar{\alpha} : C \to \mathcal{L}(X)$ such that $\bar{\alpha} \circ i = \alpha$. If $i(B)$ is an essential ideal of C and α is injective, then $\bar{\alpha}$ is injective.*

Proof. It suffices to consider the situation where $B \subseteq C$ and i is the inclusion map. Let $(e_\lambda)_{\lambda\in\Lambda}$ be an approximate identity for B. Fix $c \in C$, $b_1, \dots, b_n \in B$, and $x_1, \dots x_n \in X$. Then

$$\Big\| \sum_{i=1}^n \alpha(cb_i)(x_i) \Big\| = \lim_\lambda \Big\| \sum_{i=1}^n \alpha(ce_\lambda b_i)(x_i) \Big\| \\ \leq \lim_\lambda \Big\| \alpha(ce_\lambda) \sum_{i=1}^n \alpha(b_i)(x_i) \Big\| \\ \leq \|c\| \Big\| \sum_{i=1}^n \alpha(b_i)(x_i) \Big\|.$$

In particular if $\sum_{i=1}^n \alpha(b_i)(x_i) = \sum_{j=1}^m \alpha(b'_j)(x'_j)$, then we also have $\sum_{i=1}^n \alpha(cb_i)(x_i) = \sum_{j=1}^m \alpha(cb'_j)(x'_j)$. Thus there is a unique linear map $\bar{\alpha}(c) : \operatorname{span}\{\alpha(b)(x) : b \in B, x \in X\} \to \overline{\operatorname{span}}\{\alpha(b)(x) : b \in B, x \in X\}$ such that $\bar{\alpha}(c)(\alpha(b)(x)) = \alpha(cb)(x)$ for all b, x. Moreover, $\bar{\alpha}(c)$ is bounded with norm at most $\|c\|$ and hence extends to a bounded linear map on $\overline{\operatorname{span}}\{\alpha(b)(x) : b \in B, x \in X\}$, which is all of X because α is nondegenerate. We have

$$\langle \bar{\alpha}(c)(\alpha(b)(x)), \alpha(b')(x') \rangle_A = \langle \alpha(cb)(x), \alpha(b')(x') \rangle_A \\ = \langle x, \alpha(b^*)\alpha(c^*b')(x') \rangle_A = \langle \alpha(b)x, \bar{\alpha}(c^*)\alpha(b')(x') \rangle_A,$$

and so $\bar{\alpha}(c) \in \mathcal{L}(X)$ with adjoint $\bar{\alpha}(c^*)$. Routine calculations show that $\bar{\alpha}$ is a homomorphism. If α is injective and B is essential in C, then that $\ker(\bar{\alpha}) \cap B = \{0\}$ implies $\ker(\bar{\alpha}) = \{0\}$, and hence $\bar{\alpha}$ is injective. □

Corollary 1.56. *Let A and B be C^*-algebras and let $\phi : B \to \mathcal{M}(A)$ be a nondegenerate homomorphism. Then there is a unique homomorphism $\bar{\phi} : \mathcal{M}(B) \to \mathcal{M}(A)$ such that $\bar{\phi}(b) = \phi(b)$ for all $b \in B$. If ϕ is injective, then so is $\bar{\phi}$.*

Proof. Apply the preceding proposition to $X = A_A$ and the inclusion $i : B \to \mathcal{M}(B)$; we saw in Proposition 1.50 that $i(B)$ is essential in $\mathcal{M}(B)$. □

Lemma 1.57. *Suppose that $i : A \to B$ is a maximal unitization and that $j : A \to C$ is an injection of A onto an essential ideal of C. Then there is only one homomorphism ϕ satisfying* (1)*, and it is injective*

Proof. Suppose that $\phi, \psi : C \to B$ satisfy (1). Then for $a \in A$ we have $\psi(j(a)) = i(a) = \phi(j(a))$. Thus for each $c \in C$ and $a \in A$ we have

$$(\phi(c) - \psi(c))i(a) = \phi(c)\phi(j(a)) - \psi(c)\psi(j(a)) = \phi(cj(a)) - \psi(cj(a)).$$

Since j embeds A as an ideal of C, we have $cj(a) = j(a')$ for some $a' \in A$, and then $\phi(cj(a)) = \phi(j(a')) = i(a') = \psi(j(a')) = \psi(cj(a))$. Hence $(\phi(c) - \psi(c))i(a) = 0$. Since $i(A)$ is an essential ideal in B, Lemma 1.46 implies that $\phi(c) = \psi(c)$. Hence $\phi = \psi$.

To see that ϕ is injective, observe that $\ker(\phi) \cap j(A) = j(\ker(i)) = \{0\}$. Since $j(A)$ is essential in C, this forces $\ker(\phi) = \{0\}$. □

Proof of Theorem 1.52. To see that $\mathcal{L}(A_A)$ is a maximal unitization of A, suppose that $j : A \to C$ is an injection of A onto an essential ideal of C. Proposition 1.55 applied with $B = A$, $X = A_A$, and $\alpha := L : A \to \mathcal{K}(A_A)$ implies that there is a homormorphism $\overline{L} : C \to \mathcal{L}(A_A)$ such that $\overline{L} \circ j = L$, and so $\phi := \overline{L}$ satisfies (1).

To see that this is the unique maximal unitization of A, Suppose that $j : A \to B$ is another maximal unitization of A. The maximality of $\mathcal{L}(A_A)$ gives the homomorphism $\phi : B \to \mathcal{L}(A_A)$ such that $\phi \circ j = i$. Lemma 1.57 implies that ϕ is injective. The maximality of B gives a homomorphism $\psi : \mathcal{L}(A_A) \to B$ such that $\psi \circ i = j$. Now $\phi \circ \psi : \mathcal{L}(A_A) \to \mathcal{L}(A_A)$ satisfies $\phi \circ \psi \circ i = \phi \circ j = i$, and so uniqueness in Lemma 1.57 implies that $\phi \circ \psi = \mathrm{id}_{\mathcal{L}(A_A)}$. Similarly, $\psi \circ \phi : B \to B$ satisfies $\psi \circ \phi \circ j = \psi \circ i = j$ and uniqueness in Lemma 1.57 shows that $\psi \circ \phi = \mathrm{id}_B$. So ϕ and ψ are mutually inverse, and in particular ϕ is an isomorphism. □

The following characterisation of $\mathcal{M}(A)$ is taken from [4].

Proposition 1.58. *Suppose that A and C are C^*-algebras and that X is a Hilbert C-module. Let $\alpha : A \to \mathcal{L}(X)$ be an injective nondegenerate homomorphism. Then*

$$B := \{T \in \mathcal{L}(X) : T\alpha(A) \subseteq \alpha(A) \text{ and } \alpha(A)T \subseteq \alpha(A)\}$$

is a unital C^*-subalgebra of $\mathcal{L}(X)$, and α extends to an isomorphism $\bar{\alpha} : \mathcal{M}(A) \to B$.

Proof. That B is a linear subspace of $\mathcal{L}(X)$ follows from bilinearity of multiplication. If $S, T \in B$, then $ST\alpha(A) \subseteq S\alpha(A) \subseteq \alpha(A)$, and the inclusion $\alpha(A)ST \subseteq \alpha(A)$ follows similarly; so B is closed under multiplication. If $T \in B$ and $a \in A$, then

$$T^*\alpha(A) = (\alpha(A)^*T)^* = (\alpha(A)T)^* \subseteq \alpha(A)^* = \alpha(A),$$

and a similar calculation shows $\alpha(A)T^* \subseteq \alpha(A)$, and so B is closed under adjoints. And if $T_n \in B$ and $T_n \to T$ then continuity of multiplication and that $\alpha(A)$ is closed imply that $T \in B$. So B is a C^*-subalgebra of $\mathcal{L}(X)$ as claimed. We clearly have $1_X \in B$.

It is clear that $\alpha(A)$ is an ideal of B. We show that it is essential. For if $T\alpha(A) = \{0\}$, then $T\alpha(A)X = \{0\}$, and since α is nondegenerate, it follows that $T \cdot X = \{0\}$ and hence $T = 0$.

Now by Theorem 1.52, we just have to show that $\alpha : A \to B$ is a maximal unitization of A. So suppose that $j : A \to D$ is an injective homomorphism of A onto an essential ideal of a unital C^*-algebra D. Proposition 1.55 implies that there is an injective homomorphism $\bar{\alpha} : D \to \mathcal{L}(X)$ such that $\bar{\alpha} \circ j = \alpha$. We must show that $\bar{\alpha}(D) \subseteq B$. For $d \in D$ and $a \in A$, we have

$$\bar{\alpha}(d)\alpha(a) = \bar{\alpha}(dj(a)) = \alpha(j^{-1}(dj(a)))$$

because $j(A)$ is an ideal of D. Thus $\bar{\alpha}(d)\alpha(A) \subseteq \alpha(A)$, and similarly, $\alpha(A)\bar{\alpha}(d) \subseteq \alpha(A)$. □

To bring our discussion full circle, we show that if X is a Hilbert module then $\mathcal{L}(X)$ is the multiplier algebra of $\mathcal{K}(X)$. To do so we need a beautiful factorisation theorem, which is a particularly strong special case of the Hewitt-Cohen factorisation theorem.

Theorem 1.59. *Let A be a C^*-algebra and let X be a Hilbert A-module. For each $x \in X$ there exists $y \in X$ such that $x = y \cdot \langle y, y \rangle_A$.*

Lemma 1.60. *If X is a Hilbert A-module, then elements of the form $x \cdot a$ span a dense subset of X.*

Proof. Fix an approximate identity e_λ for the ideal $I_X = \overline{\operatorname{span}}\{\langle x, y\rangle_A : x, y \in X\}$. For $x \in X$ we have

$$\|x - x \cdot e_\lambda\|_A^2 = \big\|\langle x, x\rangle_A - \langle x, x\rangle_A e_\lambda - e_\lambda \langle x, x\rangle_A + e_\lambda \langle x, x\rangle_A e_\lambda\big\| \to 0,$$

and hence $x \cdot e_\lambda \to x$. □

To prove the result, we need to discuss adjointable operators from one Hilbert module to another. If X and Y are Hilbert A-modules, then a map $T : X \to Y$ is *adjointable* if there exists $S : Y \to X$ such that $\langle Tx, y\rangle_A = \langle x, Sy\rangle_A$ for all $x \in X$ and $y \in Y$. The same arguments we used in Lemma 1.22 show that every adjointable operator between Hilbert modules is bounded, linear and A-linear, that the adjoint is unique and that T^* is adjointable with $T^{**} = T$. We write $\mathcal{L}(X, Y)$ for the collection of adjointable operators from X to Y.

For $x \in X$ and $y \in Y$ the formula $\Theta_{x,y}(z) := x \cdot \langle y, z\rangle_A$ determines an adjointable operator $\Theta_{x,y}$ from from Y to X, with adjoint $\Theta_{y,x} : X \to Y$. We denote by $\mathcal{K}(Y, X)$ the Banach space $\overline{\operatorname{span}}\{\Theta_{x,y} : x \in X, y \in Y\} \subseteq \mathcal{L}(Y, X)$.

Let X be a Hilbert A-module. For each $x \in X$, define $D_x : X \to A_A$ by $D_x(y) = \langle x, y\rangle_A$, and define $L_x : A_A \to X$ by $L_x(a) = x \cdot a$. We have

$$\langle D_x(y), b\rangle_A = \langle x, y\rangle_A^* b = \langle y, x \cdot a\rangle_A = \langle y, L_x(a)\rangle_A,$$

So $D_x \in \mathcal{L}(X, A_A)$ and $L_x \in \mathcal{L}(A_A, X)$ with $D_x^* = L_x$.

Lemma 1.61. *Suppose that X is a Hilbert A-module. Then the map $D : x \mapsto D_x$ is an isometric conjugate-linear isomorphism of X onto $\mathcal{K}(X, A_A)$, and the map $L : x \mapsto L_x$ is an isometric linear isomorphism of X onto $\mathcal{K}(A_A, X)$.*

Proof. We have

$$D_{\alpha x + y}(z) = \langle \alpha x + y, z\rangle_A = \overline{\alpha}\langle x, z\rangle_A + \langle y, z\rangle_A = (\overline{\alpha} D_x + D_y)(z),$$

and so D is conjugate-linear. If $x = 0$ then D_x is clearly zero, and if not then

$$\|D_x\| = \sup_{\|y\|=1} \|\langle x, y\rangle_A\| = \|x\|.$$

So D is isometric and in particular has closed range. Each rank-1 operator $\theta_{a,x} : y \mapsto a \cdot \langle x, y\rangle$ from X to A_A satisfies $\Theta_{a,x}(y) = \langle x \cdot a^*, y\rangle_A = D_{x \cdot a^*}(y)$. Since $X \cdot A$ is dense in X and since D is isometric, this ensures that the range of D is contained in $\mathcal{K}(X, A_A)$. It also ensures that the range of D contains all the rank-1 operators from X to A. Since D is conjugate-linear and has closed range, the range of D therefore contains $\mathcal{K}(X, A_A)$. The assertions about L follow because L is the map $x \mapsto D_x^*$, and $T \mapsto T^*$ is an isometric conjugate-linear isomorphism of $\mathcal{K}(X, A_A)$ onto $\mathcal{K}(A_A, X)$. □

We now use a powerful 2×2-matrix trick to prove our result: in this instance, it allows us to bring to bear the functional calculus on operators from one module to another.

If X and Y are Hilbert A-modules, then the algebraic direct sum $X \oplus Y$ is a Hilbert A-module with $\langle (x, y), (x', y')\rangle_A = \langle x, x'\rangle_A + \langle y, y'\rangle_A$. Given $T \in \mathcal{K}(X, A_A)$, $S \in \mathcal{K}(A_A, X)$ and $R \in \mathcal{K}(X)$, and given an element $a \in A$, there is an operator on $A_A \oplus X$ given by

$$\begin{pmatrix} a & T \\ S & R \end{pmatrix} \begin{pmatrix} b \\ x \end{pmatrix} = \begin{pmatrix} ab + Tx \\ Sb + Rx \end{pmatrix}. \tag{2}$$

It is not difficult to check that this matrix operator is adjointable with

$$\begin{pmatrix} a & T \\ S & R \end{pmatrix}^* = \begin{pmatrix} a^* & s^* \\ T^* & R^* \end{pmatrix}$$

We claim that this matrix operator belongs to $\mathcal{K}(A_A \oplus X)$. To see this, observe that the maps

$$a \mapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}, \qquad T \mapsto \begin{pmatrix} 0 & T \\ 0 & 0 \end{pmatrix}, \qquad S \mapsto \begin{pmatrix} 0 & 0 \\ S & 0 \end{pmatrix}, \qquad \text{and} \qquad R \mapsto \begin{pmatrix} 0 & 0 \\ 0 & R \end{pmatrix}$$

are isometric embeddings of $\mathcal{K}(A_A)$, $\mathcal{K}(X, A_A)$, $\mathcal{K}(A_A, X)$ and $\mathcal{K}(X)$ into $\mathcal{K}(A_A \oplus X)$. Moreover, for $\theta_{a,x} \in \mathcal{K}(X, A_A)$, we have

$$\begin{pmatrix} 0 & \theta_{a,x} \\ 0 & 0 \end{pmatrix} = \theta_{(a,0),(0,x)},$$

and the other three embeddings likewise carry rank-1 operators to rank-1 operators. Hence each matrix operator as above belongs to $\mathcal{K}(A_A \oplus X)$. Conversely, let P_A and P_X be the projections onto A_A and X in $A_A \oplus X$. For $(a, x), (b, y) \in A_A \oplus X$, we have $P_A \Theta_{(a,x),(b,y)} P_A = \Theta_{(a,0),(b,0)}$, $P_A \Theta_{(a,x),(b,y)} P_X = \Theta_{(a,0),(0,y)}$ and so forth. So for $M \in \mathcal{K}(A_A \oplus X)$, the elements $P_A M P_A$, $P_A M P_X$, $P_X M P_A$ and $P_X M P_X$ give a decomposition of M in the form (2) and are all generalised compact operators. In short, the map sending a, S, T, R to the operator (2) is a linear isomorphism of the algebraic direct sum $\mathcal{K}(A_A) \oplus \mathcal{K}(X, A_A) \oplus \mathcal{K}(A_A, X) \oplus \mathcal{K}(X)$ onto $\mathcal{K}(A_A \oplus X)$.

Proof of Theorem 1.59. Fix $x \in X$. Then the operator $\left(\begin{smallmatrix} 0 & D_x \\ L_x & 0 \end{smallmatrix}\right)$ is a self-adjoint element of $\mathcal{K}(A_A \oplus X)$. Lemma 1.61 and the discussion above implies that every self-adjoint $M \in \mathcal{K}(A_A \oplus X)$ satisfying $M\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) = -\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)M$ is of the form $\left(\begin{smallmatrix} 0 & D_x \\ L_x & 0 \end{smallmatrix}\right)$. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^{1/3}$. By the functional calculus, $f\left(\begin{smallmatrix} 0 & D_x \\ L_x & 0 \end{smallmatrix}\right)$ belongs to $\mathcal{K}(A_A \oplus X)$ and anticommutes with $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. Hence $f\left(\begin{smallmatrix} 0 & D_x \\ L_x & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & D_y \\ L_y & 0 \end{smallmatrix}\right)$ for some $y \in X$. We now have $\left(\begin{smallmatrix} 0 & D_y \\ L_y & 0 \end{smallmatrix}\right)^3 = f\left(\begin{smallmatrix} 0 & D_x \\ L_x & 0 \end{smallmatrix}\right)^3 = \left(\begin{smallmatrix} 0 & D_x \\ L_x & 0 \end{smallmatrix}\right)$. On the other hand,

$$\left(\begin{smallmatrix} 0 & D_y \\ L_y & 0 \end{smallmatrix}\right)^3 = \left(\begin{smallmatrix} D_y L_y & 0 \\ 0 & L_y D_y \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & D_y \\ L_y & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & D_y L_y D_y \\ L_y D_y L_y & 0 \end{smallmatrix}\right).$$

In particular, $L_y D_y L_y = L_x$. We calculate $L_y D_y L_y(a) = L_y D_y(y \cdot a) = L_y(\langle y, y \cdot a\rangle_A) = y \cdot \langle y, y \cdot a\rangle_A = L_{y \cdot \langle y,y\rangle_A}(a)$, and hence $L_{y \cdot \langle y,y\rangle_A} = L_x$. Since L is isometric, it follows that $x = y \cdot \langle y, y\rangle_A$ as required. □

Corollary 1.62. *Let X be a Hilbert A-module. Then $i : \mathcal{K}(X) \to \mathcal{L}(X)$ is a maximal unitization of $\mathcal{K}(X)$, and hence $\mathcal{L}(X) = \mathcal{M}(\mathcal{K}(X))$.*

Proof. We first show that i is nondegenerate. For this, let $(T_\lambda)_{\lambda \in \Lambda}$ be an approximate identity for $\mathcal{K}(X)$. By Theorem 1.59, it suffices to show that $T_\lambda(y \cdot \langle y, y\rangle_A) \to y \cdot \langle y, y\rangle_A$ for all $y \in X$. We calculate:

$$T_\lambda(y \cdot \langle y, y\rangle_A) = (T_\lambda \Theta_{y,y})(y) \to \Theta_{y,y}(y) = y \cdot \langle y, y\rangle_A.$$

Since $\mathcal{K}(X)$ is an ideal of $\mathcal{L}(X)$, the result now follows from Proposition 1.58. □

Corollary 1.63. (1) *If $\mathcal{H}$ is a Hilbert space, then $\mathcal{M}(\mathcal{K}(\mathcal{H})) = \mathcal{B}(\mathcal{H})$; and*

(2) *If T is a locally compact Hausdorff space, then $\mathcal{M}(C_0(T))$ is isomorphic to the algebra $C_b(T)$ of bounded continuous functions on T.*

Proof. Part (1) is a special case of Corollary 1.62.

For Part (2), we already know that $C_0(T)$ is an essential ideal in $C_b(T)$. Let $\mathcal{H}$ be the Hilbert space $\ell^2(T)$, which is the closure of the finitely supported functions in $h : T \to \mathbb{C}$ with respect to $(h \mid k) = \sum_{t\in T} h(t)\overline{k(t)}$. Each $f \in C_b(T)$ defines a map $M_f : \ell^2(T) \to \ell^2(T)$ by $M_f(h)(t) = f(t)h(t)$, and M_f is adjointable with $M_f^* = M_{\overline{f}}$. Since $C_0(T) \subseteq C_b(T)$, the map $M : C_b(T) \to \mathcal{L}(\ell^2(T))$ also takes each $f \in C_0(T)$ to an adjointable operator. Suppose that $m \in \mathcal{B}(\ell^2(T))$ and that for each $f \in C_0(T)$ there exist $mf, fm \in C_0(T)$ such that $M_f m = M_{fm}$ and $mM_f = M_{mf}$. By Proposition 1.58, to see that $C_b(T) = \mathcal{M}(C_0(T))$, it suffices to show that $m = M_\phi$ for some $\phi \in C_b(T)$.

We show first that for fixed $t \in T$, if $f(t) = g(t) = 1$ then $(mf)(t) = (mg)(t)$. Indeed, if δ_t is the basis element of $\ell^2(T)$ associated to t, then

$$(mf)(t) = (M_{mf}\delta_t \mid \delta_t) = (mM_f\delta_t \mid \delta_t) = (m\delta_t \mid \delta_t),$$

and symmetrically, $(mg)(t) = (m\delta_t \mid \delta_t)$ also. So the formula $\phi(t) = (mf)(t)$ for any $f \in C_0(T)$ with $\|f\| = 1$ and $f(t) = 1$ determines a function $\phi : T \to \mathbb{C}$. To see that ϕ is continuous, fix $t \in T$. By Urysohn's lemma, there is a neighbourhood U of t and a function $f \in C_0(T)$ with $\|f\| = 1$ such that $f|_U \equiv 1$, and then $\phi(s) = (mf)(s)$ for all $s \in U$. Since mf is continuous, it follows that ϕ is continuous on U. Fix $t \in T$ and $f \in C_0(T)$ such that $\|f\| = 1$ and $f(t) = 1$. Then $|\phi(t)| = |(mf)(t)| \leq \|m\|\|f\| = \|m\|$. Hence ϕ is bounded by $\|m\|$.

It remains to show that $M_\phi = m$. For this, fix $t \in T$ and $f \in C_0(T)$ with $\|f\| = 1$ and $f(t) = 1$. Then $M_\phi\delta_t = \phi(t)\delta_t = (mf)(t)\delta_t = mM_f\delta_t = m(f(t)\delta_t) = m\delta_t$. Since the δ_t are an orthonormal basis for $\ell^2(T)$ it follows that $M_\phi = m$. □

Remark 1.64. Since $C_b(T)$ is a commutative C^*-algebra with 1, its spectrum Δ is a compact Hausdorff space. This space, usually denoted βX is a compactification of T, called the *Stone-Čech compactification*. This space is, by definition, the unique maximal compactification of T. That is, the unique compactification $i : T \to \beta T$ such that every continuous map ϕ from T to a compact Hausdorff space K induces a continuous map $\beta\phi : \beta T \to K$ satisfying $\beta\phi \circ i = \phi$. The space βT can be realised concretely as follows. Let C be the set of all continuous functions from T to $[0, 1]$ and consider the space $[0, 1]^C$ of all functions from C to $[0, 1]$; this is a compact Hausdorff space by Tychonoff's theorem. The map $i : t \mapsto (f \mapsto f(t))$ is a continuous injection of T into $[0, 1]^C$ and $\overline{i(T)}$ may be identified with βT.

So the preceding corollary says that the spectrum of the maximal unitization of a commutative C^*-algebra is the same as the maximal compactification of the spectrum.

2. Crossed Products

This section of the notes picks and chooses from Dana Williams' excellent book [9].

2.1. **Locally compact groups and Haar measure.**

We will very quickly restrict our attention to actions of the integers $\mathbb{Z}$, but at least to begin with we can maintain a reasonable level of generality.

Definition 2.1. A *topological group* is a group G endowed with a topology under which it is both Hausdorff and locally compact, and under which the operation $g \mapsto g^{-1}$ is continuous, and the operation $(g, h) \mapsto gh$ is continuous from $G \times G$ to G. To keep things straightforward, we will assume that our groups G are second countable in the sense that there is a countable base of open sets for the topology on G.

Example 2.2. The following are all examples of topological groups.

(1) Any group under the discrete topology.
(2) $\mathbb{R}^n$, $\mathbb{Q}^n$, $\mathbb{C}^n$ under the usual topologies.
(3) The group $\mathbb{T}^n$ of n-tuples of complex numbers of modulus 1 under pointwise multiplication.
(4) Finite cartesian products of topological groups.

Example 2.3. Let $\mathcal{H}$ be a Hilbert space, and let $\mathcal{U}(\mathcal{H})$ denote the collection of unitary operators on $\mathcal{H}$; that is, $\mathcal{U}(\mathcal{H}) = \{U \in \mathcal{B}(\mathcal{H}) : U^*U = UU^* = 1_{\mathcal{H}}\}$. We claim that $\mathcal{U}(\mathcal{H})$ is a group under multiplication, and is a topological group when endowed with the strong operator topology: $U_n \to U$ if and only if $U_n h \to Uh$ for all $h \in \mathcal{H}$.

To see that $\mathcal{U}(H)$ is a group under multiplication, observe that if $U, V \in \mathcal{U}(\mathcal{H})$, then

$$(UV)^*(UV) = V^*U^*UV = V^*V = 1_{\mathcal{H}} = UU^* = UVV^*U^* = (UV)(UV)^*,$$

so $\mathcal{U}(H)$ is closed under multiplication. It's closed under inverses because $U^{-1} = U^*$ for $U \in \mathcal{U}(\mathcal{H})$. It is Hausdorff because $\mathcal{B}(\mathcal{H})$ is Hausdorff in the strong topology. If $U_n \to U$ and $V_n \to V$ in the strong topology, then for $h \in \mathcal{H}$ we have

$$U_nV_nh - UVh = U_n(V_nh - Vh) + (U_n - U)Vh \to 0,$$

so multiplication is continuous. To see that inversion is continuous, we must show that if $U_n \to U$ then $U_n^* \to U^*$ in the strong topology. To see this, fix $h \in \mathcal{H}$ and calculate

$$\|U_n^*h - U^*h\|^2 = (U_n^*h|U_n^*h) - 2\Re(U^*h|U_n^*h) + (U^*h|U^*h) = 2\big(\|h\|^2 - \Re(U_nU^*h|h)\big).$$

Since $U_n(U^*h) \to U(U^*h) = h$, we therefore have $\|U_n^*h - U^*h\|^2 \to 0$. So inversion is continuous as claimed.

Example 2.4. Let A be a C^*-algebra, and let $\mathrm{Aut}(A)$ denote the collection of all automorphisms of A. Endow $\mathrm{Aut}(A)$ with the strong topology: $\alpha_n \to \alpha$ if and only if $\alpha_n(a) \to \alpha(a)$ for each $a \in A$. Then $\mathrm{Aut}(A)$ is a topological group.

Observe that if G is a topological group, then $g \mapsto g^{-1}$ is a continuous bijection of G onto itself and is its own inverse. It is therefore a homeomorphism of G. Similarly, for fixed $g \in G$, the map $h \mapsto gh$ is continuous, and has continuous inverse $h \mapsto g^{-1}h$, so is a homeomorphism of G. This implies that the topology on G is determined by the topology near e.

We will deal mainly with locally compact groups. These are the topological groups in which every point has a compact neighbourhood.

A measure μ on a locally compact space X is called a *Borel* measure if every open set is μ-measurable. It is called a *Radon* measure if: (1) for every open set V we have

$$\mu(V) = \sup\{\mu(K) : K \subseteq V \text{ is compact}\};$$

and (2) for every μ-measurable set A, we have

$$\mu(A) = \inf\{\mu(U) : U \supseteq A \text{ is open}\}.$$

If G is a locally compact group, and μ is a Radon measure on G, then μ is *left-invariant* if $\mu(gA) = \mu(A)$ for every measurable A and every $g \in G$. It is *right-invariant* if $\mu(Ag) = \mu(A)$ for every measurable A and every $g \in G$. If it is both left- and right-invariant, then we say μ is *invariant*.

Definition 2.5. Let G be a locally compact group. We say that a measure μ on G is a *Haar measure* if it is a left-invariant Radon measure. A right-invariant Radon measure is called a right Haar measure.

Theorem 2.6. *Every second-countable locally compact group G admits a Haar measure μ. For any two Haar measures μ, μ' on G, there is a strictly positive scalar k such that $\mu(S) = k\mu'(S)$ for every measurable set S. If G is compact then μ is invariant, and is assumed to be a probability measure.*

We are not going to prove the theorem or even discuss the idea of the proof. For our purposes, the following characterisation suffices. For $g \in G$, let $\lambda(g) : C_c(G) \to C_c(G)$ be given by $\lambda(g)(f)(h) = f(g^{-1}h)$. Then a Haar measure on G determines (and is determined by) a linear functional

$$I : C_c(G) \to \mathbb{C}$$

such that $I(f) > 0$ whenever f is a nonnegative function and $f(x) > 0$ for some x and such that if $I(\lambda(g)f) = I(f)$ for all $f \in C_c(G)$. So the theorem says that there is such a functional, and any two such functionals differ by a strictly positive scalar. In a sense which can be made precise, the functional I is given by

$$I(f) = \int_G f(g)\, d\mu(g).$$

Example 2.7. If G is a discrete group, then $C_c(G)$ is the collection of finitely supported functions on G, Haar measure on G is just counting measure, and the functional I is given by $I(f) = \sum_{g \in G} f(g)$. Observe that in this case, Haar measure is again invariant: we have $I(h \mapsto f(hg)) = I(f)$ for all $f \in C_c(G)$.

Example 2.8. If G is equal to $\mathbb{R}^n$ or $\mathbb{T}^n$, then Haar measure coincides with Lebesgue measure.

2.2. **Dynamical systems.** Given a topological space X, we write $\mathrm{Homeo}(X)$ for the group of homeomorphisms of X under composition. We endow it with the topology of pointwise convergence.

Definition 2.9. Let X be a locally compact Hausdorff space and G a locally compact group. An *action* of G on X is a homomorphism $\alpha : g \mapsto \alpha_g$ from G to $\mathrm{Homeo}(X)$ such that $(g, x) \mapsto \alpha_g(x)$ is continuous from $G \times X$ to X.

Remark 2.10. Observe that if α is an action of G on X as above, then the fact that it is a homomorphism means precisely that $\alpha_g(\alpha_h(x)) = \alpha_{gh}(x)$ for all $g, h \in G$ and $x \in X$. Since every group homomorphism preserves the identity element, α_e is the identity homeomorphism: that is $\alpha_e(x) = x$ for all x. Likewise, since group homomorphisms preserve inverses, $\alpha_g^{-1} = \alpha_{g^{-1}}$ for all g.

We often write $g \cdot x$ for $\alpha_g(x)$, so the axioms become $e \cdot x = x$ and $g \cdot (h \cdot x) = (gh) \cdot x$. We also often describe the third requirement above as *joint continuity* of the action.

Example 2.11. Let X be a locally compact Hausdorff space and let α be a homeomorphism of X. Then $\mathbb{Z}$ acts on X by the formula $n \cdot x = \alpha^n(x)$.

For a specific example, fix $\theta \in [0, 1)$ and define a homeomorphism r_θ of $\mathbb{T}$ by $z \mapsto e^{2\pi i\theta} z$. This is the *rotation* homeomorphism.

Let X be a locally compact Hausdorff space, let G be a locally compact group, and suppose that G acts on X. For each $g \in G$ there is a C^*-homomorphism $\alpha_g : C_0(X) \to C_0(X)$ given by

$$\alpha_g(f)(x) = f(g^{-1} \cdot x).$$

We have $\alpha_{g^{-1}} = \alpha_g^{-1}$. Moreover, for $g, h \in G$, $f \in C_0(X)$ and $x \in X$ we have

$$\alpha_g(\alpha_h(f))(x) = \alpha_h(f)(g^{-1} \cdot x) = f(h^{-1} g^{-1} \cdot x) = \alpha_{gh}(f)(x).$$

So $g \mapsto \alpha_g$ is a homomorphism of G into the group $\mathrm{Aut}(C_0(X))$ of automorphisms of $C_0(X)$.

Proposition 2.12. *Let X be a locally compact Hausdorff space, let G be a locally compact group, and suppose that G acts on X. Then the homomorphism $\alpha : G \to \mathrm{Aut}(C_0(X))$ above is continuous with respect to the strong topology on $\mathrm{Aut}(C_0(X))$.*

Proof. Fix $f \in C_0(X)$ and $\varepsilon > 0$. Suppose that $g_n \to g$ in G. Since G is locally compact, there is a compact neighbourhood V of g; and since then $V \cup \{e\}$ is also compact, we may assume that $e \in V$. Fix $N_1 \in \mathbb{N}$ such that $g_n \in V$ for $n \geq N_1$. Let $K = \{x : f(x) \geq \varepsilon/2\}$. Since $V \cdot K$ is the continuous image of the compact set $V \times K$, it is itself compact, and it contains K since $e \in V$. Since $g_n \to g$, we have $\alpha_{g_n}(f)(x) = f(g_n^{-1} \cdot x) \to f(g^{-1} \cdot x) = \alpha_g(f)(x)$ for all $x \in X$. In particular. $\alpha_{g_n}(f) \to \alpha_g(f)$ pointwise on $V \cdot K \subseteq X$. Since $V \cdot K$ is compact, pointwise convergence on K implies uniform convergence, and so there exists N_2 such that

$$\big|\big(\alpha_{g_n}(f) - \alpha_g(f)\big)(x)\big| < \varepsilon \quad \text{for all } n \geq N_2 \text{ and } x \in V \cdot K.$$

Fix $n \geq \max\{N_1, N_2\}$.

$$\begin{aligned}\|\alpha_{g_n}(f) - \alpha_g(f)\|_\infty &= \max\Big\{\sup_{x \in V \cdot K} |f(g_n^{-1} \cdot x) - f(g^{-1} \cdot x)|,\\ &\qquad\qquad \sup_{x \in X \setminus V \cdot K} |f(g_n^{-1} \cdot x) - f(g^{-1} \cdot x)|\Big\}\\ &\leq \max\Big\{\sup_{x \in V \cdot K} |\alpha_{g_n}(f)(x) - \alpha_g(f)(x)|,\\ &\qquad\qquad \sup_{x \in X \setminus V \cdot K} |f(g_n^{-1} \cdot x)| + |f(g^{-1} \cdot x)|\Big\}.\end{aligned}$$

Since $n \geq N_1$, each $g_n \in V$ and so $x \notin V \cdot K$ implies $g_n^{-1} \cdot x \notin K$, and similarly $g \cdot x \notin K$, giving $|f(g_n^{-1}x)| + |f(g^{-1}x)| < \varepsilon$, giving

$$\|\alpha_{g_n}(f) - \alpha_g(f)\|_\infty \leq \max\big\{ \sup_{x \in V \cdot K} |f(g_n^{-1} \cdot x) - f(g^{-1} \cdot x)|, \varepsilon \big\}.$$

Since $|f(g_n^{-1} \cdot x) - f(g^{-1} \cdot x)| < \varepsilon$ for all $x \in V \cdot K$ by choice of N_2, we then have $\|\alpha_{g_n}(f) - \alpha_g(f)\|_\infty < \varepsilon$, and the result follows. □

With this as our motivating example, we define a C^*-dynamical system as follows.

Definition 2.13. A *C^*-dynamical system* (A, G, α) consists of a C^*-algebra A, a locally compact Hausdorff group G and a homomorphism $\alpha : G \to \mathrm{Aut}(A)$ which is continuous with respect to the strong topology on $\mathrm{Aut}(A)$.

Example 2.14. Let $\mathbb{T}$ be the unit circle in the complex plane. Fix $\theta \in [0, 1)$ and define a homeomorphism h of $\mathbb{T}$ by "rotation by θ;" that is $h(z) := e^{2\pi i\theta} z$. By the above, there is a C^*-dynamical system $(C_0(\mathbb{T}), \mathbb{Z}, \alpha)$ determined by $\alpha_n(f)(z) = f(e^{-2n\pi i\theta} z)$.

It turns out that every C^*-dynamical system in which A is a commutative algebra arises from an action of G on a space X.

Proposition 2.15. *Let A be a commutative C^*-algebra, and let Δ denote its maximal ideal space. If α is an automorphism of A, then there is a unique homeomorphism $\alpha^* : \Delta \to \Delta$ such that $\phi \circ \alpha = \alpha^*(\phi)$ for all $\phi \in \Delta$. If (A, G, α) is a C^*-dynamical system, then there is an action of G on Δ given by $g \cdot \phi := \alpha^*_{g^{-1}}(\phi)$.*

Proof. Fix $\phi \in \Delta$. That is, ϕ is a nonzero homomorphism from A to $\mathbb{C}$. Then $\phi \circ \alpha$ is also a homomorphism from A to $\mathbb{C}$ and therefore belongs to Δ, and so there is a map $\alpha^* : \Delta \to \Delta$ defined by $\alpha^*(\phi) = \phi \circ \alpha$. Since $(\phi \circ \alpha) \circ \alpha^{-1} = \phi = (\phi \circ \alpha^{-1}) \circ \alpha$, the map α^* is bijective. By definition of the topology on Δ we have $\phi_n \to \phi$ in Δ if and only if $\phi_n(a) \to \phi(a)$ for each $a \in A$, and then

$$\alpha^*(\phi_n)(a) = \phi_n(\alpha(a)) \to \phi(\alpha(a)) = \alpha^*(\phi_n)(a)$$

for each $a \in A$. Hence α^* is continuous. Since $(\alpha^*)^{-1} = (\alpha^{-1})^*$ is also continuous, it follows that α^* is a homeomorphism.

Now suppose that (A, G, α) is a dynamical system. For $g, h \in G$ and $\phi \in \Delta$, we have

$$(gh) \cdot \phi = \alpha^*_{(gh)^{-1}}(\phi) = \phi \circ (\alpha_g \circ \alpha_h)^{-1} = \phi \circ \alpha_h^{-1} \circ \alpha_g^{-1} = \alpha^*_{g^{-1}}(\alpha^*_{h^{-1}}(\phi)) = g \cdot (h \cdot \phi),$$

and

$$\alpha^*_{g^{-1}}(\phi) = \phi \circ \alpha_{g^{-1}} = \phi \circ \alpha_g^{-1} = (\alpha^*_g)^{-1}(\phi).$$

So α^* is a homomorphism of G to $\mathrm{Homeo}(\Delta)$. For continuity, suppose that $g_n \to g$ and $\phi_n \to \phi$. Then for $a \in A$, since each ϕ_n is norm-decreasing we have

$$\begin{aligned}\|\alpha^*_{g_n}(\phi_n)(a) - \alpha^*_g(\phi)(a)\| &= \|\phi_n(\alpha_{g_n}(a)) - \phi_n(\alpha_g(a)) + \phi_n(\alpha_g(a)) - \phi(\alpha_g(a))\| \\ &\leq \|\alpha_{g_n}(a) - \alpha_g(a)\| + \|\phi_n(\alpha_g(a)) - \phi(\alpha_g(a))\|.\end{aligned}$$

Hence $\alpha^*_{g_n}(\phi_n)(a) \to \alpha^*_g(\phi)(a)$ for each $a \in A$; that is $\alpha^*_{g_n}(\phi_n) \to \alpha^*_g(\phi)$. □

Corollary 2.16. *If X is a locally compact Hausdorff space, then for every C^*-dynamical system $(C_0(X), G, \alpha)$, there is a unique action of G on X such that $\alpha_g(f)(x) = f(g^{-1} \cdot x)$ for all $f \in C_0(X)$ and $x \in X$. We call this action of G on X the action* induced *by α.*

Proof. This follows from Proposition 2.15 and Gelfand theory for the commutative C^*-algebra $C_0(X)$. □

2.3. **Covariant representations and crossed products.** To describe the crossed product C^*-algebra of a C^*-dynamical system (A, G, α), we need some background from the theory of group C^*-algebras (details can be found in Appendix C3 of [8]); we won't go into details, because we are going to specialise to the situation where G is discrete, and there things simplify significantly.

To avoid discussing modular functions on topological groups, we will suppose that Haar measure on G is invariant. This is a nontrivial assumption, but it holds trivially if G is abelian, and it also holds automatically if G is either discrete or compact.

Suppose that Haar measure on G is invariant. One can check that $C_c(G)$ becomes a *-algebra when endowed with the convolution product

$$(f * g)(t) = \int_G f(s)g(s^{-1}t)\,d\mu(s)$$

and the involution

$$f^*(s) = \overline{f(s^{-1})}.$$

Let $\mathcal{UM}(A)$ denote the group of unitary elements of the multiplier algebra of A. We endow it with the topology of strong operator convergence inherited from $\mathcal{L}(A_A)$; that is, $U_n \to U$ if and only if $U_n a \to Ua$ for all $a \in A$. This is called the strict topology. Now suppose that u is a continuous homomorphism of G into $\mathcal{UM}(A)$. Then there is a unique homomorphism $I_u : C_c(G) \to \mathcal{M}(A)$ such that for every representation π of $\mathcal{M}(A)$ on a Hilbert space $\mathcal{H}$,

$$\big(\pi(I_u(f))\xi \mid \eta\big) = \int_G f(s)\big(\pi(u_s)\xi \mid \eta\big)\,d\mu(s) \quad \text{for all } f \in C_c(G) \text{ and all } \xi, \eta \in \mathcal{H}.$$

This homomorphism I_u is bounded with respect to the 1-norm on $C_c(G)$. Conversely, every 1-norm bounded homomorphism of $C_c(G)$ arises in this way.

Definition 2.17. Let (A, G, α) be a C^*-dynamical system and B be a C^*-algebra. A *covariant homomorphism* from (A, G, α) to B is a pair (π, u) where $\pi : A \to \mathcal{M}(B)$ is a homomorphism of C^*-algebras and $u : G \to \mathcal{UM}(B)$ is a strongly continuous homomorphism of G into the unitary group of $\mathcal{M}(B)$ such that $\pi(a)I_u(f) \in B$ for all $a \in A$ and $f \in C_c(G)$ and such that

$$\pi(\alpha_g(a)) = u_g \pi(a) u_g^* \quad \text{for all } g \in G \text{ and } a \in A.$$

A covariant homomorphism (π, u) is *nondegenerate* if π is nondegenerate. We call (π, u) a covariant *representation* if $B = \mathcal{B}(\mathcal{H})$ for some Hilbert space H.

Example 2.18. Let G be a locally compact group acting on itself by translation; that is $g \cdot h = gh$. Let $\mathcal{H} := L^2(G)$. Define $\pi : C_0(G) \to \mathcal{B}(\mathcal{H})$ by $\pi(f)\xi(h) = f(h)\xi(h)$; then π is a nondegenerate representation of $C_0(G)$. Define $\lambda : G \to \mathcal{U}(\mathcal{H})$ by $\lambda_g(\xi)(h) = \xi(g^{-1}h)$. Then, with some work, one can check that (π, λ) is a nondegenerate covariant homomorphism of $(C_0(G), G, \mathrm{lt})$ into $\mathcal{B}(\mathcal{H})$.

Exercise 1. Let G be a discrete group. Then Haar measure on G is just counting measure, and $L^2(G) = \ell^2(G)$. For this example, verify that the pair (π, λ) of the preceding example really is a covariant representation.

Exercise 2. For a somewhat harder exercise, repeat the preceding exercise for an arbitrary locally compact group G.

Example 2.19. Fix $\theta \in [0,1)$, and let α be the action of $\mathbb{Z}$ on $C(\mathbb{T})$ induced by rotation by θ, so that $\alpha_n(f)(z) = f(e^{-2n\pi i\theta}z)$. Let $\mathcal{H} = \ell^2(\mathbb{Z})$. For $w \in \mathbb{T}$, define $\pi_w : C(\mathbb{T}) \to \mathcal{B}(\mathcal{H})$ by

$$\pi_w(f)\xi_n = f(e^{2n\pi i\theta}w)\xi_n.$$

Recall that $\lambda : \mathbb{Z} \to \mathcal{U}(\mathcal{H})$ is given by $\lambda_n(f)(m) = f(m-n)$. If f is the basis element ξ_p, then $\lambda_n(\xi_p)(m) = \xi_p(m-n)$ is equal to 1 if $m-n = p$ and zero otherwise; that is $\lambda_n(\xi_p)(m) = \delta_{p+n,m}$. Hence $\lambda_n(\xi_p) = \xi_{p+n}$. We check that (π_w, λ) is covariant: for $m, n \in \mathbb{Z}$ we have

$$\begin{aligned}
\lambda_n \pi_w(f) \lambda_n^* \xi_m &= \lambda_n \pi_w(f) \xi_{m-n} \\
&= \lambda_n(f(e^{2(m-n)\pi i\theta}w)\xi_{m-n}) \\
&= f(e^{-2n\pi i\theta}e^{2m\pi i\theta}w)\xi_m \\
&= \alpha_n(f)(e^{2m\pi i\theta}w)\xi_m \\
&= \pi_w(\alpha_n(f))\xi_m.
\end{aligned}$$

Theorem 2.20. *Let (A, G, α) be a C^*-dynamical system. There exist a C^*-algebra $A \times_\alpha G$ and a nondegenerate covariant homomorphism (i_A, i_G) from (A, G, α) to $\mathcal{M}(A \times_\alpha G)$ such that*

1. *$A \times_\alpha G = \overline{\operatorname{span}}\{i_A(a)I_{i_G}(f) : a \in A, f \in C_c(G)\}$, and*
2. *if (π, u) is a nondegenerate covariant homomorphism from (A, G, α) to B, then there is a homomorphism $\pi \times u : A \times_\alpha G \to B$ whose extension to $\mathcal{M}(A \times_\alpha G)$ satisfies $(\pi \times u) \circ i_A = \pi$ and $(\pi \times u) \circ i_G = u$.*

The homomorphism i_A is injective. The pair $(A \times_\alpha G, (i_A, i_G))$ is unique up to canonical isomorphism.

The C^*-algebra $A \times_\alpha G$ is called the *crossed product* of A by G, and the covariant homomorphism (i_A, i_G) is called the *universal covariant representation*, even though it is not, strictly speaking, a representation at all.

2.4. **Crossed products by discrete groups.** We will not prove Theorem 2.20 as stated. The details are formidable, but you can find them in Dana Williams' excellent book [9].

Here, we restrict our attention to discrete groups G. Let us begin by observing that if G is discrete, then for each $g \in G$, the point-mass function δ_g belongs to $C_c(G)$, and for $f \in C_c(G)$, we have $\delta_g * f(s) = \sum_{t \in G} \delta_g(t) f(t^{-1}s) = f(g^{-1}s)$, and $\delta_g^*(s) = \overline{\delta_g(s^{-1})} = \delta_{g^{-1}}(s)$. In particular, δ_e is the identity element of $C_c(G)$, and $g \mapsto \delta_g$ is a *-representation of G in $C_c(G)$.

It follows from this that if (i_A, i_G) is a covariant homomorphism of a C^*-dynamical system (A, G, α) in $\mathcal{M}(B)$, then $i_G(\delta_e) = i_G(e) = 1_{\mathcal{M}(B)}$, and hence $i_A(A) = i_A(A)i_G(\delta_e) \subseteq \overline{\operatorname{span}}\{i_A(a)I_{i_G}(f) : a \in A, f \in C_c(G)\}$.

We now have $C_c(G) = \operatorname{span}\{\delta_g : g \in G\}$, so the correspondence between unitary representations of G and representations of $C_c(G)$ described above boils down to linear extension from $\{\delta_g : g \in G\}$. The following is the specialisation of Theorem 2.20 to discrete groups.

Theorem 2.21. *Let G be a discrete group, and let (A, G, α) be a C^*-dynamical system. There exist a C^*-algebra $A \times_\alpha G$ and a nondegenerate covariant homomorphism (i_A, i_G) from (A, G, α) to $\mathcal{M}(A \times_\alpha G)$ such that*

(1) $A \times_\alpha G = \overline{\operatorname{span}}\{i_A(a)i_G(g) : a \in A, g \in G\}$, *and*
(2) *if* (π, u) *is a nondegenerate covariant homomorphism from* (A, G, α) *to* $\mathcal{M}(B)$, *then there is a homomorphism* $\pi \times u : A \times_\alpha G \to B$ *whose extension to* $\mathcal{M}(A \times_\alpha G)$ *satisfies* $(\pi \times u) \circ i_A = \pi$ *and* $(\pi \times u) \circ i_G = u$.

The pair $(A \times_\alpha G, (i_A, i_G))$ *is unique up to canonical isomorphism, and* $i_A(A) \subseteq A \times_\alpha G$.

To prove the theorem we construct a *-algebra $C_c(G, A)$ such that every covariant homomorphism of (A, G, α) determines a *-homomorphism of $C_c(G, A)$. We can then take the supremum over all such representations to obtain a seminorm on $C_c(G, A)$, and we can then complete to obtain the desired universal C^*-algebra.

Proposition 2.22. *Let* (A, G, α) *be a* C^**-dynamical system and suppose that* G *is discrete. The vector space* $C_c(G, A)$ *of finitely supported functions* f *from* G *to* A *is a complex* **-algebra under the operations*

$$(f * g)(s) = \sum_{t \in G} f(t)\alpha_t(g(t^{-1}s)) \quad \text{ and } f^*(s) = \alpha_s(f(s^{-1})^*).$$

Suppose that (π, u) *is a covariant homomorphism from* (A, G, α) *to* $\mathcal{M}(B)$. *Then there is a* **-homomorphism* $i_{\pi,u} : C_c(G, A) \to \mathcal{M}(B)$ *such that* $i_{\pi,u}(f) = \sum_{s \in G} \pi(f(s))u_s$.

Proof. It is routine to see that $(f, g) \mapsto f * g$ is a bilinear map from $C_c(G, A) \times C_c(G, A)$ to $C_c(G, A)$, and we have

$$\begin{aligned}
f * (g * h)(s) &= \sum_{t \in G} f(t)\alpha_t\Big(\sum_{r \in G} g(r)\alpha_r(h(r^{-1}t^{-1}s))\Big) \\
&= \sum_{t,r \in G} f(t)\alpha_t(g(t^{-1}tr))\alpha_{tr}(h((tr)^{-1}s)) \\
&= \sum_{r' \in G} \Big(\sum_{t \in G} f(t)\alpha_t(g(t^{-1}r'))\Big)\alpha_{r'}(h(r'^{-1}s)) \\
&= \sum_{r' \in G} (f * g)(r')\alpha_{r'}(h(r'^{-1}s)) \\
&= \big((f * g) * h\big)(s).
\end{aligned}$$

So $C_c(G, A)$ is an algebra under $(f, g) \mapsto f * g$. Clearly $f \mapsto f^*$ is conjugate-linear and self-inverse. Moreover,

$$\begin{aligned}
(f * g)^*(s)^* &= \alpha_s\big((f * g)(s^{-1})\big) \\
&= \alpha_s\Big(\sum_{t \in G} f(t)\alpha_t(g(t^{-1}s^{-1}))\Big) \\
&= \sum_{t \in G} \alpha_s(f(t))\alpha_{st}(g(t^{-1}s^{-1})) \\
&= \sum_{r \in G} \alpha_s(f(s^{-1}r))\alpha_r(g(r^{-1})) \\
&= \sum_{r \in G} \big(g^*(r)f^*(r^{-1}s)\big)^* \\
&= (g^* * f^*)(s)^*.
\end{aligned}$$

Thus $C_c(G, A)$ is a *-algebra.

Suppose that (π, u) is a covariant homomorphism from (A, G, α) to $\mathcal{M}(B)$. Then $i_{\pi,u}(f) := \sum_{s\in G} \pi(f(s))u_s$ determines is a linear map $i_{\pi,u} : C_c(G, A) \to \mathcal{M}(B)$. For $f, g \in C_c(G, A)$, we have

$$\begin{aligned} i_{\pi,u}(f)i_{\pi,u}(g) &= \sum_{s,t\in G} \pi(f(s))u_s\pi(g(t))u_t \\ &= \sum_{s,t\in G} \pi(f(s))u_s\pi(g(t))u_s^*u_su_t \\ &= \sum_{s,t\in G} \pi(f(s))\pi(\alpha_s(g(t)))u_{st} \\ &= \sum_{t'\in G} \pi\Big(\sum_{s\in G} f(s)\alpha_s(g(s^{-1}t'))\Big)u_{t'} \\ &= \sum_{t'\in G} \pi\big((f * g)(t')\big)u_{t'} \\ &= i_{\pi,u}(f * g). \end{aligned}$$

Likewise

$$\begin{aligned} i_{\pi,u}(f^*) &= \sum_{s\in G} \pi(f^*(s))u_s \\ &= \sum_{s\in G} \pi(\alpha_s(f(s^{-1})^*))u_s \\ &= \sum_{s\in G} u_s\pi(f(s^{-1})^*)u_s^*u_s \\ &= \sum_{s\in G} u_s\pi(f(s^{-1})^*) \\ &= \sum_{s\in G} \big(\pi(f(s^{-1}))u_{s^{-1}}\big)^* \\ &= \Big(\sum_{s'\in G} \pi(f(s'))u_{s'}\Big)^* \\ &= i_{\pi,u}(f)^*. \end{aligned}$$

Hence $\pi \times u$ is a *-representation of $C_c(G, A)$. □

Proof of Theorem 2.21. Let (π, u) be a covariant homomorphism of (A, G, α). Then for $f \in C_c(G, A)$, we have

$$\|i_{\pi,u}(f)\| = \Big\| \sum_{s\in G} \pi(f(s))u_s \Big\| \le \sum_{s\in G} \|\pi(f(s))\|\|u_s\| \le \sum_{s\in G} \|f(s)\|.$$

Consequently

$$\|f\| := \sup\{\|i_{\pi,u}(f)\| : (\pi, u) \text{ a covariant homomorphism}\}$$

determines a seminorm on $C_c(G, A)$. We define $A \times_\alpha G$ to be the C^*-completion of $C_c(G, A)$ with respect to this seminorm, and let $q : C_c(G, A) \to A \times_\alpha G$ be the canonical homomorphism.

For $a \in A$ and $s \in G$, let $a1_s$ denote the element of $C_c(G, A)$ such that $(a1_s)(t) = \delta_{s,t}a$. Define $i_A : A \to A \times_\alpha G$ by $i_A(a) = q(a1_e)$; then $i_A(a)i_A(b) = \pi(a)u_e\pi(b)u_e = \pi(ab)u_e = i_A(ab)$ and $i_A(a^*) = \pi(a^*) = \pi(a)^* = i_A(a)^*$, so i_A is a homomorphism from A to $A \times_\alpha G$. For $s \in G$, there is a map $\mathrm{lt}_s : C_c(G, A) \to C_c(G, A)$ given by $\mathrm{lt}_s(f)(t) = \alpha_s(f(s^{-1}t))$. For any covariant homomorphism (π, u), and any $f \in C_c(G, A)$, we have

$$\begin{aligned} i_{\pi,u}(\mathrm{lt}_s(f)) &= \sum_{t\in G} \pi(\alpha_s(f(s^{-1}t)))u_t \\ &= \sum_{t\in G} u_s f(s^{-1}t)u_{s^{-1}u_t} = \sum_{t'\in G} u_s f(t')u_{t'} = u_s i_{\pi,u}(f). \end{aligned}$$

In particular,

$$\|i_{\pi,u}(\mathrm{lt}_s(f))\| = \|u_s i_{\pi,u}(f)\| \le \|i_{\pi,u}(f)\|,$$

and hence $\|q(\mathrm{lt}_s(f))\| \le \|q(f)\|$ for all f. Thus there is a well-defined map $i_G(s) : A \times_\alpha G \to A \times_\alpha G$ given by $i_G(s)(q(f)) = q(\mathrm{lt}_s(f))$. For $f, g \in C_c(G, A)$, we have

$$\begin{aligned} \big(\mathrm{lt}_s(f)^* * g\big)(r) &= \sum_{t\in G} \mathrm{lt}_s(f)^*(t)\alpha_t(g(t^{-1}r)) \\ &= \sum_{t\in G} \alpha_t(\mathrm{lt}_s(f)(t^{-1})^*)\alpha_t(g(t^{-1}r)) \\ &= \sum_{t\in G} \alpha_t\big(\alpha_s(f(s^{-1}t^{-1})^*)g(t^{-1}r)\big) \\ &= \sum_{q\in G} \alpha_q(f(q^{-1})^*)\alpha_{qs^{-1}}(g(sq^{-1}r)) \\ &= \sum_{q\in G} f^*(q)\alpha_q(\mathrm{lt}_{s^{-1}}(g)(q^{-1}r)) \\ &= \big(f^* * \mathrm{lt}_{s^{-1}}(g)\big)(r) \end{aligned}$$

Thus, under the standard inner product on $(A \times_\alpha G)_{A\times_\alpha G}$, we have $\langle i_G(s)(q(f)), q(g)\rangle = \langle q(f), i_G(s^{-1})q(g)\rangle$. In particular, $i_G(s)$ is adjointable with adjoint $i_G(s^{-1})$. We clearly have $i_G(e) = 1$ and $i_G(st) = i_G(s)i_G(t)$. Fix $a \in A$, $s \in G$ and $f \in C_c(G, A)$. Under any covariant homomorphism (π, u) we have

$$\begin{aligned} &\|i_{\pi,u}(\mathrm{lt}_s(a1_e) * \mathrm{lt}_{s^{-1}}(f)) - i_{\pi,u}(\alpha_s(a)1_e * f)\| \\ &\qquad = \|u_s\pi(a)u_s^* i_{\pi,u}(f) - \pi(\alpha_s(a))i_{\pi,u}(f)\| = 0. \end{aligned}$$

Hence $\|q(\mathrm{lt}_s(a1_e) * \mathrm{lt}_{s^{-1}}(f)) - q(\alpha_s(a)1_e * f)\| = 0$. That is $i_G(s)i_A(a)i_G(s)^*q(f) = i_A(\alpha_s(a))q(f)$. Since $q(C_c(G, A))$ is dense in $A \times_\alpha G$, it follows that (i_A, i_G) is a covariant homomorphism.

For the universal property, observe that for any covariant homomorphism (π, u) to $\mathcal{M}(B)$ and any $f \in C_c(G)$ we have $\|i_{\pi,u}(f)\| \le \|i_{i_A,i_G}(f)\|$, so the assignment $i_{i_A,i_G}(f) \mapsto i_{\pi,u}(f)$ determines a well-defined linear map $\pi \times u : A \times_\alpha G \to \mathcal{M}(B)$ with the desired properties.

For uniqueness observe that if $(B, (\pi, u))$ has the same two properties, then the two universal properties give mutually inverse homomorphisms between $A\times_\alpha G$ and $\overline{\mathrm{span}}\{\pi(a)u_s : a \in A, s \in G\} = B$. We have $i_A(A) \subseteq A \times_\alpha G$ by construction. $\square$

Proposition 2.23. *Let G be a discrete group, and let (A, G, α) be a C^*-dynamical system. The *-homomorphism $i_{i_A,i_G} : C_c(G, A) \to A \times_\alpha G$ obtained from Proposition 2.22 is injective. In particular, i_A is injective.*

Proof. We construct a covariant representation (π, U) of (A, G, α) such that $i_{\pi,U}$ is injective. Since $i_{\pi,U} = (\pi \times U) \circ i_{i_A,i_G}$, the result follows.

Define $\langle\cdot,\cdot\rangle_A : C_c(G, A) \times C_c(G, A) \to A$ by $\langle f, g\rangle_A := \sum_{s\in G} f(s)^* g(s)$. Observe that if $f \in C_c(G, A)$, then $\langle f, f\rangle_A = \sum_s f(s)^* f(s)$ is positive, and if $f \neq 0$ then $\langle f, f\rangle_A \geq f(s)^* f(s) > 0$ for any s such that $f(s)$ is nonzero. Hence $\|f\|^2 := \|\langle f, f\rangle_A\|$ defines a norm on $C_c(G, A)$ (it's the same as the ℓ^2 norm on the algebraic direct sum $\bigoplus_{s\in G} A_A$.

Let X be the completion of $C_c(G, A)$ in this norm. Then X is a Hilbert A-module. For each $a \in A$ the operator $\pi(a) : X \to X$ determined by $(\pi(a)x)_s = \alpha_{s^{-1}}(a)x$ is adjointable with adjoint $\pi(a^*)$. So $\pi : A \to \mathcal{L}(X)$ is a homomorphism, and is clearly injective. For $t \in G$, define $U_t : C_c(G, A) \to C_c(G, A)$ by $U_t(f)(s) = f(t^{-1}s)$. Then $(U_r U_s)(f)(t) = U_s(f)(r^{-1}t) = f(s^{-1}r^{-1}t) = U_{rs}(f)(t)$, and clearly $U_e(f) = f$. We have $\langle U_s f, U_s f\rangle_A = \langle f, f\rangle_A$ for all $f \in C_c(G)$, so U_s is bounded and therefore extends to all of X. Since $\langle U_s f, g\rangle_A = \langle f, U_{s^{-1}} g\rangle_A$ for all $f, g \in C_g(G, A)$, we have $U_s \in \mathcal{L}(X)$ with $U_s^* = U_{s^{-1}}$. Thus $U : s \mapsto U_s$ is a homomorphism of G into $\mathcal{UL}(X)$ which is trivially continuous because G is discrete. For $a \in A$, $s, t \in G$, and $f \in C_c(G, A)$ we have

$$\begin{aligned}
(U_s \pi(a) U_s^* f)(t) &= (\pi(a) U_s^* f)(s^{-1}t) \\
&= \alpha_{t^{-1}s}(a)\big((U_s^* f)(s^{-1}t)\big) \\
&= \alpha_{t^{-1}s}(a) f(t) \\
&= \alpha_{t^{-1}}(\alpha_s(a)) f(t) \\
&= \big(\pi(\alpha_s(a)) f\big)(t).
\end{aligned}$$

Hence (π, U) is a covariant homomorphism of (A, G, α). Suppose that $f \in C_c(G, A)$ satisfies $i_{\pi,U}(f) = 0$. That is $\sum_{s\in G} \pi(f(s)) U_s = 0$. Fix $s \in G$. Let $f(s)^* 1_{s^{-1}} \in C_c(G, A)$ be the function that takes the value $f(s)^*$ at s^{-1} and zero elsewhere. Then

$$\begin{aligned}
0 &= i_{\pi,U}(f)(f(s)^* 1_{s^{-1}})(e) \\
&= \sum_{t\in G} \big(\pi(f(t)) U_t(f(s)^* 1_{s^{-1}})\big)(e) \\
&= \sum_{t\in G} \alpha_e(f(t))(U_t(f(s)^* 1_{s^{-1}}))(e) \\
&= \sum_{t\in G} f(t)(f(s)^* 1_{s^{-1}})(t^{-1}) \\
&= f(s) f(s)^*.
\end{aligned}$$

Thus $f(s) = 0$ and since $s \in G$ was arbitrary, $f = 0$. ∎

The universal property has many uses. Here's one. If (A, G, α) and (B, G, β) are C^*-dynamical systems, then we say that a homomorphism $\pi : A \to B$ is *equivariant* for α and β (or just G-equivariant) if $\pi(\alpha_s(a)) = \beta_s(\pi(a))$ for all $a \in A$ and $s \in G$.

Proposition 2.24. *Suppose that G is a discrete group and that (A, G, α) and (B, G, β) are C^*-dynamical systems. Suppose that $\pi : A \to B$ is a G-equivariant homomorphism.*

Then $(i_B \circ \pi, i_G^B)$ *is a covariant homomorphism of* (A, α, G)*. The induced homomorphism*

$$\pi \times 1 := (i_B \circ \pi) \times i_G^B : A \times_\alpha G \to B \times_\beta G$$

satisfies $(\pi \times 1)(i_A(a)i_G^A(s)) = i_B(\pi(a))i_G^B(s)$ *for all* $a \in A$ *and* $s \in G$*.*

Proof. We calculate:

$$i_G^B(s)(i_B \circ \pi(a))i_G^B(s)^* = i_B(\beta_s(\pi(a)) = i_B(\pi(\alpha_s(a))) = (i_B \circ \pi)(\alpha_s(a)).$$

The universal property of $A \times_\alpha G$ now implies that there is a homomorphism $\pi \times 1 : A \times_\alpha G \to B \times_\beta G$ with the desired property.

Suppose that π is surjective. Fix a spanning element $i_B(b)i_G^B(s)$ of $B \times_\beta G$. we have $b = \pi(a)$ for some a, and then $i_B(b)i_G^B(s) = (\pi \times 1)(i_A(a)i_G^A(s))$ belongs to the image of $\pi \times 1$. □

We finish by proving that our favourite example, the rotation crossed product, is a simple C^*-algebra when θ is irrational. The argument we give here is a little round-about (Davidson gives a beautiful and much more efficient proof in [1, Section VI.1]), but it emphasises the importance of the dual action on a crossed-product (Lemma 2.25) and the resulting characterisation of injectivity of homomorphisms (Corollary 2.28).

Lemma 2.25. *Let* $(A, \mathbb{Z}, \alpha)$ *be a* C^**-dynamical system. There is a strongly continuous action* $\hat{\alpha}$ *of* $\mathbb{T}$ *on* $A \times_\alpha \mathbb{Z}$ *such that* $\hat{\alpha}_z(i_A(a)i_\mathbb{Z}(n)) = z^n i_A(a)i_\mathbb{Z}(n)$ *for all* a, n*.*

Proof. One checks that for $z \in \mathbb{T}$, the pair $(a \mapsto i_A(a), n \mapsto z^n i_\mathbb{Z}(n))$ is a covariant homomorphism of $(A, \mathbb{Z}, \alpha)$ (the covariance condition is trivial because the scalars z^n and $\overline{z}^n$ cancel). The universal property of $A \times_\alpha \mathbb{Z}$ gives a homomorphism $\hat{\alpha}_z : A \times_\alpha \mathbb{Z}$ to $A \times_\alpha \mathbb{Z}$ satisfying $\hat{\alpha}_z(i_A(a)i_\mathbb{Z}(n)) = z^n i_A(a)i_\mathbb{Z}(n)$. Clearly $\hat{\alpha}_1 = \mathrm{id}$ and one checks on spanning elements that $\hat{\alpha}_w \circ \hat{\alpha}_z = \hat{\alpha}_{wz}$. This implies that $z \mapsto \hat{\alpha}_z$ is a homomorphism from $\mathbb{T}$ to $\mathrm{Aut}(A)$.

For continuity, first observe that for $a \in A$ and $n \in \mathbb{N}$, the function $z \mapsto \hat{\alpha}_z(i_A(a)i_\mathbb{Z}(n))$ is continuous. The triangle inequality then shows that for any $f \in C_c(\mathbb{Z}, A)$, the function $z \mapsto \hat{\alpha}_z(i_{i_A, i_\mathbb{Z}}(f))$ is continuous. Now fix $z_n \to z$ in $\mathbb{T}$, an element $a \in A \times_\alpha \mathbb{Z}$, and $\varepsilon > 0$. Choose $f \in C_c(\mathbb{Z}, A)$ such that $\|a - i_{i_A, i_\mathbb{Z}}(f)\| < \varepsilon/3$ and N large enough so that $\|\hat{\alpha}_{z_n}(i_{i_A, i_\mathbb{Z}}(f)) - \hat{\alpha}_z(i_{i_A, i_\mathbb{Z}}(f))\| < \varepsilon/3$ for $n \geq N$. Then for $n \geq N$ we have

$$\begin{aligned} \|\hat{\alpha}_{z_n}(a) - \hat{\alpha}_z(a)\| &\leq \|\hat{\alpha}_{z_n}(a - i_{i_A, i_\mathbb{Z}}(f))\| + \|\hat{\alpha}_z(i_{i_A, i_\mathbb{Z}}(f) - a)\| \\ &\quad + \|\hat{\alpha}_{z_n}(i_{i_A, i_\mathbb{Z}}(f)) - \hat{\alpha}_z(i_{i_A, i_\mathbb{Z}}(f))\| \qquad (3) \\ &\leq \varepsilon/3 + \varepsilon/3 + \varepsilon/3. \end{aligned}$$

□

For the next result we need to be able to integrate C^*-algebra-valued functions on $\mathbb{T}$.

Proposition 2.26. *Let* A *be a* C^**-algebra and let* $\pi : A \to \mathcal{B}(\mathcal{H})$ *be a faithful representation of* A*. There is an assignment* $f \mapsto \int_\mathbb{T} f(z)\,dz$ *from* $C(\mathbb{T}, A)$ *to* A *such that* $\|f\| \leq \|f\|_\infty$ *and*

$$\Big(\pi\Big(\int_\mathbb{T} f(z)\,dz\Big)\xi \;\Big|\; \eta\Big) = \int_\mathbb{T} (\pi(f(z))\xi \mid \eta)\,dz$$

for all $f \in C(\mathbb{T}, A)$ *and* $\xi, \eta \in \mathcal{H}$*. The assignment* $f \mapsto \int_\mathbb{T} f(z)\,dz$ *is linear and* A*-linear, carries positive-valued functions to positive elements, and is positive definite in the sense that if* f *is positive valued and nonzero, then* $\int_\mathbb{T} f(z)\,dz > 0$*.*

Proof. Fix $f \in C(\mathbb{T}, A)$. Since f is bounded, for $\xi \in \mathcal{H}$, the map $\eta \mapsto \int_{\mathbb{T}} (\eta \mid \pi(f(z))\xi)\,dz$ is a bounded linear functional on $\mathcal{H}$. So the Riesz representation theorem implies that there is a unique vector $\int f\xi$ such that $\int_{\mathbb{T}} (\pi(f(z))\xi \mid \eta) = (\int f\xi \mid \eta)$ for all η. We have

$$\begin{aligned}
\Big\| \int f\xi \Big\| &= \sup_{\|\eta\|=1} \Big| \Big(\int f\xi \mid \eta \Big) \Big| \\
&= \sup_{\|\eta\|=1} \Big| \int_{\mathbb{T}} (\pi(f(z))\xi \mid \eta)\,dz \Big| \\
&\leq \sup_{\|\eta\|=1} \int_{\mathbb{T}} \|\pi(f(z))\xi\| \|\eta\|\,dz \qquad \text{by Cauchy-Schwarz} \\
&\leq \Big(\int_{\mathbb{T}} \|f(z)\|\,dz \Big) \|\xi\| \\
&\leq \|f\|_\infty \|\xi\|. \qquad (4)
\end{aligned}$$

For ξ, ξ' and $\eta \in \mathcal{H}$, we have

$$\begin{aligned}
\big(\textstyle\int f\xi - \int f\xi' \mid \eta \big) &= \big(\textstyle\int f\xi \mid \eta \big) - \big(\textstyle\int f\xi' \mid \eta \big) \\
&= \textstyle\int_{\mathbb{T}} (\pi(f(z))\xi \mid \eta)\,dz - \int_{\mathbb{T}} (\pi(f(z))\xi' \mid \eta)\,dz \\
&= \textstyle\int_{\mathbb{T}} (\pi(f(z))(\xi - \xi') \mid \eta)\,dz \\
&= \big(\textstyle\int f(\xi - \xi') \mid \eta \big).
\end{aligned}$$

So $\int f : \xi \mapsto \int f\xi$ is a bounded linear operator on $\mathcal{H}$. We claim that $\int f \in \pi(A)$. For this, fix $\varepsilon > 0$. Fix $N \in \mathbb{N}$ such that $\sup_{x,y\in\mathbb{T}} \|f(x) - f(y)\|/N < \varepsilon/2$. For $0 \leq i \leq N$, let $U_i := \{z \in \mathbb{T} : \|f(z) - f(1)\| \in ((i-1)\varepsilon/2, (i+1)\varepsilon/2)\}$. Then each U_i is open, we have $\bigcup_{i=0}^n U_i = \mathbb{T}$ and for each i we have $\|f(z) - f(w)\| < \varepsilon$ for all $z, w \in U_i$. Choose a partition of unity ϕ_i subordinate to the U_i; that is, each ϕ_i belongs to $C_0(U_i, [0,1])$ and we have $\sum_i \phi_i(z) = 1$ for all $z \in \mathbb{T}$. Since each ϕ_i is supported on U_i, for $z \in \mathbb{T}$ we have $1 = \sum_{i=1}^N \phi_i(z) = \sum_{z\in U_i} \phi_i(z)$, and hence

$$f(z) = \sum_{z\in U_i} \phi_i(z) f(z).$$

For each i fix $z_i \in U_i$. Define $g : \mathbb{T} \to A$ by $g(z) = \sum_{i=0}^N \phi_i(z) f(z_i)$. For $z \in \mathbb{T}$ we have

$$\|g(z) - f(z)\| = \Big\| \sum_{z\in U_i} \phi_i(z)(f(z_i) - f(z)) \Big\| \leq \sum_{z\in U_i} \phi_i(z) \|f(z_i) - f(z)\| \leq \sum_{z\in U_i} \phi_i(z)\varepsilon = \varepsilon.$$

Hence $\|f - g\|_\infty < \varepsilon$. Hence (4) implies that $\| \int (f - g) \| < \varepsilon$. So it suffices to show that $\int g \in \pi(A)$: since $\pi(A)$ is closed this will force $\int f \in \pi(A)$ as well. For $0 \leq i \leq N$, let $\lambda_i := \int_{\mathbb{T}} \phi_i(z)\,dz$. For $\xi, \eta \in \mathcal{H}$, we have

$$\begin{aligned}
\Big(\int g\xi \mid \eta \Big) &= \int_{\mathbb{T}} (\pi(g(z))\xi \mid \eta)\,dz \\
&= \sum_{i=0}^N \int_{\mathbb{T}} (\phi_i(z)\pi(f(z_i))\xi \mid \eta)\,dz = \Big(\pi\Big(\sum_{i=0}^n \lambda_i f(z_i) \Big)\xi \,\Big|\, \eta \Big).
\end{aligned}$$

It follows that $\int g = \pi\Big(\sum_{i=0}^n \lambda_i f(z_i) \Big) \in \pi(A)$ as required. We define $\int_{\mathbb{T}} f(z)\,dz = \pi^{-1}(\int f)$ for every $f \in C(\mathbb{T}, A)$.

Observe that if $f(z) \geq 0 \in A$ for every $z \in \mathbb{T}$, then $(\int f\xi \mid \xi)$ is the integral of a positive function, and hence positive for every $\xi \in \mathcal{H}$ and hence $\int f$ is itself positive; moreover, if $f(a) \geq 0$ for all a and $f \neq 0$, then there exists $\xi \in \mathcal{H}$ and $z \in \mathbb{T}$ such that $(\pi(f(z))\xi \mid \xi) > 0$ and then $(\int f\xi|\xi) > 0$; so $\int f > 0$. For $\lambda \in \mathbb{C}$ and $f, g \in C(\mathbb{T}, A)$, we have

$$\begin{aligned}\big((\lambda \textstyle\int f + \int g)\xi \mid \eta\big) &= \lambda \textstyle\int_{\mathbb{T}} (\pi(f(z))\xi \mid \eta)\, dz + \int_{\mathbb{T}} (\pi(g(z))\xi \mid \eta)\, dz \\ &= \textstyle\int_{\mathbb{T}} (\pi((\lambda f + g)(z))\xi \mid \eta)\, dz = \big(\int (\lambda f + g)\xi \mid \eta\big).\end{aligned}$$

Thus $f \mapsto \int f$ is linear. For $f \in C(\mathbb{T}, A)$ and $a \in A$, the function $a \cdot f : z \mapsto af(z)$ satisfies

$$\big(\textstyle\int (af)\xi \mid \eta\big) = \int_{\mathbb{T}} (\pi(f(z))\xi \mid \pi(a)^*\eta)\, dz = \big(\int f\xi \mid \pi(a)^*\eta\big) = \big(\pi(a) \int f\xi \mid \eta\big).$$

Hence $\int (af) = \pi(a) \int (f)$. □

Proposition 2.27. *Let $(A, \mathbb{Z}, \alpha)$ be a C^*-dynamical system. There is a norm-decreasing linear map $\Phi : A \times_\alpha \mathbb{Z} \to A \times_\alpha \mathbb{Z}$ such that $\Phi(i_A(a)i_{\mathbb{Z}}(n)) = \delta_{n,0} i_A(a)$. Moreover, if $\Phi(a^*a) = 0$, then $a = 0$.*

Proof. Fix $a \in A \times_\alpha \mathbb{Z}$, then $f : z \mapsto \hat{\alpha}_z(a)$ is a continuous function from $\mathbb{T}$ to $A \times_\alpha \mathbb{Z}$. So we may define $\Phi(a) = \int_{\mathbb{T}} \hat{\alpha}_z(a)\, dz$. Since $\|\hat{\alpha}_z(a)\| = \|a\|$ for all z, we have $\|\Phi(a)\| \leq \|a\|$, and linearity follows from linearity of the $\hat{\alpha}_z$ and linearity of the assignment $f \mapsto \int_{\mathbb{T}} \hat{\alpha}_z(f)\, dz$. □

Corollary 2.28. *Let $(A, \mathbb{Z}, \alpha)$ be a C^*-dynamical system. Suppose that (π, u) is a covariant homomorphism from $(A, \mathbb{Z}, \alpha)$ to $\mathcal{M}(B)$ and that there is an action γ of $\mathbb{T}$ on $\mathcal{M}(B)$ such that $\gamma_z(\pi(a)) = \pi(a)$ and $\gamma_z(u) = zu$ for all $z \in \mathbb{T}$ and $a \in A$. If π is injective, then $\pi \times u$ is injective.*

Proof. Let $\Phi : A \times_\alpha \mathbb{Z} \to A \times_\alpha \mathbb{Z}$ be the linear map of Proposition 2.27. Let $\Psi : \mathcal{M}(B) \to \mathcal{M}(B)$ be the map given by $\Psi(b) = \int_{\mathbb{T}} \gamma_z(b)\, dz$ as in Proposition 2.26.

Since $\int_{\mathbb{T}} z^n\, dz = \delta_{n,0}$, for $a \in A$ and $n \in \mathbb{Z}$, we have

$$\Psi(\pi(a)u^n) = \int_{\mathbb{T}} z^n \pi(a) u^n\, dz = \delta_{n,0} \pi(a) u^n = (\pi \times u)\big(\Phi(i_A(a)i_{\mathbb{Z}}(n))\big).$$

Since Ψ and Φ are linear, it follows that $\Psi \circ (\pi \times u) = (\pi \times u) \circ \Phi$. Now suppose $a \in A \times_\alpha \mathbb{Z} \setminus \{0\}$. Then $\Phi(a^*a) \neq 0$. For $b \in A \setminus \{0\}$, we have $(\pi \times u)(i_A(b)) = \pi(b) \neq 0$ because π is injective. Since $\Phi(a^*a) \in i_A(A)$, it follows that $(\pi \times u)(\Phi(a^*a)) \neq 0$ and hence $\Psi((\pi \times u)(a^*a)) \neq 0$. Since Ψ is linear this forces $(\pi \times u)(a^*a) \neq 0$ and hence $(\pi \times u)(a) \neq 0$. □

Remark 2.29. To avoid discussions about dual groups, we have restricted attention here to $\mathbb{Z}$ actions. However, Corollary 2.28 is a specific instance of a very general phenomenon; there is a whole literature of dual actions and coactions which concerns what should replace the action γ in Corollary 2.28 when $\mathbb{Z}$ is replaced by a more complicated group.

To use Corollary 2.28, we need some way of verifying that the homomorphism π in a covariant homomorphism (π, u) of a system $(A, \mathbb{Z}, \alpha)$ is injective. This tends to be automatic when the action mixes things around a lot. We will describe the special situation where A is commutative because our key motivation is A_θ; but there is a version of the following result for noncommutative C^*-algebras A, which involves the action induced by α on the *primitive ideal space* $\widehat{A}$ of A.

Proposition 2.30. *Suppose that $\mathbb{Z}$ acts on a locally compact space X by homeomorphisms, and let α be the induced action on $C_0(X)$. Suppose that the action on X is* minimal *in the sense that for every $y \in X$, the orbit $\{n \cdot y : n \in \mathbb{N}\}$ is dense in X. Suppose that (π, u) is a nonzero covariant homomorphism from $(C_0(X), \mathbb{Z}, \alpha)$ to $\mathcal{M}(B)$. Then π is injective, and hence $(\pi \times u)$ restricts to an injection of $i_{C_0(X)}(C_0(X))$.*

Proof. Suppose that $\ker(\pi) \neq \{0\}$; we must show that $\ker(\pi) = C_0(X)$. Since $\ker(\pi)$ is an ideal of $C_0(X)$, there is a closed $K \subseteq X$ such that $\ker(\pi) = \{f \in C_0(X) : f|_K = 0\}$, and we must show that K is empty. So we fix $y \in X$ and we just have to show that there exists $g \in \ker(\pi)$ such that $g(y) \neq 0$.

Fix $f \in \ker(\pi) \setminus \{0\}$. Then $U := \{x \in X : f(x) \neq 0\}$ is a nonempty open set. Since the action of $\mathbb{Z}$ on X is minimal, there exists $n \in \mathbb{Z}$ such that $-n \cdot y \in U$, so $f(-n \cdot y) \neq 0$. That is $g := \alpha_n(f)$ satisfies $g(y) \neq 0$. We have

$$\pi(g) = \pi(\alpha_n(f)) = u^n \pi(f) u^{-n} = 0$$

because $\pi(f) = 0$, so $g \in \ker(\pi)$ with $g(y) \neq 0$ as required.

The final statement follows because $\pi = (\pi \times u) \circ i_{C_0(X)}$. □

Using the above, you will prove in Assignment II the following old but important theorem.

Theorem 2.31. *Fix $\theta \in [0, 1)$ and let ρ_θ be the action of $\mathbb{Z}$ on $C(\mathbb{T})$ induced by the homeomorphism $r_\theta(z) = e^{2\pi i\theta} z$ of $\mathbb{T}$. The crossed-product $A_\theta := C(\mathbb{T}) \times_{\rho_\theta} \mathbb{Z}$ is a simple C^*-algebra if and only if θ is irrational.*

We finish our discussion of crossed-products in their own right with the following straightforward corollary of Corollary 2.28, which gives you a flavour of how the latter is used in practice.

Corollary 2.32. *Let $(A, \mathbb{Z}, \alpha)$ and $(B, \mathbb{Z}, \beta)$ be C^*-dynamical systems and suppose that $\pi : A \to B$ is a $\mathbb{Z}$-equivariant homomorphism. If π is injective, then so is $\pi \times 1$, and if π is surjective, so is $\pi \times 1$.*

Proof. First suppose that π is surjective. Fix $b \in B$ and $n \in \mathbb{N}$. Since π is surjective, there exists $a \in A$ such that $b = \pi(a)$, and therefore

$$i_B(b) i_{\mathbb{Z}}^B(n) = i_B(\pi(a)) i_{\mathbb{Z}}^B(n) = (\pi \times 1)(i_A(a) i_{\mathbb{Z}}^A(n)).$$

Since $B \times_\beta \mathbb{Z} = \overline{\operatorname{span}}\{i_B(b) i_{\mathbb{Z}}^B(n) : b \in B, n \in \mathbb{Z}\}$ it follows that $(\pi \times 1)$ is surjective.

Now suppose that π is injective. Then Corollary 2.28 applied to the covariant homomorphism $(i_B \circ \pi, i_{\mathbb{Z}}^B)$ of Proposition 2.24 and the extension to $\mathcal{M}(B \times_\beta \mathbb{Z})$ of the action $\hat{\beta}$ of $\mathbb{T}$ on $B \times_\beta \mathbb{Z}$ obtained from Lemma 2.25 implies that $(\pi \times 1)$ is injective. □

2.5. **Cuntz-Pimsner algebras.** We will finish the course by discussing a class of C^*-algebras invented by Pimsner in 1997 [5]. We will show that these so-called Cuntz-Pimsner algebras include as examples all crossed products by $\mathbb{Z}$, though there are many more examples than these alone.

Let X be a right-Hilbert B-module, and let A be a C^*-algebra. We will call X a Hilbert A–B bimodule or, these days, a C^*-correspondence from A to B if it comes equipped with a left action $(a, x) \mapsto a \cdot x$ of A such that

$$\langle a \cdot x, y \rangle_B = \langle x, a^* \cdot y \rangle_B \quad \text{for all } x, y \in X \text{ and } a \in A.$$

We often call X an A–B-correspondence; if $B = A$, we also call it a correspondence over A.

Lemma 2.33. *Let X be a right-Hilbert B-module and let A be a C^*-algebra. If $\phi : A \to \mathcal{L}(X)$ is a homomorphism, then X becomes an A–B-correspondence with left action given by $a \cdot x := \phi(a)x$. Conversely, if X is an A–B-correspondence, then $\phi(a) := (x \mapsto a \cdot x)$ determines a homomorphism $\phi : A \to \mathcal{L}(X)$. We then have $(a \cdot x) \cdot b = a \cdot (x \cdot b)$ for all $a \in A$, $x \in X$ and $b \in B$.*

Proof. First suppose that $\phi : A \to \mathcal{L}(X)$ is a homomorphism, and define $a \cdot x := \phi(a)x$ for all a, x. Since elements of $\mathcal{L}(X)$ are linear, this left action distributes over the vector-space structure on X, and it distributes over the vector-space structure on A because ϕ is linear. We have $a \cdot (a' \cdot x) = \phi(a)\phi(a')x = \phi(aa')x = (aa') \cdot x$ and $(\lambda a) \cdot x = \lambda(a \cdot x) = a \cdot (\lambda x)$ for $\lambda \in \mathbb{C}$ by definition of scalar multiplication in $\mathcal{L}(X)$. We have

$$\langle a \cdot x, y\rangle_B = \langle \phi(a)x, y\rangle_B = \langle x, \phi(a)^*y\rangle_B = \langle x, \phi(a^*)y\rangle_B = \langle x, a^* \cdot y\rangle_B.$$

So this left action makes X into an A–B-correspondence.

Now suppose that X is an A–B-correspondence. Define $\phi(a) : X \to X$ by $\phi(a)x = a \cdot x$. The defining condition $\langle a \cdot x, y\rangle_B = \langle x, a^* \cdot y\rangle_B$ shows that $\langle \phi(a)x, y\rangle_B = \langle x, \phi(a^*)y\rangle_B$, so each $\phi(a)$ is adjointable with $\phi(a)^* = \phi(a^*)$. Distributivity of the action over the vector-space structure on X shows that ϕ is linear, and we have $\phi(aa') = \phi(a)\phi(a')$ because the action satisfies $a \cdot (a' \cdot x) = (aa') \cdot x$.

For the final statement, fix $a \in A$, $x \in X$ and $b \in B$. For $y \in X$, we have

$$\langle (a \cdot x) \cdot b, y\rangle_B = b^*\langle a \cdot x, y\rangle_B = b^*\langle x, a^* \cdot y\rangle_B = \langle x \cdot b, a^* \cdot y\rangle_B = \langle a \cdot (x \cdot b), y\rangle_B.$$

So Corollary 1.16 gives $(a \cdot x) \cdot b = a \cdot (x \cdot b)$. □

Example 2.34.
(1) If $\pi : A \to \mathcal{B}(\mathcal{H})$ is a representation of A, then $\mathcal{H}$ is an A–$\mathbb{C}$-correspondence with left action $a \cdot h := \pi(a)h$.
(2) If $\phi : A \to \mathcal{M}(B)$ is a homomorphism, then B_B is an A–B-correspondence with left action $a \cdot x = \phi(a)x$.
(3) If $\phi : A \to B$ is a homomorphism, then it is in particular a homomorphism from A to $\mathcal{M}(B)$, and so B_B becomes an A–B-correspondence.
(4) In particular, from each $\alpha \in \operatorname{Aut}(A)$ we obtain a correspondence A_A over A with action $a \cdot x := \alpha(a)x$. We denote this correspondence ${}_\alpha A$.
(5) The special case ${}_{\mathrm{id}}A$ is the module A_A with both left and right action given by multiplication in A.

The last example above gives us a canonical way to view a C^*-algebra as a correspondence over itself. A representation of a C^*-correspondence X over a C^*-algebra A in another C^*-algebra B should then be a bimodule map from X to B_B that preserves the inner product. The formal definition is as follows.

Definition 2.35. Let X be a correspondence over A. A *representation* of X in a C^*-algebra B is a pair (ψ, π) consisting of a linear map $\psi : X \to B$ and a homomorphism $\pi : A \to B$ such that

(1) $\pi(a)\psi(x) = \psi(a \cdot x)$ and $\psi(x)\pi(a) = \psi(x \cdot a)$ for all $a \in A$ and $x \in X$, and
(2) $\pi(\langle x, y\rangle_A) = \psi(x)^*\psi(y)$ for all $x, y \in X$.

Lemma 2.36. *Let X be a correspondence over A and (ψ, π) a representation of X in B. Then $\|\psi(x)\| \leq \|x\|$ for all x, and ψ is isometric whenever π is injective.*

Proof. We calculate

$$\|\psi(x)\|^2 = \|\psi(x)^*\psi(x)\| = \|\pi(\langle x, x\rangle_A)\| \leq \|\langle x, x\rangle_A\| = \|x\|^2,$$

and the inequality becomes equality if π is injective. □

In light of Example 2.34(3), we think of A–B-correspondences as generalised homomorphisms from A to B. It is then natural to want to be able to compose them.

Lemma 2.37. *Suppose that X is an A–B correspondence and Y is a B–C-correspondence. Then there is a C-valued sesquilinear form $[\cdot,\cdot]_C$ on the algebraic tensor product $X \odot Y$ such that*

$$[x \otimes y, x' \otimes y']_C = \langle y, \langle x, x'\rangle_B \cdot y'\rangle_C \quad \text{for all } x, x' \in X \text{ and } y, y' \in Y.$$

The set $N := \{\zeta \in X \odot Y : [\zeta, \zeta]_C = 0\}$ is a closed sub-bimodule of $X \odot Y$, and $\langle \xi + N, \eta + N\rangle_C := [\xi, \eta]_C$ satisfies the inner-product axioms, and the formula $\|\xi + N\| := \|[\xi, \xi]_C\|^{1/2}$ defines a norm on $X \odot Y/N$. The completion $X \otimes_B Y$ is an A–C-correspondence under the actions inherited from $X \odot Y$. We have $x \cdot b \otimes y = x \otimes b \cdot y$ in $X \otimes_B Y$ for all $x \in X$, $y \in Y$ and $b \in B$.

Proof. For fixed $x' \in X$ and $y' \in Y$, the map $(x, y) \mapsto \langle y, \langle x, x'\rangle_B \cdot y'\rangle_C^*$ is bilinear from $X \times Y$ to C. So the universal property of the algebraic tensor product ensures that there is a linear map $[\cdot, (x', y')]^* : X \odot Y \to C$ such that $[x \otimes y, (x', y')]_C^* = \langle y, \langle x, x'\rangle_B \cdot y'\rangle_C^*$. Taking adjoints gives a conjugate-bilinear map $[\cdot, (x', y')]$ such that $[x \otimes y, (x', y')]_C = \langle y, \langle x, x'\rangle_B \cdot y'\rangle_C$.

Hence each $\xi \in X \odot Y$ determines a bilinear map $(x', y') \mapsto [\xi, (x', y')]_C^* : X \times Y \to C$, and the universal property again extends this to a linear map $[\xi, \cdot]^* : X \odot Y \to C$ such that $[\xi, x' \otimes y']_C^* = [\xi, (x', y')]_C^*$. Composing with the adjoint map gives a sesquilinear map $[\cdot,\cdot]_C : X \odot Y \times X \odot Y \to C$ such that $[x \otimes y, x' \otimes y']_C = \langle y, \langle x, x'\rangle_B \cdot y'\rangle_C$.

The Cauchy-Schwarz inequality shows that the set N above is equal to $\{\xi \in X \odot Y : [\xi, \eta]_C = 0 \text{ for all } \eta\}$, and it is then easy to check that N is a closed sub-bimodule using the various linearity and adjointability properties of the actions and inner products on X and Y. It follows that we can form the quotient $X \odot Y/N$ and we obtain an A–C bimodule. The description of N just given shows that $[\cdot,\cdot]_C$ descends to a sesquilinear form $\langle,\rangle_C$ on $X \odot Y/N$, and the definition of N shows that this form is positive definite. We have

$$\langle x \otimes y, (x' \otimes y') \cdot c\rangle_C = \langle y, \langle x, x'\rangle_B \cdot y' \cdot c\rangle_C = \langle y, \langle x, x'\rangle_B \cdot y'\rangle_C c = \langle x \otimes y, x' \otimes y'\rangle_C c,$$

so $X \otimes_B Y$ is a right-Hilbert C-module under this inner product. We also have

$$\langle a \cdot (x \otimes y), x' \otimes y'\rangle_C = \langle y, \langle a \cdot x, x'\rangle_B \cdot y'\rangle_C = \langle y, \langle x, a^* \cdot x'\rangle_B \cdot y'\rangle_C = \langle x \otimes y, a^* \cdot (x' \otimes y')\rangle_C$$

for all x, y, a, and so $X \otimes_A Y$ is an A–C-correspondence.

For $x, x' \in X$ and $y, y' \in Y$ and for $b \in B$, we have

$$\begin{aligned}\langle x \cdot b \otimes y, x' \otimes y'\rangle_C &= \langle y, \langle x \cdot b, x'\rangle_B \cdot y'\rangle_C = \langle y, b^*\langle x, x'\rangle_B \cdot y'\rangle_C \\ &= \langle b \cdot y, \langle x, x'\rangle_B \cdot y'\rangle_C = \langle x \otimes b \cdot y, x' \otimes y'\rangle_C.\end{aligned}$$

So linearity gives $\langle x \cdot b \otimes y, \xi\rangle_C = \langle x \otimes b \cdot y, \xi\rangle_C$ for all ξ and then Corollary 1.16 implies that $x \cdot b \otimes y = x \otimes b \cdot y$. □

Example 2.38. Let X be an A–B-correspondence, and suppose that $\pi : B \to \mathcal{B}(\mathcal{H})$ is a representation of B. As we saw above, $\mathcal{H}$ is then a B–$\mathbb{C}$-correspondence. We can then form the product $X \otimes_B \mathcal{H}$. This is an A–$\mathbb{C}$-correspondence—that is, a Hilbert space, with $(y \otimes k \mid x \otimes h) = (\pi(\langle x, y\rangle_A)k \mid h)$. If $\phi : A \to \mathcal{L}(X)$ is the homomorphism inducing the left action of A on X, then the left action of A on $X \otimes_B \mathcal{H}$ is the homomorphism X-$\mathrm{Ind}(\pi)$ given by $X\text{-}\mathrm{Ind}(\pi)(a)(x \otimes h) := \phi(a)x \otimes h$. This representation X-$\mathrm{Ind}(\pi)$ is called the *induced* representation, and the process of passing from π to X-$\mathrm{Ind}(\pi)$ is called induction. This is the starting point for the theory of Morita equivalence for C^*-algebras.

If X is a correspondence over A, then we can form $X \otimes_A X$ and then $(X \otimes_A X) \otimes_A X$ and $X \otimes_A (X \otimes_A X)$. It is routine to check that the formula $(x \otimes x') \otimes x'' \mapsto x \otimes (x' \otimes x'')$ preserves inner products and so determines an isomorphism $(X \otimes_A X) \otimes_A X \cong X \otimes_A (X \otimes_A X)$. In general, up to canonical isomorphism, there is a well-defined module $X^{\otimes n} = X \otimes_A X \otimes_A \cdots \otimes_A X$ where there are n tensor factors whenever $n \geq 1$. We will write elementary tensors in this n-fold tensor power as $x_1 \otimes x_2 \otimes \cdots \otimes x_n$, once again suppressing parentheses.

As a notational convenience, we write $X^{\otimes 0} := {}_{\mathrm{id}}A$.

Exercise 3. Check that if $\phi : A \to B$ and $\psi : B \to C$ are homomorphisms, then there is an isomorphism ${}_\phi B \otimes_B {}_\psi C \cong {}_{\psi\circ\phi} C$ of A–C-correspondences given by $b \otimes c \mapsto \psi(b)c$.

Proposition 2.39. *Let X be a correspondence over A, and suppose that (ψ, π) is a representation of X in B. For each $n \geq 1$, there is a linear map $\psi_n : X^{\otimes n} \to B$ such that $\psi_n(x_1 \otimes \cdots \otimes x_n) = \psi(x_1)\psi(x_2)\cdots\psi(x_n)$ for all $x_1, \dots, x_n$, and the pair (ψ_n, π) is a representation of $X^{\otimes n}$ in B.*

Proof. We just check the case $n = 2$. The general result follows from an induction using the same argument. We calculate:

$$\begin{aligned}\langle \psi(x_1)\psi(x_2), \psi(y_1)\psi(y_2)\rangle_B &= (\psi(x_1)\psi(x_2))^*(\psi(y_1)\psi(y_2))\\ &= \psi(x_2)^*\pi(\langle x_1, y_1\rangle_A)\psi(y_2)\\ &= \psi(x_2)^*\psi(\langle x_1, y_1\rangle_A \cdot y_2)\\ &= \langle \psi(x_2), \psi(\langle x_1, y_1\rangle_A \cdot y_2)\rangle_B.\end{aligned}$$

The formula for ψ_2 preserves the inner product, and therefore extends to a bounded linear map. The remaining properties follow from easy calculations using the corresponding properties of ψ. □

We use the preceding result to describe a manageable spanning family for the C^*-algebra generated by the image of a representation of a correspondence. Given a representation (ψ, π) of a correspondence X over A in a C^*-algebra B, we define

$$C^*(\psi, \pi) := C^*(\psi(X) \cup \pi(A)) \subseteq B.$$

As a notational convention, we write $\psi_0 := \pi$. Observe that then $(\psi_0, \pi) = (\pi, \pi)$ is trivially a representation of $X^{\otimes 0} = A$ in B.

Proposition 2.40. *Let X be a correspondence over A and suppose that (ψ, π) is a representation of X in B. Under the convention that $\{\psi_n(x_1 \otimes \cdots \otimes x_n) : x_i \in X\}$ means $\pi(A)$ when $n = 0$, we have*

$$C^*(\psi, \pi) = \overline{\operatorname{span}}\{\psi_m(x_1 \otimes \cdots \otimes x_m)\psi_n(y_1 \otimes \cdots \otimes y_n)^* : m, n \geq 0, x_i, y_i \in X\}$$

$$= \overline{\operatorname{span}}\{\psi_m(\xi)\psi_n(\eta)^* : m, n \geq 0, \xi \in X^{\otimes m}, \eta \in X^{\otimes n}\}.$$

Proof. That the two spaces on the right-hand side of the displayed equation are the same is clear: the first is clearly contained in the second, and the reverse containment follows from linearity since each $X^{\otimes n} = \overline{\operatorname{span}}\{x_1 \otimes \cdots \otimes x_n : x_i \in X\}$. So we just have to establish the first equality.

The containment $\supseteq$ is trivial. For the reverse, we first show that

$$\overline{\operatorname{span}}\{\psi_m(x_1 \otimes \cdots \otimes x_m)\psi_n(y_1 \otimes \cdots \otimes y_n) : m, n \geq 0, x_i, y_i \in X\} \tag{5}$$

is a C^*-subalgebra of $C^*(\psi, \pi)$. It is a closed vector subspace by definition, and it is clearly closed under adjoints. To see that it is closed under multiplication, fix spanning elements $\psi_m(x_1 \otimes \cdots \otimes x_m)\psi_n(y_1 \otimes \cdots \otimes y_n)^*$ and $\psi_p(w_1 \otimes \cdots \otimes w_p)\psi_q(z_1 \otimes \cdots \otimes z_q)^*$. Suppose for the moment that $p \geq m$. We have

$$\begin{aligned}
&\psi_m(x_1 \otimes \cdots \otimes x_m)\psi_n(y_1 \otimes \cdots \otimes y_n)^*\psi_p(w_1 \otimes \cdots \otimes w_p)\psi_q(z_1 \otimes \cdots \otimes z_q)^* \\
&\quad = \psi_m(x_1 \otimes \cdots \otimes x_m)\pi(\langle y_1 \otimes \cdots \otimes y_n, w_1 \otimes \cdots \otimes w_n\rangle_A) \\
&\qquad \psi_{p-n}(w_{n+1} \otimes \cdots \otimes w_p)\psi_q(z_1 \otimes \cdots \otimes z_q)^* \\
&\quad = \psi_{m+p-n}(x_1 \otimes \cdots (x_m \cdot \langle y_1 \otimes \cdots \otimes y_n, w_{p-n+1} \otimes \cdots \otimes w_p\rangle_A) \otimes w_{n+1} \otimes \cdots \otimes w_p) \\
&\qquad\qquad \psi_q(z_1 \otimes \cdots \otimes z_q)^*,
\end{aligned}$$

which belongs to (5). So (5) is closed under multiplication, and is therefore a C^*-subalgebra of $C^*(\psi, \pi)$ as claimed. For $x \in X$, the Hewitt–Cohen factorisation theorem (Theorem 1.59) shows that there exists $y \in X$ such that

$$\psi(x) = \psi(y \cdot \langle y, y\rangle_A) = \psi_1(y)\psi_0(\langle y, y\rangle_A),$$

so $\psi(X)$ is contained in (5) as discussed in the first paragraph of the proof. Using the same factorisation argument on A_A shows that $A = \{bc^* : b, c \in A\}$, and so each $\pi(a)$ can be written $\pi(a) = \pi(bc^*) = \psi_0(b)\psi_0(c)^*$. Thus $\pi(A)$ is contained in (5) by the first paragraph of the proof again. So (5) is a C^*-subalgebra of B that contains all the generating elements of $C^*(\psi, \pi)$ and therefore contains $C^*(\psi, \pi)$. This completes the proof. □

It will be very convenient, henceforth, to write $\psi_\star : \bigcup_{n=0}^\infty X^{\otimes n} \to B$ for the map such that $\psi_\star|_{X^{\otimes n}} = \psi_n$.

Corollary 2.41. *Let X be a C^*-correspondence over A. If (ψ, π) is a representation of X, then for any finite subset $F \subseteq (\bigcup_{n=0}^\infty X^{\otimes}n) \times (\bigcup_{n=0}^\infty X^{\otimes n})$, we have*

$$\Big\| \sum_{(\xi,\eta)\in F} \psi_\star(\xi)\psi_\star(\eta)^* \Big\| \leq \sum_{(\xi,\eta)\in F} \|\xi\|\|\eta\|.$$

There is a universal C^*-algebra $\mathcal{T}_X$ generated by a representation (i_X, i_A) of X such that for every representation (ψ, π) of X there is a homomorphism $\psi \times \pi : \mathcal{T}_X \to C^*(\psi, \pi)$ such that $(\psi \times \pi) \circ i_X = \psi$ and $(\psi \times \pi) \circ i_A = \pi$.

Proof. We have

$$\begin{aligned}
\Big\| \sum_{(\xi,\eta)\in F} \psi_\star(\xi)\psi_\star(\eta)^* \Big\| &\leq \sum_{(\xi,\eta)\in F} \|\psi_\star(\xi)\psi_\star(\eta)^*\| \\
&\leq \sum_{(\xi,\eta)\in F} \|\psi_\star(\xi)\|\|\psi_\star(\eta)^*\| \leq \sum_{(\xi,\eta)\in F} \|\xi\|\|\eta\|
\end{aligned}$$

by Lemma 2.36 because each (ψ_n, π) is a representation.

We just sketch the proof of the existence of $\mathcal{T}_X$. Since everything in sight is assumed to be separable, each $C^*(\psi, \pi)$ is separable and so has a faithful representation on a separable Hilbert space. So the collection of representations of X on $\ell^2(\mathbb{N})$ contains (up to isomorphism) every representation of X. We let R denote the set of all representations of X on $\ell^2(\mathbb{N})$. The preceding paragraph shows that

$$i_X := \bigoplus_{(\psi,\pi)\in R} \psi \quad \text{and} \quad i_A := \bigoplus_{(\psi,\pi)\in R} \pi$$

make sense and constitute a representation of X on $\bigoplus_{(\psi,\pi)\in R} \ell^2(\mathbb{N}) = \ell^2(\mathbb{N}\times R)$. We define $\mathcal{T}_X$ to be $C^*(i_X, i_A)$. Fix a representation (ψ, π) of X, and choose a faithful representation $\rho : C^*(\psi, \pi) \to \mathcal{B}(\ell^2(\mathbb{N}))$. Then $(\rho\circ\psi, \rho\circ\pi) \in R$. By construction, for a linear combination $\sum_{(\xi,\eta)\in F} (i_X)_\star(\xi)(i_X)_\star(\eta)^* \in \mathcal{T}_X$, we have

$$\begin{aligned}\Big\| \sum_{(\xi,\eta)\in F} \psi_\star(\xi)\psi_\star(\eta)^*\Big\| &= \Big\|\rho\Big(\sum_{(\xi,\eta)\in F} \psi_\star(\xi)\psi_\star(\eta)^*\Big)\Big\| \\ &= \Big\| \sum_{(\xi,\eta)\in F} (\rho\circ\psi)_\star(\xi)(\rho\circ\psi)_\star(\eta)^*\Big\| \\ &\leq \Big\| \sum_{(\xi,\eta)\in F} (i_X)_\star(\xi)(i_X)_\star(\eta)^*\Big\|,\end{aligned}$$

so projection on the $(\rho \circ \psi, \rho \circ \pi)$-summand in $\bigoplus_{(\psi,\pi)\in R} \ell^2(\mathbb{N})$ followed by application of ρ^{-1} is a homomorphism $\psi \times \pi : \mathcal{T}_X \to C^*(\psi, \pi)$ that carries each $(i_X)_\star(\xi)(i_X)_\star(\eta)$ to $\psi_\star(\xi)\psi_\star(\eta)$. In particular, $(\psi \times \pi) \circ i_X = \psi$ and $(\psi \times \pi) \circ i_A = \pi$. □

Of course, it's possible that all the work we just did was for nothing. Maybe there *aren't* any nonzero representations of a given module X. Fortunately, that's not the case—there's one very big one, called the *Fock representation*. To describe it, we first form the C^*-algebra it maps into.

Let X be a correspondence over a C^*-algebra A. Let $\mathcal{F}_X$ denote the Hilbert-module direct sum

$$\mathcal{F}_X = \bigoplus_{n=0}^{\infty} X^{\otimes n}.$$

To be precise, we construct $\mathcal{F}_X$ by forming the algebraic direct sum $\bigcup_{n=0}^\infty \bigoplus_{i=0}^n X^{\otimes i}$ and define an A-valued inner-product on it by $\langle \oplus_{i=0}^n x_i, \oplus_{i=0}^n y_i\rangle_A = \sum_{i=0}^n \langle x_i, y_i\rangle_A$. Then $\mathcal{F}_X$ is the completion of this space in the norm $\| \oplus x_i\| = \langle \oplus x_i, \oplus x_i\rangle_A^{1/2}$.

This $\mathcal{F}_X$ is then a right-Hilbert A-module, and it becomes a correspondence over A if we define the left action to be pointwise application of the action of A on each $X^{\otimes n}$. So $a \cdot (\oplus x_i) = \oplus(a \cdot x_i)$.

Lemma 2.42. *For each $x \in X$, there is a bounded operator $l_x \in \mathcal{L}(\mathcal{F}_X)$ given by*

$$l_x\xi = \begin{cases} x \otimes \xi & \text{if } \xi \in \bigcup_{n>0} X^{\otimes n} \\ x \cdot a & \text{if } \xi = a \in A = X^{\otimes 0}.\end{cases}$$

This l_x is adjointable, with

$$l_x^* a = 0 \text{ for } a \in X^{\otimes 0} \quad \text{and} \quad l_x^*(x_1 \otimes \xi) = \langle x, x_1\rangle_A \cdot \xi \text{ for } x_1 \in X \text{ and } \xi \in \mathcal{F}_X.$$

If $\pi : A \to \mathcal{L}(\mathcal{F}_X)$ is the homomorphism induced by the left action, and $\psi : X \to \mathcal{L}(\mathcal{F}_X)$ is given by $\psi(x) = l_x$, then (ψ, π) is a representation of X in $\mathcal{F}_X$, and each ψ_n is isometric.

Proof. Fix $x \in X$, natural numbers m, n and elementary tensors $x_1 \otimes \cdots \otimes x_m$ and $y_1 \otimes \cdots \otimes y_n$. We have

$$\|l_x\xi\|^2 = \langle l_x\xi, l_x\xi\rangle_A = \langle \xi, \langle x, x\rangle_A \cdot \xi\rangle_A = \big\|\sqrt{\langle x, x\rangle_A} \cdot \xi\big\|^2 \leq \big\|\sqrt{\langle x, x\rangle_A}\big\|^2 \|\xi\|^2 = \|x\|^2\|\xi\|^2.$$

So the formula for l_x defines a bounded linear operator with norm at most $\|x\|$. If $a \in X^{\otimes 0}$, then $\langle l_x\xi, a\rangle_A = 0$ for all ξ because the range of l_x is contained in $\bigoplus_{n=1}^\infty X^{\otimes n}$. Now consider $\xi \in X^{\otimes m} \subseteq \mathcal{F}_X$ and $y \otimes \eta \in X \otimes X^{\otimes n} \subseteq X \otimes_A \mathcal{F}_X = \bigoplus_{i=1}^\infty X^{\otimes i}$. We have

$$\langle l_x\xi, y \otimes \eta\rangle_A = \langle x \otimes \xi, y \otimes \eta\rangle_A = \delta_{m,n}\langle \xi, \langle x, y\rangle_A \cdot \eta\rangle_A,$$

which shows that the given formula specifies an adjoint l_x^* for l_x.

Now let π be the homomorphism defining the left action, and define $\psi(x) = l_x$ for all x. We have $\pi(a)\psi(x)\xi = a \cdot (x \otimes \xi) = (a \cdot x) \otimes \xi$, giving $\pi(a)\psi(x) = \psi(a \cdot x)$, and a similar calculation gives $\psi(x)\pi(a) = \psi(x \cdot a)$. Furthermore,

$$\psi(x)^*\psi(y)\xi = l_x^* l_y \xi = l_x^*(y \otimes \xi) = \langle x, y\rangle_A \cdot \xi = \pi(\langle x, y\rangle_A)\xi$$

for all x, y, ξ, and so (ψ, π) is a representation as claimed.

Since $\pi(a)$ restricts on $X^{\otimes 0}$ to the action of A on itself by multiplication, π is injective, and hence isometric. Now each ψ_n is isometric by Lemma 2.36. □

This is all we need to know (and more) about $\mathcal{T}_X$. Now we need to construct the quotient $\mathcal{O}_X$. To do so, we need a technical result. The following assertion is made by Pimsner in [5], but essentially without proof. We will give the proof developed by Kajiwarra–Pinzari–Watatani in [2]. An alternative proof is given in [8], but it relies on the language of induced representations, which we have not developed here.

Proposition 2.43. *Let X be a correspondence over A and suppose that (ψ, π) is a representation of X in B. Then there is a homomorphism $\psi^{(1)} : \mathcal{K}(X) \to B$ such that $\psi^{(1)}(\theta_{x,y}) = \psi(x)\psi(y)^*$ for all x, y.*

To prove the Proposition, we first need a technical lemma. For this, observe that if $x_1, \ldots, x_n$ are elements of a right-Hilbert A-module X, then we can form the matrix

$$\big(\langle x_i, x_j\rangle_A\big)_{ij} \in M_n(A).$$

This is a self-adjoint matrix, and for $b = (b_1, \ldots, b_n)^t \in A^n$, we have

$$\begin{aligned}
\big\langle (\langle x_i, x_j\rangle_A)_{ij} b, b\big\rangle_A &= \sum_{k=1}^n ((\langle x_i, x_j\rangle_A)_{ij} b)_k^* b_k \\
&= \sum_k \sum_j (\langle x_k, x_j\rangle_A b_j)^* b_k \\
&= \sum_k \sum_j \langle x_j \cdot b_j, x_k\rangle_A b_k \\
&= \sum_{j,k} \langle x_j \cdot b_j, x_k \cdot b_k\rangle_A \\
&= \Big\langle \sum_j (x_j \cdot b_j), \sum_j (x_j \cdot b_j)\Big\rangle_A,
\end{aligned}$$

So the matrix $(\langle x_i, x_j\rangle_A)_{ij}$ is a positive element of $M_n(A)$.

Lemma 2.44 ([2, Lemma 2.1])**.** *Let X be a right-Hilbert A-module. For any finite collection of pairs $x_1, y_1,\ x_2, y_2,\ \dots,\ x_n, y_n$ in X, we have*

$$\Big\| \sum_i \theta_{x_i, y_i} \Big\| = \big\|(\langle x_i, x_j\rangle_A)_{ij}^{1/2}(\langle y_i, y_j\rangle_A)_{ij}^{1/2}\big\|. \tag{6}$$

Proof. Consider a right-Hilbert A-module Y, and fix $x, y \in Y$. We have

$$\|\theta_{x,y}\|^2 = \|\theta_{x,y}^*\theta_{x,y}\| = \|\theta_{y,x}\theta_{x,y}\| = \|\theta_{y\cdot\langle x,x\rangle_A, y}\|.$$

For $z \in Y$, we have

$$\theta_{y\cdot\langle x,x\rangle_A,y}(z) = y\cdot\langle x,x\rangle_A\langle y,z\rangle_A = y\cdot\sqrt{\langle x,x\rangle_A}\langle y\cdot\sqrt{\langle x,x\rangle_A}, z\rangle_A = \theta_{y\cdot\sqrt{\langle x,x\rangle_A}, y\cdot\sqrt{\langle x,x\rangle_A}}(z).$$

Using the notation $x \mapsto L_x$ for the isomorphism $Y \cong \mathcal{K}(A, Y)$ from Lemma 1.61, we now have

$$\begin{aligned}
\|\theta_{x,y}\|^2 &= \big\|\theta_{y\cdot\sqrt{\langle x,x\rangle_A}, y\cdot\sqrt{\langle x,x\rangle_A}}\big\| \\
&= \big\|L_{y\cdot\sqrt{\langle x,x\rangle_A}}L^*_{y\cdot\sqrt{\langle x,x\rangle_A}}\big\| \\
&= \big\|L_{y\cdot\sqrt{\langle x,x\rangle_A}}\big\|^2 \\
&= \big\|y\cdot\sqrt{\langle x,x\rangle_A}\big\|^2 \\
&= \big\|\langle y\cdot\sqrt{\langle x,x\rangle_A}, y\cdot\sqrt{\langle x,x\rangle_A}\rangle_A\big\| \\
&= \big\|\sqrt{\langle x,x\rangle_A}\langle y,y\rangle_A\sqrt{\langle x,x\rangle_A}\big\| \\
&= \big\|(\sqrt{\langle y,y\rangle_A}\sqrt{\langle x,x\rangle_A})^*\sqrt{\langle y,y\rangle_A}\sqrt{\langle x,x\rangle_A}\big\| \\
&= \big\|\sqrt{\langle x,x\rangle_A}\sqrt{\langle y,y\rangle_A}\big\|^2.
\end{aligned}$$

That is,

$$\|\theta_{x,y}\| = \|\sqrt{\langle x,x\rangle_A}\sqrt{\langle y,y\rangle_A}\|. \tag{7}$$

Now, put $X^n_{M_n(A)} := \bigoplus_{i=1}^n X$, regarded as a right-Hilbert $M_n(A)$ module with inner-product

$$\langle y, z\rangle_{M_n(A)} = (\langle y_i, z_j\rangle_A)_{ij},$$

and right action given by $(y\cdot a)_i = \sum_j (y_j \cdot a_{ji})$. Fix elements $x_1, \dots, x_n$ and $y_1, \dots, y_n$ of X, and let $x = (x_i)_{i=1}^n$ and $y = (y_i)_{i=1}^n$ be the corresponding elements of $X^n_{M_n(A)}$. For $z \in X^n_{M_n(A)}$, we have

$$\begin{aligned}
\theta_{x,y}(z) &= x\cdot\langle y, z\rangle_{M_n(A)} \\
&= (x_i)\cdot(\langle y_j, z_k\rangle_A)_{jk} \\
&= \Big(\sum_i x_i\cdot\langle y_i, z_k\rangle_A\Big)_{k=1}^n \\
&= \Big(\sum_i \theta_{x_i,y_i}(z_k)\Big)_{k=1}^n \\
&= \operatorname{diag}\Big(\sum_i \theta_{x_i,y_i}\Big)(z).
\end{aligned}$$

That is, the compact operator $\theta_{x,y}$ is given by the diagonal matrix whose diagonal entries are $\sum_i \theta_{x_i,y_i}$. Using first that the norm of a diagonal matrix is the maximum of the norms of its diagonal entries, and then Equation 7 applied to elements x, y of the module $Y = X^n_{M_n(A)}$, we obtain

$$\Big\|\sum_i \theta_{x_i,y_i}\Big\| = \|\theta_{x,y}\| = \Big\|\sqrt{\langle x,x\rangle_{M_n(A)}}\sqrt{\langle y,y\rangle_{M_n(A)}}\Big\|.$$

Now Equation (6) follows from the definition of the inner product in $X^n_{M_n(A)}$. □

Proof of Proposition 2.43. Observe that the homomorphism $\pi : A \to B$ determines a homomorphism $\pi_n : M_n(A) \to M_n(B)$ given by $\pi_n(a)_{ij} = \pi(a_{ij})$. Fix $x_1, \dots, x_n, y_1, \dots, y_n \in X$. Using Lemma 2.44 at the first step, we have

$$\begin{aligned}\Big\|\sum_i \theta_{x_i,y_i}\Big\| &= \big\|(\langle x_i,x_j\rangle_A)^{1/2}_{ij}(\langle y_i,y_j\rangle_A)^{1/2}_{ij}\big\| \\ &\geq \big\|\pi_n(\langle x_i,x_j\rangle_A)^{1/2}_{ij}\pi_n(\langle y_i,y_j\rangle_A)^{1/2}_{ij}\big\| \\ &= \big\|(\psi(x_i)^*\psi(x_j))^{1/2}_{ij}(\psi(y_i)^*\psi(y_j))^{1/2}_{ij}\big\|.\end{aligned}$$

Applying Lemma 2.44 again, this time to the elements $\psi(x_i)$ and $\psi(y_i)$ of the module B_B, we have

$$\big\|(\psi(x_i)^*\psi(x_j))^{1/2}_{ij}(\psi(y_i)^*\psi(y_j))^{1/2}_{ij}\big\| = \Big\|\sum_i \theta_{\psi(x_i),\psi(y_i)}\Big\| = \Big\|\sum_i \psi(x_i)\psi(y_i)^*\Big\|.$$

So there is a well-defined norm-decreasing linear map $\psi^{(1)} : \mathcal{K}(X) \to B$ satisfying $\psi^{(1)}(\theta_{x,y}) = \psi(x)\psi(y)^*$. Clearly $\psi^{(1)}(\theta^*_{x,y}) = \psi^{(1)}(\theta_{x,y})^*$, and so conjugate-linearity of the adjoint operation shows that $\psi^{(1)}$ preserves adjoints. We have

$$\begin{aligned}\psi^{(1)}(\theta_{x,y})\psi^{(1)}(\theta_{w,z}) &= \psi(x)\psi(y)^*\psi(w)\psi(z)^* = \psi(x)\pi(\langle y,w\rangle_A)\psi(z)^* = \psi(x\cdot\langle y,w\rangle_A)\psi(z)^* \\ &= \psi^{(1)}(\theta_{x\cdot\langle y,w\rangle_A,z}) = \psi^{(1)}(\theta_{\theta_{x,y}(w),z}) = \psi^{(1)}(\theta_{x,y}\theta_{w,z}).\end{aligned}$$

So bilinearity of multiplication shows that $\psi^{(1)}$ is a homomorphism. □

We are now ready to define the Cuntz-Pimsner algebra of a C^*-correspondence over A. For technical reasons, we will assume that the action of A on the left of the correspondence X is implemented by an *injective* homomorphism of A into $\mathcal{L}(X)$. You should be aware that without this assumption, there are two competing definitions of the Cuntz-Pimsner algebra: Pimsner's definition [5], and Katsura's modification [3].

Definition 2.45. Let X be a correspondence over A, and suppose that the homomorphism $\phi : A \to \mathcal{L}(X)$ implementing the left action is injective. We say that a representation (ψ, π) of X in B is *Cuntz-Pimsner covariant* if we have $\pi(a) = \psi^{(1)}(\phi(a))$ whenever $a \in A$ satisfies $\phi(a) \in \mathcal{K}(X)$.

Theorem 2.46. *Let X be a correspondence over A, and suppose that the homomorphism $\phi : A \to \mathcal{L}(X)$ implementing the left action is injective. There is a universal C^*-algebra $\mathcal{O}_X$ generated by a Cuntz-Pimsner covariant representation (j_X, j_A) of X in the sense that if (ψ, π) is another representation, then there is a homomorphism $\psi \times \pi : \mathcal{O}_X \to C^*(\psi, \pi)$ such that $(\psi \times \pi) \circ j_X = \psi$ and $(\psi \times \pi) \circ j_A = \pi$. The homomorphism j_A is injective, and there is an action γ of $\mathbb{T}$ on $\mathcal{O}_X$ such that $\gamma_z(j_X(x)) = zj_X(x)$ for all $x \in X$, and $\gamma_z(j_A(a)) = j_A(a)$ for all $a \in A$.*

To prove the theorem, we need to know two things. Firstly, if X is a correspondence over A and the left action of A on X is injective, we will need to know that the left action on each $X^{\otimes n}$ is injective.

Lemma 2.47. *Let X be an A–B-correspondence, and Y a B–C correspondence. Suppose that the homomorphism $\psi : B \to \mathcal{L}(Y)$ that implements the left action is injective. Then the map $T \mapsto T \otimes 1$ from $\mathcal{L}(X)$ to $\mathcal{L}(X \otimes Y)$ is injective.*

Proof. Suppose that $T \in \mathcal{L}(X) \setminus \{0\}$. Choose $x \in X$ such that $Tx \neq 0$. Then for $y \in Y$, we have

$$\begin{aligned}\big\|\langle (T \otimes 1)(x \otimes y), (T \otimes 1)(x \otimes y)\rangle_A\big\| &= \big\|\big\langle y, \langle Tx, Tx\rangle_A \cdot y\big\rangle_A\big\| \\ &= \big\|\big\langle \sqrt{\langle Tx, Tx\rangle_A} \cdot y, \sqrt{\langle Tx, Tx\rangle_A} \cdot y\big\rangle_A\big\| \\ &= \big\|\psi(\sqrt{\langle Tx, Tx\rangle_A})y\big\|^2.\end{aligned}$$

Since ψ is injective, there exists y such that $\psi(\sqrt{\langle Tx, Tx\rangle_A})y \neq 0$, and it follows that $(T \otimes 1)(x \otimes y) \neq 0$. Hence $T \otimes 1 \neq 0$. □

Secondly, we need to be able to tell non-compact operators on the Fock space from compact operators.

Lemma 2.48. *Let X be a correspondence over A. For each $T \in \mathcal{K}(X)$, we have $\|T|_{X^{\otimes j}}\| \to 0$ as $j \to \infty$.*

Proof. Fix $T \in \mathcal{K}(X)$ and $\varepsilon > 0$. Choose $\xi_i, \eta_i \in \mathcal{F}_X$ such that $\|T - \sum_{i=1}^n \theta_{\xi_i,\eta_i}\| < \varepsilon/2$. For each j, let $P_j : \mathcal{F}_X \to X^{\otimes j}$ be the projection. Since each $\xi_i, \eta_i \in \mathcal{F}_X$, there exists J such that $\|P_j\eta_i\| < \varepsilon/(2n \max_i \|\xi_i\|)$ for all $j \geq J$. We then have

$$\Big\|\Big(\sum_{i=1}^n \theta_{\xi_i,\eta_i}\Big)|_{X^{\otimes j}}\Big\| = \Big\|\Big(\sum_{i=1}^n \theta_{\xi_i,P_j\eta_i}\Big)\Big\| \leq \sum_{i=1}^n \|\xi_i\|\|P_j\eta_i\| < \varepsilon/2.$$

So for $j \geq J$, we have

$$\begin{aligned}\|T|_{X^{\otimes j}}\| &\leq \Big\|T|_{X^{\otimes j}} - \Big(\sum_{i=1}^n \theta_{\xi_i,\eta_i}\Big)|_{X^{\otimes j}}\Big\| + \Big\|\Big(\sum_{i=1}^n \theta_{\xi_i,\eta_i}\Big)|_{X^{\otimes j}}\Big\| \\ &\leq \Big\|T - \Big(\sum_{i=1}^n \theta_{\xi_i,\eta_i}\Big)\Big\| + \varepsilon/2 \\ &= \varepsilon.\end{aligned}$$

□

Proof of Theorem 2.46. Let I be the ideal of $\mathcal{T}_X$ generated by elements of the form $\pi(a) - \psi^{(1)}(\phi(a))$ where $a \in \phi^{-1}(\mathcal{K}(X))$. Then the C^*-algebra $\mathcal{O}_X := \mathcal{T}_X/I$, the linear map $j_X : x \mapsto i_X(x) + I$ and the homomorphism $j_A : a \mapsto i_A(a) + I$ have the desired universal property by definition.

To see that j_A is injective, let (ψ, π) be the Fock representation on $\mathcal{F}_X$. For $x, y \in X$ and $\xi \in \mathcal{F}_X$, we have $\psi^{(1)}(\theta_{x,y})\xi = l_x l_y^* \xi$ which is equal to zero if $\xi \in X^{\otimes 0}$, and is equal to $(\theta_{x,y} \otimes 1_{n-1})(\xi)$ if $\xi \in X^{\otimes n}$ with $n \geq 1$. In particular, $\psi^{(1)}(\phi(a))\xi = 0$ if $\xi \in X^{\otimes 0}$ and $\psi^{(1)}(\phi(a))\xi = a \cdot \xi = \pi(a)\xi$ if $\xi \in X^{\otimes n}$ for some $n \geq 1$. Hence

$$\big(\pi(a) - \psi^{(1)}(\phi(a))\big)\xi = \begin{cases} a \cdot \xi & \text{if } \xi \in X^{\otimes 0} \\ 0 & \text{otherwise.}\end{cases}$$

So, factoring $a = bc^*$ in A, and regarding b and c as elements of $X^{\otimes 0} \subseteq \mathcal{F}_X$, we have $\pi(a) - \psi^{(1)}(\phi(a)) = \theta_{b,c} \in \mathcal{K}(\mathcal{F}_X)$. On the other hand, Lemma 2.47 implies that each $\pi(a)|_{X^{\otimes n}}$ is nonzero. So $a \mapsto \pi(a)|_{X^{\otimes n}}$ is an injective C^*-homomorphism, and therefore isometric. That is $\|\pi(a)|_{X^{\otimes n}}\| = \|a\|$ for all n. Now Lemma 2.48 implies that $\pi(a) \notin \mathcal{K}(\mathcal{F}_X)$.

Let $Q : \mathcal{L}(\mathcal{F}_X) \to \mathcal{L}(\mathcal{F}_X)/\mathcal{K}(F_X)$ be the quotient map. We have just seen that $Q(\pi(a) - \psi^{(1)}(\phi(a))) = 0$ for all $a \in \phi^{-1}(\mathcal{K}(X))$, and so $(Q \circ \psi, Q \circ \pi)$ is a Cuntz-Pimsner covariant representation of X. We have also established that $Q \circ \pi$ is injective. Hence $(Q \circ \psi \times Q \circ \pi) \circ j_A$ is injective, which implies that j_A is injective.

For the final assertion, observe that (zi_X, i_A) is a Cuntz-Pimsner covariant representation in $\mathcal{O}_X$ for each z, and so induces a homomorphism $\gamma_z : \mathcal{O}_X \to \mathcal{O}_X$ satisfying the desired formula. We have $\gamma_z \circ \gamma_w = \gamma_{zw}$ on generators, and hence these two homomorphisms are equal. Since $\gamma_1 = \mathrm{id}$, it follows that $z \mapsto \gamma_z$ is a homomorphism of $\mathbb{T}$ into $\mathrm{Aut}(\mathcal{O}_X)$. The map $z \mapsto \gamma_z(\psi_\star(\xi)\psi_\star(\eta)^*)$ is clearly continuous for all $\xi, \eta \in \bigcup_n X^{\otimes n}$, and then an $\varepsilon/3$-argument shows that $z \mapsto \gamma_z(c)$ is continuous for $c \in \mathcal{O}_n$. □

Corollary 2.49. *Let $(A, \mathbb{Z}, \alpha)$ be a C^*-dynamical system, and suppose that A is unital. There is an isomorphism $\rho : A \times_\alpha \mathbb{Z} \cong \mathcal{O}_{_\alpha A}$ such that $\rho(i_A(a)) = j_A(a)$ and $\rho(i_\mathbb{Z}(1)) = j_{_\alpha A}(1_A)$.*

Proof. Let $U := j_{_\alpha A}(1_A)$. We have $UU^* = j_{_\alpha A}(1_A) j_{_\alpha A}(1_A)^* = j^{(1)}_{_\alpha A}(\theta_{1_A,1_A})$. The operator $\theta_{1_A,1_A}$ is the identity operator on ${}_\alpha A$, which is equal to $\alpha(1_A)$. Since α is the homomorphism that determines the left action on this module, we deduce that 1_A acts compactly on the module ${}_\alpha A$, and now Cuntz-Pimsner covariance gives

$$UU^* = j^{(1)}_{_\alpha A}(\phi(1_A)) = j_A(1_A) = 1_{\mathcal{O}_X}.$$

We also have

$$U^*U = j_{_\alpha A}(1_A)^* j_{_\alpha A}(1_A) = j_A(\langle 1_A, 1_A \rangle_A) = j_A(1_A^* 1_A) = 1_{\mathcal{O}_X}.$$

So U is a unitary. We calculate

$$U^* j_A(a) U = j_{_\alpha A}(1_A)^* j_A(a) j_{_\alpha A}(1_A) = j_A(\langle 1_A, a \cdot 1_A \rangle_A) = j_A(\langle 1_A, \alpha(a) 1_A \rangle_A) = j_A(\alpha(a)).$$

So, regarding U as a homomorphism $n \mapsto U^n$ of $\mathbb{Z}$ into $\mathcal{U}(\mathcal{O}_X)$, the pair (j_A, U) gives a covariant homomorphism of $(A, \mathbb{Z}, \alpha)$ in $\mathcal{O}_X$, and so induces a homomorphism $\rho = j_A \times U : A \times_\alpha \mathbb{Z} \to \mathcal{O}_X$. Theorem 2.46 shows that j_A is injective, and that there is an action γ of $\mathbb{T}$ on $\mathcal{O}_X$ such that $\gamma_z \circ j_A = j_A$ and $\gamma_z(U) = zU$. So Corollary 2.28 implies that ρ is injective.

To see that it is surjective, note that its image clearly contains $j_A(A)$, so we just have to show that it also contains $j_{_\alpha A}({}_\alpha A)$. For this, fix $x \in {}_\alpha A$; so $x = a \in A$. We can write x as $x = 1_A \cdot a$, where 1_A is regarded as an element of ${}_\alpha A$, and $a \in A$ is acting on the right. Hence

$$j_{_\alpha A}(x) = j_{_\alpha A}(1_{_\alpha A} \cdot a) = j_{_\alpha A}(1_{_\alpha A}) j_A(a) = U j_A(a) = \rho(i_\mathbb{Z}(1) i_A(a)).$$

So every generator of $\mathcal{O}_X$ belongs to the image of ρ, which implies that ρ is surjective. □

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